Chapter 1

Introduction to economic growth

This introductory lecture note is a refresher on basic concepts.

Section 1.1 defines Economic Growth as a field of economics. In Section 1.2 formulas for calculation of compound average growth rates in discrete and continuous time are presented. Section 1.3 briefly presents two sets of what is by many considered as "stylized facts" about economic growth. Finally, Section 1.4 discusses, in an informal way, the different concepts of cross-country income convergence. In his introductory Chapter 1, §1.5, Acemoglu[1] briefly touches upon these concepts.

1.1 The field

Economic growth analysis is the study of what factors and mechanisms determine the time path of *productivity* (a simple index of productivity is output per unit of labor). The focus is on

- productivity levels and
- productivity growth.

1.1.1 Economic growth theory

Economic growth theory endogenizes productivity growth via considering human capital accumulation (formal education as well as learning-by-doing)

[1] Throughout these lecture notes, "Acemoglu" refers to Daron Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press: Oxford, 2009.

and endogenous research and development. Also the conditioning role of geography and juridical, political, and cultural institutions is taken into account.

For practical reasons, economic growth theory is often stated in terms of national income and product account variables like per capita GDP. Yet the term "economic growth" may be interpreted as referring to something deeper. We could think of "economic growth" as the widening of the opportunities of human beings to lead a freer and more worthwhile life (cf. Sen, ...).

To make our complex economic environment accessible for theoretical analysis we use economic models. What *is* an economic model? It is a way of organizing one's thoughts about the economic functioning of a society. A more specific answer is to define an economic model as a conceptual structure based on a set of mathematically formulated assumptions which have an economic interpretation and from which empirically testable predictions can be derived. In particular, an economic growth model is an economic model concerned with productivity issues. The union of connected and non-contradictory models dealing with economic growth and the propositions derived from these models constitute *economic growth theory*. Occasionally, intense controversies about the validity of alternative growth theories take place.

The terms "New Growth Theory" and "endogenous growth theory" refer to theory and models which attempt at explaining sustained per capita growth as an outcome of internal mechanisms in the model rather than just a reflection of exogenous technical progress as in "Old Growth Theory".

Among the themes addressed in this course are:

- How is the world income distribution evolving?
- Why do living standards differ so much across countries and regions? Why are some countries 50 times richer than others?
- Why do per capita growth rates differ over long periods?
- What are the roles of human capital and technology innovation in economic growth? Getting the questions right.
- Catching-up and increased speed of communication and technology diffusion.
- Economic growth, natural resources, and the environment (including the climate). What are the limits to growth?
- Policies to ignite and sustain productivity growth.

- The prospects of growth in the future.

The course concentrates on *mechanisms* behind the evolution of productivity in the industrialized world. We study these mechanisms as integral parts of dynamic models.

The exam is a test of the extent to which the student has acquired understanding of these models, is able to evaluate them, from both a theoretical and empirical perspective, and is able to use them to analyze specific economic questions. The course is calculus intensive.

1.1.2 Some long-run data

Let Y denote real GDP (per year) and let N be population size. Then Y/N is GDP per capita. Further, let g_Y denote the average (compound) growth rate of Y per year since 1870 and let $g_{Y/N}$ denote the average (compound) growth rate of Y/N per year since 1870. Table 1.1 gives these growth rates for four countries. (But we should not forget that data from before WWII should be taken with a grain of salt).

	g_Y	$g_{Y/N}$
Denmark	2,67	1,87
UK	1,96	1,46
USA	3,40	1,89
Japan	3,54	2,54

Table 1.1: Average annual growth rate of GDP and GDP per capita in percent, 1870–2006. Discrete compounding. Source: Maddison, A: The World Economy: Historical Statistics, 2006, Table 1b, 1c and 5c.

Figure 1.1 displays the time path of annual GDP and GDP per capita in Denmark 1870-2006 along with regression lines estimated by OLS (logarithmic scale on the vertical axis). Figure 1.2 displays the time path of GDP per capita in UK, USA, and Japan 1870-2006. In both figures the average annual growth rates are reported. In spite of being based on exactly the same data as Table 1.1, the numbers are slightly different. Indeed, the numbers in the figures are slightly lower than those in the table. The reason is that discrete compounding is used in Table 1.1 while continuous compounding is used in the two figures. These two alternative methods of calculation are explained in the next section.

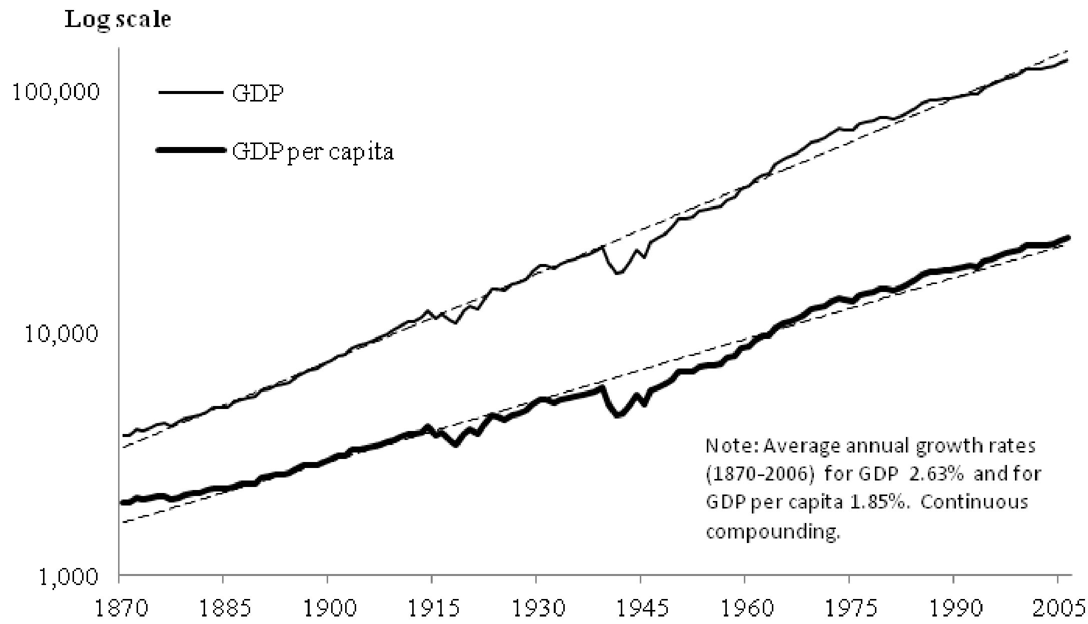


Figure 1.1: GDP and GDP per capita (1990 International Geary-Khamis dollars) in Denmark, 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

1.2 Calculation of the average growth rate

1.2.1 Discrete compounding

Let y denote aggregate labor productivity, i.e., $y \equiv Y/L$, where L is employment. The average growth rate of y from period 0 to period t, with discrete compounding, is that G which satisfies

$$\begin{aligned} y_t &= y_0(1+G)^t, \qquad t = 1, 2, ... , \qquad \text{or} && (1.1) \\ 1+G &= (\frac{y_t}{y_0})^{1/t}, \text{ i.e.,} \\ G &= (\frac{y_t}{y_0})^{1/t} - 1. && (1.2) \end{aligned}$$

"Compounding" means adding the one-period "net return" to the "principal" before adding next period's "net return" (like with interest on interest, also called "compound interest"). The growth factor $1 + G$ will generally be less than the arithmetic average of the period-by-period growth factors. To

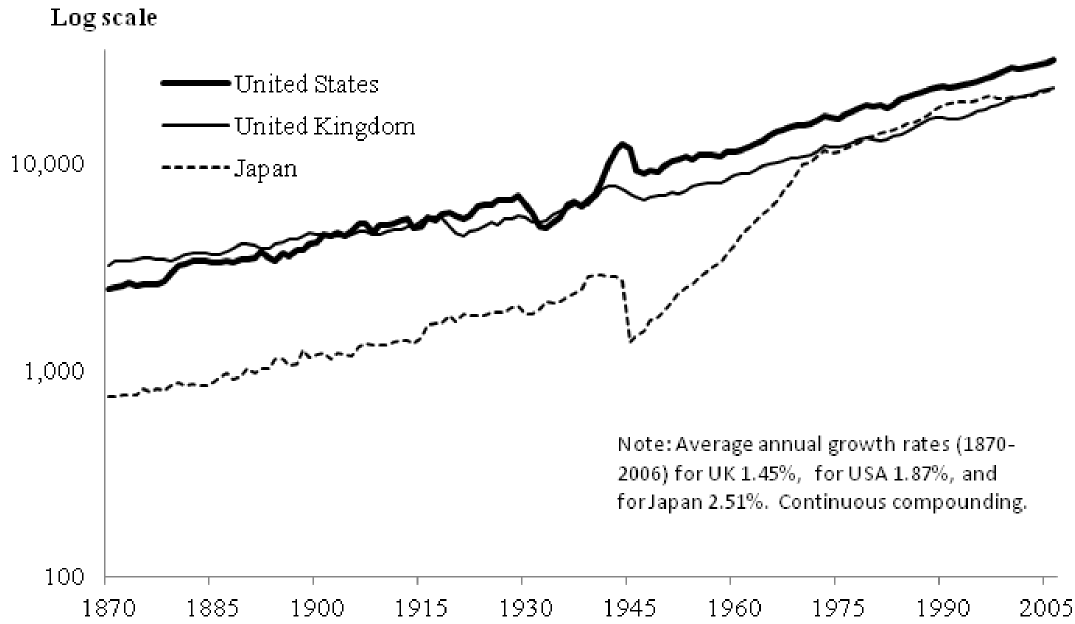


Figure 1.2: GDP per capita (1990 International Geary-Khamis dollars) in UK, USA and Japan, 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

underline this difference, $1+G$ is sometimes called the "compound average growth factor" or the "geometric average growth factor" and G itself then called the "compound average growth rate" or the "geometric average growth rate"

Using a pocket calculator, the following steps in the calculation of G may be convenient. Take logs on both sides of (1.1) to get

$$\begin{aligned}
\ln\frac{y_t}{y_0} &= t\ln(1+G) \;\Rightarrow \\
\ln(1+G) &= \frac{\ln\frac{y_t}{y_0}}{t} \Rightarrow \qquad (1.3) \\
G &= \text{antilog}\left(\frac{\ln\frac{y_t}{y_0}}{t}\right) - 1. \qquad (1.4)
\end{aligned}$$

Note that t in the formulas (1.2) and (1.4) equals the number of periods *minus 1.*

1.2.2 Continuous compounding

The average growth rate of y, with continuous compounding, is that g which satisfies

$$y_t = y_0 e^{gt}, \tag{1.5}$$

where e denotes the Euler number, i.e., the base of the natural logarithm.[2] Solving for g gives

$$g = \frac{\ln \frac{y_t}{y_0}}{t} = \frac{\ln y_t - \ln y_0}{t}. \tag{1.6}$$

The first formula in (1.6) is convenient for calculation with a pocket calculator, whereas the second formula is perhaps closer to intuition. Another name for g is the "exponential average growth rate".

Again, for discrete time data the t in the formula equals the number of periods minus 1.

Comparing with (1.3) we see that $g = \ln(1 + G) < G$ for $G \neq 0$. Yet, by a first-order Taylor approximation of $\ln(1 + G)$ about $G = 0$ we have

$$g = \ln(1 + G) \approx G \text{ for } G \text{ "small"}. \tag{1.7}$$

For a given data set the G calculated from (1.2) will be slightly above the g calculated from (1.6), cf. the mentioned difference between the growth rates in Table 1.1 and those in Figure 1.1 and Figure 1.2. The reason is that a given growth force is more powerful when compounding is continuous rather than discrete. Anyway, the difference between G and g is usually unimportant. If for example G refers to the annual GDP growth rate, it will be a small number, and the difference between G and g immaterial. For example, to $G = 0.040$ corresponds $g \approx 0.039$. Even if $G = 0.10$, the corresponding g is 0.0953. But if G stands for the inflation rate and there is high inflation, the difference between G and g will be substantial. During hyperinflation the monthly inflation rate may be, say, $G = 100\%$, but the corresponding g will be only 69%.

Which method, discrete or continuous compounding, is preferable? To some extent it is a matter of taste or convenience. In period analysis discrete compounding is most common and in continuous time analysis continuous compounding is most common.

For calculation with a pocket calculator the continuous compounding formula, (1.6), is slightly easier to use than the discrete compounding formulas, whether (1.2) or (1.4).

[2]Unless otherwise specified, whenever we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

To avoid too much sensitiveness to the initial and terminal observations, which may involve measurement error or depend on the state of the business cycle, one can use an OLS approach to the trend coefficient, g, in the following regression:

$$\ln Y_t = \alpha + gt + \varepsilon_t.$$

This is in fact what is done in Fig. 1.1.

1.2.3 Doubling time

How long time does it take for y to double if the growth rate with discrete compounding is G? Knowing G, we rewrite the formula (1.3):

$$t = \frac{\ln \frac{y_t}{y_0}}{\ln(1+G)} = \frac{\ln 2}{\ln(1+G)} \approx \frac{0.6931}{\ln(1+G)}.$$

With $G = 0.0187$, cf. Table 1.1, we find

$$t \approx 37.4 \text{ years},$$

meaning that productivity doubles every 37.4 years.

How long time does it take for y to double if the growth rate with continuous compounding is g? The answer is based on rewriting the formula (1.6):

$$t = \frac{\ln \frac{y_t}{y_0}}{g} = \frac{\ln 2}{g} \approx \frac{0.6931}{g}.$$

Maintaining the value 0.0187 also for g, we find

$$t \approx \frac{0.6931}{0.0187} \approx 37.1 \text{ years}.$$

Again, with a pocket calculator the continuous compounding formula is slightly easier to use. With a lower g, say $g = 0.01$, we find doubling time equal to 69.1 years. With $g = 0.07$ (think of China since the early 1980's), doubling time is about 10 years! Owing to the compounding, exponential growth is extremely powerful.

1.3 Some stylized facts of economic growth

1.3.1 The Kuznets facts

A well-known characteristic of modern economic growth is structural change: unbalanced sectorial growth. There is a massive reallocation of labor from

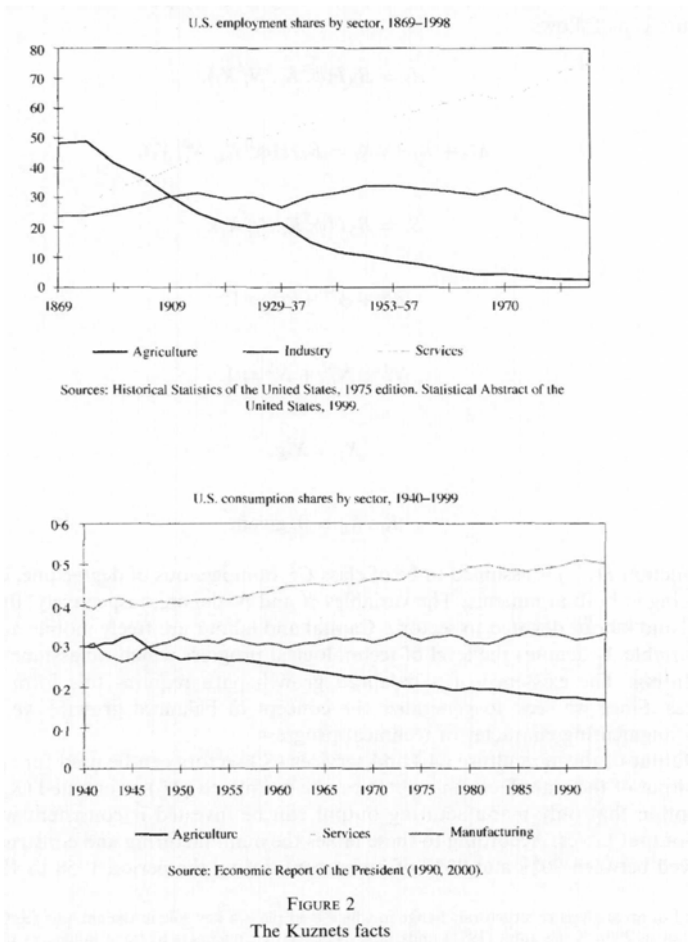


Figure 1.3: The Kuznets facts. Source: Kongsamut et al., Beyond Balanced Growth, Review of Economic Studies, vol. 68, Oct. 2001, 869-82.

agriculture into industry (manufacturing, construction, and mining) and further into services (including transport and communication). The shares of total consumption expenditure going to these three sectors have moved similarly. Differences in the demand elasticities with respect to income seem the main explanation. These observations are often referred to as the *Kuznets facts* (after Simon Kuznets, 1901-85, see, e.g., Kuznets 1957).

The two graphs in Figure 1.3 illustrate the Kuznets facts.

1.3.2 Kaldor's stylized facts

Surprisingly, in spite of the Kuznets facts, the evolution at the *aggregate* level in developed countries is by many economists seen as roughly described by what is called Kaldor's "stylized facts" (after the Hungarian-British economist Nicholas Kaldor, 1908-1986, see, e.g., Kaldor 1957, 1961)[3]:

1. Real output per man-hour grows at a more or less constant rate over fairly long periods of time. (Of course, there are short-run fluctuations superposed around this trend.)

2. The stock of physical capital per man-hour grows at a more or less constant rate over fairly long periods of time.

3. The ratio of output to capital shows no systematic trend.

4. The rate of return to capital shows no systematic trend.

5. The income shares of labor and capital (in the national accounting sense, i.e., including land and other natural resources), respectively, are nearly constant.

6. The growth rate of output per man-hour differs substantially across countries.

These claimed regularities do certainly not fit all developed countries equally well. Although Solow's growth model (Solow, 1956) can be seen as the first successful attempt at building a model consistent with Kaldor's "stylized facts", Solow once remarked about them: "There is no doubt that they are stylized, though it is possible to question whether they are facts" (Solow, 1970). Yet, for instance the study by Attfield and Temple (2010) of US and UK data since the Second World War concludes with support for Kaldor's "facts". Recently, several empiricists[4] have questioned "fact" 5, however, referring to the inadequacy of the methods which standard national income accounting applies to separate the income of entrepreneurs, sole proprietors, and unincorporated businesses into labor and capital income. It is claimed that these methods obscure a tendency in recent decades of the labor income share to fall.

The sixth Kaldor fact is, of course, generally accepted as a well documented observation (a nice summary is contained in Pritchett, 1997).

Kaldor also proposed hypotheses about the links between growth in the different sectors (see, e.g., Kaldor 1967):

a. Productivity growth in the manufacturing and construction sectors is enhanced by output growth in these sectors (this is also known as Verdoorn's Law). Increasing returns to scale and learning by doing are the main factors behind this.

[3] Kaldor presented his six regularities as "a stylised view of the facts".

[4] E.g., Gollin (2002), Elsby et al. (2013), and Karabarbounis and Neiman (2014).

b. Productivity growth in agriculture and services is enhanced by output growth in the manufacturing and construction sectors.

Kongsamut et al. (2001) and Foellmi and Zweimüller (2008) offer theoretical explanations of why the Kuznets facts and the Kaldor facts can coexist.

1.4 Concepts of income convergence

The two most popular across-country income convergence concepts are "β convergence" and "σ convergence".

1.4.1 β convergence vs. σ convergence

Definition 1 *We say that β convergence occurs for a given selection of countries if there is a tendency for the poor (those with low income per capita or low output per worker) to subsequently grow faster than the rich.*

By "grow faster" is meant that the growth rate of per capita income (or per worker output) is systematically higher.

In many contexts, a more appropriate convergence concept is the following:

Definition 2 *We say that σ convergence, with respect to a given measure of dispersion, occurs for a given collection of countries if this measure of dispersion, applied to income per capita or output per worker across the countries, declines systematically over time. On the other hand, σ divergence occurs, if the dispersion increases systematically over time.*

The reason that σ convergence must be considered the more appropriate concept is the following. In the end, it is the question of increasing or decreasing dispersion across countries that we are interested in. From a superficial point of view one might think that β convergence implies decreasing dispersion and vice versa, so that β convergence and σ convergence are more or less equivalent concepts. But since the world is not deterministic, but stochastic, this is not true. Indeed, β convergence is only a necessary, not a sufficient condition for σ convergence. This is because over time some reshuffling among the countries is always taking place, and this implies that there will always be some extreme countries (those initially far away from the mean) that move closer to the mean, thus creating a negative correlation between initial level and subsequent growth, in spite of equally many countries moving from a middle position toward one of the extremes.[5] In this way β convergence may be observed at the same time as there is no σ convergence; the mere presence of random measurement errors implies a bias in this direction because a growth rate depends negatively on the initial measurement and positively on the later measurement. In fact, β convergence may be consistent with σ *divergence* (for a formal proof of this claim, see Barro and Sala-i-Martin, 2004, pp. 50-51 and 462 ff.; see also Valdés, 1999, p. 49-50, and Romer, 2001, p. 32-34).

Hence, it is wrong to conclude from β convergence (poor countries tend to grow faster than rich ones) to σ convergence (reduced dispersion of per capita income) without any further investigation. The mistake is called "regression towards the mean" or "Galton's fallacy". Francis Galton was an anthropologist (and a cousin of Darwin), who in the late nineteenth century observed that tall fathers tended to have not as tall sons and small fathers tended to have taller sons. From this he falsely concluded that there was a tendency to averaging out of the differences in height in the population. Indeed, being a true aristocrat, Galton found this tendency pitiable. But since his conclusion was mistaken, he did not really have to worry.

Since σ convergence comes closer to what we are ultimately looking for, from now, when we speak of just "income convergence", σ convergence is understood.

In the above definitions of σ convergence and β convergence, respectively, we were vague as to what kind of selection of countries is considered. In principle we would like it to be a representative sample of the "population" of countries that we are interested in. The population could be all countries in the world. Or it could be the countries that a century ago had obtained a certain level of development.

One should be aware that historical GDP data are constructed retrospectively. Long time series data have only been constructed for those countries that became relatively rich during the after-WWII period. Thus, if we as our sample select the countries for which long data series exist, what is known as *selection bias* is involved which generates a spurious convergence. A country which was poor a century ago will only appear in the sample if it grew rapidly over the next 100 years. A country which was relatively rich a century ago will appear in the sample unconditionally. This selection bias problem was

[5] As an intuitive analogy, think of the ordinal rankings of the sports teams in a league. The dispersion of rankings is constant by definition. Yet, no doubt there will allways be some tendency for weak teams to rebound toward the mean and of champions to revert to mediocrity. (This example is taken from the first edition of Barro and Sala-i-Martin, *Economic Growth*, 1995; I do not know why, but the example was deleted in the second edition from 2004.)

pointed out by DeLong (1988) in a criticism of widespread false interpretations of Maddison's long data series (Maddison 1982).

