Economic Growth, June 2012.

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A suggested solution to the problem set at the exam in Economic Growth, June 15, 2012

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (20 %)

The economy is described by:

$$Y_t = A_t^{\sigma} L_t^{\alpha} Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, \tag{1.1}$$

$$\dot{A}_t = \lambda A_t^{\varepsilon} L_t, \qquad \lambda > 0, 0 < \varepsilon < 1, \quad A_0 > 0 \text{ given}, \tag{1.2}$$

$$L_t = \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, \tag{1.3}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), and Z the amount of land (fixed). Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene; that is, (1.3) should be seen as a short-cut.

a) We consider a pre-industrial economy. (1.1) is an aggregate production function and indicates that the only production factors are labor and land (capital thus ignored). In accordance with the replication argument, there are CRS w.r.t. these rival inputs. The factor A_t^{σ} measures total factor productivity. In view of (1.2), the technology level, A_t, is rising over time. The increase in A_t per time unit is seen to be an increasing function of the size of the population. This reflects the hypothesis that population breeds ideas; these are non-rival and enter the pool of technical knowledge available for society as a whole. The rate per capita, λA^{ε}, by which population breeds ideas is an increasing function of the

[1] The solution below contains *more* details and more precision than can be expected at a three hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

already existing level of technical knowledge. This reflects the hypothesis that the larger is the stock of ideas, the easier do new ideas arise (perhaps by combination of existing ideas).

We are told that (1.3) is a short-cut description of a Malthusian population mechanism. Suppose the true mechanism is

$$\dot{L}_t = \beta(y_t - \bar{y})L_t, \qquad \beta > 0, \bar{y} > 0, \tag{1.4}$$

where $y_t \equiv Y_t/L_t$ is per capita income and $\bar{y}$ is subsistence minimum (so φ in (1.3) is just a parameter measuring the inverse of the subsistence minimum). A rise in y_t above $\bar{y}$ will lead to increases in L_t, thereby generating downward pressure on Y_t/L_t and perhaps end up pushing y_t below $\bar{y}$. When this happens, population will be decreasing for a while and so return towards its sustainable level, $Y_t/\bar{y}$.

The assumption (1.3) is thus a short-cut as if the population instantaneously adjusts to its sustainable level (as if $\beta \to \infty$). The model hereby gives a long-run picture, ignoring the Malthusian ups and downs in population and per capita income about the subsistence minimum. The important feature is that the technology level and thereby Y_t as well as the sustainable population will be rising over time. This speeds up the arrival of new ideas and so raises Y_t even faster.

b) By (1.3), $L_t = \varphi Y_t = \varphi A_t^{\sigma} L_t^{\alpha}$, normalizing the constant Z to 1. Consequently $L_t^{1-\alpha}$ $= \varphi A_t^{\sigma}$ so that

$$L_t = \varphi^{\frac{1}{1-\alpha}} A_t^{\frac{\sigma}{1-\alpha}}.$$

Substituting this into (1.2) gives

$$\dot{A}_t = \lambda \varphi^{\frac{1}{1-\alpha}} A_t^{\varepsilon + \frac{\sigma}{1-\alpha}} \equiv \hat{\lambda} A_t^{\varepsilon + \frac{\sigma}{1-\alpha}}. \tag{1.5}$$

c) We define $\mu \equiv \varepsilon + \frac{\sigma}{1-\alpha}$ and assume $\mu > 1$. Then (1.5) can be written

$$\dot{A}_t = \hat{\lambda} A_t^{\mu},$$

which is a nonlinear differential equation in A. Let $x \equiv A^{1-\mu}$. Then

$$\dot{x}_t = (1-\mu)A_t^{-\mu}\hat{\lambda}A_t^{\mu} = (1-\mu)\hat{\lambda}, \tag{1.6}$$

a constant. To find x_t from this, we only need simple integration (see below). The hint to 1c) is unfortunately misleading since the formula offered is only valid for the case of a linear differential equation with constant coefficients where the coefficient to x_t *differs* from zero. In (1.6) the coefficient to x_t is zero. Hence the offered formula is useless

