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DIFFERENT INDECOMPOSABILITY CONCEPTS FOR A VON NEUMANN TECHNOLOGY: A NOTE (*)

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This paper surveys some of the different indecomposability concepts for a von Neumann technology given in the literature, namely the concepts introduced by Gale (1956, 1960), Weil (1968) and Roemer (1980). Gale's concept was defined to give a sufficient condition for uniqueness of the equilibrium growth rate in a von Neumann growth model. Yet, Weil gave a weaker sufficient condition for this and he defined indecomposability accordingly without, however, relating his definition to Gale's definition. In fact, the peculiarity of Gale's definition is the requirement that all goods are necessary, directly or indirectly, as input for the production of any good. This, however, is not necessary for uniqueness of the equilibrium growth rate.

1. PRELIMINARIES

We consider the closed von Neumann model. There are n goods and m elementary processes. The n goods include fixed capital goods at their different stages of weak and tear. A commodity can serve both as a capital good and as a consumption good. There is constant returns to scale. Every good is an output of at least one process. Time is treated as discrete, $t = 1, 2, \ldots$. Output appears one period after the corre-

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sponding input is applied. The basic notations are as follows:

A:	$(n \times m)$ input matrix;
B:	$(n \times m)$ output matrix;
a_{ij}:	characteristic element of A;
b_{ij}:	characteristic element of B;
a_i, b_i:	characteristic row vectors of A and B respectively;
a^j, b^j:	characteristic column vectors of A and B respectively;
x:	intensity vector (an m-vector);
p:	price vector (an n-vector);
$(Bx)_i$:	i-th coordinate in the vector Bx;
α:	growth factor (1+ growth rate);
β:	interest factor (1+ interest rate) (¹);
α_0:	maximum growth factor;
β_0:	minimum interest factor;
N:	the set $\{1, \ldots, n\}$;
M:	the set $\{1, \ldots, m\}$.

As to the mathematical notations:

$x \geqq y$:	$x_i \geqq y_i,\ i = 1, \ldots, m$;
$x \geq y$:	$x \geqq y,\ x \neq y$;
$x > y$:	$x_i > y_i,\ i = 1, \ldots, m$;
xy:	the scalar product of x and y;
$\mathrm{supp}(x)$:	the set $\{i \mid x_i > 0\}$ (called the support of the nonnegative vector x).

The pair (A, B) of two nonnegative $(n \times m)$ matrices is called a *von Neumann technology* if they satisfy the following two assumptions:

Ass. 1: $a^j \geq 0,\ j = 1, \ldots, m$ (i.e., every process needs at least one good as input).

Ass. 2: $b_i \geq 0,\ i = 1, \ldots, n$ (i.e., every good is an output of some process).

Throughout this paper the matrices A and B are assumed to satisfy these assumptions.

(¹) β is sometimes in the literature called the profit factor.

We say that the vector x is an *intensity vector at the growth factor* $\alpha > 0$ if

$$Bx \geqq aAx\,, \qquad x \geq 0\,. \tag{1}$$

(1) is a necessary condition for balanced growth at α $(x_{t+1} = \alpha x_t,$ $t = 1, 2, \ldots)$, when the inputs for one period can only be supplied from the outputs from the preceding period. The *maximum growth factor*, denoted α_0, is the maximum α with respect to which an intensity vector exists, i.e.,

$$\alpha_0 = \max\{\alpha \in R \mid \exists x \geq 0\colon Bx \geqq \alpha Ax\}\,.$$

The vector p is called a *price vector at the interest factor* $\beta > 0$ *if*

$$pB \leqq \beta pA\,, \qquad p \geq 0\,. \tag{2}$$

(2) is a necessary condition for competitive equilibrium at β (as under constant returns to scale no process can in equilibrium give positive pure profits). The *minimum interest factor*, denoted β_0, is the minimum β with respect to which a price vector exists, i.e.,

$$\beta_0 = \min\{\beta \in R \mid \exists p \geq 0\colon pB \leqq \beta pA\}\,.$$

It is well known that the numbers α_0 and β_0 exist and satisfy the inequalities

$$0 < \beta_0 \leqq \alpha_0 \tag{3}$$

(see, e.g., Gale 1960).

