Chapter 5

Growth accounting and the concept of TFP: Some limitations

5.1 Introduction

This chapter addresses the concepts of Total Factor Productivity, TFP, and TFP *growth*.[1] We underline the distinction between descriptive accounting and causal analysis. The chapter ends up with a warning regarding careless use of the concept of TFP growth in cross-country comparisons − and a suggested alternative approach.

For convenience, we treat time as continuous (although the timing of the variables is indicated merely by a subscript).

5.2 TFP growth and TFP level

Let Y_t denote aggregate output, in the sense of value added in fixed prices, at time t in a sector or the economy as a whole. Suppose Y_t is determined via the function

$$Y_t = F(K_t, H_t, t), \tag{5.1}$$

where K_t is an index of the physical capital input and H_t an index of quality-adjusted labor input. Natural resources (land, oil wells, coal in the ground, etc.) constitute a third primary production factor. The role of this factor is in growth accounting often subsumed under K.

[1] I thank Niklas Brønager for useful discussions.

The "quality-adjustment" of the input of labor (man-hours per year) aims at taking different levels of education and work experience into account. The heterogeneity of both types of input, and of output as well, implies huge measurement and conceptual difficulties. Here we ignore these problems. The third argument in (5.1) is time, t, indicating that the production function $F(\cdot\,,\cdot\,,t)$ is time-dependent. This is to open up for "shifts in the production function", due to new technology. We assume F is a neoclassical production function. When the partial derivative of F w.r.t. the third argument is positive, i.e., $\partial F/\partial t > 0$, technical change amounts to technical *progress*.[2]

To simplify, we shall here address TFP and TFP growth without taking the heterogeneity of the labor input into account. So we just count delivered work hours per time unit. Then (5.1) is reduced to the simpler case,

$$Y_t = F(K_t, L_t, t), \tag{5.2}$$

where L_t is the number of man-hours per year. As to measurement of K_t, some adaptation of the *perpetual inventory method*[3] is typically used, with some correction for under-estimated quality improvements of investment goods in national income accounting. Similarly, the output measure is (or at least should be) corrected for under-estimated quality improvements of consumption goods.

The notion of Total Factor Productivity at time t, TFP_t, is intended to indicate the *level* of productivity of the joint input (K_t, L_t). Generally, *productivity* of a given input is defined as the output per time unit divided by this input per time unit. So, considering (5.2), (average) labor productivity is simply Y_t/L_t. The concept of Total Factor Productivity is more complex, however, because it does not refer to a single input, but to a combination of several distinct inputs, in the present case two. And these distinct inputs may over time change their quantitative interrelationship, here the *ratio* K_t/L_t. It is then not obvious what can be meant by the "productivity" of the vector (K_t, L_t).

It is common in the literature to circumvent the problem of a direct definition of the PTF level and instead go straight away to a *decomposition* of output growth and on this basis define TFP *growth*. This is the approach we also follow here.

[2] Sometimes in growth accounting the left-hand side variable, Y, in (5.2) is the gross product rather than value added. Then non-durable intermediate inputs should be taken into account as a third production factor and enter as an additional argument of $\tilde{F}$ in (5.2). Since non-market production is difficult to measure, the government sector is often excluded from Y in (5.2). An alternative name in the literature for "total factor productivity" is *"multifactor productivity"*, abbreviated MFP.

[3] Cf. Section 2.2 in Chapter 2.

5.2.1 TFP growth

Let the growth rate of a variable Z at time t be written g_{Zt} (could also be written with a comma, as $g_{Z,t}$, but to save notation, we skip the comma unless needed for clarity). Take the total derivative w.r.t. t in (5.2) to get

$$\dot{Y}_t = F_K(K_t, L_t, t)\dot{K}_t + F_L(K_t, L_t, t)\dot{L}_t + F_t(K_t, L_t, t) \cdot 1.$$

Dividing through by Y_t gives

$$\begin{aligned} g_{Yt} &\equiv \frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left[F_K(K_t, L_t, t)\dot{K}_t + F_L(K_t, L_t, t)\dot{L}_t + F_t(K_t, L_t, t) \cdot 1\right] \\ &= \frac{K_t F_K(K_t, L_t, t)}{Y_t} g_{Kt} + \frac{L_t F_L(K_t, L_t, t)}{Y_t} g_{Lt} + \frac{F_t(K_t, L_t, t)}{Y_t} \\ &\equiv \varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt} + \frac{F_t(K_t, L_t, t)}{Y_t}, \end{aligned} \tag{5.3}$$

where ε_{Kt} and ε_{Lt} are shorthands for $\varepsilon_K(K_t, L_t, t) \equiv \frac{K_t F_K(K_t, L_t, t)}{F(K_t, L_t, t)}$ and $\varepsilon_L(K_t, L_t, t) \equiv \frac{L_t F_L(K_t, L_t, t)}{F(K_t, L_t, t)}$, respectively, that is, the partial output elasticities w.r.t. the two production factors, evaluated at the factor combination (K_t, L_t) at time t. Finally, $F_t(K_t, L_t, t) \equiv \partial F/\partial t$, that is, the partial derivative w.r.t. the third argument of the function F, evaluated at the point (K_t, L_t, t).

The equation (5.3) is the basic *growth-accounting relation*, showing how the output growth rate can be decomposed into the "contribution" from growth in each of the inputs and a *residual,* $F_t(K_t, L_t, t)/Y_t$, which is not directly measurable. The equation was introduced already by Solow (1957), and the residual became known as the *Solow residual.* We have:

$$\text{Solow residual } \equiv g_{Yt} - (\varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt}) = \frac{F_t(K_t, L_t, t)}{Y_t}, \tag{5.4}$$

The Solow residual thus indicates what is left when from the output growth rate is subtracted the contribution from growth in the factor inputs weighted by the output elasticities w.r.t. these inputs. In brief:

> The Solow residual at time t reveals that part of time-t output growth which is *not attributable* to time-t growth in the factor inputs.

How can the Solow residual be calculated on the basis of empirical data? The output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}, will, under perfect competition and absence of externalities in equilibrium equal the income shares of capital and labor, respectively. Time series for these income shares and for Y, K, and L, hence also for g_{Yt}, g_{Kt}, and g_{Lt}, can be obtained (directly or with some adaptation) from national income accounts. This allows an indirect measurement of the residual in (5.4). Of course, data are in discrete time. So to make the calculations in practice, we have to translate (5.4) into discrete time. The weights ε_{Kt} and ε_{Lt} can then be quantified as two-years moving averages of the output elasticities w.r.t. capital and labor, respectively, and thus approximated by the respective factor income shares.[4]

It is not uncommon to *identify* the TFP growth rate with the Solow residual. This is unfortunate since, being a residual, the calculated Solow residual may reflect the contribution of many things. Some of these are what we want to measure, like effects of current technical innovation in a broad sense including organizational improvement. But, as Solow himself was quick to point out, the calculated Solow residual may also reflect the influence of other factors like absence of perfect competition, varying capacity utilization, labor hoarding during downturns, measurement errors, and aggregation bias.

Nevertheless, let us assume we have been able to control for these other factors by extraction of the business cycle elements in the data.[5] So we are ready to replace "Solow residual" in (5.4) with TFP growth rate and write

$$g_{\text{TFP}t} = g_{Yt} - (\varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt}) = \frac{F_t(K_t, L_t, t)}{Y_t}. \tag{5.5}$$

Interpretation:

> The TFP growth rate at time t reveals the contribution to time-t output growth from time-t technical change (in a broad sense including learning by doing and organizational improvement).

Let y_t denote output per unit of labor, i.e., $Y_t \equiv y_t L_t$, and let k_t denote capital per unit of labor, i.e., $K_t \equiv k_t L_t$. Then, $g_{Yt} = g_{yt} + g_{Lt}$ and $g_{Kt} = g_{kt} + g_{Lt}$. Under constant returns to scale (CRS), we have $\varepsilon_{Lt} = 1 - \varepsilon_{Kt}$. Hence, under CRS, (5.5) can be written

$$\begin{aligned} g_{\text{TFP}t} &= g_{yt} + g_{Lt} - (\varepsilon_{Kt}(g_{kt} + g_{Lt}) + (1 - \varepsilon_{Kt}) g_{Lt}) \\ &= g_{yt} - \varepsilon_{Kt} g_{kt}. \end{aligned} \tag{5.6}$$

Under CRS, the TFP growth rate at time t thus reveals, under CRS, that part of time-t labor productivity growth which is *not attributable* to time-t growth in the capital-labor ratio. Interpretation:

[4] See, e.g., Acemoglu (2009, p. 79).

[5] Solow (1957) adjusted his capital data by assuming that idle capital as a fraction of total capital was the same as the rate of unemployment.

Under CRS, the TFP growth rate at time t reveals the contribution to time-t labor productivity growth from time-t technical change (in a broad sense including learning by doing and organizational improvement).

So far we have only addressed the instantaneous Solow residual and the instantaneous TFP growth rate. To get measures of interest for growth analysis, one needs to consider these things over long time intervals, preferably more than a decade. We come back to this aspect at the end of the next sub-section.

5.2.2 The TFP level

Let us see what can be said about the *level* of TFP, that "something" for which we have calculated a growth rate without having defined what it actually is.[6]

Suppose we know the instantaneous growth rate, $g(t)$, of a variable, $x(t)$, over the time interval $[0, T]$, i.e.,

$$\frac{dx(t)/dt}{x(t)} = g(t) \quad \text{for} \quad t \in [0, T]. \tag{5.7}$$

This makes up a simple linear differential equation in x, usually written in the form $dx(t)/dt = g(t)x(t)$. For a given initial value, $x(0)$, the solution is

$$x(t) = x(0)e^{\int_0^t g(\tau)d\tau}. \tag{5.8}$$

This formula applies to TFP as well. Suppose we for all t in the interval $[0, T]$ have calculated the growth rate of TFP. Then, in (5.7) we can replace $x(t)$ by TFP_t and $g(t)$ by $g_{\text{TFP}t}$. Applying the solution formula (5.8), we get

$$\text{TFP}_t = \text{TFP}_0 e^{\int_0^t g_{\text{TFP}\tau} d\tau}. \tag{5.9}$$

For a given initial value $\text{TFP}_0 > 0$, the *level* of TFP at any time t within the given time interval $[0, T]$ is determined by the right-hand side of (5.9). Considering discrete time and interpreting $g_{\text{TFP}\tau}$ as one-period growth rates, we similarly have

$$\text{TFP}_t = \text{TFP}_0(1 + g_{\text{TFP}0})(1 + g_{\text{TFP}1}) \dots (1 + g_{\text{TFP}t-1}). \tag{5.10}$$

These two formulas at least give us an overall growth factor for TFP from time 0 to time t:

[6] It happens that authors make no clear *terminological* distinction between TFP *level* and TFP *growth*, denoting both just "TFP". That is bound to cause confusion.

> The TFP level at time t relative to that at time 0 reveals the cumulative "*direct* contribution" to output growth since time 0 from technical change since time 0.