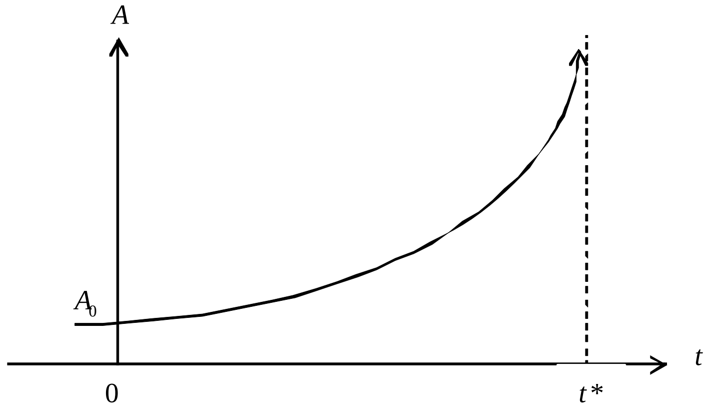

Figure 1.1:

and thereby misleading. In the grading this will be taken into account since it may have delayed some examinees' problem solving. In addition all examinees are offered the option to participate at the re-exam in Economic Growth August 15, 2012, including the option to choose the best grade among the two then obtained.

We now integrate (1.6):

$$x_t = x_0 + \int_0^t \dot{x}_\tau d\tau = x_0 + (1-\mu)\hat{\lambda}t.$$

As $A = x^{\frac{1}{1-\mu}}$ and $x_0 = A_0^{1-\mu}$, this implies

$$A_t = x_t^{\frac{1}{1-\mu}} = \left[A_0^{1-\mu} + (1-\mu)\hat{\lambda}t\right]^{\frac{1}{1-\mu}} = \frac{1}{\left[A_0^{1-\mu} + (1-\mu)\hat{\lambda}t\right]^{\frac{1}{\mu-1}}}. \tag{1.7}$$

d) Yes, we can!

Recall that $\mu > 1$. Although to begin with, A_t may grow extremely slowly, the growth in A_t will be accelerating because of the positive feedback (visible in (1.2)) from both rising population and rising A_t. Indeed, since $\mu > 1$, the denominator in (1.7) will be decreasing over time and approach zero in finite time, namely as t approaches the finite value $t^* = A_0^{1-\mu}/((\mu-1)\hat{\lambda})$. Fig. 1.1 illustrates.

It follows from (1.1) that explosive growth in A implies explosive growth in Y. The acceleration in the evolution of Y will sooner or later make Y move fast enough so that the Malthusian population mechanism (which for biological reasons has to be slow) can not catch up. Then, what was earlier only a transitory excess of y_t over $\bar{y}$ becomes a permanent excess and takes the form of sustained growth in y_t. As economic history has testified and as there are economic theories to explain (higher opportunity costs of raising children, skill-biased technical change, the quality-quantity trade-off, improved contraception technology, etc.), this resulted in the demographic transition with fertility declining faster than mortality.

So the story told by the model (that is, the story told by Michael Kremer, 1993) is the following. When $\mu > 1$, the Malthusian regime has to come to an end because the battle between scarcity of land (or natural resources more generally) and technological progress will inevitably be won by the latter.

2. Solution to Problem 2 (50 %)

For convenience, key equations are repeated here:

$$\dot{K}_t = Y_t - c_t L - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ is given.} \tag{2.1}$$

The production function of firm i is

$$Y_{it} = F(K_{it}, A_t L_{it}), \tag{2.2}$$

where F is neoclassical and has CRS. The variable A_t evolves according to

$$A_t = K_t, \tag{2.3}$$

where $K_t = \sum_i K_{it}$.

a) A_t can be interpreted as the economy-wide technology level. According to the general hypothesis of *learning-by-investing* the economy-wide technology level is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$A_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{2.4}$$

where I_s^n is aggregate net investment and λ is the "learning parameter".