1.4.2 Measures of dispersion

Our next problem is: *what* measure of dispersion is to be used as a useful descriptive statistics for σ convergence? Here there are different possibilities. To be precise about this we need some notation. Let

$$\begin{aligned} y &\equiv \frac{Y}{L}, \qquad \text{and} \\ q &\equiv \frac{Y}{N}, \end{aligned}$$

where Y = real GDP, L = employment, and N = population. If the focus is on living standards, Y/N, is the relevant variable.[6] But if the focus is on (labor) productivity, it is Y/L, that is relevant. Since most growth models focus on Y/L rather than Y/N, let os take y as our example.

One might think that the standard deviation of y could be a relevant measure of dispersion when discussing whether σ convergence is present or not. The *standard deviation* of y across n countries in a given year is

$$\sigma_y \equiv \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}, \tag{1.8}$$

where

$$\bar{y} \equiv \frac{\sum_i y_i}{n}, \tag{1.9}$$

i.e., $\bar{y}$ is the average output per worker. However, if this measure were used, it would be hard to find *any* group of countries for which there is income convergence. This is because y tends to grow over time for most countries, and then there is an inherent tendency for the variance also to grow; hence also the square root of the variance, σ_y, tends to grow. Indeed, suppose that for all countries, y is doubled from time t_1 to time t_2. Then, automatically, σ_y is also doubled. But hardly anyone would interpret this as an increase in the income inequality across the countries.

Hence, it is more adequate to look at the standard deviation of *relative* income levels:

$$\sigma_{y/\bar{y}} \equiv \sqrt{\frac{1}{n}\sum_i(\frac{y_i}{\bar{y}} - 1)^2}. \tag{1.10}$$

[6] Or perhaps better, Q/N, where $Q \equiv GNP \equiv GDP - rD - wF$. Here, rD, denotes net interest payments on foreign debt and wF denotes net labor income of foreign workers in the country.

This measure is the same as what is called the *coefficient of variation*, CV_y, usually defined as

$$CV_y \equiv \frac{\sigma_y}{\bar{y}}, \tag{1.11}$$

that is, the standard deviation of y standardized by the mean. That the two measures are identical can be seen in this way:

$$\frac{\sigma_y}{\bar{y}} \equiv \frac{\sqrt{\frac{1}{n}\sum_i (y_i - \bar{y})^2}}{\bar{y}} = \sqrt{\frac{1}{n}\sum_i (\frac{y_i - \bar{y}}{\bar{y}})^2} = \sqrt{\frac{1}{n}\sum_i (\frac{y_i}{\bar{y}} - 1)^2} \equiv \sigma_{y/\bar{y}}.$$

The point is that the coefficient of variation is "scale free", which the standard deviation itself is not.

Instead of the coefficient of variation, another scale free measure is often used, namely the standard deviation of $\ln y$, i.e.,

$$\sigma_{\ln y} \equiv \sqrt{\frac{1}{n}\sum_i (\ln y_i - \ln y^*)^2}, \tag{1.12}$$

where

$$\ln y^* \equiv \frac{\sum_i \ln y_i}{n}. \tag{1.13}$$

Note that y^* is the geometric average, i.e., $y^* \equiv \sqrt[n]{y_1 y_2 \cdots y_n}$. Now, by a first-order Taylor approximation of $\ln y$ around $y = \bar{y}$, we have

$$\ln y \approx \ln \bar{y} + \frac{1}{\bar{y}}(y - \bar{y})$$

Hence, as a very rough approximation we have $\sigma_{\ln y} \approx \sigma_{y/\bar{y}} = CV_y$, though this approximation can be quite poor (cf. Dalgaard and Vastrup, 2001). It may be possible, however, to defend the use of $\sigma_{\ln y}$ in its own right to the extent that y tends to be approximately lognormally distributed across countries.

Yet another possible measure of income dispersion across countries is the *Gini index* (see for example Cowell, 1995).

1.4.3 Weighting by size of population

Another important issue is whether the applied dispersion measure is based on a *weighting of the countries by size of population.* For the world as a whole, when no weighting by size of population is used, then there is a slight tendency to income divergence according to the $\sigma_{\ln q}$ criterion (Acemoglu, 2009, p. 4), where q is per capita income ($\equiv Y/N$). As seen by Fig. 4 below, this tendency is not so clear according to the CV_q criterion. Anyway, when there *is* weighting by size of population, then in the last twenty years there has been a tendency to income convergence at the global level (Sala-i-Martin 2006; Acemoglu, 2009, p. 6). With weighting by size of population (1.12) is modified to

$$\sigma^w_{\ln q} \equiv \sqrt{\sum_i w_i(\ln q_i - \ln q^*)^2},$$

where

$$w_i = \frac{N_i}{N} \quad \text{and} \quad \ln q^* \equiv \sum_i w_i \ln q_i.$$

1.4.4 Unconditional vs. conditional convergence

Yet another distinction in the study of income convergence is that between unconditional (or absolute) and conditional convergence. We say that a large heterogeneous group of countries (say the countries in the world) show *unconditional* income convergence if income convergence occurs for the whole group without conditioning on specific characteristics of the countries. If income convergence occurs only for a subgroup of the countries, namely those countries that in advance share the same "structural characteristics", then we say there is *conditional* income convergence. As noted earlier, when we speak of just income "convergence", income "σ convergence" is understood. If in a given context there might be doubt, one should of course be explicit and speak of unconditional or conditional σ convergence. Similarly, if the focus for some reason is on β convergence, we should distinguish between unconditional and conditional β convergence.

What the precise meaning of "structural characteristics" is, will depend on what model of the countries the researcher has in mind. According to the Solow model, a set of relevant "structural characteristics" are: the aggregate production function, the initial level of technology, the rate of technical progress, the capital depreciation rate, the saving rate, and the population growth rate. But the Solow model, as well as its extension with human capital (Mankiw et al., 1992), is a model of a closed economy with exogenous technical progress. The model deals with "within-country" convergence in the sense that the model predicts that a closed economy being initially below or above its steady state path, will over time converge towards its steady state path. It is far from obvious that this kind of model is a good model of cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and some countries are

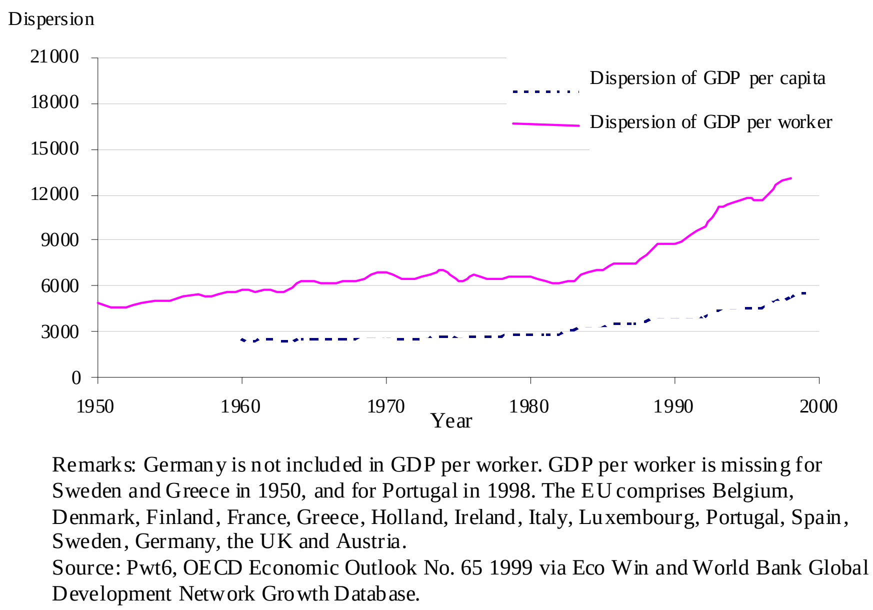


Figure 1.4: Standard deviation of GDP per capita and per worker across 12 EU countries, 1950-1998.

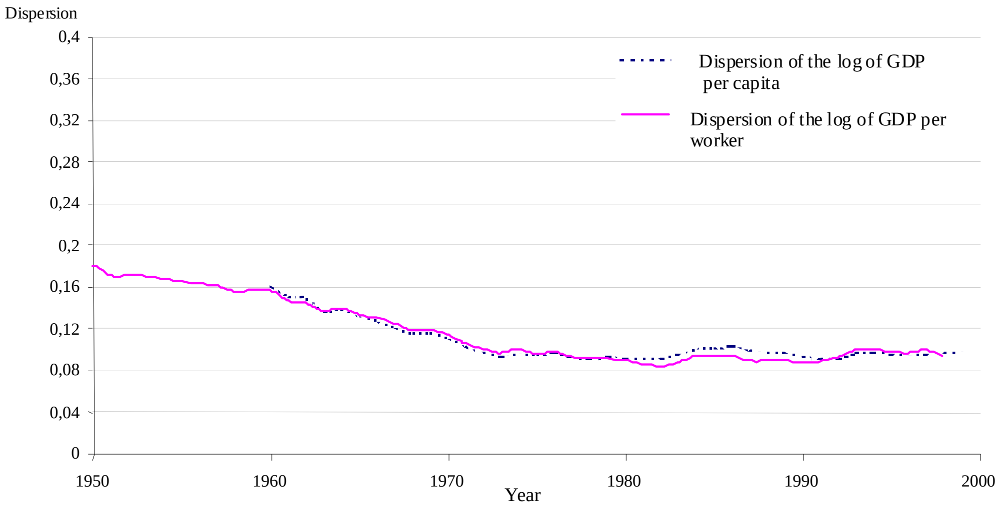
pushing the technological frontier further out, while others try to imitate and catch up.

1.4.5 A bird's-eye view of the data

In the following no serious econometrics is attempted. We use the term "trend" in an admittedly loose sense.

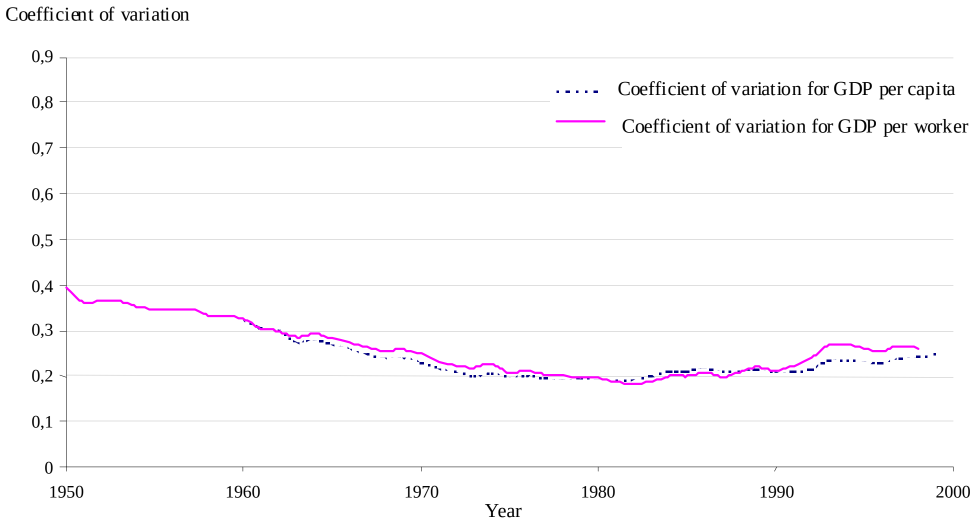
Figure 1.4 shows the time profile for the standard deviation of y itself for 12 EU countries, whereas Figure 1.5 and Figure 1.6 show the time profile of the standard deviation of $\log y$ and the time profile of the coefficient of variation, respectively. Comparing the upward trend in Figure 1.4 with the downward trend in the two other figures, we have an illustration of the fact that the movement of the standard deviation of y itself does not capture income convergence. To put it another way: although there seems to be conditional income convergence with respect to the two scale-free measures, Figure 1.4 shows that this tendency to convergence is *not* so strong as to produce a narrowing of the absolute distance between the EU countries.[7]

[7]Unfortunately, sometimes misleading graphs or texts to graphs about across-country



Remarks: Germany is not included in GDP per worker. GDP per worker is missing for Sweden and Greece in 1950, and for Portugal in 1998. The EU comprises Belgium, Denmark, Finland, France, Greece, Holland, Ireland, Italy, Luxembourg, Portugal, Spain, Sweden, Germany, the UK and Austria.
Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

Figure 1.5: Standard deviation of the log of GDP per capita and per worker across 12 EU countries, 1950-1998.



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Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

Figure 1.6: Coefficient of variation of GDP per capita and GDP per worker across 12 EU countries, 1950-1998.

Figure 1.7 shows the time path of the coefficient of variation across 121 countries in the world, 22 OECD countries and 12 EU countries, respectively. We see the lack of unconditional income convergence, but the presence of conditional income convergence. One should not over-interpret the observation of convergence for the 22 OECD countries over the period 1950-1990. It is likely that this observation suffer from the selection bias problem mentioned in Section 1.4.1. A country that was poor in 1950 will typically have become a member of OECD only if it grew relatively fast afterwards.

1.4.6 Other convergence concepts

Of course, just considering the time profile of the first and second moments of a distribution may sometimes be a poor characterization of the evolution of the distribution. For example, there are signs that the distribution has polarized into *twin peaks* of rich and poor countries (Quah, 1996a; Jones,

income convergence are published. In the collection of exercises, Chapter 1, you are asked to discuss some examples of this.

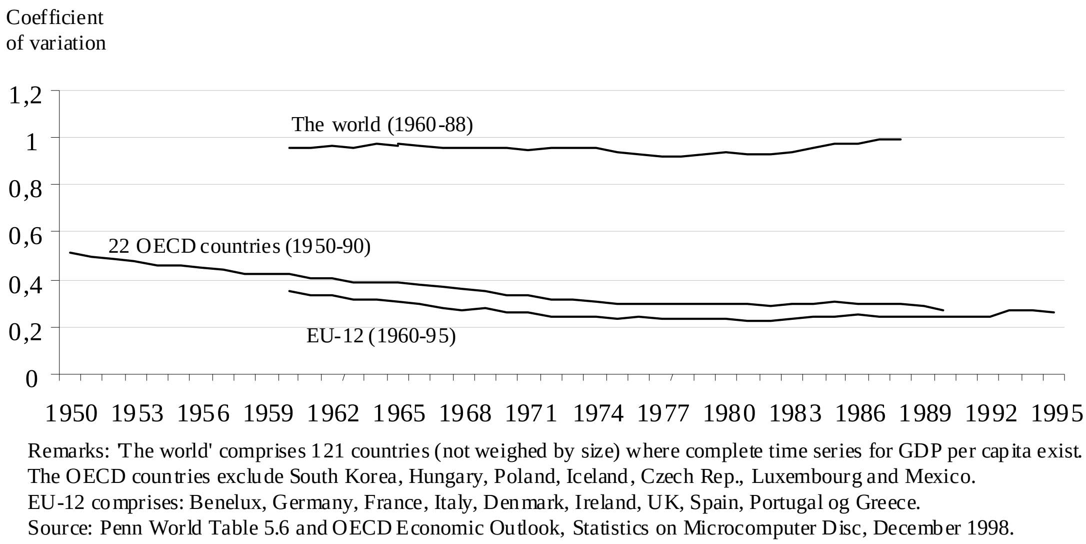

Figure 1.7: Coefficient of variation of income per capita across different sets of countries.

1997). Related to this observation is the notion of club convergence. If income convergence occurs *only* among a subgroup of the countries that to some extent share the same initial conditions, then we say there is *club-convergence.* This concept is relevant in a setting where there are *multiple* steady states toward which countries can converge. At least at the theoretical level multiple steady states can easily arise in overlapping generations models. Then the initial condition for a given country matters for which of these steady states this country is heading to. Similarly, we may say that *conditional club-convergence* is present, if income convergence occurs *only* for a subgroup of the countries, namely countries sharing similar structural characteristics (this may to some extent be true for the OECD countries) *and*, within an interval, similar initial conditions.

Instead of focusing on income convergence, one could study *TFP convergence* at aggregate or industry level.[8] Sometimes the less demanding concept of *growth rate convergence* is the focus.

The above considerations are only of a very elementary nature and are only about descriptive statistics. The reader is referred to the large existing literature on concepts and econometric methods of relevance for character-

[8]See, for instance, Bernard and Jones 1996a and 1996b.

izing the evolution of world income distribution (see Quah, 1996b, 1996c, 1997, and for a survey, see Islam 2003).

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Chapter 2

Review of technology and factor shares of income

The aim of this chapter is, first, to introduce the terminology concerning firms' technology and technological change used in the lectures and exercises of this course. At a few points I deviate somewhat from definitions in Acemoglu's book. Section 1.3 can be used as a formula manual for the case of CRS.

Second, the chapter contains a brief discussion of the notions of a representative firm and an aggregate production function. The distinction between long-run versus short-run production functions is also commented on. The last sections introduce the concept of elasticity of substitution between capital and labour and its role for the direction of movement over time of the income shares of capital and labor under perfect competition.

Regarding the distinction between discrete and continuous time analysis, most of the definitions contained in this chapter are applicable to both.

2.1 The production technology

Consider a two-factor production function given by

$$Y = F(K, L), \tag{2.1}$$

where Y is output (value added) per time unit, K is capital input per time unit, and L is labor input per time unit ($K \geq 0$, $L \geq 0$). We may think of (2.1) as describing the output of a firm, a sector, or the economy as a whole. It is in any case a very simplified description, ignoring the heterogeneity of output, capital, and labor. Yet, for many macroeconomic questions it may be a useful first approach. Note that in (2.1) not only Y but also K and L

represent *flows,* that is, quantities per unit of time. If the time unit is one year, we think of K as measured in machine hours per year. Similarly, we think of L as measured in labor hours per year. Unless otherwise specified, it is understood that the rate of utilization of the production factors is constant over time and normalized to one for each production factor. As explained in Chapter 1, we can then use the same symbol, K, for the *flow* of capital services as for the *stock* of capital. Similarly with L.

2.1.1 A neoclassical production function

By definition, K and L are non-negative. It is generally understood that a production function, $Y = F(K, L)$, is *continuous* and that $F(0, 0) = 0$ (no input, no output). Sometimes, when specific functional forms are used to represent a production function, that function may not be defined at points where $K = 0$ or $L = 0$ or both. In such a case we adopt the convention that the domain of the function is understood extended to include such boundary points whenever it is possible to assign function values to them such that continuity is maintained. For instance the function $F(K, L) = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$, is not defined at $(K, L) = (0, 0)$. But by assigning the function value 0 to the point $(0, 0)$, we maintain both continuity and the "no input, no output" property.

We call the production function *neoclassical* if for all (K, L), with $K > 0$ and $L > 0$, the following additional conditions are satisfied:

(a) $F(K, L)$ has continuous first- and second-order partial derivatives satisfying:

$$F_K > 0, \quad F_L > 0, \tag{2.2}$$
$$F_{KK} < 0, \quad F_{LL} < 0. \tag{2.3}$$

(b) $F(K, L)$ is strictly quasiconcave (i.e., the level curves, also called isoquants, are strictly convex to the origin).

In words: (a) says that a neoclassical production function has continuous substitution possibilities between K and L and the *marginal productivities* are positive, but diminishing in own factor. Thus, for a given number of machines, adding one more unit of labor, adds to output, but less so, the higher is already the labor input. And (b) says that every isoquant, $F(K, L) = \bar{Y}$, has a strictly convex form qualitatively similar to that shown in Figure 2.1.[1]

[1] For any fixed $\bar{Y} \geq 0$, the associated *isoquant* is the level set $\{(K, L) \in \mathbb{R}_+ |\ F(K, L) = \bar{Y}\}$.

When we speak of for example F_L as the marginal *productivity* of labor, it is because the "pure" partial derivative, $\partial Y/\partial L = F_L$, has the denomination of a productivity (output units/yr)/(man-yrs/yr). It is quite common, however, to refer to F_L as the marginal *product* of labor. Then a unit marginal increase in the labor input is understood: $\Delta Y \approx (\partial Y/\partial L)\Delta L = \partial Y/\partial L$ when $\Delta L = 1$. Similarly, F_K can be interpreted as the marginal *productivity* of capital or as the marginal *product* of capital. In the latter case it is understood that $\Delta K = 1$, so that $\Delta Y \approx (\partial Y/\partial K)\Delta K = \partial Y/\partial K$.

The definition of a neoclassical production function can be extended to the case of n inputs. Let the input quantities be $X_1, X_2, \ldots, X_n$ and consider a production function $Y = F(X_1, X_2, \ldots, X_n)$. Then F is called neoclassical if all the marginal productivities are positive, but diminishing, and F is strictly quasiconcave (i.e., the upper contour sets are strictly convex, cf. Appendix A).

Returning to the two-factor case, since $F(K, L)$ presumably depends on the level of technical knowledge and this level depends on time, t, we might want to replace (2.1) by

$$Y_t = F^t(K_t, L_t), \tag{2.4}$$

where the superscript on F indicates that the production function may shift over time, due to changes in technology. We then say that $F^t(\cdot)$ is a neoclassical production function if it satisfies the conditions (a) and (b) for all pairs (K_t, L_t). *Technological progress* can then be said to occur when, for K_t and L_t held constant, output increases with t.

For convenience, to begin with we skip the explicit reference to time and level of technology.

The marginal rate of substitution Given a neoclassical production function F, we consider the isoquant defined by $F(K, L) = \bar{Y}$, where $\bar{Y}$ is a positive constant. The *marginal rate of substitution*, MRS_{KL}, of K for L at the point (K, L) is defined as the absolute slope of the isoquant at that point, cf. Figure 2.1. The equation $F(K, L) = \bar{Y}$ defines K as an implicit function of L. By implicit differentiation we find $F_K(K, L)dK/dL + F_L(K, L) = 0$, from which follows

$$MRS_{KL} \equiv -\frac{dK}{dL}_{|Y=\bar{Y}} = \frac{F_L(K, L)}{F_K(K, L)} > 0. \tag{2.5}$$

That is, MRS_{KL} measures the amount of K that can be saved (approximately) by applying an extra unit of labor. In turn, this equals the ratio

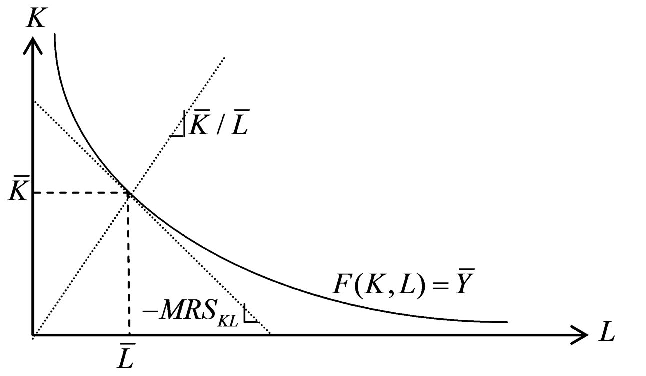


Figure 2.1: MRS_{KL} as the absolute slope of the isoquant.

of the marginal productivities of labor and capital, respectively.[2] Since F is neoclassical, by definition F is strictly quasi-concave and so the marginal rate of substitution is diminishing as substitution proceeds, i.e., as the labor input is further increased along a given isoquant. Notice that this feature characterizes the marginal rate of substitution for any neoclassical production function, whatever the returns to scale (see below).