Why do we say "*direct* contribution"? The reason is that the cumulative technical change since time 0 may also have an *indirect* effect on output growth, namely via affecting the output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}. Through this channel cumulative technical change affects the weights attached to the growth of inputs before the residual is obtained. This possible indirect effect over time of technical change is not included in the concept of TFP growth.

Anyway, suppose we are interested in the *average* annual TFP growth rate calculated on data for, say, T years. Then we may normalize TFP_0 to equal 1 and on the basis of (5.10) calculate TFP_t. Next, we look for a $\bar{g}_{\text{TFP}}$ satisfying the equation

$$\text{TFP}_t = 1 \cdot (1 + \bar{g}_{\text{TFP}})^T.$$

The solution for $\bar{g}_{\text{TFP}}$ is

$$\bar{g}_{\text{TFP}} = \text{antilog}\left(\frac{\log \text{TFP}_T}{T}\right) - 1.$$

This is the annual compound TFP growth rate from year 0 to year T, using discrete compounding. If we want the annual compound TFP growth rate from year 0 to year T, using continuous compounding, we consider (5.9) with $t = T$, and solve the equation

$$\text{TFP}_T = 1 \cdot e^{\hat{g}_{\text{TFP}} \cdot T},$$

which gives

$$\hat{g}_{\text{TFP}} = \frac{\log \text{TFP}_T}{T}.$$

Because continuous compounding is more powerful, for a given terminal value of TFP, we will get $\hat{g}_{\text{TFP}} < \bar{g}_{\text{TFP}}$ (whenever $\bar{g}_{\text{TFP}} \neq 0$), but the difference will be negligible (since $\log(1 + x) \lessapprox x$ for x "small", where "$\lessapprox$" means "close to", but "less than" unless $x = 0$).

Jones and Vollrath (2013, p. 47) present growth accounting results for the US 1948-2010, exposing, among other things, the "productivity slowdown" that occurred after 1973. Growth accounting results for Denmark and other countries, 1981-2006, are reported in De økonomiske Råd (2010).

Before proceeding, we note that some analysts take a quick approach to growth accounting and assume beforehand that the output elasticities

ε_{Kt} and ε_{Lt} are constant over time apart from small random disturbances. This could be because the economy is assumed to be in steady state or the aggregate production is assumed to be Cobb-Douglas, usually with the addition of CRS. In the latter case, $y_t = B_t k_t^{\alpha}$, $0 < \alpha < 1$, and (5.6) gives $g_{\text{TFP}t} = g_{yt} - \alpha g_{kt} = g_{Bt}$. Then, under balanced growth with $g_{yt} = g_{kt} = g$, we have

$$g_{\text{TFP}t} = (1 - \alpha)g \quad \text{for all } t. \tag{5.11}$$

Besides exposing a simple way of measuring TFP growth (under certain conditions), this formula may serve as a prelude to the following reminder about how *not* to interpret growth accounting.

5.2.3 Accounting versus causality

Sometimes people interpret growth accounting as telling how much of output growth is *explained* by technical change and how much is explained by the contribution from factor growth. Such an identification of a descriptive accounting relationship with deeper causality is misleading. Without a complete dynamic model it makes no sense to talk about "explanation" and "causality".

The result (5.11) illustrates this. On the one hand, one finds from growth accounting a TFP growth rate equal to $(1-\alpha)g$, while the remainder, αg, of labor productivity growth is attributed to growth in the capital-labor ratio. On the other hand, if for instance a Solow growth model is the theoretical framework within which the variables are assumed generated, then g will be the exogenous rate of labor-aumenting technical progress which determines *both* $g_{\text{TFP}t}$ and $g_{kt} = \alpha g$. Here the TFP growth rate understates the "contribution" of technical change to productivity growth by a factor $1 - \alpha$. The *whole* of g_y is determined – explained – by the assumed rate, g, of exogenous technical progress. If g were nil, we would have $g_{kt} = 0$ as well as $g_{\text{TFP}t} = 0$.

Or suppose the theoretical framework within which the variables are assumed generated is the Arrow model of learning by investing.[7] Then it is the interaction between endogenous learning and endogenous investment that explains both g_{kt}, g, and $g_{\text{TFP}t}$. There is no one-way causal link involved. There is a mutual relationship between learning and investment, one presupposes the other. It is like "which comes first, the chicken or the egg?".

Let us now return to the intricate question what TFP actually measures in economic terms. We start with a convenient special case.

[7] Arrow (1962). The model is outlined in Chapter 12 below.

5.3 The case of Hicks-neutrality*

In the case of Hicks neutrality, by definition, technical change can be represented by the evolution of a one-dimensional variable, B_t, and the production function in (5.2) can be specified as

$$Y_t = F(K_t, L_t, t) = B_t \bar{F}(K_t, L_t). \tag{5.12}$$

Here the TFP level is at any time, t, identical to the level of B_t if we normalize the initial values of both B and TFP to be the same, i.e., $\text{TFP}_0 = B_0 > 0$. Indeed, calculating the TFP growth rate implied by (5.12) gives

$$g_{\text{TFP}t} = \frac{F_t(K_t, L_t, t)}{Y_t} = \frac{\dot{B}_t \bar{F}(K_t, L_t)}{B_t \bar{F}(K_t, L_t)} = \frac{\dot{B}_t}{B_t} \equiv g_{Bt}, \tag{5.13}$$

where the second equality comes from the fact that K_t and L_t are kept fixed when the *partial* derivative of F w.r.t. t is calculated. The formula (5.9) now gives

$$\text{TFP}_t = B_0 \cdot e^{\int_0^t g_{B\tau} d\tau} = B_t.$$

The convenient feature of Hicks neutrality is thus that we can write

$$\text{TFP}_t = \frac{F(K_t, L_t, t)}{F(K_t, L_t, 0)} = \frac{B_t \bar{F}(K_t, L_t)}{B_0 \bar{F}(K_t, L_t)} = B_t, \tag{5.14}$$

using the normalization $B_0 = 1$. That is:

> *Under Hicks neutrality, TFP_t appears as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.*

So in the case of Hicks neutrality the economic meaning of the TFP level is straightforward. The reason is that under Hicks neutrality the output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}, are *independent* of technical change. Moreover, the relationship also holds the opposite way: if the output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}, are independent of technical change, then technical change is Hicks neutral.

We now turn to difficulties regarding interpretation of TFP that arise in the general case.

5.4 Absence of Hicks-neutrality*

The above straightforward economic interpretation of TFP only holds under Hicks-neutral technical change. Neither under general technical change nor even under Harrod- or Solow-neutral technical change, will the current TFP level appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0. This is so unless the production function is Cobb-Douglas in which case both Harrod and Solow neutrality *imply* Hicks-neutrality.

To see this, let us return to the general time-dependent production function in (5.2). Let X_t denote the ratio between the current output level at time t and the hypothetical output level, $F(K_t, L_t, 0)$, that would have obtained with the current inputs of capital and labor in case of no change in the technology since time 0, i.e.,

$$X_t \equiv \frac{F(K_t, L_t, t)}{F(K_t, L_t, 0)}. \tag{5.15}$$

So X_t can be seen as a factor of "joint-productivity" growth from time 0 to time t evaluated at the time-t input combination.

If this X_t should always indicate the level of TFP at time t, the growth rate of X_t should equal the growth rate of TFP. Generally, it does not, however. Indeed, defining $G(K_t, L_t) \equiv F(K_t, L_t, 0)$, by the rule for the time derivative of fractions,[8] we have

$$\begin{aligned} g_{X,t} &\equiv \frac{dF(K_t, L_t, t)/dt}{F(K_t, L_t, t)} - \frac{dG(K_t, L_t)/dt}{G(K_t, L_t)} \\ &= \frac{1}{Y_t}\left[F_K(K_t, L_t, t)\dot{K}_t + F_L(K_t, L_t, t)\dot{L}_t + F_t(K_t, L_t, t)\cdot 1\right] \\ &\quad -\frac{1}{G(K_t, L_t)}\left[G_K(K_t, L_t)\dot{K}_t + G_L(K_t, L_t)\dot{L}_t\right] \\ &= \varepsilon_K(K_t, L_t, t)g_{Kt} + \varepsilon_L(K_t, L_t, t)g_{Lt} + \frac{F_t(K_t, L_t, t)}{Y_t} \\ &\quad -(\varepsilon_K(K_t, L_t, 0)g_{Kt} + \varepsilon_L(K_t, L_t, 0)g_{Lt}) \qquad (5.16) \\ &= (\varepsilon_K(K_t, L_t, t) - \varepsilon_K(K_t, L_t, 0))\, g_{Kt} \\ &\quad +(\varepsilon_L(K_t, L_t, t) - \varepsilon_L(K_t, L_t, 0))g_{Lt} + g_{\mathrm{TFP}t} \\ &\neq g_{\mathrm{TFP}t} \qquad \text{generally,} \end{aligned}$$

where $g_{\mathrm{TFP}t}$ is given in (5.5). We see that:

[8] See Appendix A to Chapter 3.

> The time-t growth rate of the joint-productivity index X equals the time-t TFP growth rate plus the cumulative impact of technical change since time 0 on the direct contribution to time-t output growth from time-t input growth.

Unless the partial output elasticities w.r.t. capital and labor, respectively, are unaffected by technical change, the conclusion is that TFP_t tend to differ from our X_t defined in (5.15). So:

> *In the absence of Hicks neutrality, current TFP does not generally appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.*

Consider the difference between $g_{X,t}$ and $g_{\mathrm{TFP}t}$:

$$g_{X,t} - g_{\mathrm{TFP}t} = (\varepsilon_K(K_t, L_t, t) - \varepsilon_K(K_t, L_t, 0))\, g_{Kt} + (\varepsilon_L(K_t, L_t, t) - \varepsilon_L(K_t, L_t, 0)) g_{Lt}.$$

Under CRS, the coefficients to the growth rates in K and L will be of the same absolute value but have opposite sign. This is an implication of $\varepsilon_K(K_t, L_t, \cdot) + \varepsilon_L(K_t, L_t, \cdot) = 1$ under CRS. Since usually g_{Kt} exceeds g_{Lt} considerably, the difference between $g_{X,t}$ and $g_{\mathrm{TFP}t}$ may be substantial.

Balanced growth at the aggregate level, hence Harrod neutrality, seems to characterize the growth experience of the UK and US over at least a century (Kongsamut et al., 2001; Attfield and Temple, 2010). At the same time the aggregate elasticity of factor substitution is generally estimated to be significantly less than one (cf. Chapter 2.7). This amounts to rejection of the Cobb-Douglas specification of the aggregate production function. So, at the aggregate level, Harrod neutrality rules out Hicks neutrality.

Since at least at the aggregate level Hicks-neutrality is empirically doubtful, the level of TFP can usually *not* be identified with the intuitive joint-productivity measure X_t, defined in (5.15) above. Then, to my knowledge there is no simple economic interpretation of what the TFP level actually measures.