The idea is that investment – the production of capital goods – as an unintended *by-product* results in *experience* or what we may alternatively call on-the-job *learning.* This adds to the knowledge about how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive and better satisfy the needs of the users. The learning is assumed to benefit essentially all firms in the economy. There are knowledge spillovers across firms and these spillovers are reasonably fast relative to the time horizon relevant for growth theory.

b) We suppress the time index when not needed for clarity. Consider firm i. We are told that each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables. Hence maximization of profits, $\Pi_i = F(K_i, AL_i) - (r + \delta)K_i - wL_i$, leads to the first-order conditions

$$\begin{aligned} \partial \Pi_i / \partial K_i &= F_1(K_i, AL_i) - (r + \delta) = 0, \\ \partial \Pi_i / \partial L_i &= F_2(K_i, AL_i)A - w = 0. \end{aligned} \tag{2.5}$$

CRS implies that F is homogeneous of degree one. Hence, by Euler's theorem, the first-order partial derivatives, F_1 and F_2, are homogeneous of degree 0. Thus, we can write (2.5) as

$$F_1(k_i, A) = r + \delta, \tag{2.6}$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (2.6) determines k_i uniquely. From this follows that the chosen k_i will be the same for all firms, say $\bar{k}$. In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $K = \sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital intensity, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{2.7}$$

As a consequence we can use (2.6) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, A_t) - \delta = F_1(k_t, K_t) - \delta = F_1(1, L) - \delta \equiv \bar{r}, \tag{2.8}$$

where we have inserted (2.3) and used homogeneity of degree zero once again.

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, A) L_i = \sum_i F(k, A) L_i \quad \text{(by (2.2) and (2.7))} \\ &= F(k, A) \sum_i L_i = F(k, A)L = F(K, AL) = F(K, KL) = F(1, L)K, \quad \text{(by (2.3))}, \end{aligned}$$

where we have several times used that F is homogeneous of degree one.

c) The representative household faces the flow budget identity

$$\dot{a}_t = \bar{r} a_t + w_t - c_t, \qquad a_0 \text{ given},$$

and the NPG condition

$$\lim_{t \to \infty} a_t e^{-\bar{r}t} \geq 0,$$

where a_t is per capita financial wealth. Given the specified preferences, the resulting consumption-saving plan implies that per capita consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\bar{r} - \rho) \equiv g,$$

and the transversality condition that the NPG condition is satisfied with strict equality. To ensure a positive growth rate we need that $\bar{r} > \rho$, i.e.,

$$F_1(1, L) - \delta > \rho. \tag{A1}$$

We are informed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A2}$$

This ensures boundedness of discounted utility so that the optimization problem of the household does indeed have a solution.

d) In our closed economy without natural resources and government debt, we have $a_t = k_t$. By proportionality between Y and K and constancy of the interest rate (cf. b)) follows that the model is a reduced-form AK Ramsey model. We know that in such models there will be no transitional dynamics; the transversality condition ensures that k ($\equiv K/L$) from the beginning will grow at the same rate as c, that is, the rate g. And so will y ($\equiv Y/L$) in view of $y = F(1, L)k$.

e) In this model, if L were rising, also the interest rate, hence also g, would be rising over time. This follows from (2.8) since for a two-factor neoclassical CRS production function F we always have $F_{12}(1, L) > 0$ (complementarity between K and L). Rising L thus implies that the total marginal productivity of capital will be rising over time. Note, however, that there is no growth "explosion" in the sense of infinite output in finite time. That phenomenon only arises if the exponent, λ, on the state variable, here K, is *larger* than one - as is the situation in (1.5) (where the state variable is A, not K, however).

f) The social planner faces the aggregate production function $Y_t = F(1, L)K_t$ or, in per capita terms, $y_t = F(1, L)k_t$. The social planner's problem is to choose $(c_t)_{=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= F(1, L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, && (2.9) \\ k_t &\geq 0 \text{ for all } t > 0. && (2.10) \end{aligned}$$

The current-value Hamiltonian is

$$H(k,c,\eta,t) = \frac{c^{1-\theta}}{1-\theta} + \eta\left(F(1,L)k - c - \delta k\right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \tag{2.11}$$

$$\frac{\partial H}{\partial k} = \eta(F(1,L) - \delta) = -\dot{\eta} + \rho\eta. \tag{2.12}$$

We guess that also the transversality condition,

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \tag{2.13}$$

must be satisfied by an optimal solution. This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really a solution.