When we want to draw attention to the dependency of the marginal rate of substitution on the factor combination considered, we write $MRS_{KL}(K, L)$. Sometimes in the literature, the marginal rate of substitution between two production factors, K and L, is called the *technical* rate of substitution (or the technical rate of transformation) in order to distinguish from a consumer's marginal rate of substitution between two consumption goods.

As is well-known from microeconomics, a firm that minimizes production costs for a given output level and given factor prices, will choose a factor combination such that MRS_{KL} equals the ratio of the factor prices. If $F(K, L)$ is homogeneous of degree q, then the marginal rate of substitution depends only on the factor proportion and is thus the same at any point on the ray $K = (\bar{K}/\bar{L})L$. That is, in this case the expansion path is a straight line.

The Inada conditions A continuously differentiable production function is said to satisfy the *Inada conditions*[3] if

$$\lim_{K\to 0} F_K(K, L) = \infty, \lim_{K\to\infty} F_K(K, L) = 0, \tag{2.6}$$

$$\lim_{L\to 0} F_L(K, L) = \infty, \lim_{L\to\infty} F_L(K, L) = 0. \tag{2.7}$$

[2] The subscript $|Y = \bar{Y}$ in (2.5) indicates that we are moving along a given isoquant, $F(K, L) = \bar{Y}$. Expressions like, e.g., $F_L(K, L)$ or $F_2(K, L)$ mean the partial derivative of F w.r.t. the second argument, evaluated at the point (K, L).

[3] After the Japanese economist Ken-Ichi Inada, 1925-2002.

In this case, the marginal productivity of either production factor has no upper bound when the input of the factor becomes infinitely small. And the marginal productivity is gradually vanishing when the input of the factor increases without bound. Actually, (2.6) and (2.7) express *four* conditions, which it is preferable to consider separately and label one by one. In (2.6) we have two *Inada conditions for* MPK (the marginal productivity of capital), the first being a *lower*, the second an *upper* Inada condition for MPK. And in (2.7) we have two *Inada conditions for* MPL (the marginal productivity of labor), the first being a *lower*, the second an *upper* Inada condition for MPL. In the literature, when a sentence like "the Inada conditions are assumed" appears, it is sometimes not made clear which, and how many, of the four are meant. Unless it is evident from the context, it is better to be explicit about what is meant.

The definition of a neoclassical production function we gave above is quite common in macroeconomic journal articles and convenient because of its flexibility. There are textbooks that define a neoclassical production function more narrowly by including the Inada conditions as a requirement for calling the production function neoclassical. In contrast, in this course, when in a given context we need one or another Inada condition, we state it explicitly as an additional assumption.

2.1.2 Returns to scale

If all the inputs are multiplied by some factor, is output then multiplied by the same factor? There may be different answers to this question, depending on circumstances. We consider a production function $F(K, L)$ where $K > 0$ and $L > 0$. Then F is said to have *constant returns to scale* (CRS for short) if it is homogeneous of degree one, i.e., if for all (K, L) and all $\lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

As all inputs are scaled up or down by some factor > 1, output is scaled up or down by the same factor.[4] The assumption of CRS is often defended by the *replication argument.* Before discussing this argument, lets us define the two alternative "pure" cases.

The production function $F(K, L)$ is said to have *increasing returns to scale* (IRS for short) if, for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) > \lambda F(K, L).$$

[4] In their definition of a neoclassical production function some textbooks add constant returns to scale as a requirement besides (a) and (b). This course follows the alternative terminology where, if in a given context an assumption of constant returns to scale is needed, this is stated as an additional assumption.

That is, IRS is present if, when all inputs are scaled up by some factor $>$ 1, output is scaled up by *more* than this factor. The existence of gains by specialization and division of labor, synergy effects, etc. sometimes speak in support of this assumption, at least up to a certain level of production. The assumption is also called the *economies of scale* assumption.

Another possibility is *decreasing returns to scale* (DRS). This is said to occur when for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) < \lambda F(K, L).$$

That is, DRS is present if, when all inputs are scaled up by some factor, output is scaled up by *less* than this factor. This assumption is also called the *diseconomies of scale* assumption. The underlying hypothesis may be that control and coordination problems confine the expansion of size. Or, considering the "replication argument" below, DRS may simply reflect that behind the scene there is an additional production factor, for example land or a irreplaceable quality of management, which is tacitly held fixed, when the factors of production are varied.

EXAMPLE 1 The production function

$$Y = AK^{\alpha}L^{\beta}, \qquad A > 0, 0 < \alpha < 1, 0 < \beta < 1, \tag{2.8}$$

where A, α, and β are given parameters, is called a *Cobb-Douglas production function*. The parameter A depends on the choice of measurement units; for a given such choice it reflects "efficiency", also called the "total factor productivity". As an exercise the reader may verify that (2.8) satisfies (a) and (b) above and is therefore a neoclassical production function. The function is homogeneous of degree $\alpha + \beta$. If $\alpha + \beta = 1$, there are CRS. If $\alpha + \beta < 1$, there are DRS, and if $\alpha + \beta > 1$, there are IRS. Note that α and β must be less than 1 in order not to violate the diminishing marginal productivity condition. □

EXAMPLE 2 The production function

$$Y = \min(AK, BL), \qquad A > 0, B > 0, \tag{2.9}$$

where A and B are given parameters, is called a *Leontief production function* or a *fixed-coefficients production function;* A and B are called the *technical coefficients*. The function is not neoclassical, since the conditions (a) and (b) are not satisfied. Indeed, with this production function the production factors are not substitutable at all. This case is also known as the case of *perfect complementarity* between the production factors. The interpretation is that already installed production equipment requires a fixed number of workers to operate it. The inverse of the parameters A and B indicate the required capital input per unit of output and the required labor input per unit of output, respectively. Extended to many inputs, this type of production function is often used in multi-sector input-output models (also called Leontief models). In aggregate analysis neoclassical production functions, allowing substitution between capital and labor, are more popular than Leontief functions. But sometimes the latter are preferred, in particular in short-run analysis with focus on the use of already installed equipment where the substitution possibilities are limited.[5] As (2.9) reads, the function has CRS. A generalized form of the Leontief function is $Y = \min(AK^{\gamma}, BL^{\gamma})$, where $\gamma > 0$. When $\gamma < 1$, there are DRS, and when $\gamma > 1$, there are IRS. □

The replication argument The assumption of CRS is widely used in macroeconomics. The model builder may appeal to the *replication argument.* This is the argument saying that by doubling all the inputs, we should always be able to double the output, since we are just "replicating" what we are already doing. Suppose we want to double the production of cars. We may then build another factory identical to the one we already have, man it with identical workers and deploy the same material inputs. Then it is reasonable to assume output is doubled.

In this context it is important that the CRS assumption is about *technology* in the sense of functions linking outputs to inputs. Limits to the *availability* of input resources is an entirely different matter. The fact that for example managerial talent may be in limited supply does not preclude the thought experiment that *if* a firm could double all its inputs, including the number of talented managers, then the output level could also be doubled.

The replication argument presupposes, first, that *all* the relevant inputs are explicit as arguments in the production function; second, that these are changed equiproportionately. This, however, exhibits the weakness of the replication argument as a defence for assuming CRS of our present production function, $F(\cdot)$. One could easily make the case that besides capital and labor, also land is a necessary input and should appear as a separate argument.[6] If an industrial firm decides to duplicate what it has been doing, it needs a piece of land to build another plant like the first. Then, on the basis of the replication argument we should in fact expect DRS w.r.t. capital and labor alone. In manufacturing and services, empirically, this and other possible

[5] Cf. Section 2.4.

[6] We think of "capital" as producible means of production, whereas "land" refers to non-producible natural resources, including for example building sites.

sources for departure from CRS may be minor and so many macroeconomists feel comfortable enough with assuming CRS w.r.t. K and L alone, at least as a first approximation. This approximation is, however, less applicable to poor countries, where natural resources may be a quantitatively important production factor.

There is a further problem with the replication argument. Strictly speaking, the CRS claim is that by changing all the inputs equiproportionately by *any* positive factor, λ, which does not have to be an integer, the firm should be able to get output changed by the same factor. Hence, the replication argument requires that indivisibilities are negligible, which is certainly not always the case. In fact, the replication argument is more an argument *against* DRS than *for* CRS in particular. The argument does not rule out IRS due to synergy effects as size is increased.

Sometimes the replication line of reasoning is given a more subtle form. This builds on a useful *local* measure of returns to scale, named the *elasticity of scale*.

The elasticity of scale* To allow for indivisibilities and mixed cases (for example IRS at low levels of production and CRS or DRS at higher levels), we need a local measure of returns to scale. One defines the *elasticity of scale*, $\eta(K, L)$, of F at the point (K, L), where $F(K, L) > 0$, as

$$\eta(K, L) = \frac{\lambda}{F(K, L)} \frac{dF(\lambda K, \lambda L)}{d\lambda} \approx \frac{\Delta F(\lambda K, \lambda L)/F(K, L)}{\Delta\lambda/\lambda}, \text{ evaluated at } \lambda = 1. \tag{2.10}$$

So the elasticity of scale at a point (K, L) indicates the (approximate) percentage increase in output when both inputs are increased by 1 percent. We say that

$$\text{if } \eta(K, L) \begin{cases} > 1, & \text{then there are locally } \textit{IRS}, \\ = 1, & \text{then there are locally } \textit{CRS}, \\ < 1, & \text{then there are locally } \textit{DRS}. \end{cases} \tag{2.11}$$

The production function *may* have the same elasticity of scale everywhere. This is the case if and only if the production function is homogeneous. If F is homogeneous of degree h, then $\eta(K, L) = h$ and h is called the *elasticity of scale parameter*.

Note that the elasticity of scale at a point (K, L) will always equal the sum of the partial output elasticities at that point:

$$\eta(K, L) = \frac{F_K(K, L)K}{F(K, L)} + \frac{F_L(K, L)L}{F(K, L)}. \tag{2.12}$$

This follows from the definition in (2.10) by taking into account that

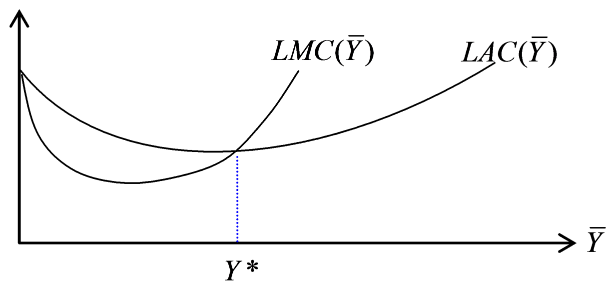


Figure 2.2: Locally CRS at optimal plant size.

$$\begin{aligned}\frac{dF(\lambda K, \lambda L)}{d\lambda} &= F_K(\lambda K, \lambda L)K + F_L(\lambda K, \lambda L)L \\ &= F_K(K, L)K + F_L(K, L)L, \text{ when evaluated at } \lambda = 1.\end{aligned}$$

Figure 2.2 illustrates a popular case from introductory economics, an average cost curve which from the perspective of the individual firm (or plant) is U-shaped: at low levels of output there are falling average costs (thus IRS), at higher levels rising average costs (thus DRS).[7] Given the input prices, w_K and w_L, and a specified output level, $\bar{Y}$, we know that the cost minimizing factor combination $(\bar{K}, \bar{L})$ is such that $F_L(\bar{K}, \bar{L})/F_K(\bar{K}, \bar{L}) = w_L/w_K$. It is shown in Appendix A that the elasticity of scale at $(\bar{K}, \bar{L})$ will satisfy:

$$\eta(\bar{K}, \bar{L}) = \frac{LAC(\bar{Y})}{LMC(\bar{Y})}, \tag{2.13}$$

where $LAC(\bar{Y})$ is average costs (the minimum unit cost associated with producing $\bar{Y}$) and $LMC(\bar{Y})$ is marginal costs at the output level $\bar{Y}$. The L in LAC and LMC stands for "long-run", indicating that both capital and labor are considered variable production factors within the period considered. At the optimal plant size, Y^*, there is equality between LAC and LMC, implying a unit elasticity of scale, that is, locally we have CRS. That the long-run average costs are here portrayed as rising for $\bar{Y} > Y^*$, is not essential for the argument but may reflect either that coordination difficulties are inevitable or that some additional production factor, say the building site of the plant, is tacitly held fixed.

Anyway, we have here a more subtle replication argument for CRS w.r.t. K and L at the aggregate level. Even though technologies may differ across plants, the surviving plants in a competitive market will have the same average costs at the optimal plant size. In the medium and long run, changes in

[7]By a "firm" is generally meant the company as a whole. A company may have several "manufacturing plants" placed at different locations.

aggregate output will take place primarily by entry and exit of optimal-size plants. Then, with a large number of relatively small plants, each producing at approximately constant unit costs for small output variations, we can without substantial error assume constant returns to scale at the aggregate level. So the argument goes. Notice, however, that even in this form the replication argument is not entirely convincing since the question of indivisibility remains. The optimal plant size may be large relative to the market − and is in fact so in many industries. Besides, in this case also the perfect competition premise breaks down.

2.1.3 Properties of the production function under CRS

The empirical evidence concerning returns to scale is mixed. Notwithstanding the theoretical and empirical ambiguities, the assumption of CRS w.r.t. capital and labor has a prominent role in macroeconomics. In many contexts it is regarded as an acceptable approximation and a convenient simple background for studying the question at hand.

Expedient inferences of the CRS assumption include:

(i) marginal costs are constant and equal to average costs (so the right-hand side of (2.13) equals unity);

(ii) if production factors are paid according to their marginal productivities, factor payments exactly exhaust total output so that pure profits are neither positive nor negative (so the right-hand side of (2.12) equals unity);

(iii) a production function known to exhibit CRS and satisfy property (a) from the definition of a neoclassical production function above, will automatically satisfy also property (b) and consequently *be* neoclassical;

(iv) a neoclassical two-factor production function with CRS has always $F_{KL} > 0$, i.e., it exhibits "direct complementarity" between K and L;

(v) a two-factor production function known to have CRS and to be twice continuously differentiable with positive marginal productivity of each factor everywhere in such a way that all isoquants are strictly convex to the origin, *must* have *diminishing* marginal productivities everywhere.[8]

[8] Proofs of these claims can be found in intermediate microeconomics textbooks and in the Appendix to Chapter 2 of my Lecture Notes in Macroeconomics.

A principal implication of the CRS assumption is that it allows a reduction of dimensionality. Considering a neoclassical production function, $Y = F(K, L)$ with $L > 0$, we can under CRS write $F(K, L) = LF(K/L, 1)$ $\equiv Lf(k)$, where $k \equiv K/L$ is called the *capital-labor ratio* (sometimes the *capital intensity*) and $f(k)$ is the *production function in intensive form* (sometimes named the per capita production function). Thus output per unit of labor depends only on the capital intensity:

$$y \equiv \frac{Y}{L} = f(k).$$

When the original production function F is neoclassical, under CRS the expression for the marginal productivity of capital simplifies:

$$F_K(K, L) = \frac{\partial Y}{\partial K} = \frac{\partial\left[Lf(k)\right]}{\partial K} = Lf'(k)\frac{\partial k}{\partial K} = f'(k). \tag{2.14}$$

And the marginal productivity of labor can be written

$$\begin{aligned} F_L(K, L) &= \frac{\partial Y}{\partial L} = \frac{\partial\left[Lf(k)\right]}{\partial L} = f(k) + Lf'(k)\frac{\partial k}{\partial L} \\ &= f(k) + Lf'(k)K(-L^{-2}) = f(k) - f'(k)k. \end{aligned} \tag{2.15}$$

A neoclassical CRS production function in intensive form always has a positive first derivative and a negative second derivative, i.e., $f' > 0$ and $f'' < 0$. The property $f' > 0$ follows from (2.14) and (2.2). And the property $f'' < 0$ follows from (2.3) combined with

$$F_{KK}(K, L) = \frac{\partial f'(k)}{\partial K} = f''(k)\frac{\partial k}{\partial K} = f''(k)\frac{1}{L}.$$

For a neoclassical production function with CRS, we also have

$$f(k) - f'(k)k > 0 \text{ for all } k > 0, \tag{2.16}$$

in view of $f(0) \geq 0$ and $f'' < 0$. Moreover,

$$\lim_{k \to 0}\left[f(k) - f'(k)k\right] = f(0). \tag{2.17}$$

Indeed, from the *mean value theorem*[9] we know that for any $k > 0$ there exists a number $a \in (0, 1)$ such that $f'(ak) = (f(k) - f(0))/k$. For this a we thus have $f(k) - f'(ak)k = f(0) < f(k) - f'(k)k$, where the inequality

[9] This theorem says that if f is continuous in $[\alpha, \beta]$ and differentiable in (α, β), then there exists at least one point γ in (α, β) such that $f'(\gamma) = (f(\beta) - f(\alpha))/(\beta - \alpha)$.

follows from $f'(ak) > f'(k)$, by $f'' < 0$. In view of $f(0) \geq 0$, this establishes (2.16). And from $f(k) > f(k) - f'(k)k > f(0)$ and continuity of f (so that $\lim_{k\to 0^+} f(k) = f(0)$) follows (2.17).

Under CRS the Inada conditions for MPK can be written

$$\lim_{k\to 0} f'(k) = \infty, \qquad \lim_{k\to \infty} f'(k) = 0. \tag{2.18}$$

In this case standard parlance is just to say that "f satisfies the Inada conditions".

An input which must be positive for positive output to arise is called an *essential input*; an input which is not essential is called an *inessential input.* The second part of (2.18), representing the upper Inada condition for MPK under CRS, has the implication that *labor* is an essential input; but capital need not be, as the production function $f(k) = a + bk/(1 + k)$, $a > 0, b > 0$, illustrates. Similarly, under CRS the upper Inada condition for MPL implies that *capital* is an essential input. These claims are proved in Appendix C. Combining these results, when *both* the upper Inada conditions hold and CRS obtain, then both capital and labor are essential inputs.[10]

Figure 2.3 is drawn to provide an intuitive understanding of a neoclassical CRS production function and at the same time illustrate that the lower Inada conditions are more questionable than the upper Inada conditions. The left panel of Figure 2.3 shows output per unit of labor for a *CRS neoclassical production function* satisfying the Inada conditions for MPK. The $f(k)$ in the diagram could for instance represent the Cobb-Douglas function in Example 1 with $\beta = 1-\alpha$, i.e., $f(k) = Ak^{\alpha}$. The right panel of Figure 2.3 shows a non-neoclassical case where only two alternative *Leontief techniques* are available, technique 1: $y = \min(A_1 k, B_1)$, and technique 2: $y = \min(A_2 k, B_2)$. In the exposed case it is assumed that $B_2 > B_1$ and $A_2 < A_1$ (if $A_2 \geq A_1$ at the same time as $B_2 > B_1$, technique 1 would not be efficient, because the same output could be obtained with less input of at least one of the factors by shifting to technique 2). If the available K and L are such that $k < B_1/A_1$ or $k > B_2/A_2$, some of either L or K, respectively, is idle. If, however, the available K and L are such that $B_1/A_1 < k < B_2/A_2$, it is efficient to *combine* the two techniques and use the fraction μ of K and L in technique 1 and the remainder in technique 2, where $\mu = (B_2/A_2 - k)/(B_2/A_2 - B_1/A_1)$. In this way we get the "labor productivity curve" OPQR (the envelope of the two techniques) in Figure 2.3. Note that for $k \to 0$, MPK stays equal to $A_1 < \infty$, whereas for all $k > B_2/A_2$, $MPK = 0$. A similar feature remains true, when we consider *many,* say n, alternative efficient Leontief techniques available.

[10] Given a Cobb-Douglas production function, both production factors are essential whether we have DRS, CRS, or IRS.

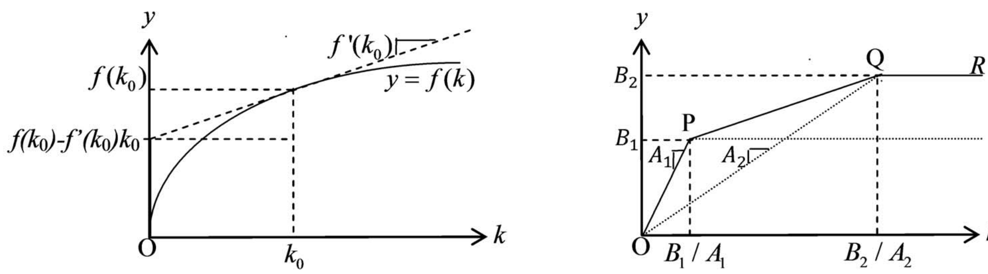


Figure 2.3: Two labor productivity curves based on CRS technologies. Left: neoclassical technology with Inada conditions for MPK satisfied; the graphical representation of MPK and MPL at $k = k_0$ as $f'(k_0)$ and $f(k_0) - f'(k_0)k_0$ are indicated. Right: a combination of two efficient Leontief techniques.

Assuming these techniques cover a considerable range w.r.t. the B/A ratios, we get a labor productivity curve looking more like that of a neoclassical CRS production function. On the one hand, this gives some intuition of what lies behind the assumption of a neoclassical CRS production function. On the other hand, it remains true that for all $k > B_n/A_n$, $MPK = 0$,[11] whereas for $k \rightarrow 0$, MPK stays equal to $A_1 < \infty$, thus questioning the lower Inada condition.

The implausibility of the lower Inada conditions is also underlined if we look at their implication in combination with the more reasonable upper Inada conditions. Indeed, the four Inada conditions taken *together* imply, under CRS, that output has no upper bound when either input goes to infinity for fixed amount of the other input (see Appendix C).

2.2 Technological change

When considering the movement over time of the economy, we shall often take into account the existence of *technological change*. When technological change occurs, the production function becomes time-dependent. Over time the production factors tend to become more productive: more output for given inputs. To put it differently: the isoquants move inward. When this is the case, we say that the technological change displays *technological progress*.