A closer look at X_t vs. TFP_t

The fact that in the absence of Hicks-neutrality, TFP and the index X differ is the reason that we in Section 2.2 characterized the time-t TFP level relative to the time-0 level as the cumulative "*direct* contribution" on output growth since time 0 from cumulative technical change, thus excluding the possible indirect contribution coming about via the potential effect of technical change on the output elasticities w.r.t. capital and labor and thereby on the contribution to output from input growth.

Given that the joint-productivity index X is the more intuitive joint-productivity measure, why is TFP the more popular measure? There are at least two reasons for this. First, it can be shown that the TFP measure has more convenient balanced growth properties. Second, X is more difficult to measure. To see the reason for this, we substitute (5.3) into (5.16) to get

$$g_{Xt} = g_{Yt} - (\varepsilon_K(K_t, L_t, 0)g_{Kt} + \varepsilon_L(K_t, L_t, 0)g_{Lt}). \tag{5.17}$$

The relevant output elasticities, $\varepsilon_K(K_t, L_t, 0) \equiv \frac{K_t F_K(K_t,L_t,0)}{F(K_t,L_t,0)}$ and $\varepsilon_L(K_t, L_t, 0) \equiv \frac{L_t F_L(K_t,L_t,0)}{F(K_t,L_t,0)}$, are hypothetical constructs, referring to the technology as it was at time 0, but with the factor combination observed at time t, not at time 0. The nice thing about the Solow residual is that under the assumptions of perfect competition and absence of externalities, it allows measurement by using data on prices and quantities alone, that is, without knowledge of the production function. To evaluate g_X, however, we need estimates of the hypothetical output elasticities, $\varepsilon_K(K_t, L_t, 0)$ and $\varepsilon_L(K_t, L_t, 0)$. This requires knowledge about how the output elasticities depend on the factor combination and time, respectively, that is, knowledge about the production function.

5.5 A warning regarding cross-country TFP growth comparisons

When Harrod neutrality applies, relative TFP growth rates across sectors or countries can be quite deceptive. Consider a group of n countries that share some structural characteristics. Country i has the aggregate production function

$$Y_{it} = F^{(i)}(K_{it}, A_t L_{it}) \qquad i = 1, 2, ..., n,$$

where $F^{(i)}$ is a neoclassical production function with CRS, and A_t is the level of labor-augmenting technology which we assume shared by all the countries (these are open and "close" to each other). Technical progress is thus Harrod-neutral. Let the growth rate of A be a constant $g > 0$.

Define $\tilde{k}_{it} \equiv K_{it}/(A_t L_{it}) \equiv k_{it}/A_t$ and $\tilde{y}_{it} \equiv Y_{it}/(A_t L_{it}) \equiv y_{it}/A_t$. Suppose the countries feature (within-country) convergence, i.e.,

$$\tilde{k}_{it} \to \tilde{k}_i^* \quad \text{and} \quad \tilde{y}_{it} \to \tilde{y}_i^* = f^{(i)}(\tilde{k}_i^*) \quad \text{for} \quad t \to \infty,$$

where $f^{(i)}$ is the production function in intensive form. Since $k_{it} \equiv \tilde{k}_{it} A_t$ and $y_{it} \equiv \tilde{y}_{it} A_t$, we thus have

$$g_{k_i} \rightarrow g_A \ (= g) \quad \text{and} \quad g_{y_i} \rightarrow g_A \quad \text{for} \quad t \rightarrow \infty.$$

So in the long run g_{k_i} and g_{y_i} tend to the constant g.

Formula (5.6) then gives the TFP growth rate of country i in the long run as

$$g_{\text{TFP}_i} = g_{y_i} - \alpha_i^* g_{k_i} = (1 - \alpha_i^*) g, \tag{5.18}$$

where α_i^* is the output elasticity w.r.t. capital, $f^{(i)\prime}(\tilde{k}_i)\tilde{k}_i / f^{(i)}(\tilde{k}_i)$, evaluated at $\tilde{k}_i = \tilde{k}_i^*$. Under labor-augmenting technical progress, the TFP growth rate thus varies negatively with the output elasticity w.r.t. capital (the capital income share under perfect competition). Owing to differences in product and industry composition, the countries have different α_i^*'s. In view of (5.18), for two different countries, i and j, we therefore get

$$\frac{TFP_i}{TFP_j} \rightarrow \begin{cases} \infty \text{ if } \alpha_i^* < \alpha_j^*, \\ 1 \text{ if } \alpha_i^* = \alpha_j^*, \\ 0 \text{ if } \alpha_i^* > \alpha_j^*, \end{cases} \tag{5.19}$$

for $t \rightarrow \infty$.[9]

In spite of long-run growth in the essential variable, y, being the same across the countries, their TFP growth rates are very different. Countries with low α^* appear to be technologically very dynamic and countries with high α^* appear to be lagging behind. The explanation is simply that a higher α^* means that a larger fraction of $g_y = g_k = g$ becomes driven by ("explained by") g_k in the growth accounting (5.18), leaving a smaller residual. But it is the exogenous technology growth rate g that drives both g_k and g_{TFP}. The level of α^* is just the long-run output elasticity w.r.t. capital and reflects neither technological dynamism nor its opposite. Notwithstanding the countries' different α^*, their long-run growth rates of per capita consumption will be the same, namely g. Moreover, if the economies can be described, for instance, by a Solow model with the same s, δ, and n (standard notation) across the countries, and the ratio $s/(\delta + g + n)$ happens to equal 1, then even the *level* of per capita consumption in the countries will in the long run be the same growth rate. Nevertheless there will be persistent differences in their TFP growth rates, and (5.19) remains true.

We conclude that comparison of TFP growth rates across countries may misrepresent the intuitive meaning of productivity and technical progress

[9]If F is Cobb-Douglas with output elasticity w.r.t. capital equal to α_i, the key result, (5.18), can be derived more directly by first defining $B_t = A_t^{1-\alpha_i}$, then writing the production function in the Hicks-neutral form (5.12), and finally use (5.13).

when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral (even if technical progress were at the same time Hicks-neutral as is the case with a Cobb-Douglas specification).

On this background let us briefly consider a different decomposition than the one made in standard growth accounting. Under CRS, equation (5.10) holds. Subtracting $\varepsilon_{Kt}g_{yt}$ on both sides, dividing through by $1 - \varepsilon_{Kt}$, and rearranging gives

$$g_{yt} = \frac{\varepsilon_{Kt}}{1-\varepsilon_{Kt}} g_{\frac{k}{y}t} + \frac{1}{1-\varepsilon_{Kt}} g_{\text{TFP}t}. \tag{5.20}$$

This says that increases in the capital-output ratio as well as TFP contribute to growth in labor productivity via the "multipliers" $\varepsilon_{Kt}/(1-\varepsilon_{Kt})$ and $1/(1-\varepsilon_{Kt})$, respectively. This may speak for focusing on $g_{\text{TFP}t}/(1-\varepsilon_{Kt})$ rather than $g_{\text{TFP}t}$ it self. A growth path along which the capital-output ratio is constant (as it tends to be in the long run according to Kaldor's 'stylized facts") will feature labor productivity growth equal to the TFP growth rate multiplied by the inverse of the output elasticity w.r.t. labor (since, under CRS, $1 - \varepsilon_{Kt} = \varepsilon_{Lt}$).[10]

Thus, in the comparison of the n countries above, where in the long run the capital-output ratios are indeed constant ($\tilde{k}_i^*/f(\tilde{k}_i^*)$ is constant), it makes sense to focus on

$$\frac{g_{\text{TFP}_i}}{1-\alpha_{it}^*} = g_{yt} - \frac{\varepsilon_{Kt}}{1-\varepsilon_{Kt}} g_{\frac{k}{y}t} = \frac{1}{1-\varepsilon_{Kt}} g_{yt} - \frac{\varepsilon_{Kt}}{1-\varepsilon_{Kt}} g_{kt}. \tag{5.21}$$

This measure of the contribution of technical change ends up in the long run equal to the rate of Harrod-neutral technical progress, g, cf. (5.18).[11]

Since this "corrected TFP growth rate" in many models, including the one considered above, ultimately becomes identical to the labor productivity growth rate in the long run, it is not unreasonable in simple international comparisons to just compare levels and growth rates of Y/L across countries.

Remark on levels accounting

In growth accounting we consider productivity of a single country at different points in time. Another discipline is named *levels accounting*, where one compares productivity across different countries at a single point in time. See Caselli (2005), Acemoglu (2009, Chapter 3.5), and Jones and Vollrath (2013, Chapter 3).

[10] The labor productivity growth rate along a path along which the capital-output ratio is constant has occationally been called the *Harrod-productivity growth rate*.

[11] To focus on this, or a similar, measure was suggested by ....

5.6 Summing up

Growth accounting is – as the name indicates – a descriptive way of presenting growth data. So we should not confuse growth *accounting* with *causality* in growth analysis. To talk about causality we need a theoretical model supported by the data. On the basis of such a model we can say that this or that set of exogenous factors through the propagation mechanisms of the model cause this or that phenomenon, including economic growth.

In a complete model with exogenous technical progress, g_{kt} will be *induced* by this technical progress. If technical progress is endogenous through learning by investing, as in Arrow (1962), there is mutual causation between g_{kt} and technical progress. Yet other kinds of models explain both technical progress and capital accumulation through R&D, cf. Barro (1999) and Fernald and Jones (2014).

When technical change is not Hicks-neutral, the *level* of TFP can at best be *approximated* by the intuitive joint-productivity measure X_t, defined in (5.15) above. The approximation may not be good. And in absence of Hicks-neutrality, there seems to exist no simple economic interpretation of what the TFP *level* actually measures.

We also observed that relative TFP growth rates across sectors or countries can be quite deceptive when output elasticities w.r.t. capital differ. It may be more reasonable to compare the "corrected TFP growth rate" defined in (5.21) above or just compare levels and growth rates of Y/L across countries.

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Chapter 6

Transitional dynamics. Barro-style growth regressions

In this chapter we discuss three issues, all of which are related to the transitional dynamics of a growth model:

- Do poor countries necessarily tend to approach their steady state from below?
- How fast (or rather how slow) are the transitional dynamics in a growth model?
- What exactly is the theoretical foundation for a Barro-style growth regression analysis?

The Solow growth model may serve as the analytical point of departure for the first two issues and to some extent also for the third.