We call the quadruplet (p, x, α, β) a *competitive equilibrium* if it satisfies (1) and (2) as well as the following two rules:

$$\begin{aligned} &p(B - \alpha A)x = 0 \qquad \text{(the Rule of Free Goods)}\,, \\ &p(B - \beta A)x = 0 \qquad \text{(the Rule of Unprofitable Processes)}\,. \end{aligned} \tag{4}$$

One easily sees that if the total value of output is positive, i.e.,

$$pBx > 0\,, \tag{5}$$

which is a natural condition for an economic system, then $\beta = \alpha$ in any competitive equilibrium for this model. For this reason and because the condition $\beta = \alpha$ (interest factor equal to growth factor) is in accordance with the Golden Rule, we are especially interested in von Neumann equilibria. A *von Neumann equilibrium* is a triplet (p, x, α) satisfying (1) and (2) with $\beta = \alpha$. Notice, that a von Neumann equilibrium necessarily satisfies the two rules in (4) and is thus a competitive equilibrium. The number α in a von Neumann equilibrium is called an *equilibrium growth factor*. An *economic von Neumann equilibrium* is a von Neumann equilibrium satisfying (5).

The price-interest pair (p, α) of a von Neumann equilibrium is said to *sustain* or *support* growth at the rate $\alpha - 1$ because it will permit continual growth at this rate. There will be no loss incurred in meeting the interest charge (assumed to be the only obstacle to investment) and there exists no other intensity vector which, by yielding a positive profit, would lure resources away from this growth path (²).

The von Neumann-KMT-Theorem (³): If the $(n \times m)$ matrices A and B satisfy Ass. 1 and Ass. 2, then:

(i) For any $\alpha \in [\beta_0, \alpha_0]$ there exists a von Neumann equilibrium.

(ii) For $\alpha = \beta_0$ and for $\alpha = \alpha_0$ there exists an economic von Neumann equilibrium.

(iii) There are at most $\min(n, m)$ different α's for which an economic von Neumann equilibrium exists.

Proof: See, e.g., Nikaido 1968, Morgenstern and Thompson 1976 (⁴).

A lemma which is many contexts is very useful is the following.

Lemma 1: If x is an intensity vector at α and p is a price vector at β, where $\beta < \alpha$, then for any $\gamma \in [\beta, \alpha]$ the triplet (p, x, γ) is a von Neumann equilibrium.

Proof: For $\gamma \in [\beta, \alpha]$ x satisfies (1) with $\alpha = \gamma$, and p satisfies (2) with $\beta = \gamma$. Q.E.D.

(²) Koopmanns (1964) and Morishima (1969) contain detailed introductions to the von Neumann model, viewing it in a broader economic context.

(³) « KMT » stands for Kemeny, Morgenstern and Thompson (cf. their fundamental paper from 1956).

(⁴) The same authors also prove that there exist *central* von Neumann solutions, i.e. solutions where a good has zero price *only if* the good is overproduced and where a process is not used *only if* it involves a loss.

2. DIFFERENT INDECOMPOSABILITY CONCEPTS

On certain conditions we have that $\alpha_0 = \beta_0$ so that the equilibrium growth factor, α, where $\alpha_0 \leqq \alpha \leqq \beta_0$, is *unique*. In Gale (1956, 1960) it was shown that a sufficient condition for uniqueness is that the technology is irreducible in the following sense:

Definition 1: Given a von Neumann technology (A, B) the nonempty set of goods $S \subset N$ is called an *independant subset* if there exists a set $T \subset M$ such that $a_{ij} = 0$ for $i \notin S$ and $j \in T$ and for all $i \in S$, $b_{ij} > 0$ for some $j \in T$. An independant subset $S \subset N$ is called *irreducible* if there is no proper independant subset contained in S. The technology is called *irreducible* if the total set of goods, N, is irreducible. Otherwise the technology is called *reducible*.

In economic terms: a proper subset of goods is independant if these goods can be produced without consuming (directly or indirectly) goods outside the set. If the technology is reducible we can reorder the columns of A and B and the rows of A and B in such a way that A and B take the form

$$A = \begin{array}{c} \\ S\{ \\ \\ \end{array} \overbrace{\quad}^{T} \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline 0 & A_{22} \end{array}\right] \qquad B = \begin{array}{c} \\ S\{ \\ \\ \end{array} \overbrace{\quad}^{T} \left[\begin{array}{c:c} B_{11} & B_{12} \\ \hdashline B_{21} & B_{22} \end{array}\right] \tag{6}$$

where the elements of the submatrix 0 are all zero and the rows of B_{11} are all semipositive. To say that the technology is irreducible is equivalent to saying that such a decomposition is not possible. Such impossibility derives from *all* goods being necessary (directly or indirectly) as input in order that the economy can sustain itself (see Proposition 4 below).

Although Gale's irreducibility is sufficient for $\alpha_0 = \beta_0$, Weil (1968, 1970) provided a slightly weaker condition, called indecomposability, which is also sufficient. This condition is related to the concept of a subtechnology.

Definition 2: Given a von Neumann technology (A, B) and the nonempty subsets $S \subset N$, $T \subset M$, the restriction of (A, B) to $S \times T$, i.e.