[11] Here we assume the techniques are numbered according to ranking with respect to the size of B.

Concepts of neutral technological change

A first step in taking technological change into account is to replace (2.1) by (2.4). Empirical studies often specialize (2.4) by assuming that technological change take a form known as *factor-augmenting* technological change:

$$Y_t = F(A_t K_t, B_t L_t), \tag{2.19}$$

where F is a (time-independent) neoclassical production function, Y_t, K_t, and L_t are output, capital, and labor input, respectively, at time t, while A_t and B_t are time-dependent "efficiencies" of capital and labor, respectively, reflecting technological change.

In macroeconomics an even more specific form is often assumed, namely the form of *Harrod-neutral technological change.*[12] This amounts to assuming that A_t in (2.19) is a constant (which we can then normalize to one). So only B_t, which is then conveniently denoted T_t, is changing over time, and we have

$$Y_t = F(K_t, T_t L_t). \tag{2.20}$$

The efficiency of labor, T_t, is then said to indicate the *technology level.* Although one can imagine natural disasters implying a fall in T_t, generally T_t tends to rise over time and then we say that (2.20) represents *Harrod-neutral technological progress.* An alternative name often used for this is *labor-augmenting* technological progress. The names "factor-augmenting" and, as here, "labor-augmenting" have become standard and we shall use them when convenient, although they may easily be misunderstood. To say that a change in T_t is labor-augmenting might be understood as meaning that more labor is required to reach a given output level for given capital. In fact, the opposite is the case, namely that T_t has risen so that less labor input is required. The idea is that the technological change affects the output level *as if* the labor input had been increased exactly by the factor by which T was increased, and nothing else had happened. (We might be tempted to say that (2.20) reflects "labor saving" technological change. But also this can be misunderstood. Indeed, keeping L unchanged in response to a rise in T implies that the same output level requires *less capital* and thus the technological change is "capital saving".)

If the function F in (2.20) is homogeneous of degree one (so that the technology exhibits CRS w.r.t. capital and labor), we may write

$$\tilde{y}_t \equiv \frac{Y_t}{T_t L_t} = F(\frac{K_t}{T_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0.$$

[12] After the English economist Roy F. Harrod, 1900-1978.

where $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$ (habitually called the "effective" capital intensity or, if there is no risk of confusion, just the capital intensity). In rough accordance with a general trend in aggregate productivity data for industrialized countries we often assume that T grows at a constant rate, g, so that in discrete time $T_t = T_0(1+g)^t$ and in continuous time $T_t = T_0 e^{gt}$, where $g > 0$. The popularity in macroeconomics of the hypothesis of labor-augmenting technological progress derives from its consistency with Kaldor's "stylized facts", cf. Chapter 4.

There exists two alternative concepts of neutral technological progress. *Hicks-neutral* technological progress is said to occur if technological development is such that the production function can be written in the form

$$Y_t = T_t F(K_t, L_t), \tag{2.21}$$

where, again, F is a (time-independent) neoclassical production function, while T_t is the growing technology level.[13] The assumption of Hicks-neutrality has been used more in microeconomics and partial equilibrium analysis than in macroeconomics. If F has CRS, we can write (2.21) as $Y_t = F(T_t K_t, T_t L_t)$. Comparing with (2.19), we see that in this case Hicks-neutrality is equivalent to $A_t = B_t$ in (2.19), whereby technological change is said to be *equally factor-augmenting.*

Finally, in a symmetric analogy with (2.20), what is known as *capital-augmenting* technological progress is present when

$$Y_t = F(T_t K_t, L_t). \tag{2.22}$$

Here technological change acts as if the capital input were augmented. For some reason this form is sometimes called *Solow-neutral* technological progress.[14] This association of (2.22) to Solow's name is misleading, however. In his famous growth model,[15] Solow assumed Harrod-neutral technological progress. And in another famous contribution, Solow generalized the concept of Harrod-neutrality to the case of *embodied* technological change and capital of *different vintages,* see below.

It is easily shown (Exercise I.9) that the Cobb-Douglas production function (2.8) (with time-independent output elasticities w.r.t. K and L) satisfies all three neutrality criteria at the same time, if it satisfies one of them (which it does if technological change does not affect α and β). It can also be shown that within the class of neoclassical CRS production functions the Cobb-Douglas function is the only one with this property (see Exercise ??).

[13] After the English economist and Nobel Prize laureate John R. Hicks, 1904-1989.
[14] After the American economist and Nobel Prize laureate Robert Solow (1924-).
[15] Solow (1956).

Note that the neutrality concepts do not say anything about the *source* of technological progress, only about the quantitative form in which it materializes. For instance, the occurrence of Harrod-neutrality should not be interpreted as indicating that the technological change emanates specifically from the labor input in some sense. Harrod-neutrality only means that technological innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* in the form (2.20), that is, *as if* an improvement in the quality of the labor input had occurred. (Even when improvement in the quality of the labor input is on the agenda, the result may be a reorganization of the production process ending up in a higher B_t along with, or instead of, a higher A_t in the expression (2.19).)

Rival versus nonrival goods

When a production function (or more generally a production possibility set) is specified, a given level of technical knowledge is presumed. As this level changes over time, the production function changes. In (2.4) this dependency on the level of knowledge was represented indirectly by the time dependency of the production function. Sometimes it is useful to let the knowledge dependency be explicit by perceiving knowledge as an additional production factor and write, for instance,

$$Y_t = F(X_t, T_t), \tag{2.23}$$

where T_t is now an index of the amount of knowledge, while X_t is a vector of ordinary inputs like raw materials, machines, labor etc. In this context the distinction between rival and nonrival inputs or more generally the distinction between rival and nonrival goods is important. A good is *rival* if its character is such that one agent's use of it inhibits other agents' use of it at the same time. A pencil is thus rival. Many production inputs like raw materials, machines, labor etc. have this property. They are elements of the vector X_t. By contrast, however, technical knowledge is a *nonrival* good. An arbitrary number of factories can simultaneously use the same piece of technical knowledge in the sense of a *list of instructions about how different inputs can be combined to produce a certain output*. An engineering principle or a farmaceutical formula are examples. (Note that the distinction rival-nonrival is different from the distinction excludable-nonexcludable. A good is *excludable* if other agents, firms or households, can be excluded from using it. Other firms can thus be excluded from commercial use of a certain piece of technical knowledge if it is patented. The existence of a patent concerns the legal status of a piece of knowledge and does not interfere with its economic character as a nonrival input.).

What the replication argument really says is that by, conceptually, doubling all the *rival* inputs, we should always be able to double the output, since we just "replicate" what we are already doing. This is then an argument for (at least) CRS w.r.t. the elements of X_t in (2.23). The point is that because of its nonrivalry, we do not need to increase the stock of knowledge. Now let us imagine that the stock of knowledge *is* doubled at the same time as the rival inputs are doubled. Then *more* than a doubling of output should occur. In this sense we may speak of IRS w.r.t. the rival inputs and T taken together.

Before proceeding, we briefly comment on how the capital stock, K_t, is typically measured. While data on gross investment, I_t, is available in national income and product accounts, data on K_t usually is not. One approach to the measurement of K_t is the *perpetual inventory method* which builds upon the accounting relationship

$$K_t = I_{t-1} + (1-\delta)K_{t-1}. \tag{2.24}$$

Assuming a constant capital depreciation rate δ, backward substitution gives

$$K_t = I_{t-1}+(1-\delta)\left[I_{t-2} + (1-\delta)K_{t-2}\right] = \ldots = \sum_{i=1}^{N}(1-\delta)^{i-1}I_{t-i}+(1-\delta)^N K_{t-N}. \tag{2.25}$$

Based on a long time series for I and an estimate of δ, one can insert these observed values in the formula and calculate K_t, starting from a rough conjecture about the initial value K_{t-N}. The result will not be very sensitive to this conjecture since for large N the last term in (2.25) becomes very small.

Embodied vs. disembodied technological progress

There exists an additional taxonomy of technological change. We say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The new technology is incorporated in the design of newly produced equipment, but this equipment will not participate in subsequent technological progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Thus investment becomes an important driving force in productivity increases.

We way formalize embodied technological progress by writing capital accumulation in the following way:

$$K_{t+1} - K_t = q_t I_t - \delta K_t, \tag{2.26}$$

where I_t is gross investment in period t, i.e., $I_t = Y_t - C_t$, and q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising q so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in *efficiency units*. In aggregate models C and I are produced with the same technology, the aggregate production function. From this together with (2.26) follows that q capital goods can be produced at the same minimum cost as one consumption good. Hence, the equilibrium price, p, of capital goods in terms of the consumption good must equal the inverse of q, i.e., $p = 1/q$. The output-capital ratio in value terms is $Y/(pK) = qY/K$.

Note that even if technological change does not directly appear in the production function, that is, even if for instance (2.20) is replaced by $Y_t = F(K_t, L_t)$, the economy may experience a rising standard of living when q is growing over time.

In contrast, *disembodied technological change* occurs when new technical and organizational knowledge increases the combined productivity of the production factors independently of when they were constructed or educated. If the K_t appearing in (2.20), (2.21), and (2.22) above refers to the total, historically accumulated capital stock as calculated by (2.25), then the evolution of T in these expressions can be seen as representing disembodied technological change. All vintages of the capital equipment benefit from a rise in the technology level T_t. No new investment is needed to benefit.

Based on data for the U.S. 1950-1990, and taking quality improvements into account, Greenwood et al. (1997) estimate that embodied technological progress explains about 60% of the growth in output per man hour. So, empirically, *embodied* technological progress seems to play a dominant role. As this tends not to be fully incorporated in national income accounting at fixed prices, there is a need to adjust the investment levels in (2.25) to better take estimated quality improvements into account. Otherwise the resulting K will not indicate the capital stock measured in efficiency units.

2.3 The concepts of representative firm and aggregate production function*

Many macroeconomic models make use of the simplifying notion of a *representative firm.* By this is meant a fictional firm whose production "represents" aggregate production (value added) in a sector or in society as a whole.

Suppose there are n firms in the sector considered or in society as a whole. Let F^i be the production function for firm i so that $Y_i = F^i(K_i, L_i)$, where Y_i, K_i, and L_i are output, capital input, and labor input, respectively, $i = 1, 2, \ldots, n$. Further, let $Y = \Sigma_{i=1}^n Y_i$, $K = \Sigma_{i=1}^n K_i$, and $L = \Sigma_{i=1}^n L_i$. Ignoring technological change, suppose the aggregate variables are related through some function, F^*, such that we can write

$$Y = F^*(K, L),$$

and such that the choices of a single firm facing this production function coincide with the aggregate outcomes, $\Sigma_{i=1}^n Y_i$, $\Sigma_{i=1}^n K_i$, and $\Sigma_{i=1}^n L_i$, in the original economy. Then $F^*(K, L)$ is called the *aggregate production function* or the production function of the *representative* firm. It is *as if* aggregate production is the result of the behavior of such a single firm.

A simple example where the aggregate production function is well-defined is the following. Suppose that all firms have the *same* production function so that $Y_i = F(K_i, L_i)$, $i = 1, 2, \ldots, n$. If in addition F has CRS, we have

$$Y_i = F(K_i, L_i) = L_i F(k_i, 1) \equiv L_i f(k_i),$$

where $k_i \equiv K_i/L_i$. Hence, facing given factor prices, cost-minimizing firms will choose the same capital intensity $k_i = k$ for all i. From $K_i = kL_i$ then follows $\sum_i K_i = k \sum_i L_i$ so that $k = K/L$. Thence,

$$Y \equiv \sum Y_i = \sum L_i f(k_i) = f(k) \sum L_i = f(k)L = F(k, 1)L = F(K, L).$$

In this (trivial) case the aggregate production function is well-defined and turns out to be exactly the same as the identical CRS production functions of the individual firms. Moreover, given CRS and $k_i = k$ for all i, we have $\partial Y_i/\partial K_i = f'(k_i) = f'(k) = F_K(K, L)$ for all i. So each firm's marginal productivity of capital is the same as the marginal productivity of capital on the basis of the aggregate production function.

Allowing for the existence of *different* production functions at firm level, we may define the aggregate production function as

$$\begin{aligned} F(K, L) &= \max_{(K_1, L_1, \ldots, K_n, L_n) \geq 0} F^1(K_1, L_1) + \cdots + F^n(K_n, L_n) \\ \text{s.t. } \sum_i K_i &\leq K, \quad \sum_i L_i \leq L. \end{aligned}$$

Allowing also for existence of different output goods, different capital goods, and different types of labor makes the issue more intricate, of course.

Yet, if firms are price taking profit maximizers and there are nonincreasing returns to scale, we at least know that the aggregate outcome is *as if,* for given prices, the firms jointly maximize aggregate profit on the basis of their combined production technology (Mas-Colell et al., 1955). The problem is, however, that the conditions needed for this to imply existence of an aggregate production function which is *well-behaved* (in the sense of inheriting simple qualitative properties from its constituent parts) are restrictive.

Nevertheless macroeconomics often treats aggregate output as a single homogeneous good and capital and labor as being two single and homogeneous inputs. There was in the 1960s a heated debate about the problems involved in this, with particular emphasis on the aggregation of different kinds of equipment into one variable, the capital stock "K". The debate is known as the "Cambridge controversy" because the dispute was between a group of economists from Cambridge University, UK, and a group from Massachusetts Institute of Technology (MIT), which is located in Cambridge, USA. The former group questioned the theoretical robustness of several of the neoclassical tenets, including the proposition that rising aggregate capital intensity tends to be associated with a falling rate of interest. Starting at the disaggregate level, an association of this sort is not a logical necessity because, with different production functions across the industries, the relative prices of produced inputs tend to change, when the interest rate changes. While acknowledging the possibility of "paradoxical" relationships, the latter group maintained that in a macroeconomic context they are likely to cause devastating problems only under exceptional circumstances. In the end this is a matter of empirical assessment.[16]

To avoid complexity and because, for many important issues in growth theory, there is today no well-tried alternative, we shall in this course most of the time use aggregate constructs like "Y", "K", and "L" as simplifying devices, hopefully acceptable in a first approximation. There are cases, however, where some disaggregation is pertinent. When for example the role of imperfect competition is in focus, we shall be ready to disaggregate the production side of the economy into several product lines, each producing its own differentiated product. We shall also touch upon a type of growth models where a key ingredient is the phenomenon of "creative destruction" meaning that an incumbent technological leader is competed out by an entrant with a qualitatively new technology.

Like the representative firm, the *representative household* and the *aggre-*

[16] In his review of the Cambridge controversy Mas-Colell (1989) concluded that: "What the 'paradoxical' comparative statics [of disaggregate capital theory] has taught us is simply that modelling the world as having a single capital good is not *a priori* justified. So be it."

gate consumption function are simplifying notions that should be applied only when they do not get in the way of the issue to be studied. The importance of budget constraints may make it even more difficult to aggregate over households than over firms. Yet, *if* (and that is a big if) all households have the *same, constant* marginal propensity to consume out of income, aggregation is straightforward and the representative household is a meaningful concept. On the other hand, if we aim at understanding, say, the *interaction* between lending and borrowing households, perhaps via financial intermediaries, the representative household is not a useful starting point. Similarly, if the theme is conflicts of interests between firm owners and employees, the existence of *different* types of households should be taken into account.

2.4 Long-run vs. short-run production functions*

Is the substitutability between capital and labor the same "ex ante" and "ex post"? By ex ante is meant "when plant and machinery are to be decided upon" and by ex post is meant "after the equipment is designed and constructed". In the standard neoclassical competitive setup, of for instance the Solow or the Ramsey model, there is a presumption that also after the construction and installation of the equipment in the firm, the ratio of the factor inputs can be fully adjusted to a change in the relative factor price. In practice, however, when some machinery has been constructed and installed, its functioning will often require a more or less fixed number of machine operators. What can be varied is just the *degree of utilization* of the machinery. That is, after construction and installation of the machinery, the choice opportunities are no longer described by the neoclassical production function but by a Leontief production function,

$$Y = \min(Au\bar{K}, BL), \qquad A > 0, B > 0, \tag{2.27}$$

where $\bar{K}$ is the size of the installed machinery (a fixed factor in the short run) measured in efficiency units, u is its utilization rate ($0 \leq u \leq 1$), and A and B are given technical coefficients measuring efficiency.

So in the short run the choice variables are u and L. In fact, essentially only u is a choice variable since efficient production trivially requires $L = Au\bar{K}/B$. Under "full capacity utilization" we have $u = 1$ (each machine is used 24 hours per day seven days per week). "Capacity" is given as $A\bar{K}$ per week. Producing efficiently at capacity requires $L = A\bar{K}/B$ and the marginal product by increasing labor input is here nil. But if demand, Y^d, is *less* than capacity, satisfying this demand efficiently requires $u = Y^d/(A\bar{K}) < 1$ and $L = Y^d/B$. As long as $u < 1$, the marginal productivity of labor is a *constant*, B.

The various efficient input proportions that are possible *ex ante* may be approximately described by a neoclassical CRS production function. Let this function on intensive form be denoted $y = f(k)$. When investment is decided upon and undertaken, there is thus a choice between alternative efficient pairs of the technical coefficients A and B in (2.27). These pairs satisfy

$$f(k) = Ak = B. \tag{2.28}$$

So, for an increasing sequence of k's, $k_1, k_2, \ldots, k_i, \ldots$, the corresponding pairs are $(A_i, B_i) = (f(k_i)/k_i, f(k_i))$, $i = 1, 2, \ldots$.[17] We say that ex ante, depending on the relative factor prices as they are "now" and are expected to evolve in the future, a suitable technique, (A_i, B_i), is chosen from an opportunity set described by the given neoclassical production function. But ex post, i.e., when the equipment corresponding to this technique is installed, the production opportunities are described by a Leontief production function with $(A, B) = (A_i, B_i)$.

In the picturesque language of Phelps (1963), technology is in this case *putty-clay.* Ex ante the technology involves capital which is "putty" in the sense of being in a malleable state which can be transformed into a range of various machinery requiring capital-labor ratios of different magnitude. But once the machinery is constructed, it enters a "hardened" state and becomes "clay". Then factor substitution is no longer possible; the capital-labor ratio at full capacity utilization is fixed at the level $k = B_i/A_i$, as in (2.27). Following the terminology of Johansen (1972), we say that a putty-clay technology involves a "long-run production function" which is neoclassical and a "short-run production function" which is Leontief.

In contrast, the standard neoclassical setup assumes the same range of substitutability between capital and labor ex ante and ex post. Then the technology is called *putty-putty.* This term may also be used if ex post there is at least *some* substitutability although less than ex ante. At the opposite pole of putty-putty we may consider a technology which is *clay-clay.* Here neither ex ante nor ex post is factor substitution possible. Table 2.1 gives an overview of the alternative cases.

Table 2.1. Technologies classified according to

[17] The points P and Q in the right-hand panel of Fig. 2.3 can be interpreted as constructed this way from the neoclassical production function in the left-hand panel of the figure.

factor substitutability ex ante and ex post

	Ex post substitution	
Ex ante substitution	possible	impossible
possible	putty-putty	putty-clay
impossible		clay-clay

The putty-clay case is generally considered the realistic case. As time proceeds, technological progress occurs. To take this into account, we may replace (2.28) and (2.27) by $f(k_t, t) = A_t k_t = B_t$ and $Y_t = \min(A_t u_t \bar{K}_t, B_t L_t)$, respectively. If a new pair of Leontief coefficients, (A_{t_2}, B_{t_2}), efficiency-dominates its predecessor (by satisfying $A_{t_2} \geq A_{t_1}$ and $B_{t_2} \geq B_{t_1}$ with at least one strict equality), it may pay the firm to invest in the new technology at the same time as some old machinery is scrapped. Real wages tend to rise along with technological progress and the scrapping occurs because the revenue from using the old machinery in production no longer covers the associated labor costs.

The clay property ex-post of many technologies is important for short-run analysis. It implies that there may be non-decreasing marginal productivity of labor up to a certain point. It also implies that in its investment decision the firm will have to take expected future technologies and future factor prices into account. For many issues in long-run analysis the clay property ex-post may be less important, since over time adjustment takes place through new investment.

2.5 The neoclassical theory of factor income shares

To begin with, we ignore technological progress and write aggregate output as $Y = F(K, L)$, where F is neoclassical with CRS. From Euler's theorem follows that $F(K, L) = F_1 K + F_2 L = f'(k)K + (f(k) - kf'(k))L$, where $k \equiv K/L$. In equilibrium under perfect competition we have

$$Y = \hat{r}K + wL,$$

where $\hat{r} = r + \delta = f'(k) \equiv \hat{r}(k)$ is the cost per unit of capital input and $w = f(k) - kf'(k) \equiv w(k)$ is the real wage, i.e., the cost per unit of labor input. We have $\hat{r}'(k) = f''(k) < 0$ and $w'(k) = -kf''(k) > 0$.

The *labor income share* is

$$\frac{wL}{Y} = \frac{f(k) - kf'(k)}{f(k)} \equiv \frac{w(k)}{f(k)} \equiv SL(k) = \frac{wL}{\hat{r}K + wL} = \frac{\frac{w/\hat{r}}{k}}{1 + \frac{w/\hat{r}}{k}}, \qquad (2.29)$$

where the function $SL(\cdot)$ is the income share of labor function, $w/\hat{r}$ is the *factor price ratio*, and $(w/\hat{r})/k = w/(\hat{r}k)$ is the *factor income ratio.* As $\hat{r}'(k) = f''(k) < 0$ and $w'(k) = -kf''(k) > 0$, the factor price ratio, $w/\hat{r}$, is an increasing function of k.

Suppose that capital tends to grow faster than labor so that k rises over time. Unless the production function is Cobb-Douglas, this will under perfect competition affect the labor income share. But apriori it is not obvious in what direction. By (2.29) we see that the labor income share moves in the same direction as the factor *income* ratio, $(w/\hat{r})/k$. The latter goes up (down) depending on whether the percentage rise in the factor price ratio $w/\hat{r}$ is greater (smaller) than the percentage rise in k. So, if we let $\mathrm{E}\ell_x g(x)$ denote the elasticity of a function $g(x)$ w.r.t. x, we can only say that

$$SL'(k) \gtreqless 0 \text{ for } \mathrm{E}\ell_k \frac{w}{\hat{r}} \gtreqless 1, \tag{2.30}$$

respectively. In words: if the production function is such that the ratio of the marginal productivities of the two production factors is strongly (weakly) sensitive to the capital-labor ratio, then the labor income share rises (falls) along with a rise in K/L.