6.1 Point of departure: the Solow model

As is well-known, the fundamental differential equation for the Solow model is

$$\dot{\tilde{k}}(t) = sf(\tilde{k}(t)) - (\delta + g + n)\tilde{k}(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \tag{6.1}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $f(\tilde{k}(t)) \equiv F(\tilde{k}(t), 1)$, $A(t) = A_0 e^{gt}$, and $L(t) = L_0 e^{nt}$ (standard notation). The production function F is neoclassical with CRS and the parameters satisfy $0 < s < 1$ and $\delta + g + n > 0$. The production function on intensive form, f, therefore satisfies $f(0) \geq 0$, $f' > 0$, $f'' < 0$, and

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \frac{\delta + g + n}{s} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

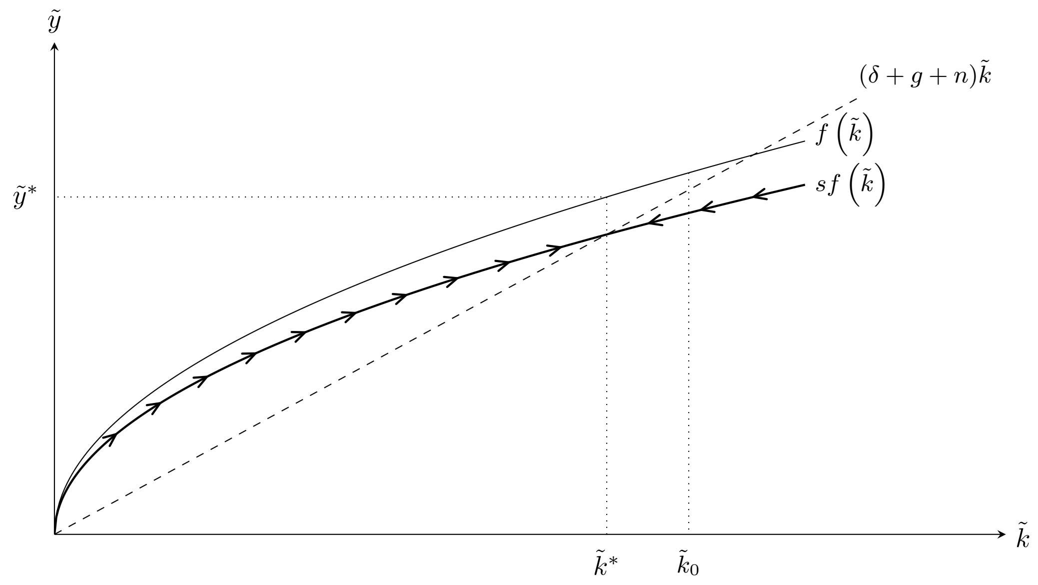

Figure 6.1: Phase diagram 1 (capital essential).

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, that is, a unique positive solution to the equation

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{6.2}$$

Furthermore, given an arbitrary $\tilde{k}_0 > 0$, we have for all $t \geq 0$,

$$\dot{\tilde{k}}(t) \gtreqless 0 \text{ for } \tilde{k}(t) \lesseqgtr \tilde{k}^*, \tag{6.3}$$

respectively. The steady state, $\tilde{k}^*$, is thus *globally asymptotically stable* in the sense that for all $\tilde{k}_0 > 0$, $\lim_{t\to\infty} \tilde{k}(t) = k^*$, and this convergence is *monotonic* (in the sense that $\tilde{k}(t) - \tilde{k}^*$ does not change sign during the adjustment process).

Figure 6.1 illustrates the dynamics as seen from the perspective of (6.1): $\tilde{k}$ is rising (falling) when saving per unit of effective labor, AL, is greater (less) than the amount needed to maintain the effective capital-labor ratio constant in spite of capital depreciation, more labor and better technology.

Figure 6.2 illustrates the dynamics emerging when we rewrite (6.1) this way:

$$\dot{\tilde{k}}(t) = s\left(f(\tilde{k}(t)) - \frac{\delta+g+n}{s}\tilde{k}(t)\right) \gtreqless 0 \text{ for } \tilde{k}(t) \lesseqgtr \tilde{k}^*.$$

In Figure 6.3 yet another illustration is exhibited, based on rewriting (6.1)

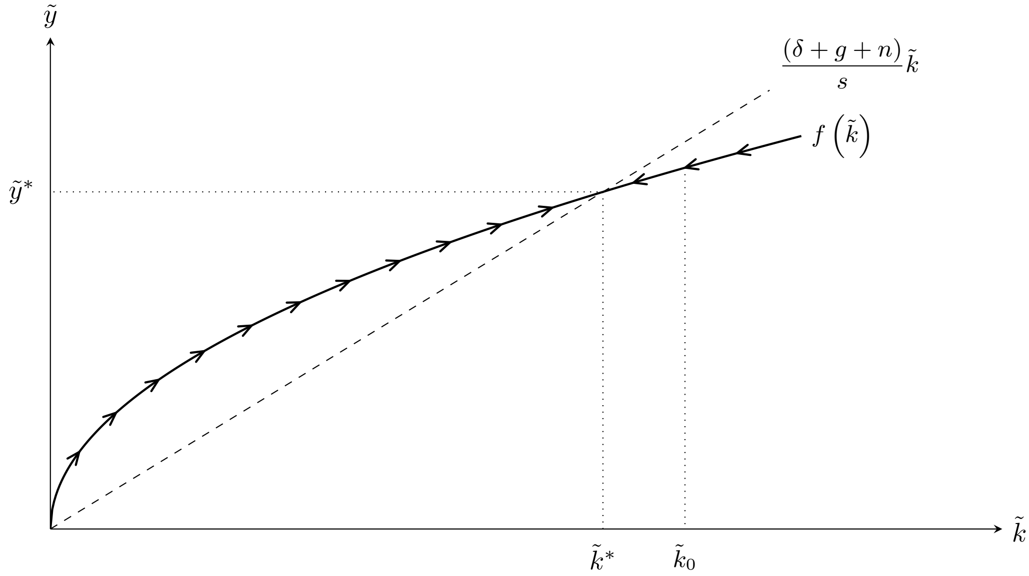

Figure 6.2: Phase diagram 2.

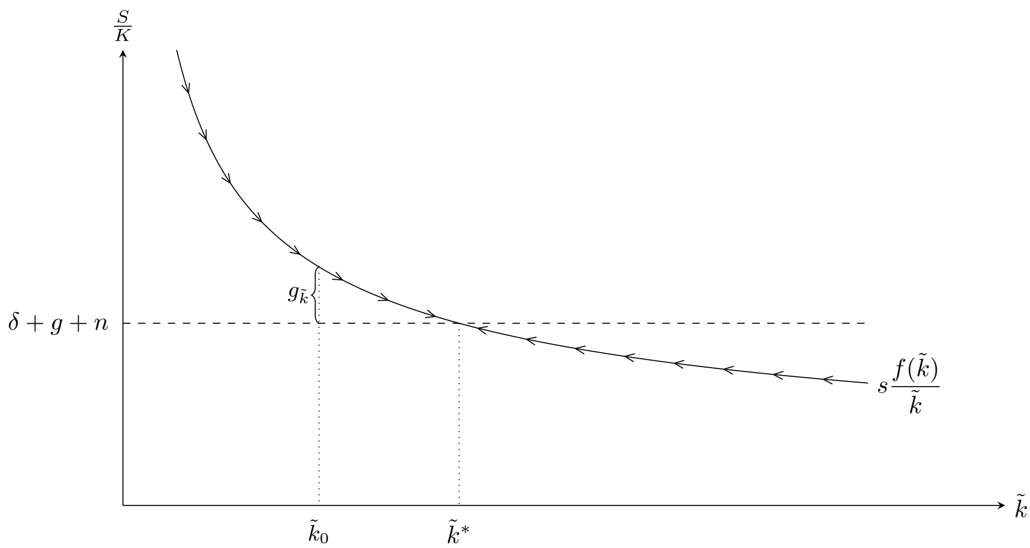

Figure 6.3: Phase diagram 3.

this way:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\frac{f(\tilde{k}(t))}{\tilde{k}(t)} - (\delta + g + n),$$

where $sf(\tilde{k}(t))/\tilde{k}(t)$ is gross saving per unit of capital, $S(t)/K(t) \equiv (Y(t) - C(t))/K(t)$.

From now on the dating of the variables is suppressed unless needed for clarity.

6.2 Do poor countries tend to approach their steady state from below?

From some textbooks (for instance Barro and Sala-i-Martin, 2004) one gets the impression that poor countries tend to approach their steady state *from below*. But this is *not* what the Penn World Table data seems to indicate. And from a theoretical point of view the size of $\tilde{k}_0$ relative to $\tilde{k}^*$ is certainly ambiguous, whether the country is rich or poor. To see this, consider a poor country with initial effective capital intensity

$$\tilde{k}_0 \equiv \frac{K_0}{A_0 L_0}.$$

Here K_0/L_0 will typically be small for a poor country (the country has not yet accumulated much capital relative to its fast-growing population). The technology level, A_0, however, *also* tends to be small for a poor country. Hence, whether we should expect $\tilde{k}_0 < \tilde{k}^*$ or $\tilde{k}_0 > \tilde{k}^*$ is not obvious *apriori*. Or equivalently: whether we should expect that a poor country's GDP at an arbitrary point in time grows at a rate higher or lower than the country's steady-state growth rate, $g + n$, is not obvious *apriori*.

While Figure 6.3 illustrates the case where the inequality $\tilde{k}_0 < \tilde{k}^*$ holds, Figure 6.1 and 6.2 illustrate the opposite case. There *exists* some empirical evidence indicating that poor countries tend to approach their steady state *from above*. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

The prejudice that poor countries *apriori* should tend to approach their steady state from below seems to come from a confusion of conditional and unconditional β convergence. The Solow model predicts - and data supports - that within a group of countries with similar structural characteristics (approximately the same f, A_0, g, s, n, and δ), the initially poorer countries will grow faster than the richer countries. This is because the poorer countries (small $y(0) = f(\tilde{k}_0)A_0$) will be the countries with relatively small initial capital-labor ratio, k_0. As all the countries in the group have approximately the same A_0, the poorer countries thus have $\tilde{k}_0 \equiv k_0/A_0$ relatively small, i.e., $\tilde{k}_0 < \tilde{k}^*$. From $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k})A$ follows that the growth rate in output per worker of these poor countries tends to exceed g. Indeed, we have generally (for instance in the Solow model as well as the Ramsey model)

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + g = \frac{f'(\tilde{k})\dot{\tilde{k}}}{f(\tilde{k})} + g \gtreqless g \text{ for } \dot{\tilde{k}} \gtreqless 0, \text{ i.e., for } \tilde{k} \lesseqgtr \tilde{k}^*.$$

So, *within* the group, the poor countries tend to approach the steady state, $\tilde{k}^*$, *from below*.

The countries in the world as a whole, however, differ a lot w.r.t. their structural characteristics, including their A_0. Unconditional β convergence is definitely rejected by the data. Then there is no reason to expect the poorer countries to have $\tilde{k}_0 < \tilde{k}^*$ rather than $\tilde{k}_0 > \tilde{k}^*$. Indeed, according to the mentioned study by Cho and Graham (1996), it turns out that the data for the relatively poor countries favors the latter inequality rather than the first.

6.3 Within-country convergence speed and adjustment time

Our next issue is: How fast (or rather how slow) are the transitional dynamics in a growth model? To put it another way: according to a given growth model with convergence, how fast does the economy approach its steady state? The answer turns out to be: not very fast - to say the least. This is a rather general conclusion and is confirmed by the empirics: adjustment processes in a growth context are quite time consuming.