$(A, B|S, T)$, is called a *subtechnology* if $a_{ij} = b_{ij} = 0$ for $i \notin S$ and $j \in T$ and for all $i \in S$, $b_{ij} > 0$ for some $j \in T$. A subtechnology $(A, B|S, T)$ is a *proper subtechnology* if $S \neq N$ (and therefore $T \neq M$).

Existence of a proper subtechnology is illustrated in (7), where the rows of B_{11} are all semipotive.

$$A = S\left\{\begin{bmatrix} \overbrace{A_{11}}^{T} & A_{12} \\ 0 & A_{22} \end{bmatrix}\right. \qquad B = S\left\{\begin{bmatrix} \overbrace{B_{11}}^{T} & B_{12} \\ 0 & B_{22} \end{bmatrix}\right. \tag{7}$$

In economic terms the goods in S and the processes in T constitute a subtechnology if all goods used as input in these processes taken together are also produced by them and all goods produced by them belong to S. This means that a subtechnology is itself a von Neumann technology and as such can function technologically independently from the rest of the economy ([5]).

Definition 3: A subtechnology $(A, B|S, T)$ (or the original von Neumann technology) is called *indecomposable* if it contains no proper subtechnology (otherwise it is called *decomposable*). If furthermore for all $j \notin T$, $a_{ij} + b_{ij} > 0$ for some $i \notin S$, then we have a *maximum indecomposable subtechnology*. If a technology (A, B) can be decomposed into two disjoint subtechnologies, i.e. two subtechnologies, $(A, B|S_1, T_1)$ and $(A, B|S_2, T_2)$, where $S_1 \cap S_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$, and $T_1 \cup T_2 = M$, then (A, B) is called *completely decomposable*.

Thus an indecomposable subtechnology is maximum if it is not possible to include more of the existing elementary processes without rendering it decomposable. Furthermore, the decomposable technology (A, B)

([5]) Kemeny, Morgenstern and Thompson (1956) (KMT) showed that a von Neumann technology can always be decomposed into a finite number of subtechnologies, one for each growth factor, α, to which an economic von Neumann equilibrium exists. This can be done in such a way that each subtechnology has an economic von Neumann equilibrium $(\alpha, \bar{p}, \bar{x})$ which by adding the appropriate number of zero coordinates to $\bar{p}$ and $\bar{x}$ can be made identical to an economic von Neumann equilibrium for the original von Neumann technology. Such subtechnologies are by KMT called *subeconomies*. (Some goods or processes may occur in more than one subeconomy.) This « KMT-decomposition » is thus based not only on the zero-or-not-character of the matrix entries, but also on their actual sizes (cf. also Morgenstern and Thompson 1976, and Moeschlin 1974).

in (7) is completely decomposable if $A_{12} = B_{12} = 0$. It should also be noted that any von Neumann technology with only one good or only one process is, assording to Definition 3, indecomposable. Statements on indecomposable von Neumann technologies usually include this border case of a one-good or one-process technology, if Ass. 1 is satisfied. But if we, contrary to Ass. 1, had a one-good or one-process technology with $A = 0$, then assertions on indecomposable technologies would not necessarily apply. This is so because technologies with $A = 0$ have properties more akin to decomposable technologies in general than to indecomposable technologies.

Now, let us look at the inclusion relationship between irreducibility and indecomposability. Since $a_{ij} > 0$ implies $a_{ij} + b_{ij} > 0$, while the converse does not hold, we have:

Proposition 1: A von Neumann technology which is irreducible is also indecomposable, but the converse is not true.

For example the technology

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}, \tag{8}$$

is reducible, but still indecomposable (as $A + B > 0$).

Roemer's condition of indecomposability (Roemer 1980), which serves to secure price uniqueness, is even more restrictive than Gale's irreducibility. In order to avoid confusion with Weil's concept we will use the term « connected » when speaking of a technology which is indecomposable in the sense of Roemer.

Definition 4: A von Neumann technology (A, B) is *connected* at β_0 if all intensity vectors at β_0 use at least n processes where n is the total number of goods.

Since for $\alpha \geqq \beta_0$ $\{x \geq 0 | Bx \geqq \alpha Ax\} \subset \{x \geq 0 | Bx \geqq \beta_0 Ax\}$ connectedness at β_0 means that selfsustaining growth at any growth factor $\alpha \in [\beta_0, \alpha_0]$ requires that at least n processes are active. As Roemer (1980, p. 450) shows we have:

Proposition 2: A von Neumann technology which is connected at β_0 is also irreducible, but the converse is not true.

In the special case of the Leontief technology (where A is square and B is the identity matrix) all three concepts, irreducibility, indecomposability, and connectedness, reduce to the same, the ordinary indecomposability concept from the Frobenius theory on nonnegative square matrices.