Usually, however, the inverse elasticity is considered, namely $\mathrm{E}\ell_{w/\hat{r}}k$ $(= 1/\mathrm{E}\ell_k \frac{w}{\hat{r}})$. This elasticity indicates how sensitive the cost minimizing capital-labor ratio, k, is to a given factor price ratio $w/\hat{r}$. Under perfect competition $\mathrm{E}\ell_{w/\hat{r}}k$ coincides with what is known as the *elasticity of factor substitution* (for a general definition, see below). The latter is often denoted σ. In the CRS case, σ will be a function of only k so that we can write $\mathrm{E}\ell_{w/\hat{r}}k = \sigma(k)$. By (2.30), we therefore have

$$SL'(k) \gtreqless 0 \text{ for } \sigma(k) \lesseqgtr 1,$$

respectively.

The size of the elasticity of factor substitution is a property of the production function, hence of the technology. In special cases the elasticity of factor substitution is a constant, i.e., independent of k. For instance, if F is Cobb-Douglas, i.e., $Y = K^\alpha L^{1-\alpha}, 0 < \alpha < 1$, we have $\sigma(k) \equiv 1$, as shown in Section 2.7. In this case variation in k does not change the labor income share under perfect competition. Empirically there is not agreement about the "normal" size of the elasticity of factor substitution for industrialized economies, but the bulk of studies seems to conclude with $\sigma(k) < 1$ (see below).

Adding Harrod-neutral technical progress We now add Harrod-neutral technical progress. We write aggregate output as $Y = F(K, TL)$, where F

is neoclassical with CRS, and $T = T_t = T_0(1+g)^t$. Then the labor income share is

$$\frac{wL}{Y} = \frac{w/T}{Y/(TL)} \equiv \frac{\tilde{w}}{\tilde{y}}.$$

The above formulas still hold if we replace k by $\tilde{k} \equiv K/(TL)$ and w by $\tilde{w} \equiv w/T$. We get

$$SL'(\tilde{k}) \gtreqless 0 \text{ for } \sigma(\tilde{k}) \lesseqgtr 1,$$

respectively. We see that if $\sigma(\tilde{k}) < 1$ in the relevant range for $\tilde{k}$, then market forces tend to *increase* the income share of the factor that is becoming relatively more scarce, which is efficiency-adjusted labor, TL, if $\tilde{k}$ is increasing. And if instead $\sigma(\tilde{k}) > 1$ in the relevant range for $\tilde{k}$, then market forces tend to *decrease* the income share of the factor that is becoming relatively more scarce.

While k empirically is clearly growing, $\tilde{k} \equiv k/T$ is not necessarily so because also T is increasing. Indeed, according to Kaldor's "stylized facts", apart from short- and medium-term fluctuations, $\tilde{k}$ − and therefore also $\hat{r}$ and the labor income share − tend to be more or less constant over time. This can happen whatever the sign of $\sigma(\tilde{k}^*) - 1$, where $\tilde{k}^*$ is the long-run value of the effective capital-labor ratio $\tilde{k}$. Given CRS and the production function f, the elasticity of substitution between capital and labor does not depend on whether $g = 0$ or $g > 0$, but only on the function f itself and the level of $K/(TL)$.

As alluded to earlier, there are empiricists who reject Kaldor's "facts" as a general tendency. For instance Piketty (2014) essentially claims that in the very long run the effective capital-labor ratio $\tilde{k}$ has an upward trend, temporarily braked by two world wars and the Great Depression in the 1930s. If so, the sign of $\sigma(\tilde{k}) - 1$ becomes decisive for in what direction wL/Y will move. Piketty interprets the econometric literature as favoring $\sigma(\tilde{k}) > 1$, which means there should be downward pressure on wL/Y. This particular source behind a falling wL/Y can be questioned, however. Indeed, $\sigma(\tilde{k}) > 1$ contradicts the more general empirical view referred to above.[18]

Immigration

Here is another example that illustrates the importance of the size of $\sigma(\tilde{k})$. Consider an economy with perfect competition and a given aggregate capital stock K and technology level T (entering the production function in the labor-augmenting way as above). Suppose that for some reason, immigration,

[18] According to Summers (2014), Piketty's interpretation relies on conflating gross and net returns to capital.

say, aggregate labor supply, L, shifts up and full employment is maintained by the needed real wage adjustment. Given the present model, in what direction will aggregate labor income $wL = \tilde{w}(\tilde{k})TL$ then change? The effect of the larger L is to some extent offset by a lower w brought about by the lower effective capital-labor ratio. Indeed, in view of $d\tilde{w}/d\tilde{k} = -\tilde{k}f''(\tilde{k}) > 0$, we have $\tilde{k} \downarrow$ implies $w \downarrow$ for fixed T. So we cannot apriori sign the change in wL. The following relationship can be shown (Exercise ??), however:

$$\frac{\partial(wL)}{\partial L} = (1 - \frac{\alpha(\tilde{k})}{\sigma(\tilde{k})})w \gtreqless 0 \text{ for } \alpha(\tilde{k}) \lesseqgtr \sigma(\tilde{k}), \tag{2.31}$$

respectively, where $a(\tilde{k}) \equiv \tilde{k}f'(\tilde{k})/f(\tilde{k})$ is the output elasticity w.r.t. capital which under perfect competition equals the gross capital income share. It follows that the larger L will not be fully offset by the lower w as long as the elasticity of factor substitution, $\sigma(\tilde{k})$, exceeds the gross capital income share, $\alpha(\tilde{k})$. This condition seems confirmed by most of the empirical evidence (see Section 2.7).

The next section describes the concept of the elasticity of factor substitution at a more general setting. The subsequent section introduces the special case known as the CES production function.

2.6 The elasticity of factor substitution*

We shall here discuss the concept of elasticity of factor substitution at a more general level. Fig. 2.4 depicts an isoquant, $F(K, L) = \bar{Y}$, for a given neoclassical production function, $F(K, L)$, which need not have CRS. Let MRS denote the marginal rate of substitution of K for L, i.e., $MRS = F_L(K, L)/F_K(K, L)$.[19] At a given point (K, L) on the isoquant curve, MRS is given by the absolute value of the slope of the tangent to the isoquant at that point. This tangent coincides with that isocost line which, given the factor prices, has minimal intercept with the vertical axis while at the same time touching the isoquant. In view of $F(\cdot)$ being neoclassical, the isoquants are by definition strictly convex to the origin. Consequently, MRS is rising along the curve when L decreases and thereby K increases. Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio K/L, as a function of MRS. If we let MRS rise along the given isoquant, the corresponding value of the ratio K/L will also rise.

[19] When there is no risk of confusion as to what is up and what is down, we use MRS as a shorthand for the more precise expression MRS_{KL}.

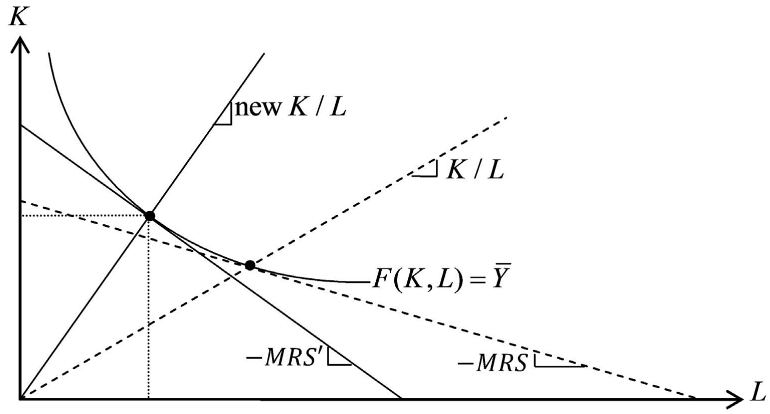


Figure 2.4: Substitution of capital for labor as the marginal rate of substitution increases from MRS to MRS'.

The *elasticity of substitution* between capital and labor is defined as the elasticity of the ratio K/L with respect to MRS when we move along a given isoquant, evaluated at the point (K, L). Let this elasticity be denoted $\tilde{\sigma}(K, L)$. Thus,

$$\tilde{\sigma}(K,L) = \frac{MRS}{K/L}\frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} = \frac{\frac{d(K/L)}{K/L}}{\frac{dMRS}{MRS}}_{|Y=\bar{Y}}. \tag{2.32}$$

Although the elasticity of factor substitution is a characteristic of the technology as such and is here defined without reference to markets and factor prices, it helps the intuition to refer to factor prices. At a cost-minimizing point, MRS equals the factor price ratio $w/\hat{r}$. Thus, the *elasticity of factor substitution* will under cost minimization coincide with *the percentage increase in the ratio of the cost-minimizing factor ratio induced by a one percentage increase in the inverse factor price ratio, holding the output level unchanged.*[20] The elasticity of factor substitution is thus a positive number and reflects how sensitive the capital-labor ratio K/L is under cost minimization to an increase in the factor price ratio $w/\hat{r}$ for a given output level. The less curvature the isoquant has, the greater is the elasticity of factor substitution. In an analogue way, in consumer theory one considers the elasticity of substitution between two consumption goods or between consumption today and consumption tomorrow. In that context the role of the given isoquant

[20]This characterization is equivalent to interpreting the elasticity of substitution as the percentage *decrease* in the factor ratio (when moving along a given isoquant) induced by a one-percentage *increase* in the *corresponding* factor price ratio.

is taken over by an indifference curve. That is also the case when we consider the intertemporal elasticity of substitution in labor supply, cf. the next chapter.

Calculating the elasticity of substitution between K and L at the point (K, L), we get

$$\tilde{\sigma}(K, L) = -\frac{F_K F_L (F_K K + F_L L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]}, \tag{2.33}$$

where all the derivatives are evaluated at the point (K, L). When $F(K, L)$ has CRS, the formula (2.33) simplifies to

$$\tilde{\sigma}(K, L) = \frac{F_K(K, L)F_L(K, L)}{F_{KL}(K, L)F(K, L)} = -\frac{f'(k)\left(f(k) - f'(k)k\right)}{f''(k)kf(k)} \equiv \sigma(k), \tag{2.34}$$

where $k \equiv K/L$.[21] We see that under CRS, the elasticity of substitution depends only on the capital-labor ratio k, not on the output level. We will now consider the case where the elasticity of substitution is independent also of the capital-labor ratio.

2.7 The CES production function

It can be shown[22] that if a neoclassical production function with CRS has a constant elasticity of factor substitution different from one, it must be of the form

$$Y = A\left[\alpha K^\beta + (1 - \alpha)L^\beta\right]^{\frac{1}{\beta}}, \tag{2.35}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function* (CES for Constant Elasticity of Substitution). For a given choice of measurement units, the parameter A reflects efficiency (or what is known as *total factor productivity*) and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (16.32) is not defined when either K or L (or both) equal 0. We can circumvent this problem by extending the domain of the CES function and assign the function value 0 to these points when $\beta < 0$. Continuity is maintained in the extended domain (see Appendix E).

[21] The formulas (2.33) and (2.34) are derived in Appendix D of Chapter 4 of Groth, Lecture Notes in Macroeconomics..

[22] See, e.g., Arrow et al. (1961).

By taking partial derivatives in (16.32) and substituting back we get

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta}\left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}\left(\frac{Y}{L}\right)^{1-\beta}, \tag{2.36}$$

where $Y/K = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and $Y/L = A\left[\alpha k^{\beta} + 1 - \alpha\right]^{\frac{1}{\beta}}$. The marginal rate of substitution of K for L therefore is

$$MRS = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha}k^{1-\beta} > 0.$$

Consequently,

$$\frac{dMRS}{dk} = \frac{1-\alpha}{\alpha}(1-\beta)k^{-\beta},$$

where the inverse of the right-hand side is the value of $dk/dMRS$. Substituting these expressions into (16.34) gives

$$\tilde{\sigma}(K,L) = \frac{1}{1-\beta} \equiv \sigma, \tag{2.37}$$

confirming the constancy of the elasticity of substitution. Since $\beta < 1$, $\sigma > 0$ always. A higher substitution parameter, β, results in a higher elasticity of factor substitution, σ. And $\sigma \lesseqgtr 1$ for $\beta \lesseqgtr 0$, respectively.

Since $\beta = 0$ is not allowed in (16.32), at first sight we cannot get $\sigma = 1$ from this formula. Yet, $\sigma = 1$ can be introduced as the *limiting* case of (16.32) when $\beta \to 0$, which turns out to be the Cobb-Douglas function. Indeed, one can show[23] that, for fixed K and L,

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to AK^{\alpha}L^{1-\alpha}, \text{ for } \beta \to 0.$$

By a similar procedure as above we find that a Cobb-Douglas function always has elasticity of substitution equal to 1; this is exactly the value taken by σ in (16.35) when $\beta = 0$. In addition, the Cobb-Douglas function is the *only* production function that has unit elasticity of substitution whatever the capital-labor ratio.

Another interesting limiting case of the CES function appears when, for fixed K and L, we let $\beta \to -\infty$ so that $\sigma \to 0$. We get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\min(K,L), \text{ for } \beta \to -\infty. \tag{2.38}$$

[23] For proofs of this and the further claims below, see Appendix E of Chapter 4 of Groth, Lecture Notes in Macroeconomics.

So in this case the CES function approaches a Leontief production function, the isoquants of which form a right angle, cf. Fig. 2.5. In the limit there is *no* possibility of substitution between capital and labor. In accordance with this the elasticity of substitution calculated from (16.35) approaches zero when β goes to $-\infty$.

Finally, let us consider the "opposite" transition. For fixed K and L we let the substitution parameter rise towards 1 and get

$$A\left[\alpha K^{\beta}+(1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \rightarrow A\left[\alpha K+(1-\alpha)L\right], \text{ for } \beta \rightarrow 1.$$

Here the elasticity of substitution calculated from (16.35) tends to ∞ and the isoquants tend to straight lines with slope $-(1-\alpha)/\alpha$. In the limit, the production function thus becomes linear and capital and labor become *perfect substitutes.*

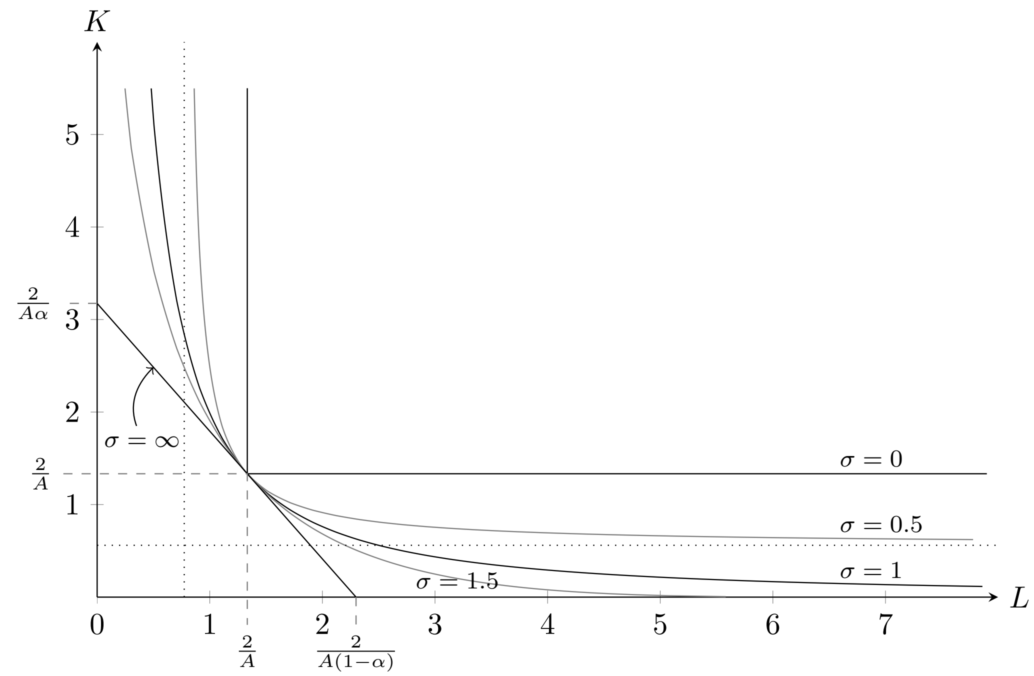


Figure 2.5: Isoquants for the CES function for alternative values of σ ($A = 1.5$, $\bar{Y} = 2$, and $\alpha = 0.42$).

Fig. 2.5 depicts isoquants for alternative CES production functions and their limiting cases. In the Cobb-Douglas case, $\sigma = 1$, the horizontal and vertical asymptotes of the isoquant coincide with the coordinate axes. When $\sigma < 1$, the horizontal and vertical asymptotes of the isoquant belong to the interior of the positive quadrant. This implies that both capital and labor are essential inputs. When $\sigma > 1$, the isoquant terminates in points *on* the coordinate axes. Then neither capital, nor labor are essential inputs. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for industrialized economies. The elasticity also differs across the production sectors. A thorough econometric study (Antràs, 2004) of U.S. data indicate the aggregate elasticity of substitution to be in the interval $(0.5, 1.0)$. The survey by Chirinko (2008) concludes with the interval $(0.4, 0.6)$. Starting from micro data, a recent study by Oberfield and Raval (2014) finds that the elasticity of factor substitution for the US manufacturing sector as a whole has been stable since 1970 at about 0.7.

The CES production function in intensive form

Dividing through by L on both sides of (16.32), we obtain the CES production function in intensive form,

$$y \equiv \frac{Y}{L} = A(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}, \tag{2.39}$$

where $k \equiv K/L$. The marginal productivity of capital can be written

$$MPK = \frac{dy}{dk} = \alpha A \left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}} = \alpha A^{\beta} \left(\frac{y}{k}\right)^{1-\beta},$$

which of course equals $\partial Y / \partial K$ in (16.33). We see that the CES function violates either the lower or the upper Inada condition for MPK, depending on the sign of β. Indeed, when $\beta < 0$ (i.e., $\sigma < 1$), then for $k \to 0$ both y/k and dy/dk approach an upper bound equal to $A\alpha^{1/\beta} < \infty$, thus violating the lower Inada condition for MPK (see the right-hand panel of Fig. 2.3). It is also noteworthy that in this case, for $k \to \infty$, y approaches an upper bound equal to $A(1-\alpha)^{1/\beta} < \infty$. These features reflect the low degree of substitutability when $\beta < 0$.

When instead $\beta > 0$, there is a high degree of substitutability ($\sigma > 1$). Then, for $k \to \infty$ both y/k and $dy/dk \to A\alpha^{1/\beta} > 0$, thus violating the upper Inada condition for MPK (see right panel of Fig. 2.6). It is also noteworthy that for $k \to 0$, y approaches a positive lower bound equal to $A(1-\alpha)^{1/\beta} > 0$. Thus, in this case capital is not essential. At the same time $dy/dk \to \infty$ for $k \to 0$ (so the lower Inada condition for the marginal productivity of capital holds). Details are in Appendix E.

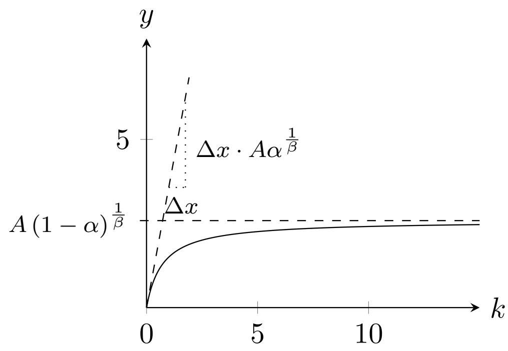
The marginal productivity of labor is

$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}y^{1-\beta} = (1-\alpha)A(\alpha k^{\beta} + 1 - \alpha)^{(1-\beta)/\beta} \equiv w(k),$$

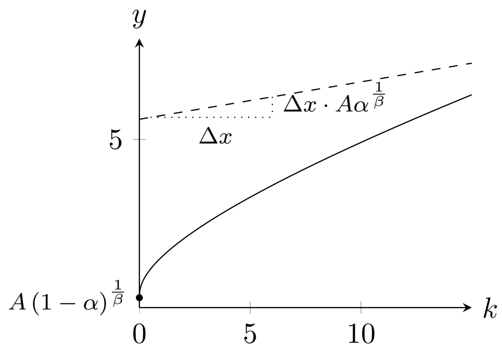
from (16.33).

Since (16.32) is symmetric in K and L, we get a series of symmetric results by considering output per unit of capital as $x \equiv Y/K = A\left[\alpha + (1-\alpha)(L/K)^{\beta}\right]^{1/\beta}$. In total, therefore, when there is low substitutability ($\beta < 0$), for fixed input of either of the production factors, there is an upper bound for how much an unlimited input of the other production factor can increase output. And when there is high substitutability ($\beta > 0$), there is no such bound and an unlimited input of either production factor take output to infinity.

The Cobb-Douglas case, i.e., the limiting case for $\beta \to 0$, constitutes in several respects an intermediate case in that *all* four Inada conditions are satisfied and we have $y \to 0$ for $k \to 0$, and $y \to \infty$ for $k \to \infty$.



a) The case of $\sigma < 1$.



a) The case of $\sigma > 1$.

Figure 2.6: The CES production function in intensive form, $\sigma = 1/(1-\beta)$, $\beta < 1$.

Generalizations

The CES production function considered above has CRS. By adding an elasticity of scale parameter, γ, we get the generalized form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \qquad \gamma > 0. \tag{2.40}$$

In this form the CES function is homogeneous of degree γ. For $0 < \gamma < 1$, there are DRS, for $\gamma = 1$ CRS, and for $\gamma > 1$ IRS. If $\gamma \neq 1$, it may be convenient to consider $Q \equiv Y^{1/\gamma} = A^{1/\gamma}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}$ and $q \equiv Q/L = A^{1/\gamma}(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$.

The elasticity of substitution between K and L is $\sigma = 1/(1-\beta)$ whatever the value of γ. So including the limiting cases as well as non-constant returns to scale in the "family" of production functions with constant elasticity of substitution, we have the simple classification displayed in Table 2.2.

Table 2.2 The family of production functions with constant elasticity of substitution.

$\sigma = 0$	$0 < \sigma < 1$	$\sigma = 1$	$\sigma > 1$
Leontief	CES	Cobb-Douglas	CES

Note that only for $\gamma \leq 1$ is (16.38) a *neoclassical* production function. This is because, when $\gamma > 1$, the conditions $F_{KK} < 0$ and $F_{NN} < 0$ do not hold everywhere.