Log-differentiating w.r.t. t in (2.11) and combining with (2.12) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1,L) - \delta - \rho) \equiv g_{SP}. \tag{2.14}$$

By Euler's theorem $F(1,L) = F_1(1,L) + F_2(1,L)L > F_1(1,L)$. It follows that $g_{SP} > g$. The interpretation is that the social planner internalizes the economy-wide learning effect associated with capital investment. To ensure bounded intertemporal utility we sharpen (A2) to

$$\rho > (1-\theta)g_{SP}. \tag{A2'}$$

By a reasoning similar to the one at d), the transversality condition together with the reduced-form AK structure implies that also in the candidate SP solution does k_t and y_t already from the beginning grow at the same rate as c_t. Moreover, by derivation or directly from the hint to g), we have that $c_0 = (F(1,L) - \delta - g_{SP})k_0$.

Hereby we have completely characterized a candidate solution to the present resource allocation problem. The next step is to argue that this candidate solution does indeed satisfy a set of *sufficient* conditions for an optimal solution. It is enough here to just refer to *Mangasarian's sufficiency theorem.*

We now return to the market economy. Given the investment subsidy, σ, the firms' capital costs are $(1-\sigma)(r+\delta)$ per unit of capital per time unit. Firm i then chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-\sigma)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-\sigma)(r+\delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-\sigma)(r+\delta).$$

Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k,K)}{1-\sigma} - \delta = \frac{F_1(1,L)}{1-\sigma} - \delta, \tag{2.15}$$

again using Euler's theorem.

It follows that σ should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1,L) - \delta$. If we can obtain $r = F(1,L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required σ thus satisfies

$$r = \frac{F_1(1,L)}{1-\sigma} - \delta = F(1,L) - \delta,$$

so that

$$\sigma = 1 - \frac{F_1(1,L)}{F(1,L)} = \frac{F(1,L) - F_1(1,L)}{F(1,L)} = \frac{F_2(1,L)L}{F(1,L)}.$$

g) With consumption taxation the tax revenue will be τcL, which with a balanced budget must equal the *required* tax revenue:

$$\tau cL = \sigma(r+\delta)K = \left(F(1,L) - F_1(1,L)\right)K.$$

Thus, the required consumption tax rate is

$$\tau = \frac{F(1,L) - F_1(1,L)}{c/k} = \frac{F(1,L) - F_1(1,L)}{F(1,L) - \delta - \gamma_{SP}} = \frac{F_2(1,L)L}{F(1,L) - \delta - \gamma_{SP}} > 0, \tag{2.16}$$

where we have used the hint. Substituting (2.14) into (2.16), the solution for τ can be written

$$\tau = \frac{\theta\left[F(1,L) - F_1(1,L)\right]}{(\theta-1)(F(1,L)-\delta)+\rho} = \frac{\theta F_2(1,L)L}{(\theta-1)(F(1,L)-\delta)+\rho}.$$

h) The required tax rate on consumption is thus a constant. Since there is no utility from leisure, the consumption tax therefore does not distort the consumption/saving decision on the margin. The resulting resource allocation is seen to replicate the SP solution from f). It follows that the allocation obtained by this subsidy-tax policy *is* optimal.

i) An alternative policy would be to pay firms a production subsidy $s > 0$ so that if firm i produces and sells Y_i, its revenue is $(1+s)Y_i$. Another alternative would be to pay households a subsidy, ξ, to raise the rate of return on households' saving from r_t to $(1+\xi)r_t$. Such subsidy policies can do the job if s and ξ, respectively, have the optimal size and the consumption tax rate is adjusted so as to provide the needed tax revenue in the new situation.

An alternative non-distortionary tax in this model would be a labor income tax (since labor supply is inelastic).

j) Compared with the Solow growth model, a strength of the present model is that it endogenizes technological change. More, the learning-by-investing hypothesis (in the form $0 < \lambda < 1$) has some support in the data.

Among the drawbacks are the following. By relying on the case $\lambda = 1$, the model assumes a much stronger and permanent learning effect than supported by the data. Moreover, the case $\lambda = 1$ is a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which *apriori* can take any value within an interval. Second, the imposed value leads to non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system.

An additional drawback is that if a growing population is included in the model, counter-factual growing growth rates appear, cf. e) above.