In Acemoglu (2009) we meet the concept of speed of convergence at p. 54 (under an alternative name, rate of adjustment) and p. 81 (in connection with Barro-style growth regressions). Here we shall go more into detail with the issue of speed of convergence.

Again the Solow model is our frame of reference. We search for a formula for the *speed of convergence* of $\tilde{k}(t)$ and $y(t)/y^*(t)$ in a closed economy described by the Solow model. So our analysis is concerned with *within-country convergence*: how fast do variables such as $\tilde{k}$ and y approach their steady state paths in a closed economy? The key adjustment mechanism is linked to diminishing returns to capital (falling marginal productivity of capital) in the process of capital accumulation. The problem of *cross-country convergence* (which is what "β convergence" and "σ convergence" are about) is in principle more complex because also such mechanisms as technological catching-up and cross-country factor movements are involved.

6.3.1 Convergence speed for $\tilde{k}(t)$

The ratio of $\dot{\tilde{k}}(t)$ to $(\tilde{k}(t) - \tilde{k}^*) \neq 0$ can be written

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t) - \tilde{k}^*} = \frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*}, \tag{6.4}$$

since $d\tilde{k}^*/dt = 0$. We define the *instantaneous speed of convergence* at time t as the (proportionate) rate of *decline* of the distance $\left|\tilde{k}(t) - \tilde{k}^*\right|$ at time t and we denote it $\mathrm{SOC}_t(\tilde{k})$.[1] Thus,

$$\mathrm{SOC}_t(\tilde{k}) \equiv -\frac{d\left(\left|\tilde{k}(t) - \tilde{k}^*\right|\right)/dt}{\left|\tilde{k}(t) - \tilde{k}^*\right|} = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \tag{6.5}$$

per time unit, where the equality sign is valid for monotonic convergence.

Generally, $\mathrm{SOC}_t(\tilde{k})$ depends on both the absolute size of the difference $\tilde{k} - \tilde{k}^*$ at time t and its sign. But if the difference is already "small", $\mathrm{SOC}_t(\tilde{k})$ will be "almost" constant for increasing t and we can find an approximate measure for it. Let the function $\varphi(\tilde{k})$ be defined by $\dot{\tilde{k}} = sf(\tilde{k}) - m\tilde{k} \equiv \varphi(\tilde{k})$, where $m \equiv \delta + g + n$. A first-order Taylor approximation of $\varphi(\tilde{k})$ around $\tilde{k} = \tilde{k}^*$ gives

$$\varphi(\tilde{k}) \approx \varphi(\tilde{k}^*) + \varphi'(\tilde{k}^*)(\tilde{k} - \tilde{k}^*) = 0 + (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*).$$

For $\tilde{k}$ in a small neighborhood of the steady state, $\tilde{k}^*$, we thus have

$$\begin{aligned}
\dot{\tilde{k}} &= \varphi(\tilde{k}) \approx (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*) \\
&= (\frac{sf'(\tilde{k}^*)}{m} - 1)m(\tilde{k} - \tilde{k}^*) \\
&= (\frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (6.2))} \\
&\equiv (\varepsilon(\tilde{k}^*) - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (6.6))},
\end{aligned}$$

[1] Synonyms for speed of convergence are *rate of convergence, rate of adjustment* or *adjustment speed.*

where $\varepsilon(\tilde{k}^*)$ is the output elasticity w.r.t. capital, evaluated in the steady state. So

$$\frac{K}{Y}\frac{\partial Y}{\partial K} = \frac{\tilde{k}}{f(\tilde{k})} f'(\tilde{k}) \equiv \varepsilon(\tilde{k}), \tag{6.6}$$

where $0 < \varepsilon(\tilde{k}) < 1$ for all $\tilde{k} > 0$.

Applying the definition (6.5) and the identity $m \equiv \delta + g + n$, we now get

$$\text{SOC}_t(\tilde{k}) = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} = \frac{-\dot{\tilde{k}}(t)}{\tilde{k}(t) - \tilde{k}^*} \approx (1-\varepsilon(\tilde{k}^*))(\delta+g+n) \equiv \beta(\tilde{k}^*) > 0. \tag{6.7}$$

This result tells us how fast, approximately, the economy approaches its steady state if it starts "close" to it. If, for example, $\beta(\tilde{k}^*) = 0.02$ per year, then 2 percent of the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ vanishes per year. We also see that everything else equal, a higher output elasticity w.r.t. capital implies a lower speed of convergence.

In the limit, for $\left|\tilde{k} - \tilde{k}^*\right| \to 0$, the instantaneous speed of convergence coincides with what is called the *asymptotic speed of convergence*, defined as

$$\text{SOC}^*(\tilde{k}) \equiv \lim_{\left|\tilde{k}-\tilde{k}^*\right| \to 0} \text{SOC}_t(\tilde{k}) = \beta(\tilde{k}^*). \tag{6.8}$$

Multiplying through by $-(\tilde{k}(t) - \tilde{k}^*)$, the equation (6.7) takes the form of a homogeneous linear differential equation (with constant coefficient), $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. With $x(t) = \tilde{k}(t) - \tilde{k}^*$ and "=" replaced by "≈", we get in the present case

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t} \to 0 \text{ for } t \to \infty. \tag{6.9}$$

This is the approximative time path for the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ and shows how the gap becomes smaller and smaller at the rate $\beta(\tilde{k}^*)$.

One of the reasons that the speed of convergence is important is that it indicates what weight should be placed on transitional dynamics of a growth model relative to the steady-state behavior. The speed of convergence matters for instance for the evaluation of growth-promoting policies. In growth models with diminishing marginal productivity of production factors, successful growth-promoting policies have transitory growth effects and permanent level effects. Slower convergence implies that the full benefits are slower to arrive.

6.3.2 Convergence speed for $\log \tilde{k}(t)$*

We have found an approximate expression for the convergence speed of $\tilde{k}$. Since models in empirical analysis and applied theory are often based on log-linearization, we might ask what the speed of convergence of $\log \tilde{k}$ is. The answer is: approximately the same and asymptotically exactly the same as that of $\tilde{k}$ itself! Let us see why.

A first-order Taylor approximation of $\log \tilde{k}(t)$ around $\tilde{k} = \tilde{k}^*$ gives

$$\log \tilde{k}(t) \approx \log \tilde{k}^* + \frac{1}{\tilde{k}^*}(\tilde{k}(t) - \tilde{k}^*). \tag{6.10}$$

By definition

$$\begin{aligned}
\mathrm{SOC}_t(\log \tilde{k}) &= -\frac{d(\log \tilde{k}(t) - \log \tilde{k}^*)/dt}{\log \tilde{k}(t) - \log \tilde{k}^*} = -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)(\log \tilde{k}(t) - \log \tilde{k}^*)} \\
&\approx -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)\frac{\tilde{k}(t)-\tilde{k}^*}{\tilde{k}^*}} = \frac{\tilde{k}^*}{\tilde{k}(t)}\mathrm{SOC}_t(\tilde{k}) \to \mathrm{SOC}^*(\tilde{k}) = \beta(\tilde{k}^*) \quad (6.11) \\
\text{for } \tilde{k}(t) &\to \tilde{k}^*,
\end{aligned}$$

where in the second line we have used, first, the approximation (6.10), second, the definition in (6.7), and third, the definition in (6.8).

So, at least in a neighborhood of the steady state, the instantaneous rate of decline of the logarithmic distance of $\tilde{k}$ to the steady-state value of $\tilde{k}$ approximates the instantaneous rate of decline of the distance of $\tilde{k}$ itself to its steady-state value. The asymptotic speed of convergence of $\log \tilde{k}$ coincides with that of $\tilde{k}$ itself and is exactly $\beta(\tilde{k}^*)$.

In the Cobb-Douglas case (where $\varepsilon(\tilde{k}^*)$ is a constant, say α) it is possible to find an explicit solution to the Solow model, see Acemoglu (2009, p. 53) and Exercise II.2. It turns out that the instantaneous speed of convergence in a finite distance from the steady state is a constant and equals the asymptotic speed of convergence, $(1 - \alpha)(\delta + g + n)$.

6.3.3 Convergence speed for $y(t)/y^*(t)$*

The variable which we are interested in is usually not so much $\tilde{k}$ in itself, but rather labor productivity, $y(t) \equiv \tilde{y}(t)A(t)$. In the interesting case where $g > 0$, labor productivity does not converge towards a constant. We therefore focus on the ratio $y(t)/y^*(t)$, where $y^*(t)$ denotes the hypothetical value of labor productivity at time t, conditional on the economy being on its steady-state path, i.e.,

$$y^*(t) \equiv \tilde{y}^* A(t). \tag{6.12}$$

We have

$$\frac{y(t)}{y^*(t)} \equiv \frac{\tilde{y}(t)A(t)}{\tilde{y}^*A(t)} = \frac{\tilde{y}(t)}{\tilde{y}^*}. \tag{6.13}$$

As $\tilde{y}(t) \to \tilde{y}^*$ for $t \to \infty$, the ratio $y(t)/y^*(t)$ converges towards 1 for $t \to \infty$.

Taking logs on both sides of (6.13), we get

$$\begin{aligned}
\log \frac{y(t)}{y^*(t)} &= \log \frac{\tilde{y}(t)}{\tilde{y}^*} = \log \tilde{y}(t) - \log \tilde{y}^* \\
&\approx \log \tilde{y}^* + \frac{1}{\tilde{y}^*}(\tilde{y}(t) - y^*) - \log \tilde{y}^* \text{ (first-order Taylor approx. of } \log \tilde{y}\text{)} \\
&= \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}(t)) - f(\tilde{k}^*)) \\
&\approx \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}^*) + f'(\tilde{k}^*)(\tilde{k}(t) - \tilde{k}^*) - f(\tilde{k}^*)) \text{ (first-order approx. of } f(\tilde{k})) \\
&= \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \equiv \varepsilon(\tilde{k}^*)\frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \\
&\approx \varepsilon(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \qquad \text{(by (6.10))}.
\end{aligned} \tag{6.14}$$

Multiplying through by $-(\log \tilde{k}(t) - \log \tilde{k}^*)$ in (6.11) and carrying out the differentiation w.r.t. time, we find an approximate expression for the growth rate of $\tilde{k}$,

$$\begin{aligned}
\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)} &\equiv g_{\tilde{k}}(t) \approx -\frac{\tilde{k}^*}{\tilde{k}(t)}\mathrm{SOC}_t(\tilde{k})(\log \tilde{k}(t) - \log \tilde{k}^*) \\
&\to -\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \quad \text{for } \tilde{k}(t) \to \tilde{k}^*,
\end{aligned} \tag{6.15}$$

where the convergence follows from the last part of (6.11). We now calculate the time derivative on both sides of (6.14) to get

$$\begin{aligned}
d(\log \frac{y(t)}{y^*(t)})/dt &= d(\log \frac{\tilde{y}(t)}{\tilde{y}^*})/dt = \frac{d\tilde{y}(t)/dt}{\tilde{y}(t)} \equiv g_{\tilde{y}}(t) \\
&\approx \varepsilon(\tilde{k}^*)g_{\tilde{k}}(t) \approx -\varepsilon(\tilde{k}^*)\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*).
\end{aligned} \tag{6.16}$$

from (6.15). Dividing through by $-\log(y(t)/y^*(t))$ in this expression, taking (6.14) into account, gives

$$-\frac{d(\log \frac{y(t)}{y^*(t)})/dt}{\log \frac{y(t)}{y^*(t)}} = -\frac{d(\log \frac{y(t)}{y^*(t)} - \log 1)/dt}{\log \frac{y(t)}{y^*(t)} - \log 1} \equiv \mathrm{SOC}_t(\log \frac{y}{y^*}) \approx \beta(\tilde{k}^*), \tag{6.17}$$

in view of $\log 1 = 0$. So the logarithmic distance of y from its value on the steady-state path at time t has approximately the same rate of decline as the logarithmic distance of $\tilde{k}$ from $\tilde{k}$'s value on the steady-state path at time t. The asymptotic speed of convergence for $\log y(t)/y^*(t)$ is exactly the same as that for $\tilde{k}$, namely $\beta(\tilde{k}^*)$.