We may generalize further by assuming there are n inputs, in the amounts $X_1, X_2, ..., X_n$. Then the CES production function takes the form

$$Y = A\left[\alpha_1 {X_1}^{\beta} + \alpha_2 {X_2}^{\beta} + ...\alpha_n {X_n}^{\beta}\right]^{\frac{\gamma}{\beta}}, \ \alpha_i > 0 \text{ for all } i, \sum_i \alpha_i = 1, \gamma > 0. \tag{2.41}$$

In analogy with (16.34), for an n-factor production function the *partial elasticity of substitution* between factor i and factor j is defined as

$$\sigma_{ij} = \frac{MRS_{ij}}{X_i/X_j} \frac{d(X_i/X_j)}{dMRS_{ij}}_{|Y=\bar{Y}},$$

where it is understood that not only the output level but also all X_k, $k \neq i, j$, are kept constant. Note that $\sigma_{ji} = \sigma_{ij}$. In the CES case considered in (16.39), all the partial elasticities of substitution take the same value, $1/(1-\beta)$.

2.8 Literature notes

As to the question of the empirical validity of the constant returns to scale assumption, Malinvaud (1998) offers an account of the econometric difficulties associated with estimating production functions. Studies by Basu (1996) and Basu and Fernald (1997) suggest returns to scale are about constant or decreasing. Studies by Hall (1990), Caballero and Lyons (1992), Harris and Lau (1992), Antweiler and Treffler (2002), and Harrison (2003) suggest there are quantitatively significant increasing returns, either internal or external. On this background it is not surprising that the case of IRS (at least at industry level), together with market forms different from perfect competition, has in recent years received more attention in macroeconomics and in the theory of economic growth.

Macroeconomists' use of the value-laden term "technological progress" in connection with technological change may seem suspect. But the term should be interpreted as merely a label for certain types of shifts of isoquants in an abstract universe. At a more concrete and disaggregate level analysts of course make use of more refined notions about technological change, recognizing for example not only benefits of new technologies, but also the risks, including risk of fundamental mistakes (think of the introduction and later abandonment of asbestos in the construction industry).

Informative history of technology is contained in Ruttan (2001) and Smil (2003). For more general economic history, see, e.g., Clark (2008) and Persson (2010). Forecasts of technological development in the next decades are contained in, for instance, Brynjolfsson and McAfee (2014).

Embodied technological progress, sometimes called investment-specific technological progress, is explored in, for instance, Solow (1960), Greenwood et al. (1997), and Groth and Wendner (2014). Hulten (2001) surveys the literature and issues related to measurement of the direct contribution of capital accumulation and technological change, respectively, to productivity growth.

Conditions ensuring that a representative household is admitted and the concept of Gorman preferences are discussed in Acemoglu (2009). Another useful source, also concerning the conditions for the representative firm to be a meaningful notion, is Mas-Colell et al. (1995). For general discussions of the limitations of representative agent approaches, see Kirman (1992) and Gallegati and Kirman (1999). Reviews of the "Cambridge Controversy" are contained in Mas-Colell (1989) and Felipe and Fisher (2003). The last-mentioned authors find the conditions required for the well-behavedness of these constructs so stringent that it is difficult to believe that actual economies are in any sense close to satisfy them. For a less distrustful view, see for instance Ferguson (1969), Johansen (1972), Malinvaud (1998), Jorgenson et al. (2005), and Jones (2005).

It is often assumed that capital depreciation can be described as geometric (in continuous time exponential) evaporation of the capital stock. This formula is popular in macroeconomics, more so because of its simplicity than its realism. An introduction to more general approaches to depreciation is contained in, e.g., Nickell (1978).

2.9 References

(incomplete)

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Chapter 3

Continuous time analysis

Because dynamic analysis is generally easier in continuous time, growth models are often stated in continuous time. This chapter gives an account of the conceptual aspects of continuous time analysis. Appendix A considers simple growth arithmetic in continuous time. And Appendix B provides solution formulas for linear first-order differential equations.

3.1 The transition from discrete time to continuous time

We start from a discrete time framework. The run of time is divided into successive periods of equal length, taken as the time-unit. Let us here index the periods by $i = 0, 1, 2, ...$ Thus financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

3.1.1 Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t + \Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t + \Delta t)$

approach zero, i.e., let $\Delta t \to 0$. When a is a differentiable function of t, we have

$$\lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t + \Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *derivative of* $a(\cdot)$ at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given,} \tag{3.1}$$

where $s(t)$ is the saving flow at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit choose one year. Going back to discrete time, if wealth grows at a constant rate $g > 0$ per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, \ldots . \tag{3.2}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period,

$$a_i = a_0(1 + \frac{g}{n})^i. \tag{3.3}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (3.3) gives

$$a(t) = a_{nt} = a_0(1 + \frac{g}{n})^{nt} = a_0 \left[(1 + \frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$. Thus $m \to \infty$. Then, in the limit there is continuous compounding and it can be shown that

$$a(t) = a_0 e^{gt}, \tag{3.4}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1 + 1/m)^m \simeq 2.7182818285....$

The formula (3.4) is the continuous-time analogue to the discrete time formula (3.2) with annual compounding. A geometric growth factor is replaced by an exponential growth factor, e^{gt}, and this growth factor is valid for any t in the time interval $(-\tau_1, \tau_2)$ for which the growth rate of a equals the constant g (τ_1 and τ_2 being some positive real numbers).

We can also view the formulas (3.2) and (3.4) as the solutions to a difference equation and a differential equation, respectively. Thus, (3.2) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value

a_0. And (3.4) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{3.5}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (3.5) is called the *basic accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.

3.1.2 Compound interest and discounting

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro on a drawing account in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{3.6}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *interest intensity*, respectively, or the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (3.5) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time analysis, we use compound discounting. We reverse the accumulation formula and go from the compounded or terminal value to the present value $a(t_0)$. Similarly, given a consumption plan, $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time

t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt} dt, \tag{3.7}$$

presupposing a constant interest rate. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (3.7) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau) d\tau} dt.$$

In (3.7) $c(t)$ is discounted by $e^{-rt} \approx (1+r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (**??**) where it is c_{t-1} that is discounted by $(1+r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t-1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

3.2 The allowed range for parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{3.8}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (3.8) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

3.3 Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations*. What is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these elementary units we can form *composite measurement units*. Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t)+I(t)$ or $K(t)+\dot{K}(t)$ are thus illegitimate. But one can write $K(t+\Delta t) \approx K(t)+(I(t)-\delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0,\ T)$ or perhaps

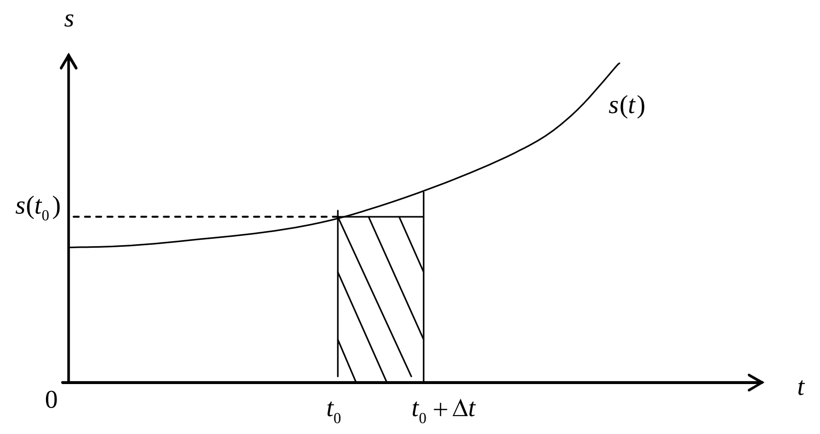

Figure 3.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0+\Delta t$ is $\approx$ the hatched area.

$[0,\ \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0+\Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0+\Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0)\cdot\Delta t$, cf. the hatched area in Figure 3.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

3.4 The choice between discrete and continuous time formulation

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. And, as for instance Allen (1967) argued, it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. And centralized asset markets equilibrate very fast and respond immediately to new information. For such markets a formulation in continuous time seems a better approximation.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{3.9}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1 + \alpha)x_t$, where $1 + \alpha < 0$. The solution is $x_t = (1 + \alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1 + \alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise (together with divergence if $\alpha < -2$) in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

It should be added, however, that this potential problem *can* always be avoided within discrete time models by choosing a sufficiently *short* period length. Indeed, the solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (3.9) the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (3.9). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. Yet, everything else equal, the clearer distinction between stocks and flows in continuous time than in discrete time speaks for the former. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.

3.5 Appendix A: Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small.

3.6 Appendix B: Solution formulas for linear differential equations of first order

For a general differential equation of first order, $\dot{x}(t) = \varphi(x(t), t)$, with $x(t_0) = x_{t_0}$ and where φ is a continuous function, we have, at least for t in an interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$,

$$x(t) = x_{t_0} + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau. \qquad (*)$$

To get a confirmation, calculate $\dot{x}(t)$ from (*).

For the special case of a linear differential equation of first order, $\dot{x}(t) + a(t)x(t) = b(t)$, we can specify the solution. Three sub-cases of rising complexity are:

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = (x_{t_0} - x^*)e^{-a(t-t_0)} + x^*, \text{ where } x^* = \frac{b}{a}.$$

If $a = 0$, we get, directly from (*), the solution $x(t) = x_{t_0} + bt$.[1]

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0} e^{-a(t-t_0)} + e^{-a(t-t_0)} \int_{t_0}^{t} b(s) e^{a(s-t_0)} ds.$$

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0} e^{-a(t-t_0)} + e^{-a(t-t_0)} c \int_{t_0}^{t} e^{(a+h)(s-t_0)} ds = (x_{t_0} - \frac{c}{a+h}) e^{-a(t-t_0)} + \frac{c}{a+h} e^{h(t-t_0)}.$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0} e^{-\int_{t_0}^{t} a(\tau)d\tau} + e^{-\int_{t_0}^{t} a(\tau)d\tau} \int_{t_0}^{t} b(s) e^{\int_{t_0}^{s} a(\tau)d\tau} ds.$$

[1] Some non-linear differential equations can be transformed into this simple case. For simplicity let $t_0 = 0$. Consider the equation $\dot{y}(t) = \alpha y(t)^{\beta}$, $y_0 > 0$ given, $\alpha \neq 0, \beta \neq 1$ (a *Bernoulli equation*). To find the solution for $y(t)$, let $x(t) \equiv y(t)^{1-\beta}$. Then, $\dot{x}(t) = (1-\beta)y(t)^{-\beta}\dot{y}(t) = (1-\beta)y(t)^{-\beta}\alpha y(t)^{\beta} = (1-\beta)\alpha$. The solution for this is $x(t) = x_0 + (1-\beta)\alpha t$, where $x_0 = y_0^{1-\beta}$. Thereby the solution for $y(t)$ is $y(t) = x(t)^{1/(1-\beta)} = \left(y_0^{1-\beta} + (1-\beta)\alpha t\right)^{1/(1-\beta)}$, which is defined for $t > -y_0^{1-\beta}/((1-\beta)\alpha$.

Special case: $b(t) = 0$. Solution:

$$x(t) = x_{t_0} e^{-\int_{t_0}^{t} a(\tau)d\tau}.$$

Even more special case: $b(t) = 0$ and $a(t) = a$, a constant. Solution:

$$x(t) = x_{t_0} e^{-a(t-t_0)}.$$

Remark 1 For $t_0 = 0$, most of the formulas will look simpler.

Remark 2 To check whether a suggested solution *is* a solution, calculate the time derivative of the suggested solution and add an arbitrary constant. By appropriate adjustment of the constant, the final result should be a replication of the original differential equation together with its initial condition.

Chapter 4

Skill-biased technical change. Balanced growth theorems

This chapter is both an alternative and a supplement to the pages 60-64 in Acemoglu, where the concepts of neutral technical change and balanced growth, including Uzawa's theorem, are discussed.

Since "neutral" technical change should be seen in relation to "biased" technical change, Section 1 below introduces the concept of "biased" technical change. Also concerning biased technical change do three different definitions, Hicks', Harrod's, and what the literature has dubbed "Solow's". Below we concentrate on Hick's definition − with an application to how technical change affects the evolution of the skill premium. So the focus is on the production factors skilled and unskilled labor rather than capital and labor. While regarding capital and labor it is Harrod's classifications that are most used in macroeconomics, regarding skilled and unskilled labor it is Hicks'.

The remaining sections discuss the concept of balanced growth and present three fundamental propositions about balanced growth. In view of the generality of the propositions, they have a broad field of application. Our propositions 1 and 2 are slight extensions of part 1 and 2, respectively, of what Acemoglu calls Uzawa's Theorem I (Acemoglu, 2009, p. 60). Our Proposition 3 essentially corresponds to what Acemoglu calls Uzawa's Theorem II (Acemoglu, 2009, p. 63).

4.1 The rising skill premium

4.1.1 Skill-biased technical change in the sense of Hicks: An example

Let aggregate output be produced through a differentiable three-factor production function $\tilde{F}$:

$$Y = \tilde{F}(K, L_1, L_2, t),$$

where K is capital input, L_1 is input of unskilled labor (also called blue-collar labor below), and L_2 is input of skilled labor. Suppose technological change is such that the production function can be rewritten

$$\tilde{F}(K, L_1, L_2, t) = F(K, H(L_1, L_2, t)), \tag{4.1}$$

where the "nested" function $H(L_1, L_2, t)$ represents input of a "human capital" aggregate. Let F be CRS-neoclassical w.r.t. K and H and let H be CRS-neoclassical w.r.t. (L_1, L_2). Finally, let $\partial H/\partial t > 0$. So "technical change" amounts to "technical progress".

In equilibrium under perfect competition in the labor markets the relative wage, often called the "skill premium", will be

$$\frac{w_2}{w_1} = \frac{\partial Y/\partial L_2}{\partial Y/\partial L_1} = \frac{F_H \partial H/\partial L_2}{F_H \partial H/\partial L_1} = \frac{H_2(L_1, L_2, t)}{H_1(L_1, L_2, t)} = \frac{H_2(1, L_2/L_1, t)}{H_1(1, L_2/L_1, t)}, \tag{4.2}$$

where we have used Euler's theorem (saying that if H is homogeneous of degree one in its first two arguments, then the partial derivatives of H are homogeneous of degree zero w.r.t. these arguments).

Time is continuous (nevertheless the time argument of a variable, x, is in this section written as a subscript t). Hicks' definitions are now: If for all $L_2/L_1 > 0$,

$$\frac{d\left(\frac{H_2(1,L_2/L_1,t)}{H_1(1,L_2/L_1,t)}\right)}{dt}\Bigg|_{\frac{L_2}{L_1}\text{ constant}} \gtreqless 0, \text{ then technical change is}$$

$$\begin{cases} \text{skill-biased in the sense of Hicks,} \\ \text{skill-neutral in the sense of Hicks.} \\ \text{blue collar-biased in the sense of Hicks,} \end{cases} \tag{4.3}$$

respectively.

In the US the skill premium (measured by the wage ratio for college grads vis-a-vis high school grads) has had an upward trend since 1950 (see for instance Jones and Romer, 2010).[1] If in the same period the relative supply of skilled labor had been roughly constant, by (4.3) in combination with (4.2), a possible explanation could be that technological change has been skill-biased in the sense of Hicks. In reality, in the same period also the relative supply of skilled labor has been rising (in fact even faster than the skill premium). Since in spite of this the skill premium has *risen,* it suggests that the extend of "skill-biasedness" has been even stronger.

We may alternatively put it this way. As the H function is CRS-neoclassical w.r.t. L_1 and L_2, we have $H_{22} < 0$ *and* $H_{12} > 0$, cf. Chapter 2. Hence, by (4.2), a rising L_2/L_1 without technical change would imply a *declining* skill premium. That the opposite has happened must, within our simple model, be due to (a) there *has* been technical change, and (b) technical change has *favoured skilled labor* (which means that technical change has been skill-biased in the sense of Hicks).

An additional aspect of the story is that skill-biasedness helps *explain* the observed increase in the relative *supply* of skilled labor. If for a constant relative supply of skilled labor, the skill premium is increasing, this increase strengthens the incentive to go to college. Thereby the relative supply of skilled labor (reflecting the fraction of skilled labor in the labor force) tends to increase.

4.1.2 Capital-skill complementarity

An additional potential source of a rising skill premium is *capital-skill complementarity.* Let the aggregate production function be

$$Y = \tilde{F}(K, L_1, L_2, t) = F(K, A_{1t}L_1, A_{2t}L_2) = (K+A_{1t}L_1)^{\alpha}(A_{2t}L_2)^{1-\alpha}, \ 0 < \alpha < 1,$$

where A_{1t} and A_{2t} are technical coefficients that may be rising over time. In this production function capital and unskilled labor are perfectly substitutable (the partial elasticity of factor substitution between them is $+\infty$). On the other hand there is *direct complementarity* between capital and skilled labor, i.e., $\partial^2 Y/(\partial L_2 \partial K) > 0$.

Under perfect competition the skill premium is

$$\begin{aligned} \frac{w_2}{w_1} &= \frac{\partial Y/\partial L_2}{\partial Y/\partial L_2} = \frac{(K + A_{1t}L_1)^{\alpha}(1-\alpha)(A_{2t}L_2)^{-\alpha}A_{2t}}{\alpha(K + A_{1t}L_1)^{\alpha-1}A_{1t}(A_{2t}L_2)^{1-\alpha}} \\ &= \frac{1-\alpha}{\alpha}\left(\frac{K + A_{1t}L_1}{A_{2t}L_2}\right)\frac{A_{2t}}{A_{1t}}. \end{aligned} \tag{4.4}$$

[1] On the other hand, over the years 1915 - 1950 the skill premium had a downward trend (Jones and Romer, 2010).

Here, if technical change is absent (A_{1t} and A_{2t} constant), a rising capital stock will, *for fixed* L_1 *and* L_2, raise the skill premium.

A more realistic scenario is, however, a situation with an approximately constant real interest rate, cf. Kaldor's stylized facts. We have, again by perfect competition,

$$\frac{\partial Y}{\partial K} = \alpha(K + A_{1t}L_1)^{\alpha-1}(A_{2t}L_2)^{1-\alpha} = \alpha\left(\frac{K + A_{1t}L_1}{A_{2t}L_2}\right)^{\alpha-1} = r_t + \delta, \quad (4.5)$$

where r_t is the real interest rate at time t and δ is the (constant) capital depreciation rate. For $r_t = r$, a constant, (4.5) gives

$$\frac{K + A_{1t}L_1}{A_{2t}L_2} = \left(\frac{r+\delta}{\alpha}\right)^{-\frac{1}{1-\alpha}} \equiv c, \quad (4.6)$$

a constant. In this case, (4.4) shows that capital-skill complementarity is *not sufficient* for a rising skill premium. A rising skill premium requires that technical change brings about a rising A_{2t}/A_{1t}. So again an observed rising skill premium, along with a more or less constant real interest rate, suggests that technical change is skill-biased.

We may rewrite (4.6) as

$$\frac{K}{A_{2t}L_2} = c - \frac{A_{1t}L_1}{A_{2t}L_2},$$

where the conjecture is that $A_{1t}L_1/(A_{2t}L_2) \to 0$ for $t \to \infty$. The analysis suggests the following story. Skill-biased technical progress generates rising productivity as well as a rising skill premium. The latter induces more and more people to go to college. The rising level of education in the labor force raises productivity further. This is a basis for further capital accumulation, continuing to replace unskilled labor, and so on.

In particular since the early 1980s the skill premium has been sharply increasing in the US (see Acemoglu, p. 498). This is also the period where ICT technologies took off.

4.2 Balanced growth and constancy of key ratios

The focus now shifts to homogeneous labor vis-a-vis capital.

We shall state general definitions of the concepts of "steady state" and "balanced growth", concepts that are related but not identical. With respect to "balanced growth" this implies a minor deviation from the way Acemoglu briefly defines it informally on his page 57. The main purpose of the present chapter is to lay bare the connections between these two concepts as well as their relation to the hypothesis of Harrod-neutral technical progress and Kaldor's stylized facts.

4.2.1 The concepts of steady state and balanced growth

A basic equation in many one-sector growth models for a closed economy in continuous time is

$$\dot{K} = I - \delta K = Y - C - \delta K \equiv S - \delta K, \tag{4.7}$$

where K is aggregate capital, I aggregate gross investment, Y aggregate output, C aggregate consumption, S aggregate gross saving ($\equiv Y - C$), and $\delta \geq 0$ is a constant physical capital depreciation rate.

Usually, in the theoretical literature on dynamic models, a *steady state* is defined in the following way:

Definition 3 *A steady state of a dynamic model is a stationary solution to the fundamental differential equation(s) of the model.*

Or briefly: a steady state is a stationary point of a dynamic process.

Let us take the Solow growth model as an example. Here gross saving equals sY, where s is a constant, $0 < s < 1$. Aggregate output is given by a neoclassical production function, F, with CRS and Harrod-neutral technical progress: $Y = F(K, AL) = ALF(\tilde{k}, 1) \equiv ALf(\tilde{k})$, where L is the labor force, A is the level of technology, and $\tilde{k} \equiv K/(AL)$ is the (effective) capital intensity. Moreover, $f' > 0$ and $f'' < 0$. Solow assumes $L(t) = L(0)e^{nt}$ and $A(t) = A(0)e^{gt}$, where n and $g \geq 0$ are the constant growth rates of the labor force and technology, respectively. By log-differentiating $\tilde{k}$ w.r.t. t,[2] we end up with the *fundamental differential equation* ("law of motion") of the Solow model:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + g + n)\tilde{k}. \tag{4.8}$$

Thus, in the Solow model, a (non-trivial) steady state is a $\tilde{k}^* > 0$ such that, if $\tilde{k} = \tilde{k}^*$, then $\dot{\tilde{k}} = 0$. In passing we note that, by (4.8), such a $\tilde{k}^*$ must satisfy the equation $f(\tilde{k}^*)/\tilde{k}^* = (\delta + g + n)/s$, and in view of $f'' < 0$, it is unique and globally asymptotically stable if it exists. A sufficient condition

[2] Or by directly using the fraction rule, see Appendix A to Chapter 3.

for its existence is that $\delta + g + n > 0$ and f satisfies the Inada conditions $\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0$.

The most common definition in the literature of *balanced growth* for an aggregate economy is the following:

Definition 4 *A balanced growth path is a path $(Y, K, C)_{t=0}^{\infty}$ along which the quantities $Y, K,$ and C are positive and grow at constant rates (not necessarily positive and not necessarily the same).*

Acemoglu, however, defines (Acemoglu, 2009, p. 57) balanced growth in the following way: "balanced growth refers to an allocation where output grows at a constant rate and capital-output ratio, the interest rate, and factor shares remain constant". My problem with this definition is that it mixes growth of aggregate quantities with income distribution aspects (interest rate and factor income shares). And it is not made clear what is meant by the output-capital ratio if the relative price of capital goods is changing over time. So I stick to the definition above which is quite standard and is known to function well in many different contexts.