The model should just be seen as a simple abstract example of endogenous growth.

3. Solution to Problem 3 (30 %)

a) The suggested catching-up hypothesis is that

$$\frac{\dot{A}_t}{A_t} = \xi \frac{\tilde{A}_t}{A_t}, \qquad \xi > 0,$$

and

$$\xi = \varphi(h), \qquad \varphi' > 0,$$

where A_t is the technology level of the considered country, $\tilde{A}_t$ is the world frontier technology level, and h is the average human capital, h, in the country.

A general health improvement in the country may in the longer run raise h and thereby the catching-up ability ξ. This may be because a health improvement tends to raise life expectancy and thereby the incentive to invest in human capital. Or it could be due to the fact that undernourishment in the embryonic stage may impede the cognitive abilities of the child.

b) Yes. Poor countries differ from rich countries in several factors that are complementary to physical capital, for instance the technology level and the human capital level.

Consider a set of countries, $j = 1, 2, ..., N$. Country j has the aggregate production function

$$Y_j = F(K_j, A_j h_j L_j) = A_j h_j L_j F(\frac{K_j}{A_j h_j L_j}, 1) \equiv A_j h_j L_j f(\tilde{k}_j), \quad f' > 0, f'' < 0,$$

where F is neoclassical with CRS (standard notation). Let r_j denote the equilibrium net rate of return on capital in country j. Then, under perfect competition,

$$r_j = \frac{\partial Y_j}{\partial K_j} - \delta = f(\tilde{k}_j) - \delta.$$

Can the countries have the same r in spite of widely differing K_j/L_j? Yes. Differing K_j/L_j does not rule out that $\tilde{k}_j = \tilde{k}$ for all j. Indeed, as

$$\tilde{k}_j \equiv \frac{K_j}{A_j h_j L_j} = \frac{K_j/L_j}{A_j h_j},$$

and as countries with low K_j/L_j (the poor countries) also tend to have low A_j and h_j, the $\tilde{k}_j$'s - and therefore also the r_j's - may be more or less of the same size.

c) In the Acemoglu textbook there is only one model where long-run productivity growth is driven by a combination of physical and human capital accumulation, namely the reduced-form AK model with physical and human capital presented in Chapter 11.2.[2] The aggregate production function of the model is

$$Y_t = F(K_t, H_t) \equiv F(K_t, h_t L_t), \tag{3.1}$$

where F is neoclassical with CRS (standard notation).

Whether such an aggregate production function can be defended depends on how human capital is assumed to be formed. The model in question assumes that human capital is formed in a way similar to physical capital. Ignoring the explicit timing, the model assumes that

$$\begin{aligned} Y &= C + I_K + I_H, \\ \dot{K} &= I_K - \delta_K K, \qquad \delta_K > 0, \\ \dot{H} &= I_H - \delta_H H, \qquad \delta_H > 0, \end{aligned} \tag{3.2}$$

[2] The MRW (1992) model mentioned in several of Acemoglu's chapters is not a good example, since in the MRW model long-run per capita growth is not driven by a combination of physical and human capital accumulation, but by exogenous technical progress.

where I_K and I_H denote gross investment in physical and human capital, respectively.

One problem is, however, that there is neither theoretical nor empirical reason to believe that the exponent on h in (3.1) should be exactly 1 when human capital is formed as in (3.2). The replication argument is of no help here. And from an empirical point of view, it is difficult to find support for (3.1) when the "stuff' called human capital is formed as in (3.2). With such a "production function" for human capital, according to the Mankiw, Romer and Weil (1992) study the exponent on h should be close to $\frac{1}{2}$.

The model ends up in a fully endogenous growth model which after an initial adjustment process features positive balanced per capita growth forever in a way quite similar to the simple AK model. This occurs without any improvements in technology. This seems implausible.

As also mentioned by Acemoglu, a small change in income taxation will in this model imply a large and permanent change in the per capita growth rate. Also implausible.

Yet, a *strength* of the model *relative to* the simple AK model is, of course, that it is somewhat less extreme since it maintains diminishing marginal productivity of each kind of capital separately and does not imply zero labor income. Indeed, the model is consistent with Kaldor's stylized facts.

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