What about the speed of convergence of $y(t)/y^*(t)$ itself? Here the same principle as in (6.11) applies. The asymptotic speed of convergence for $\log(y(t)/y^*(t))$ is the same as that for $y(t)/y^*(t)$ (and vice versa), namely $\beta(\tilde{k}^*)$.

With one year as our time unit, standard parameter values are: $g = 0.02$, $n = 0.01$, $\delta = 0.05$, and $\varepsilon(\tilde{k}^*) = 1/3$. We then get $\beta(\tilde{k}^*) = (1-\varepsilon(\tilde{k}^*))(\delta+g+n)$ $= 0.053$ per year. In the empirical Chapter 11 of Barro and Sala-i-Martin (2004), it is argued that a lower value of $\beta(\tilde{k}^*)$, say 0.02 per year, fits the data better. This requires $\varepsilon(\tilde{k}^*) = 0.75$. Such a high value of $\varepsilon(\tilde{k}^*)$ ($\approx$ the income share of capital) may seem difficult to defend. But if we reinterpret K in the Solow model so as to include *human* capital (skills embodied in human beings and acquired through education and learning by doing), a value of $\varepsilon(\tilde{k}^*)$ at that level may not be far out.

6.3.4 Adjustment time

Let τ_ω be the time that it takes for the fraction $\omega \in (0,1)$ of the initial gap between $\tilde{k}$ and $\tilde{k}^*$ to be eliminated, i.e., τ_ω satisfies the equation

$$\frac{\left|\tilde{k}(\tau_\omega) - \tilde{k}^*\right|}{\left|\tilde{k}(0) - \tilde{k}^*\right|} = \frac{\tilde{k}(\tau_\omega) - \tilde{k}^*}{\tilde{k}(0) - \tilde{k}^*} = 1 - \omega, \tag{6.18}$$

where $1-\omega$ is the fraction of the initial gap still remaining at time τ_ω. In (6.18) we have applied that $sign(\tilde{k}(t) - \tilde{k}^*) = sign(\tilde{k}(0) - \tilde{k}^*)$ in view of monotonic convergence.

By (6.9), we have

$$\tilde{k}(\tau_\omega) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

In view of (6.18), this implies

$$1 - \omega \approx e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

Taking logs on both sides and solving for τ_ω gives

$$\tau_\omega \approx -\frac{\log(1-\omega)}{\beta(\tilde{k}^*)}. \tag{6.19}$$

This is the approximate *adjustment time* required for $\tilde{k}$ to eliminate the fraction ω of the initial distance of $\tilde{k}$ to its steady-state value, $\tilde{k}^*$, when the adjustment speed (speed of convergence) is $\beta(\tilde{k}^*)$.

Often we consider the *half-life* of the adjustment, that is, the time it takes for half of the initial gap to be eliminated. To find the half-life of the adjustment of $\tilde{k}$, we put $\omega = \frac{1}{2}$ in (6.19). Again we use one year as our time unit. With the parameter values from Section 6.3.3, we have $\beta(\tilde{k}^*) = 0.053$ per year and thus

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.053} \approx \frac{0.69}{0.053} = 13,1 \text{ years.}$$

As noted above, Barro and Sala-i-Martin (2004) estimate the asymptotic speed of convergence to be $\beta(\tilde{k}^*) = 0.02$ per year. With this value, the half-life is approximately

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.02} \approx \frac{0.69}{0.02} = 34.7 \text{ years.}$$

And the time needed to eliminate three quarters of the initial distance to steady state, $\tau_{3/4}$, will then be about 70 years ($= 2 \cdot 35$ years, since $1 - 3/4 = \frac{1}{2} \cdot \frac{1}{2}$).

Among empirical analysts there is not general agreement about the size of $\beta(\tilde{k}^*)$. Some authors, for example Islam (1995), using a panel data approach, find speeds of convergence considerably larger, between 0.05 and 0.09. McQuinne and Whelan (2007) get similar results. There is a growing realization that the speed of convergence differs across periods and groups of countries. Perhaps an empirically reasonable range is $0.02 < \beta(\tilde{k}^*) < 0.09$. Correspondingly, a reasonable range for the half-life of the adjustment will be 7.6 years $< \tau_{\frac{1}{2}} < 34.7$ years.

Most of the empirical studies of convergence use a variety of cross-country regression analysis of the kind described in the next section. Yet the theoretical frame of reference is often the Solow model - or its extension with human capital (Mankiw et al., 1992). These models are closed economy models with exogenous technical progress and deal with "within-country" convergence. It is not obvious that they constitute an appropriate framework for studying cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and where some countries are pushing the technological frontier further out, while others try to imitate and catch up. At least one should be aware that the empirical estimates obtained may reflect mechanisms in addition to the falling marginal productivity of capital in the process of capital accumulation.

6.4 Barro-style growth regressions*

Barro-style growth regression analysis, which became very popular in the 1990s, draws upon transitional dynamics aspects (including the speed of convergence) as well as steady state aspects of neoclassical growth theory (for instance the Solow model or the Ramsey model).

Chapter 3.2 in Acemoglu (2009) presents Barro's growth regression equations in an unconventional form, see Acemoglu's equations (3.12), (3.13), and (3.14). The left-hand side appears as if it is just the growth rate of y (output per unit of labor) from one year to the next. But the true left-hand side of a Barro equation is the average compound annual growth rate of y over many years. Moreover, since Acemoglu's text is very brief about the formal links to the underlying neoclassical theory of transitional dynamics, we will spell the details out here.

Most of the preparatory work has already been done above. The point of departure is a neoclassical one-sector growth model for a closed economy:

$$\dot{\tilde{k}}(t) = s(\tilde{k}(t))f(\tilde{k}(t)) - (\delta + g + n)k(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \text{ given}, \quad (6.20)$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $A(t) = A_0e^{gt}$, and $L(t) = L_0e^{nt}$ as above. The Solow model is the special case where the saving-income ratio, $s(\tilde{k}(t))$, is a constant $s \in (0, 1)$.

It is assumed that the model, (6.20), generates monotonic convergence, i.e., $\tilde{k}(t) \to \tilde{k}^* > 0$ for $t \to \infty$. Applying again a first-order Taylor approximation, as in Section 3.1, and taking into account that $s(\tilde{k})$ now may depend on $\tilde{k}$, as for instance it generally does in the Ramsey model, we find the asymptotic speed of convergence for $\tilde{k}$ to be

$$\text{SOC}^*(\tilde{k}) = (1 - \varepsilon(\tilde{k}^*) - \eta(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*) > 0, \quad (*)$$

where $\eta(\tilde{k}^*) \equiv \tilde{k}^*s'(\tilde{k}^*)/s(\tilde{k}^*)$ is the elasticity of the saving-income ratio w.r.t. the effective capital intensity, evaluated at $\tilde{k} = \tilde{k}^*$. (In case of the Ramsey model, one can alternatively use the fact that $\text{SOC}^*(\tilde{k})$ equals the absolute value of the negative eigenvalue of the Jacobian matrix associated with the dynamic system of the model, evaluated in the steady state. For a fully specified Ramsey model this eigenvalue can be numerically calculated by an appropriate computer algorithm; in the Cobb-Douglas case there exists even an explicit algebraic formula for the eigenvalue, see Barro and Sala-i-Martin, 2004). In a neighborhood of the steady state, the previous formulas remain valid with $\beta(\tilde{k}^*)$ defined as in (*). The asymptotic speed of convergence of for example $y(t)/y^*(t)$ is thus $\beta(\tilde{k}^*)$ as given in (*). For notational convenience, we will just denote it β, interpreted as a derived parameter, i.e.,

$$\beta = (1 - \varepsilon(\tilde{k}^*) - \eta(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*). \quad (6.21)$$

In case of the Solow model, $\eta(\tilde{k}^*) = 0$ and we are back in Section 3.

In view of $y(t) \equiv \tilde{y}(t)A(t)$, we have $g_y(t) = g_{\tilde{y}}(t) + g$. By (6.16) and the definition of β,

$$g_y(t) \approx g - \varepsilon(\tilde{k}^*)\beta(\log \tilde{k}(t) - \log \tilde{k}^*) \approx g - \beta(\log y(t) - \log y^*(t)), \quad (6.22)$$

where the last approximation comes from (6.14). This generalizes Acemoglu's Equation (3.10) (recall that Acemoglu concentrates on the Solow model and that his k^* is the same as our $\tilde{k}^*$).

With the horizontal axis representing time, Figure 6.4 gives an illustration of these transitional dynamics. As $g_y(t) = d\log y(t)/dt$ and $g = d\log y^*(t)/dt$, (6.22) is equivalent to

$$\frac{d(\log y(t) - \log y^*(t))}{dt} \approx -\beta(\log y(t) - \log y^*(t)). \quad (6.23)$$

So again we have a simple differential equation of the form $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. The solution of (6.23) is thus

$$\log y(t) - \log y^*(t) \approx (\log y(0) - \log y^*(0))e^{-\beta t}.$$

As $y^*(t) = y^*(0)e^{gt}$, this can be written

$$\log y(t) \approx \log y^*(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t}. \quad (6.24)$$

The solid curve in Figure 6.4 depicts the evolution of $\log y(t)$ in the case where $\tilde{k}_0 < \tilde{k}^*$ (note that $\log y^*(0) = \log f(\tilde{k}^*) + \log A_0$). The dotted curve exemplifies the case where $\tilde{k}_0 > \tilde{k}^*$. The figure illustrates per capita income convergence: low initial income is associated with a high subsequent growth rate which, however, diminishes along with the diminishing logarithmic distance of per capita income to its level on the steady state path.