Note that in the Solow model (as well as in many other models) we have that if the economy is in a steady state, $\tilde{k} = \tilde{k}^*$, then the economy features balanced growth. Indeed, a steady state of the Solow model implies by definition that $\tilde{k} \equiv K/(AL)$ is constant. Hence K must grow at the same *constant* rate as AL, namely $g + n$. In addition, $Y = f(\tilde{k}^*)AL$ in a steady state, showing that also Y must grow at the constant rate $g + n$. And so must then $C = (1 - s)Y$. So in a steady state of the Solow model the path followed by $(Y, K, C)_{t=0}^{\infty}$ is a balanced growth path.

As we shall see in the next section, in the Solow model (and many other models) the reverse also holds: if the economy features balanced growth, then it is in a steady state. But this equivalence between steady state and balanced growth does not hold in all models.

4.2.2 A general result about balanced growth

An interesting fact is that, given the dynamic resource constraint (4.7), we have *always* that if there is balanced growth with positive gross saving, then the ratios Y/K and C/Y are constant (by "*always*" is meant: independently of how saving is determined and how the labor force and technology evolve). And also the other way round: as long as gross saving is positive, constancy of the Y/K and C/Y ratios is enough to ensure balanced growth. So balanced growth and constancy of certain key ratios are essentially equivalent.

This is a very practical general observation. And since Acemoglu does not state any balanced growth theorem at this general level, we shall do it here, together with a proof. Letting g_x denote the growth rate of the (positively valued) variable x, i.e., $g_x \equiv \dot{x}/x$, we claim:

Proposition 1 *(the balanced growth equivalence theorem). Let* $(Y, K, C)_{t=0}^{\infty}$ *be a path along which* Y, K*,* C*, and* $S \equiv Y - C$ *are positive for all* $t \geq 0$*. Then, given the accumulation equation (4.7), the following holds:*

(i) if there is balanced growth, then $g_Y = g_K = g_C$*, and the ratios* Y/K *and* C/Y *are constant;*

(ii) if Y/K *and* C/Y *are constant, then* Y, K*, and* C *grow at the same constant rate, i.e., not only is there balanced growth, but the growth rates of* Y, K*, and* C *are the same.*

Proof Consider a path $(Y, K, C)_{t=0}^{\infty}$ along which Y, K, C, and $S \equiv Y - C$ are positive for all $t \geq 0$. (i) Assume there is balanced growth. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (4.7), we have that $S/K = g_K + \delta$ is constant, implying

$$g_S = g_K. \tag{*}$$

Further, since $Y = C + S$,

$$\begin{aligned} g_Y &= \frac{\dot{Y}}{Y} = \frac{\dot{C}}{Y} + \frac{\dot{S}}{Y} = g_C \frac{C}{Y} + g_S \frac{S}{Y} = g_C \frac{C}{Y} + g_K \frac{S}{Y} \qquad \text{(by (*))} \\ &= g_C \frac{C}{Y} + g_K \frac{Y - C}{Y} = \frac{C}{Y}(g_C - g_K) + g_K. \qquad (**) \end{aligned}$$

Now, let us provisionally assume that $g_K \neq g_C$. Then (**) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{***}$$

which is a constant since g_Y, g_K, and g_C are constant. Constancy of C/Y requires that $g_C = g_Y$, hence, by (***), $C/Y = 1$, i.e., $C = Y$. In view of $Y = C + S$, however, this outcome contradicts the given condition that $S > 0$. Hence, our provisional assumption and its implication, (***), are falsified. Instead we have $g_K = g_C$. By (**), this implies $g_Y = g_K = g_C$, but now without the condition $C/Y = 1$ being implied. It follows that Y/K and C/Y are constant.

(ii) Suppose Y/K and C/Y are constant. Then $g_Y = g_K = g_C$, so that C/K is a constant. We now show that this implies that g_K is constant. Indeed, from (4.7), $S/Y = 1 - C/Y$, so that also S/Y is constant. It follows that $g_S = g_Y = g_K$, so that S/K is constant. By (4.7),

$$\frac{S}{K} = \frac{\dot{K} + \delta K}{K} = g_K + \delta,$$

so that g_K is constant. This, together with constancy of Y/K and C/Y, implies that also g_Y and g_C are constant. □

Remark. It is part (i) of the proposition which requires the assumption $S > 0$ for all $t \geq 0$. If $S = 0$, we would have $g_K = -\delta$ and $C \equiv Y - S = Y$, hence $g_C = g_Y$ for all $t \geq 0$. Then there would be balanced growth if the common value of g_C and g_Y had a constant growth rate. This growth rate, however, could easily differ from that of K. Suppose $Y = AK^{\alpha}L^{1-\alpha}$, $g_A = \gamma$ and $g_L = n$ (γ and n constants). Then we would have $g_C = g_Y = \gamma - \alpha\delta + (1-\alpha)n$, which could easily be strictly positive and thereby different from $g_K = -\delta \leq 0$ so that (i) no longer holds. □

The nice feature is that this proposition holds for *any* model for which the simple dynamic resource constraint (4.7) is valid. No assumptions about for example CRS and other technology aspects or about market form are involved. Note also that Proposition 1 suggests a link from balanced growth to steady state. And such a link *is* present in for instance the Solow model. Indeed, by (i) of Proposition 1, balanced growth implies constancy of Y/K, which in the Solow model implies that $f(\tilde{k})/\tilde{k}$ is constant. In turn, the latter is only possible if $\tilde{k}$ is constant, that is, if the economy is in steady state.

There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technological change, see Groth et al., 2010). Therefore, it is recommendable always to maintain a distinction between the terms steady state and balanced growth.

4.3 The crucial role of Harrod-neutrality

Proposition 1 suggests that if one accepts Kaldor's stylized facts (see Chapter 1) as a characterization of the past century's growth experience, and if one wants a model consistent with them, one should construct the model such that it can generate balanced growth. For a model to be capable of generating balanced growth, however, technological progress must be of the Harrod-neutral type (i.e., be labor-augmenting), at least in a neighborhood of the balanced growth path. For a fairly general context (but of course not as general as that of Proposition 1), this was shown already by Uzawa (1961). We now present a modernized version of Uzawa's contribution.

Let the aggregate production function be

$$Y(t) = \tilde{F}(K(t), BL(t), t), \qquad B > 0, \tag{4.9}$$

where B is a constant that depends on measurement units. The only technology assumption needed is that $\tilde{F}$ has CRS w.r.t. the first two arguments

($\tilde{F}$ need not be neoclassical for example). As a representation of technical progress, we assume $\partial\tilde{F}/\partial t > 0$ for all $t \geq 0$ (i.e., as time proceeds, unchanged inputs result in more and more output). We also assume that the labor force evolves according to

$$L(t) = L(0)e^{nt}, \tag{4.10}$$

where n is a constant. Further, non-consumed output is invested and so (4.7) is the dynamic resource constraint of the economy.

Proposition 2 *(Uzawa's balanced growth theorem) Let* $P = (Y(t), K(t), C(t))_{t=0}^{\infty}$, *where* $0 < C(t) < Y(t)$ *for all* $t \geq 0$, *be a path satisfying the capital accumulation equation (4.7), given the CRS-production function (4.9) and the labor force path in (4.10). Then:*

(i) a necessary condition for this path to be a balanced growth path is that along the path it holds that

$$Y(t) = \tilde{F}(K(t), BL(t), t) = \tilde{F}(K(t), A(t)L(t), 0), \tag{4.11}$$

where $A(t) = Be^{gt}$ *with* $g \equiv g_Y - n$;

(ii) for any $g > 0$ *such that there is a* $q > \delta + g + n$ *with the property that the production function* $\tilde{F}$ *in (4.9) allows an output-capital ratio equal to* q *at* $t = 0$ *(i.e.,* $\tilde{F}(1, \tilde{k}^{-1}, 0) = q$ *for some real number* $\tilde{k} > 0$*), a sufficient condition for* $\tilde{F}$ *to be compatible with a balanced growth path with output-capital ratio* q*, is that* $\tilde{F}$ *can be written as in (4.11) with* $A(t) = Be^{gt}$.

Proof (i)[3] Suppose the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path. By definition, g_K and g_Y are then constant, so that $K(t) = K(0)e^{g_K t}$ and $Y(t) = Y(0)e^{g_Y t}$. We then have

$$Y(t)e^{-g_Y t} = Y(0) = \tilde{F}(K(0), BL(0), 0) = \tilde{F}(K(t)e^{-g_K t}, BL(t)e^{-nt}, 0), \quad (*)$$

where we have used (4.9) with $t = 0$. In view of the precondition that $S(t) \equiv Y(t) - C(t) > 0$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (*) then implies

$$Y(t) = \tilde{F}(K(t)e^{g_Y t}e^{-g_K t}, BL(t)e^{g_Y t}e^{-nt}, 0) = \tilde{F}(K(t), Be^{(g_Y - n)t}L(t), 0).$$

[3] This part draws upon Schlicht (2006), who generalized a proof in Wan (1971, p. 59) for the special case of a constant saving rate.

We see that (4.11) holds for $A(t) = Be^{gt}$ with $g \equiv g_Y - n$.

(ii) Suppose (4.11) holds with $A(t) = Be^{gt}$. Let $g \geq 0$ be given such that there is a $q > g + n + \delta > 0$ with the property that

$$\tilde{F}(1, \tilde{k}^{-1}, 0) = q \qquad (**)$$

for some constant $\tilde{k} > 0$. Our strategy is to prove the claim in (ii) by construction of a path $P = (Y(t), K(t), C(t))_{t=0}^{\infty}$ which satisfies it. We let P be such that the saving-income ratio is a constant $s \equiv (\delta+g+n)/q \in (0,1)$, i.e., $Y(t) - C(t) \equiv S(t) = sY(t)$ for all $t \geq 0$. Inserting this, together with $Y(t) = f(\tilde{k}(t))A(t)L(t)$, where $f(\tilde{k}(t)) \equiv \tilde{F}(\tilde{k}(t), 1, 0)$ and $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, into (4.7), rearranging gives the Solow equation (4.8). Hence $\tilde{k}(t)$ is constant if and only if $\tilde{k}(t)$ satisfies the equation $f(\tilde{k}(t))/\tilde{k}(t) = (\delta + g + n)/s \equiv q$. By (**) and the definition of f, the required value of $\tilde{k}(t)$ is $\tilde{k}$, which is consequently the unique steady state for the constructed Solow equation. Letting $K(0)$ satisfy $K(0) = \tilde{k}BL(0)$, where $B = A(0)$, we thus have $\tilde{k}(0) = K(0)/(A(0)L(0)) = \tilde{k}$. So that the initial value of $\tilde{k}(t)$ equals the steady state value. It follows that $\tilde{k}(t) = \tilde{k}$ for all $t \geq 0$, and so $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t) = f(\tilde{k})/\tilde{k} = q$ for all $t \geq 0$. In addition, $C(t) = (1-s)Y(t)$, so that $C(t)/Y(t)$ is constant along the path P. As both Y/K and C/Y are thus constant along the path P, by (ii) of Proposition 1 follows that P is a balanced growth path, as was to be proved. □

The form (4.11) indicates that along a balanced growth path, technical progress must be purely "labor augmenting", that is, Harrod-neutral. It is in this case convenient to define a new CRS function, F, by $F(K(t), A(t)L(t)) \equiv \tilde{F}(K(t), A(t)L(t), 0)$. Then (i) of the proposition implies that at least along the balanced growth path, we can rewrite the production function this way:

$$Y(t) = \tilde{F}(K(t), A(0)L(t), t) = F(K(t), A(t)L(t)), \qquad (4.12)$$

where $A(0) = B$ and $A(t) = A(0)e^{gt}$ with $g \equiv g_Y - n$.

It is important to recognize that the occurrence of Harrod-neutrality says nothing about what the *source* of technological progress is. Harrod-neutrality should not be interpreted as indicating that the technological progress emanates specifically from the labor input. Harrod-neutrality only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* at the aggregate level in the form (4.12).[4]

[4] For a CRS Cobb-Douglas production function with technological progress, Harrod-neutrality is present whenever the output elasticity w.r.t capital (often denoted α) is constant over time.

What is the intuition behind the Uzawa result that for balanced growth to be possible, technical progress must have the purely labor-augmenting form? First, notice that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output. In contrast, in simple macro models labor is a non-produced production factor which (at least in the context of (4.10)) grows in an exogenous way. Second, because of CRS, the original formulation, (4.9), of the production function implies that

$$1 = \tilde{F}(\frac{K(t)}{Y(t)}, \frac{L(t)}{Y(t)}, t). \tag{4.13}$$

Now, since capital is accumulated non-consumed output, it tends to inherit the trend in output such that $K(t)/Y(t)$ must be constant along a balanced growth path (this is what Proposition 1 is about). Labor does not inherit the trend in output; indeed, the ratio $L(t)/Y(t)$ is free to adjust as time proceeds. When there is technical progress ($\partial\tilde{F}/\partial t > 0$) along a balanced growth path, this progress must manifest itself in the form of a changing $L(t)/Y(t)$ in (13.5) as t proceeds, precisely because $K(t)/Y(t)$ *must* be constant along the path. In the "normal" case where $\partial\tilde{F}/\partial L > 0$, the needed change in $L(t)/Y(t)$ is a *fall* (i.e., a rise in $Y(t)/L(t)$). This is what (13.5) shows. Indeed, the fall in $L(t)/Y(t)$ must exactly offset the effect on $\tilde{F}$ of the rising t, when there is a fixed capital-output ratio.[5] It follows that along the balanced growth path, $Y(t)/L(t)$ is an increasing implicit function of t. If we denote this function $A(t)$, we end up with (4.12) with specified properties (given by g and q).

The generality of Uzawa's theorem is noteworthy. The theorem assumes CRS, but does not presuppose that the technology is neoclassical, not to speak of satisfying the Inada conditions.[6] And the theorem holds for exogenous as well as endogenous technological progress. It is also worth mentioning that the proof of the sufficiency part of the theorem is *constructive*. It provides a method to construct a hypothetical balanced growth path (BGP from now).[7]

A simple implication of the Uzawa theorem is the following. Interpreting the $A(t)$ in (4.11) as the "level of technology", we have:

COROLLARY Along a BGP with positive gross saving and the technology level, $A(t)$, growing at the rate $g \geq 0$, output grows at the rate $g + n$ while labor productivity, $y \equiv Y/L$, and consumption per unit of labor, $c \equiv C/L$, grow at the rate g.

[5] This way of presenting the intuition behind the Uzawa result draws upon Jones and Scrimgeour (2008).

[6] Many accounts of the Uzawa theorem, including Jones and Scrimgeour (2008), presume a neoclassical production function, but the theorem is much more general.

[7] Part (ii) of Proposition 2 is left out in Acemoglu's book.

Proof That $g_Y = g + n$ follows from (i) of Proposition 2. As to the growth rate of labor productivity we have

$$y_t = \frac{Y(0)e^{g_Y t}}{L(0)e^{nt}} = y(0)e^{(g_Y - n)t} = y(0)e^{gt}.$$

Finally, by Proposition 1, along a BGP with $S > 0$, $c \equiv C/L$ must grow at the same rate as y. □

We shall now consider the implication of Harrod-neutrality for the income shares of capital and labor when the technology is neoclassical and markets are perfectly competitive.

4.4 Harrod-neutrality and the functional income distribution

There is one facet of Kaldor's stylized facts we have so far not related to Harrod-neutral technical progress, namely the long-run "approximate" constancy of both the income share of labor, wL/Y, and the rate of return to capital. At least with neoclassical technology, profit maximizing firms, and perfect competition in the output and factor markets, these properties are inherent in the combination of constant returns to scale, balanced growth, and the assumption that the relative price of capital goods (relative to consumption goods) is constant over time. The latter condition holds in models where the capital good is nothing but non-consumed output, cf. (4.7).[8]

To see this, we start out from a neoclassical CRS production function with Harrod-neutral technological progress,

$$Y(t) = F(K(t), A(t)L(t)). \tag{4.14}$$

With $w(t)$ denoting the real wage at time t, in equilibrium under perfect competition the labor income share will be

$$\frac{w(t)L(t)}{Y(t)} = \frac{\frac{\partial Y(t)}{\partial L(t)}L(t)}{Y(t)} = \frac{F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)}. \tag{4.15}$$

In this simple model, without natural resources, (gross) capital income equals non-labor income, $Y(t) - w(t)L(t)$. Hence, if $r(t)$ denotes the (net) rate of return to capital at time t, then

$$r(t) = \frac{Y(t) - w(t)L(t) - \delta K(t)}{K(t)}. \tag{4.16}$$

[8] The reader may think of the "corn economy" example in Acemoglu, p. 28.

Denoting the (gross) capital income share by $\alpha(t)$, we can write this $\alpha(t)$ (in equilibrium) in three ways:

$$\begin{aligned}
\alpha(t) &\equiv \frac{Y(t) - w(t)L(t)}{Y(t)} = \frac{(r(t)+\delta)K(t)}{Y(t)}, \\
\alpha(t) &= \frac{F(K(t), A(t)L(t)) - F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)} = \frac{F_1(K(t), A(t)L(t))K(t)}{Y(t)}, \\
\alpha(t) &= \frac{\frac{\partial Y(t)}{\partial K(t)} K(t)}{Y(t)}, \qquad (4.17)
\end{aligned}$$

where the first row comes from (4.16), the second from (4.14) and (4.15), the third from the second together with Euler's theorem.[9] Comparing the first and the last row, we see that in equilibrium

$$\frac{\partial Y(t)}{\partial K(t)} = r(t) + \delta.$$

In this condition we recognize one of the first-order conditions in the representative firm's profit maximization problem under perfect competition, since $r(t) + \delta$ can be seen as the firm's required gross rate of return.[10]

In the absence of uncertainty, the equilibrium real interest rate in the bond market must equal the rate of return on capital, $r(t)$. And $r(t) + \delta$ can then be seen as the firm's cost of disposal over capital per unit of capital per time unit, consisting of interest cost plus capital depreciation.

Proposition 3 *(factor income shares and rate of return under balanced growth) Let the path $(K(t), Y(t), C(t))_{t=0}^{\infty}$ be a BGP in a competitive economy with the production function (4.14) and with positive saving. Then, along the BGP, the $\alpha(t)$ in (4.17) is a constant, $\alpha \in (0, 1)$. The labor income share will be $1 - \alpha$ and the (net) rate of return on capital will be $r = \alpha q - \delta$, where q is the constant output-capital ratio along the BGP.*

Proof By CRS we have $Y(t) = F(K(t), A(t)L(t)) = A(t)L(t)F(\tilde{k}(t), 1) \equiv A(t)L(t)f(\tilde{k}(t))$. In view of part (i) of Proposition 2, by balanced growth, $Y(t)/K(t)$ is some constant, q. Since $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t)$ and $f'' < 0$, this implies $\tilde{k}(t)$ constant, say equal to $\tilde{k}^*$. But $\partial Y(t)/\partial K(t) = f'(\tilde{k}(t))$, which

[9] From Euler's theorem, $F_1 K + F_2 AL = F(K, AL)$, when F is homogeneous of degree one.

[10] With natural resources, say land, entering the set of production factors, the formula, (4.16), for the rate of return to capital should be modified by subtracting land rents from the numerator.

then equals the constant $f'(\tilde{k}^*)$ along the BGP. It then follows from (4.17) that $\alpha(t) = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, where $0 < \alpha$ follows from $f' > 0$ and $\alpha < 1$ from the fact that $q = Y/K = f(\tilde{k}^*)/\tilde{k}^* > f'(\tilde{k}^*)$, in view of $f'' < 0$ and $f(0) \geq 0$. Then, by the first equality in (4.17), $w(t)L(t)/Y(t) = 1 - \alpha(t) = 1 - \alpha$. Finally, by (4.16), the (net) rate of return on capital is $r = (1 - w(t)L(t)/Y(t))Y(t)/K(t) - \delta = \alpha q - \delta$. □

This proposition is of interest by displaying a link from balanced growth to constancy of factor income shares and the rate of return, that is, some of the "stylized facts" claimed by Kaldor. Note, however, that although the proposition implies constancy of the income shares and the rate of return, it does not *determine* them, except in terms of α and q. But both q and, generally, α are endogenous and depend on $\tilde{k}^*$,[11] which will generally be unknown as long as we have not specified a theory of saving. This takes us to theories of aggregate saving, for example the simple Ramsey model, cf. Chapter 8 in Acemoglu's book.

4.5 What if technological change is embodied?

In our presentation of technological progress above we have implicitly assumed that all technological change is *disembodied*. And the way the propositions 1, 2, and 3, are formulated assume this.

As noted in Chapter 2, *disembodied technological change* occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again the aggregate dynamic resource constraint (4.7) and the production function (4.9):

$$\dot{K}(t) = I(t) - \delta K(t), \tag{4.18}$$

$$Y(t) = \tilde{F}(K(t), BL(t), t), \qquad \partial \tilde{F}/\partial t > 0. \tag{4.19}$$

Here $Y(t) - C(t)$ is aggregate gross investment, $I(t)$. For a given level of $I(t)$, the resulting amount of new capital goods per time unit $(\dot{K}(t) + \delta K(t))$, measured in efficiency units, is independent of *when* this investment occurs. It is thereby not affected by technological progress. Similarly, the interpretation of $\partial \tilde{F}/\partial t > 0$ in (4.19) is that the higher technology level obtained as time proceeds results in higher productivity of *all* capital and labor. Thus also

[11] As to α, there is of course a trivial exception, namely the case where the production function is Cobb-Douglas and α therefore is a given parameter.

firms that have only old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.[12]

In contrast, we say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.

As also noted in Chapter 2, we may represent embodied technological progress by writing capital accumulation in the following way,

$$\dot{K}(t) = q(t)I(t) - \delta K(t), \tag{4.20}$$

where $I(t)$ is gross investment at time t and $q(t)$ measures the "quality" (productivity) of newly produced investment goods. The increasing level of technology implies increasing $q(t)$ so that a given level of investment gives rise to a greater and greater additions to the capital stock, K, measured in efficiency units. As in our aggregate framework, q capital goods can be produced at the same minimum cost as one consumption good, we have $p \cdot q = 1$, where p is the equilibrium price of capital goods in terms of consumption goods. So embodied technological progress is likely to result in a steady decline in the relative price of capital equipment, a prediction confirmed by the data (see, e.g., Greenwood et al., 1997).