Returning now to the example (8) above we see that although the technology is reducible, it has $\alpha_0 = \beta_0 = 3$ (6). That is, the growth factor consistent with von Neumann equilibrium is unique. Actually, as Weil (1970) observes, the following holds:

Proposition 3: A sufficient, though not necessary, condition for the equilibrium growth factor to be unique, i.e. $\alpha_0 = \beta_0$, is that the technology is indecomposable.

Proof: $0 < \beta_0 \leqq \alpha_0$; to prove sufficiency we need only show that $\beta_0 \geqq \alpha_0$. Let x be an intensity vector at α_0 and p a price vector at β_0; then $Bx \geqq \alpha_0 Ax$ and $pB \leqq \beta_0 pA$. Therefore

$$\beta_0 pAx \geqq pBx \geqq \alpha_0 pAx \,. \tag{9}$$

Now, as $Bx \geqq \alpha_0 Ax$, supp $(Ax) \subset$ supp (Bx), so that when the technology is indecomposable, $Bx > 0$ by Lemma 2 below. Since $p \geq 0$, it follows that $pBx > 0$ and therefore, from (9), $pAx > 0$, i.e. $\beta_0 \geqq \alpha_0$.

That indecomposability is not a necessary condition can be seen in the following way. Change the example (8) by putting $b_{21} = 0$. Then the technology is decomposable; but it still has $\alpha_0 = \beta_0 = 3$ (this example is taken from Weil 1970). Q.E.D.

Lemma 2: For an indecomposable von Neumann technology (A, B), if $x \geq 0$ and for this x supp $(Ax) \subset$ supp (Bx), then $Bx > 0$ (i.e., in self-sustaining growth all commodities are produced).

Proof (adapted from Gale 1960, p. 315): Let $S =$ supp (Bx). As $x \geq 0$, $Ax \geq 0$ according to Ass. 1. From supp $(Ax) \subset$ supp (Bx), therefore, $Bx \geq 0$ so that $S \neq \emptyset$. Let $T =$ supp (x) and $c_{ij} = a_{ij} + b_{ij}$; then $c_{ij} = 0$ for $i \notin S$, $j \in T$, because otherwise $c_i x > 0$, $b_i x = 0$, for $i \notin S$, and we could not have supp $(Ax) \subset$ supp (Bx). From indecomposability, then, $S = N$, i.e. $Bx > 0$. Q.E.D.

(6) As only one process (the second) need be active to attain growth at α_0, the technology is not connected.

Thus Gale's irreducibility concept is unnecessarily restrictive. The restrictiveness of the concept lies in the fact that it requires that *all* goods are necessary (directly or indirectly) as input for the production of any good. We may call input goods having this property *basic goods* (cf. Sraffa 1960). Formally:

Definition 5: Given a von Neumann technology (A, B) the i-th good is called a *basic good* if for any $x \geq 0$ such that supp $(Ax) \subset$ supp (Bx) we have $(Ax)_i > 0$. If the i-th good is not a basic good it is called a *non-basic good.*

Proposition 4: A von Neumann technology (A, B) is irreducible if and only if all goods are basic goods.

Proof: « Only if ». Let $x \geq 0$ be such that supp $(Ax) \subset$ supp (Bx). The assertion is: irreducibility implies $Ax > 0$. From Ass. 1 $Ax \geq 0$. The assertion is trivially true for $n = 1$. Assume $n > 1$. Letting $S =$ supp (Ax) we have $\emptyset \neq S \subset$ supp (Bx). Therefore, letting $T =$ supp (x) we have for all $i \in S$, $b_{ij} > 0$ for some $j \in T$. At the same time,

$$\text{if } i \notin S \,, \quad \text{then } (Ax)_i = \sum_{j=1}^{m} a_{ij} x_j = 0 \,,$$

i.e. $a_{ij} = 0$ for $i \notin S$, $j \in T$. But this contradicts irreducibility unless $S = N$, i.e. $Ax > 0$.

« If ». Assume the technology is reducible. Then there exist nonempty sets $S \subset N$, and $T \subset M$ such that $a_{ij} = 0$ for $i \notin S$, $j \in T$, and for all $i \in S$, $b_{ij} > 0$ for some $j \in T$. Thus goods not in S cannot be basic goods. Q.E.D.

It should be mentioned, that when considering von Neumann models *with consumption* one has that the growth rate compatible with a von Neumann equilibrium with a *positive* value of consumption is unique, whether or not the technology (A, B) is indecomposable (Fujimoto 1975, Morishima 1976). But the question of indecomposability or decomposability may, of course, be of importance in other contexts, e.g. when deciding which processes should be intensified to meet a given change in final demand.

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