For convenience, we will from now on treat (6.24) as an equality. Subtracting $\log y(0)$ on both sides, we get

$$\begin{aligned} \log y(t) - \log y(0) &= \log y^*(0) - \log y(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t} \\ &= gt - (1 - e^{-\beta t})(\log y(0) - \log y^*(0)). \end{aligned}$$

Dividing through by $t > 0$ gives

$$\frac{\log y(t) - \log y(0)}{t} = g - \frac{1 - e^{-\beta t}}{t}(\log y(0) - \log y^*(0)). \quad (6.25)$$

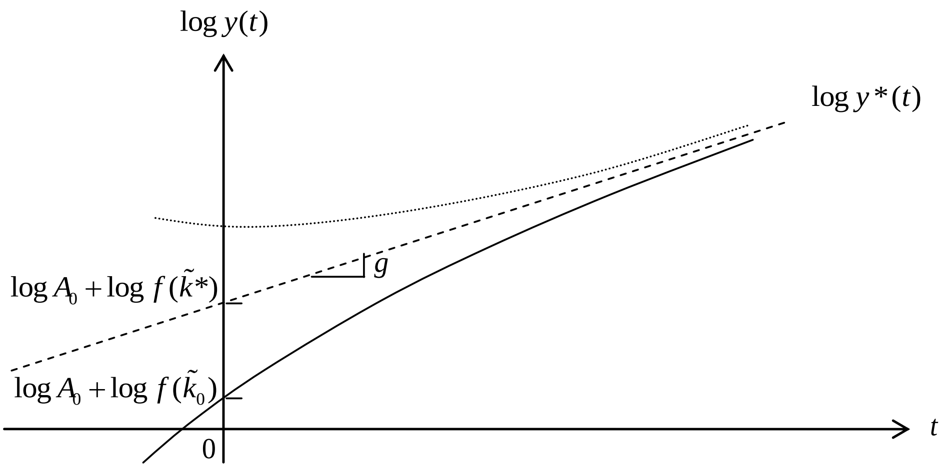

Figure 6.4: Evolution of log $y(t)$. Solid curve: the case $\tilde{k}_0 < \tilde{k}^*$. Dotted curve: the case $\tilde{k}_0 > \tilde{k}^*$. Stippled line: the steady-state path.

On the left-hand side appears the average compound annual growth rate of y from period 0 to period t, which we will denote $\bar{g}_y(0,t)$. On the right-hand side appears the initial distance of $\log y$ to its hypothetical level along the steady state path. The coefficient, $-(1-e^{-\beta t})/t$, to this distance is negative and approaches zero for $t \to \infty$. Thus (6.25) is a translation into growth form of the convergence of $\log y_t$ towards the steady-state path, $\log y_t^*$, in the theoretical model without shocks. Rearranging the right-hand side, we get

$$\bar{g}_y(0,t) = g + \frac{1-e^{-\beta t}}{t}\log y^*(0) - \frac{1-e^{-\beta t}}{t}\log y(0) \equiv b^0 + b^1 \log y(0),$$

where both the constant $b^0 \equiv g + \left[(1-e^{-\beta t})/t\right]\log y^*(0)$ and the coefficient $b^1 \equiv -(1-e^{-\beta t})/t$ are determined by "structural characteristics". Indeed, β is determined by δ, g, n, $\varepsilon(\tilde{k}^*)$, and $\eta(\tilde{k}^*)$ through (6.21), and $y^*(0)$ is determined by A_0 and $f(\tilde{k}^*)$ through (6.12), where, in turn, $\tilde{k}^*$ is determined by the steady state condition $s(\tilde{k}^*)f(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*$, $s(\tilde{k}^*)$ being the saving-income ratio in the steady state.

With data for N countries, $i = 1, 2, \ldots, N$, a test of the *unconditional convergence hypothesis* may be based on the regression equation

$$\bar{g}_{y_i}(0,t) = b^0 + b^1 \log y_i(0) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2), \tag{6.26}$$

where ϵ_i is the error term. This can be seen as a Barro growth regression equation in its simplest form. For countries in the entire world, the theoretical hypothesis $b^1 < 0$ is clearly not supported (or, to use the language of statistics, the null hypothesis, $b^1 = 0$, is not rejected).[2]

Allowing for the considered countries having different structural characteristics, the Barro growth regression equation takes the form

$$\bar{g}_{y_i}(0,t) = b_i^0 + b^1 \log y_i(0) + \epsilon_i, \qquad b^1 < 0, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2). \tag{6.27}$$

In this "fixed effects" form, the equation has been applied for a test of the *conditional convergence hypothesis,* $b^1 < 0$, often supporting this hypothesis. That is, within groups of countries with similar characteristics (like, e.g., the OECD countries), there is a tendency to convergence.

From the estimate of b^1 the implied estimate of the asymptotic speed of convergence, β, is readily obtained through the formula $b^1 \equiv (1 - e^{-\beta t})/t$. Even β, and therefore also the slope, b^1, does depend, theoretically, on country-specific structural characteristics. But the sensitivity on these do not generally seem large enough to blur the analysis based on (6.27) which abstracts from this dependency.

With the aim of testing hypotheses about growth determinants, Barro (1991) and Barro and Sala-i-Martin (1992, 2004) decompose b_i^0 so as to reflect the role of a set of potentially causal measurable variables,

$$b_i^0 = \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \ldots + \alpha_m x_{im},$$

where the α's are the coefficients and the x's are the potentially causal variables.[3] These variables could be measurable Solow-type parameters among those appearing in (6.20) or a broader set of determinants, including for instance the educational level in the labor force, and institutional variables like rule of law and democracy. Some studies include the initial within-country inequality in income or wealth among the x's and extend the theoretical framework correspondingly.[4]

From an econometric point of view there are several problematic features in regressions of Barro's form (also called the β convergence approach). These problems are discussed in Acemoglu pp. 82-85.

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[2] Cf. Acemoglu, p. 16. For the OECD countries, however, b^1 is definitely found to be negative (cf. Acemoglu, p. 17).

[3] Note that our α vector is called β in Acemoglu, pp. 83-84. So Acemoglu's β is to be distinquished from our β which denotes the asymptotic speed of convergence.

[4] See, e.g., Alesina and Rodrik (1994) and Perotti (1996), who argue for a negative relationship between inequality and growth. Forbes (2000), however, rejects that there should be a robust negative correlation between the two.

6.5 References

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Chapter 7

Why the Malthusian era must come to an end

This chapter presents the *population-breeds-ideas model* by Michael Kremer (Kremer, 1993). The point of the model is to show that under certain conditions, the cumulative and nonrival character of technical knowledge makes it almost inevitable that the Malthusian regime of stagnating income per capita, close to subsistence minimum, will sooner or later in the historical evolution be surpassed.

This topic relates to Section 8.2 of Jones and Vollrath (2013). Section 4.2 of Acemoglu (2009) briefly discuss two special cases of the Kremer model.

7.1 The general model

Suppose a pre-industrial economy can be described by:

$$\begin{aligned} Y_t &= A_t^\sigma L_t^\alpha Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, & (7.1) \\ \dot{A}_t &= \lambda A_t^\varepsilon L_t, \qquad \lambda > 0, \quad A_0 > 0 \text{ given}, & (7.2) \\ L_t &= \frac{Y_t}{\bar{y}}, \qquad \bar{y} > 0, & (7.3) \end{aligned}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), Z the amount of land (fixed), and $\bar{y}$ subsistence minimum. By this is not meant some point almost at starvation, but an income level sufficient for food, clothing, shelter etc. to the worker, including family and offspring, thereby enabling reproduction of the labor force.

Both Z and $\bar{y}$ are considered as constant parameters. Time is continuous and it is understood that a kind of Malthusian population mechanism (see below) is operative behind the scene.

The exclusion of capital from the aggregate production function, (7.1), reflects the presumption that capital (tools etc.) is quantitatively of minor importance in a pre-industrial economy. In accordance with the replication argument, the production function has CRS w.r.t. the rival inputs, labor and land. The factor A_t^{σ} measures total factor productivity. As the right-hand side of (7.2) is positive, the technology level, A_t, is rising over time (although far back in time very very slowly). The increase in A_t per time unit is seen to be an increasing function of the size of the population. This reflects the hypothesis that population breeds ideas; these are *nonrival* and enter the pool of technical knowledge available for society as a whole. Indeed, the use of an idea by one agent does not preclude others' use of the same idea. Dividing through by L in (7.1) we see that $y \equiv Y_t/L_t = A_t^{\sigma}(Z/L_t)^{1-\alpha}$. The nonrival character is displayed by labor productivity being dependent on the total stock of knowledge, not on this stock per worker. In contrast, labor productivity depends on *land per worker*.

The rate per capita by which population breeds ideas is λA^{ε}. In case $\varepsilon > 0$, this rate is an increasing function of the already existing level of technical knowledge. This case reflects the hypothesis that the larger is the stock of ideas the easier do new ideas arise (perhaps by combination of existing ideas). The opposite case, $\varepsilon < 0$, is the one where "the easiest ideas are found first" or "the low-hanging fruits are picked first".

Equation (7.3) is a shortcut description of a Malthusian population mechanism. Suppose the true mechanism is

$$\dot{L}_t = \beta(y_t - \bar{y})L_t \gtreqless 0 \quad \text{for} \quad y_t \gtreqless \bar{y}, \tag{7.4}$$

where $\beta > 0$ is the speed of adjustment, y_t is per capita income, and $\bar{y} > 0$ is subsistence minimum. A rise in y_t above $\bar{y}$ will lead to increases in L_t through earlier marriage, higher fertility, and lower mortality. Thereby downward pressure on Y_t/L_t is generated, perhaps pushing y_t below $\bar{y}$. When this happens, population will be decreasing for a while and so return towards its sustainable level, $Y_t/\bar{y}$. Equation (7.3) treats this mechanism as if the population instantaneously adjusts to its sustainable level (i.e., as if $\beta \to \infty$). The model hereby gives a long-run picture, ignoring the Malthusian ups and downs in population and per capita income about the subsistence minimum. The important feature is that the technology level, and thereby Y_t, as well as the sustainable population will be *rising* over time. This speeds up the arrival of new ideas and so Y_t is raised even faster although per-capita income remains at its long-run level, $\bar{y}$.[1]

For simplicity, we now normalize the constant Z to be 1.

[1]Extending the model with the institution of private ownership and competitive markets, the absence of a growing standard of living corresponds to the doctrine from classical

7.2 Law of motion

The dynamics of the model can be reduced to one differential equation, the law of motion of technical knowledge. By (7.3) and (7.1), $L_t = Y_t/\bar{y} = A_t^{\sigma} L_t^{\alpha}/\bar{y}$. Consequently $L_t^{1-\alpha} = A_t^{\sigma}/\bar{y}$ so that

$$L_t = \bar{y}^{\frac{1}{\alpha-1}} A_t^{\frac{\sigma}{1-\alpha}}. \tag{7.5}$$

Substituting this into (7.2) gives the law of motion of technical knowledge:

$$\dot{A}_t = \lambda \bar{y}^{\frac{1}{\alpha-1}} A_t^{\varepsilon+\frac{\sigma}{1-\alpha}} \equiv \hat{\lambda} A_t^{\mu}, \tag{7.6}$$

where we have defined $\hat{\lambda} \equiv \lambda \bar{y}^{1/(\alpha-1)}$ and $\mu \equiv \varepsilon + \sigma/(1-\alpha)$. As will appear in the remainder, the "feedback parameter" μ is of key importance for the dynamics. We immediately see that if $\mu = 1$, the differential equation (7.6) is linear, while otherwise it is nonlinear.

The case $\mu = 1$: When $\mu = 1$, there will be a constant growth rate $g_A = \hat{\lambda}$ in technical knowledge. By (7.5), this results in a constant population growth rate $g_L = [\sigma/(1-\alpha)]\,\hat{\lambda}$, which is also the growth rate of output in view of (7.3). By the definition of $\hat{\lambda}$ in (7.6), we see that, as expected, the population and output growth rate is an increasing function of the creativity parameter λ and a decreasing function of the subsistence minimum.[2]

In this case the economy never leaves the Malthusian regime of a more or less constant standard of living close to existence minimum. Takeoff never occurs.

The case $\mu \neq 1$. Then (7.6) can be written

$$\dot{A}_t = \hat{\lambda} A_t^{\mu}, \tag{7.7}$$

which is a nonlinear differential equation in A.[3] Let $x \equiv A^{1-\mu}$. Then

$$\dot{x}_t = (1-\mu) A_t^{-\mu} \hat{\lambda} A_t^{\mu} = (1-\mu)\hat{\lambda}, \tag{7.8}$$

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economics called the *iron law of wages.* This is the theory (from Malthus and Ricardo) that scarce natural resources and the pressure from population growth causes real wages to remain at subsistende level. There may occationally occur a technological improvement, which leads to a transitory real wage increase, triggering of an increase in population which ultimately brings down wages.

These classical economists did not recognize any tendency to sustained technical progress and therefore missed the immanent tendency to sustained population growth at the pre-industrial stage of economic development. Karl Marx was the first among the classical economists to really see and emphasize sustained technical progress.

[2] If $\sigma = 1-\alpha$ as in Acemoglu's analysis, $\mu = 1$ requires $\varepsilon = 0$, and in this case L and Y grow at the same rate as knowledge.

[3] The differential equation, (7.7), is a special case of what is known as the *Bernoulli equation.* In spite of being a non-linear differential equation, the Bernoulli equation always has an explicit solution.

a constant. To find x_t from this, we only need simple integration:

$$x_t = x_0 + \int_0^t \dot{x}_\tau d\tau = x_0 + (1-\mu)\hat{\lambda}t.$$

As $A = x^{\frac{1}{1-\mu}}$ and $x_0 = A_0^{1-\mu}$, this implies

$$A_t = x_t^{\frac{1}{1-\mu}} = \left[A_0^{1-\mu} + (1-\mu)\hat{\lambda}t\right]^{\frac{1}{1-\mu}} = \frac{1}{\left[A_0^{1-\mu} - (\mu-1)\hat{\lambda}t\right]^{\frac{1}{\mu-1}}}. \tag{7.9}$$

There are now two sub-cases, $\mu > 1$ and $\mu < 1$. The latter sub-case leads to permanent but decelerating growth in knowledge and population and the Malthusian regime is never transcended (see Exercise III.3). The former sub-case is the interesting one.

7.3 The inevitable ending of the Malthusian regime when $\mu > 1$

Assume $\mu > 1$. In this case the result (7.9) implies that the Malthusian regime *must* come to an end.

Although to begin with, A_t may grow extremely slowly, the growth in A_t will be *accelerating* because of the *positive feedback* (visible in (7.2)) from both rising population and rising A_t. Indeed, since $\mu > 1$, the denominator in (7.9) will be decreasing over time and approach zero in finite time, namely as t approaches the finite value $t^* = A_0^{1-\mu}/((\mu-1)\hat{\lambda})$. As an implication, according to (7.9), A_t goes towards *infinity* in *finite* time. The stylized graph in Fig. 7.1 illustrates. The evolution of technical knowledge becomes *explosive* as t approaches t^*.

It follows from (7.5) and (7.1) that explosive growth in A implies explosive growth in L and Y, respectively. The acceleration in the evolution of Y will sooner or later make Y rise fast enough so that the Malthusian population mechanism (which for biological reasons has to be slow) can not catch up. Then, what was in the Malthusian regime only a transitory excess of y_t over $\bar{y}$, will at some $t = \hat{t} < t^*$ become a permanent excess and take the form of sustained growth in y_t.

We may think of this post-Malthusian phase as describing pre-industrial Britain. Technological innovations speeded up, helped by market-friendly institutions, intellectual property rights, and deliberate and systematic application of science and engineering. This lead to the *takeoff* known as the *industrial revolution*.

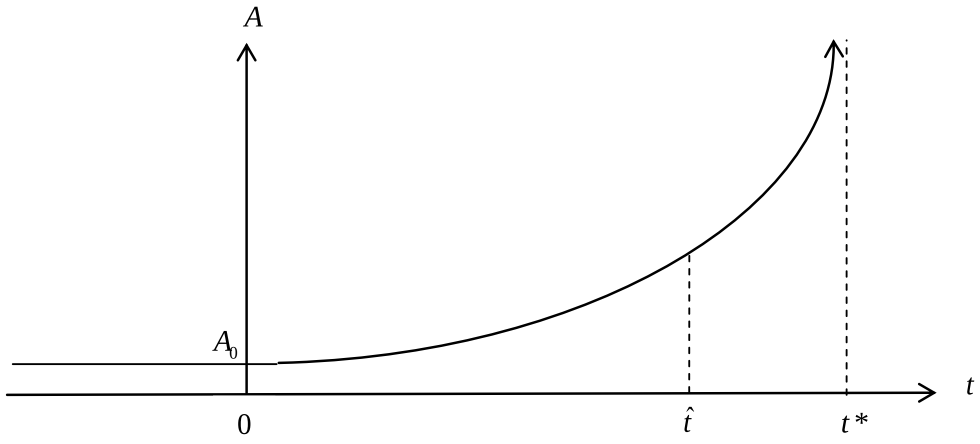

Figure 7.1: Accelerating growth in A when the feedback parameter μ exceeds one.

Note that Fig. 7.1 illustrates only what the process (7.7), with $\mu > 1$, implies *as long as it rules*, namely that knowledge goes towards infinity in *finite* time. The process necessarily ceases to rule long before time t^* is reached, however. This is because the process presupposes that the Malthusian population mechanism keeps track with output growth so as to maintain (7.3) which at some point before t^* becomes impossible because of the acceleration in the latter.

In a neighborhood of this point the takeoff will occur, featuring sustained growth in output per capita. According to equation (7.4), the takeoff should also feature a permanently rising population growth rate. As economic history has testified, however, along with the rising standard of living the demographics changed radically (in the U.K. during the 19th century). The *demographic transition* took place with fertility declining faster than mortality. This results in completely different dynamics, hence the model as it stands no longer fits.[4] As to the demographic transition as such, explanations suggested by economists include: higher real wages mean higher opportunity costs of raising children instead of producing; reduced use of child labor; the trade-off between "quality" (educational level) of the offspring and their "quantity" (Becker, Galor)[5]; skill-biased technical change; and improved contraception technology.

[4]Kremer (1993), however, also includes an extended model taking some of these changed dynamics into account.

[5]See Acemoglu, Section 21.2.

7.4 Closing remarks

The population-breeds-ideas model is about dynamics in the Malthusian regime of the pre-industrial epoch. The story told by the model is the following. When the feedback parameter, μ, is above one, the Malthusian regime has to come to an end because the battle between scarcity of land (or natural resources more generally) and technological progress (absent natural catastrophes) will inevitably be won by the latter. The reason is the cumulative and nonrival character of technical knowledge. This nonrivalry implies economies of scale. Moreover, the stock of knowledge is growing *endogenously.* This knowledge growth generates output growth and, through the demographic mechanism (7.3), growth in the stock of people, which implies a *positive feedback* to the growth of knowledge and so on. On top of this, if $\varepsilon > 0$, knowledge growth has a direct positive feedback on itself through (7.2). When the total positive feedback is strong enough ($\mu > 1$), it generates an explosive process.[6]

On the basis of demographers' estimates of the growth in global population over most of human history, Kremer (1993) finds empirical support for $\mu > 1$. Indeed, in the opposite case, $\mu \leq 1$, there would *not* have been a rising world population growth rate since one million years B.C. to the industrial revolution. The data in Kremer (1993, p. 682) indicates that the world population growth rate has been more or less proportional to the size of population until recently.

Final remark. Compared with Kremer's version of the model, we have allowed $\sigma \neq 1$, but at the same time introduced a simplification relative to Kremer's setup. Kremer starts from a slightly more general ideas-formation equation, namely $\dot{A}_t = \lambda A_t^{\varepsilon} L_t^{\psi}$ with $\psi > 0$, while in our (7.2) we have assumed $\psi = 1$. If $\psi > 1$, the ideas-creating brains reinforce one another. This only fortifies the acceleration in knowledge creation and thereby "supports" the case $\mu > 1$.[7] If on the other hand $0 < \psi < 1$, the idea-creating brains partly offset one another, for instance by simultaneously coming up with more or less the same ideas (the case of "overlap"). This generalization does not change the qualitative results. By assuming that the number of new ideas per time unit is proportional to the stock of brains, we have chosen to focus on an intermediate case in order to avoid secondary factors blurring the main mechanism.

[6] In the appendix the explosion result is considered in a general mathematical context.

[7] Kremer's calibration suggests $\psi \approx 6/5$.

7.5 Appendix

A. The mathematical background

Mathematically, the background for the explosion result is that the solution to a first-order differential equation of the form $\dot{x}(t) = \alpha + bx(t)^c$, $c > 1$, $b \neq 0$, $x(0) = x_0$ given, is always explosive. Indeed, the solution, $x = x(t)$, will have the property that $x(t) \rightarrow \pm\infty$ for $t \rightarrow t^*$ for some $t^* > 0$ where t^* depends on the initial conditions; and thereby the solution is defined only on a bounded time interval which depends on the initial condition.

Take the differential equation $\dot{x}(t) = 1 + x(t)^2$, $x(0) = 0$,as an example. As is well-known, the solution is $x(t) = \tan t = \sin t / \cos t$, defined for $t \in (-\pi/2, \pi/2)$.

B. Comparison with the two special cases considered in Acemoglu (2009)

At pp. 113-14 Acemoglu presents two versions of this framework, both of which assume $\sigma = 1 - \alpha$. This assumption is arbitrary; it is included as a special case in our formulation above. As to the other parameter relating to the role of knowledge, ε, Acemuglu assumes $\varepsilon = 0$ in his first version of the framework. This leads to constant population growth but a forever stagnating standard of living (Acemoglu, p. 113). In his second version, Acemoglu assumes $\varepsilon = 1$. This leads to many centuries of slow but (weakly) accelerating population growth and then ultimately a "takeoff" with sustained rise in the standard of living, to be followed by the "demographic transition" (outside the model). This latter outcome arises for a much larger set of parameter values than $\varepsilon = 1$ and is therefore theoretically more robust than appears in Acemoglu's exposition.

7.6 References

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