This raises the question how the propositions 1, 2, and 3 fare in the case of embodied technological progress. The answer is that a generalized version of Proposition 1 goes through. Essentially, we only need to replace (4.7) by (13.13) and interpret K in Proposition 1 as the *value* of the capital stock, i.e., we have to replace K by $\tilde{K} = pK$.

But the concept of Harrod-neutrality no longer fits the situation without further elaboration. Hence to obtain analogies to Proposition 2 and Proposition 3 is a more complicated matter. Suffice it to say that with embodied technological progress, the class of production functions that are consistent with balanced growth is smaller than with disembodied technological progress.

[12] In the standard versions of the Solow model and the Ramsey model it is assumed that all technological progress has this form - for no other reason than that this is by far the simplest case to analyze.

4.6 Concluding remarks

In the Solow model as well as in many other models with disembodied technological progress, a steady state and a balanced growth path imply each other. Indeed, they are in that model, as well as many others, two sides of the same process. There *exist* exceptions, however, that is, cases where steady state and a balanced growth are not equivalent (some open economy models and some models with *embodied* technical change). So the two concepts should be held apart.[13]

Note that the definition of balanced growth refers to *aggregate* variables. At the same time as there is balanced growth at the aggregate level, *structural change* may occur. That is, a changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009, Chapter 20).

In view of the key importance of Harrod-neutrality, a natural question is: has growth theory uncovered any *endogenous* tendency for technical progress to converge to Harrod-neutrality? Fortunately, in his Chapter 15 Acemoglu outlines a theory about a mechanism entailing such a tendency, the theory of "directed technical change". Jones (2005) suggests an alternative mechanism.

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[13] Here we deviate from Acemoglu, p. 65, where he says that he will use the two terms "interchangingly". We also deviate from Barro and Sala-i-Martin (2004, pp. 33-34) who *define* a steady state as synonymous with a balanced growth path as the latter was defined above.

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Chapter 5

Growth accounting and the concept of TFP: Some limitations

5.1 Introduction

This chapter addresses the concepts of Total Factor Productivity, TFP, and TFP *growth*.[1] We underline the distinction between descriptive accounting and causal analysis. The chapter ends up with a warning regarding careless use of the concept of TFP growth in cross-country comparisons − and a suggested alternative approach.

For convenience, we treat time as continuous (although the timing of the variables is indicated merely by a subscript).

5.2 TFP growth and TFP level

Let Y_t denote aggregate output, in the sense of value added in fixed prices, at time t in a sector or the economy as a whole. Suppose Y_t is determined via the function

$$Y_t = F(K_t, H_t, t), \tag{5.1}$$

where K_t is an index of the physical capital input and H_t an index of quality-adjusted labor input. Natural resources (land, oil wells, coal in the ground, etc.) constitute a third primary production factor. The role of this factor is in growth accounting often subsumed under K.

[1] I thank Niklas Brønager for useful discussions.

The "quality-adjustment" of the input of labor (man-hours per year) aims at taking different levels of education and work experience into account. The heterogeneity of both types of input, and of output as well, implies huge measurement and conceptual difficulties. Here we ignore these problems. The third argument in (5.1) is time, t, indicating that the production function $F(\cdot\,,\cdot\,,t)$ is time-dependent. This is to open up for "shifts in the production function", due to new technology. We assume F is a neoclassical production function. When the partial derivative of F w.r.t. the third argument is positive, i.e., $\partial F/\partial t > 0$, technical change amounts to technical *progress*.[2]

To simplify, we shall here address TFP and TFP growth without taking the heterogeneity of the labor input into account. So we just count delivered work hours per time unit. Then (5.1) is reduced to the simpler case,

$$Y_t = F(K_t, L_t, t), \tag{5.2}$$

where L_t is the number of man-hours per year. As to measurement of K_t, some adaptation of the *perpetual inventory method*[3] is typically used, with some correction for under-estimated quality improvements of investment goods in national income accounting. Similarly, the output measure is (or at least should be) corrected for under-estimated quality improvements of consumption goods.

The notion of Total Factor Productivity at time t, TFP_t, is intended to indicate the *level* of productivity of the joint input (K_t, L_t). Generally, *productivity* of a given input is defined as the output per time unit divided by this input per time unit. So, considering (5.2), (average) labor productivity is simply Y_t/L_t. The concept of Total Factor Productivity is more complex, however, because it does not refer to a single input, but to a combination of several distinct inputs, in the present case two. And these distinct inputs may over time change their quantitative interrelationship, here the *ratio* K_t/L_t. It is then not obvious what can be meant by the "productivity" of the vector (K_t, L_t).

It is common in the literature to circumvent the problem of a direct definition of the PTF level and instead go straight away to a *decomposition* of output growth and on this basis define TFP *growth.* This is the approach we also follow here.

[2]Sometimes in growth accounting the left-hand side variable, Y, in (5.2) is the gross product rather than value added. Then non-durable intermediate inputs should be taken into account as a third production factor and enter as an additional argument of $\tilde{F}$ in (5.2). Since non-market production is difficult to measure, the government sector is often excluded from Y in (5.2). An alternative name in the literature for "total factor productivity" is *"multifactor productivity",* abbreviated MFP.

[3]Cf. Section 2.2 in Chapter 2.

5.2.1 TFP growth

Let the growth rate of a variable Z at time t be written g_{Zt} (could also be written with a comma, as $g_{Z,t}$, but to save notation, we skip the comma unless needed for clarity). Take the total derivative w.r.t. t in (5.2) to get

$$\dot{Y}_t = F_K(K_t, L_t, t)\dot{K}_t + F_L(K_t, L_t, t)\dot{L}_t + F_t(K_t, L_t, t) \cdot 1.$$

Dividing through by Y_t gives

$$\begin{aligned} g_{Yt} &\equiv \frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left[F_K(K_t, L_t, t)\dot{K}_t + F_L(K_t, L_t, t)\dot{L}_t + F_t(K_t, L_t, t) \cdot 1\right] \\ &= \frac{K_t F_K(K_t, L_t, t)}{Y_t} g_{Kt} + \frac{L_t F_L(K_t, L_t, t)}{Y_t} g_{Lt} + \frac{F_t(K_t, L_t, t)}{Y_t} \\ &\equiv \varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt} + \frac{F_t(K_t, L_t, t)}{Y_t}, \end{aligned} \tag{5.3}$$

where ε_{Kt} and ε_{Lt} are shorthands for $\varepsilon_K(K_t, L_t, t) \equiv \frac{K_t F_K(K_t, L_t, t)}{F(K_t, L_t, t)}$ and $\varepsilon_L(K_t, L_t, t) \equiv \frac{L_t F_L(K_t, L_t, t)}{F(K_t, L_t, t)}$, respectively, that is, the partial output elasticities w.r.t. the two production factors, evaluated at the factor combination (K_t, L_t) at time t. Finally, $F_t(K_t, L_t, t) \equiv \partial F/\partial t$, that is, the partial derivative w.r.t. the third argument of the function F, evaluated at the point (K_t, L_t, t).

The equation (5.3) is the basic *growth-accounting relation*, showing how the output growth rate can be decomposed into the "contribution" from growth in each of the inputs and a *residual,* $F_t(K_t, L_t, t)/Y_t$, which is not directly measurable. The equation was introduced already by Solow (1957), and the residual became known as the *Solow residual.* We have:

$$\text{Solow residual } \equiv g_{Yt} - (\varepsilon_{Kt} g_{Kt} + \varepsilon_{Lt} g_{Lt}) = \frac{F_t(K_t, L_t, t)}{Y_t}, \tag{5.4}$$

The Solow residual thus indicates what is left when from the output growth rate is subtracted the contribution from growth in the factor inputs weighted by the output elasticities w.r.t. these inputs. In brief:

> The Solow residual at time t reveals that part of time-t output growth which is *not attributable* to time-t growth in the factor inputs.

How can the Solow residual be calculated on the basis of empirical data? The output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}, will, under perfect competition and absence of externalities in equilibrium equal the income shares of capital and labor, respectively. Time series for these income shares and for Y, K, and L, hence also for g_{Yt}, g_{Kt}, and g_{Lt}, can be obtained (directly or with some adaptation) from national income accounts. This allows an indirect measurement of the residual in (5.4). Of course, data are in discrete time. So to make the calculations in practice, we have to translate (5.4) into discrete time. The weights ε_{Kt} and ε_{Lt} can then be quantified as two-years moving averages of the output elasticities w.r.t. capital and labor, respectively, and thus approximated by the respective factor income shares.[4]

It is not uncommon to *identify* the TFP growth rate with the Solow residual. This is unfortunate since, being a residual, the calculated Solow residual may reflect the contribution of many things. Some of these are what we want to measure, like effects of current technical innovation in a broad sense including organizational improvement. But, as Solow himself was quick to point out, the calculated Solow residual may also reflect the influence of other factors like absence of perfect competition, varying capacity utilization, labor hoarding during downturns, measurement errors, and aggregation bias.

Nevertheless, let us assume we have been able to control for these other factors by extraction of the business cycle elements in the data.[5] So we are ready to replace "Solow residual" in (5.4) with TFP growth rate and write

$$g_{\text{TFP}t} = g_{Yt} - (\varepsilon_{Kt}g_{Kt} + \varepsilon_{Lt}g_{Lt}) = \frac{F_t(K_t, L_t, t)}{Y_t}. \tag{5.5}$$

Interpretation:

> The TFP growth rate at time t reveals the contribution to time-t output growth from time-t technical change (in a broad sense including learning by doing and organizational improvement).

Let y_t denote output per unit of labor, i.e., $Y_t \equiv y_t L_t$, and let k_t denote capital per unit of labor, i.e., $K_t \equiv k_t L_t$. Then, $g_{Yt} = g_{yt} + g_{Lt}$ and $g_{Kt} = g_{kt} + g_{Lt}$. Under constant returns to scale (CRS), we have $\varepsilon_{Lt} = 1 - \varepsilon_{Kt}$. Hence, under CRS, (5.5) can be written

$$\begin{aligned} g_{\text{TFP}t} &= g_{yt} + g_{Lt} - (\varepsilon_{Kt}(g_{kt} + g_{Lt}) + (1 - \varepsilon_{Kt})g_{Lt}) \\ &= g_{yt} - \varepsilon_{Kt}g_{kt}. \end{aligned} \tag{5.6}$$

Under CRS, the TFP growth rate at time t thus reveals, under CRS, that part of time-t labor productivity growth which is *not attributable* to time-t growth in the capital-labor ratio. Interpretation:

[4] See, e.g., Acemoglu (2009, p. 79).

[5] Solow (1957) adjusted his capital data by assuming that idle capital as a fraction of total capital was the same as the rate of unemployment.

> Under CRS, the TFP growth rate at time t reveals the contribution to time-t labor productivity growth from time-t technical change (in a broad sense including learning by doing and organizational improvement).

So far we have only addressed the instantaneous Solow residual and the instantaneous TFP growth rate. To get measures of interest for growth analysis, one needs to consider these things over long time intervals, preferably more than a decade. We come back to this aspect at the end of the next sub-section.

5.2.2 The TFP level

Let us see what can be said about the *level* of TFP, that "something" for which we have calculated a growth rate without having defined what it actually is.[6]

Suppose we know the instantaneous growth rate, $g(t)$, of a variable, $x(t)$, over the time interval $[0, T]$, i.e.,

$$\frac{dx(t)/dt}{x(t)} = g(t) \quad \text{for} \quad t \in [0, T]\,. \tag{5.7}$$

This makes up a simple linear differential equation in x, usually written in the form $dx(t)/dt = g(t)x(t)$. For a given initial value, $x(0)$, the solution is

$$x(t) = x(0)e^{\int_0^t g(\tau)d\tau}. \tag{5.8}$$

This formula applies to TFP as well. Suppose we for all t in the interval $[0, T]$ have calculated the growth rate of TFP. Then, in (5.7) we can replace $x(t)$ by TFP_t and $g(t)$ by $g_{\text{TFP}t}$. Applying the solution formula (5.8), we get

$$\text{TFP}_t = \text{TFP}_0 e^{\int_0^t g_{\text{TFP}\tau}d\tau}. \tag{5.9}$$

For a given initial value $\text{TFP}_0 > 0$, the *level* of TFP at any time t within the given time interval $[0, T]$ is determined by the right-hand side of (5.9). Considering discrete time and interpreting $g_{\text{TFP}\tau}$ as one-period growth rates, we similarly have

$$\text{TFP}_t = \text{TFP}_0(1 + g_{\text{TFP}0})(1 + g_{\text{TFP}1}) \dots (1 + g_{\text{TFP}t-1}). \tag{5.10}$$

These two formulas at least give us an overall growth factor for TFP from time 0 to time t:

[6] It happens that authors make no clear *terminological* distinction between TFP *level* and TFP *growth*, denoting both just "TFP". That is bound to cause confusion.

> The TFP level at time t relative to that at time 0 reveals the cumulative "*direct* contribution" to output growth since time 0 from technical change since time 0.

Why do we say "*direct* contribution"? The reason is that the cumulative technical change since time 0 may also have an *indirect* effect on output growth, namely via affecting the output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}. Through this channel cumulative technical change affects the weights attached to the growth of inputs before the residual is obtained. This possible indirect effect over time of technical change is not included in the concept of TFP growth.

Anyway, suppose we are interested in the *average* annual TFP growth rate calculated on data for, say, T years. Then we may normalize TFP_0 to equal 1 and on the basis of (5.10) calculate TFP_t. Next, we look for a $\bar{g}_{\text{TFP}}$ satisfying the equation

$$\text{TFP}_t = 1 \cdot (1 + \bar{g}_{\text{TFP}})^T.$$

The solution for $\bar{g}_{\text{TFP}}$ is

$$\bar{g}_{\text{TFP}} = \text{antilog}\left(\frac{\log \text{TFP}_T}{T}\right) - 1.$$

This is the annual compound TFP growth rate from year 0 to year T, using discrete compounding. If we want the annual compound TFP growth rate from year 0 to year T, using continuous compounding, we consider (5.9) with $t = T$, and solve the equation

$$\text{TFP}_T = 1 \cdot e^{\hat{g}_{\text{TFP}} \cdot T},$$

which gives

$$\hat{g}_{\text{TFP}} = \frac{\log \text{TFP}_T}{T}.$$

Because continuous compounding is more powerful, for a given terminal value of TFP, we will get $\hat{g}_{\text{TFP}} < \bar{g}_{\text{TFP}}$ (whenever $\bar{g}_{\text{TFP}} \neq 0$), but the difference will be negligible (since $\log(1 + x) \lessapprox x$ for x "small", where "$\lessapprox$" means "close to", but "less than" unless $x = 0$).

Jones and Vollrath (2013, p. 47) present growth accounting results for the US 1948-2010, exposing, among other things, the "productivity slowdown" that occurred after 1973. Growth accounting results for Denmark and other countries, 1981-2006, are reported in De økonomiske Råd (2010).

Before proceeding, we note that some analysts take a quick approach to growth accounting and assume beforehand that the output elasticities

ε_{Kt} and ε_{Lt} are constant over time apart from small random disturbances. This could be because the economy is assumed to be in steady state or the aggregate production is assumed to be Cobb-Douglas, usually with the addition of CRS. In the latter case, $y_t = B_t k_t^\alpha$, $0 < \alpha < 1$, and (5.6) gives $g_{\text{TFP}t} = g_{yt} - \alpha g_{kt} = g_{Bt}$. Then, under balanced growth with $g_{yt} = g_{kt} = g$, we have

$$g_{\text{TFP}t} = (1 - \alpha) g \quad \text{for all } t. \tag{5.11}$$

Besides exposing a simple way of measuring TFP growth (under certain conditions), this formula may serve as a prelude to the following reminder about how *not* to interpret growth accounting.

5.2.3 Accounting versus causality

Sometimes people interpret growth accounting as telling how much of output growth is *explained* by technical change and how much is explained by the contribution from factor growth. Such an identification of a descriptive accounting relationship with deeper causality is misleading. Without a complete dynamic model it makes no sense to talk about "explanation" and "causality".

The result (5.11) illustrates this. On the one hand, one finds from growth accounting a TFP growth rate equal to $(1-\alpha)g$, while the remainder, αg, of labor productivity growth is attributed to growth in the capital-labor ratio. On the other hand, if for instance a Solow growth model is the theoretical framework within which the variables are assumed generated, then g will be the exogenous rate of labor-aumenting technical progress which determines *both* $g_{\text{TFP}t}$ and $g_{kt} = \alpha g$. Here the TFP growth rate understates the "contribution" of technical change to productivity growth by a factor $1-\alpha$. The *whole* of g_y is determined – explained – by the assumed rate, g, of exogenous technical progress. If g were nil, we would have $g_{kt} = 0$ as well as $g_{\text{TFP}t} = 0$.

Or suppose the theoretical framework within which the variables are assumed generated is the Arrow model of learning by investing.[7] Then it is the interaction between endogenous learning and endogenous investment that explains both g_{kt}, g, and $g_{\text{TFP}t}$. There is no one-way causal link involved. There is a mutual relationship between learning and investment, one presupposes the other. It is like "which comes first, the chicken or the egg?".

Let us now return to the intricate question what TFP actually measures in economic terms. We start with a convenient special case.

[7] Arrow (1962). The model is outlined in Chapter 12 below.

5.3 The case of Hicks-neutrality*

In the case of Hicks neutrality, by definition, technical change can be represented by the evolution of a one-dimensional variable, B_t, and the production function in (5.2) can be specified as

$$Y_t = F(K_t, L_t, t) = B_t \bar{F}(K_t, L_t). \tag{5.12}$$

Here the TFP level is at any time, t, identical to the level of B_t if we normalize the initial values of both B and TFP to be the same, i.e., $\text{TFP}_0 = B_0 > 0$. Indeed, calculating the TFP growth rate implied by (5.12) gives

$$g_{\text{TFP}t} = \frac{F_t(K_t, L_t, t)}{Y_t} = \frac{\dot{B}_t \bar{F}(K_t, L_t)}{B_t \bar{F}(K_t, L_t)} = \frac{\dot{B}_t}{B_t} \equiv g_{Bt}, \tag{5.13}$$

where the second equality comes from the fact that K_t and L_t are kept fixed when the *partial* derivative of F w.r.t. t is calculated. The formula (5.9) now gives

$$\text{TFP}_t = B_0 \cdot e^{\int_0^t g_{B\tau} d\tau} = B_t.$$

The convenient feature of Hicks neutrality is thus that we can write

$$\text{TFP}_t = \frac{F(K_t, L_t, t)}{F(K_t, L_t, 0)} = \frac{B_t \bar{F}(K_t, L_t)}{B_0 \bar{F}(K_t, L_t)} = B_t, \tag{5.14}$$

using the normalization $B_0 = 1$. That is:

> *Under Hicks neutrality, TFP$_t$ appears as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.*

So in the case of Hicks neutrality the economic meaning of the TFP level is straightforward. The reason is that under Hicks neutrality the output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}, are *independent* of technical change. Moreover, the relationship also holds the opposite way: if the output elasticities w.r.t. capital and labor, ε_{Kt} and ε_{Lt}, are independent of technical change, then technical change is Hicks neutral.

We now turn to difficulties regarding interpretation of TFP that arise in the general case.

5.4 Absence of Hicks-neutrality*

The above straightforward economic interpretation of TFP only holds under Hicks-neutral technical change. Neither under general technical change nor even under Harrod- or Solow-neutral technical change, will the current TFP level appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0. This is so unless the production function is Cobb-Douglas in which case both Harrod and Solow neutrality *imply* Hicks-neutrality.

To see this, let us return to the general time-dependent production function in (5.2). Let X_t denote the ratio between the current output level at time t and the hypothetical output level, $F(K_t, L_t, 0)$, that would have obtained with the current inputs of capital and labor in case of no change in the technology since time 0, i.e.,

$$X_t \equiv \frac{F(K_t, L_t, t)}{F(K_t, L_t, 0)}. \tag{5.15}$$

So X_t can be seen as a factor of "joint-productivity" growth from time 0 to time t evaluated at the time-t input combination.

If this X_t should always indicate the level of TFP at time t, the growth rate of X_t should equal the growth rate of TFP. Generally, it does not, however. Indeed, defining $G(K_t, L_t) \equiv F(K_t, L_t, 0)$, by the rule for the time derivative of fractions,[8] we have

$$\begin{aligned} g_{X,t} &\equiv \frac{dF(K_t, L_t, t)/dt}{F(K_t, L_t, t)} - \frac{dG(K_t, L_t)/dt}{G(K_t, L_t)} \\ &= \frac{1}{Y_t}\left[F_K(K_t, L_t, t)\dot{K}_t + F_L(K_t, L_t, t)\dot{L}_t + F_t(K_t, L_t, t)\cdot 1\right] \\ &\quad -\frac{1}{G(K_t, L_t)}\left[G_K(K_t, L_t)\dot{K}_t + G_L(K_t, L_t)\dot{L}_t\right] \\ &= \varepsilon_K(K_t, L_t, t)g_{Kt} + \varepsilon_L(K_t, L_t, t)g_{Lt} + \frac{F_t(K_t, L_t, t)}{Y_t} \\ &\quad -(\varepsilon_K(K_t, L_t, 0)g_{Kt} + \varepsilon_L(K_t, L_t, 0)g_{Lt}) \qquad (5.16) \\ &= (\varepsilon_K(K_t, L_t, t) - \varepsilon_K(K_t, L_t, 0))\, g_{Kt} \\ &\quad +(\varepsilon_L(K_t, L_t, t) - \varepsilon_L(K_t, L_t, 0))g_{Lt} + g_{\text{TFP}t} \\ &\neq g_{\text{TFP}t} \qquad \text{generally,} \end{aligned}$$

where $g_{\text{TFP}t}$ is given in (5.5). We see that:

[8] See Appendix A to Chapter 3.

> The time-t growth rate of the joint-productivity index X equals the time-t TFP growth rate plus the cumulative impact of technical change since time 0 on the direct contribution to time-t output growth from time-t input growth.

Unless the partial output elasticities w.r.t. capital and labor, respectively, are unaffected by technical change, the conclusion is that TFP_t tend to differ from our X_t defined in (5.15). So:

> *In the absence of Hicks neutrality, current TFP does not generally appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.*

Consider the difference between $g_{X,t}$ and $g_{\text{TFP}t}$:

$$g_{X,t} - g_{\text{TFP}t} = (\varepsilon_K(K_t, L_t, t) - \varepsilon_K(K_t, L_t, 0))\, g_{Kt} + (\varepsilon_L(K_t, L_t, t) - \varepsilon_L(K_t, L_t, 0)) g_{Lt}.$$

Under CRS, the coefficients to the growth rates in K and L will be of the same absolute value but have opposite sign. This is an implication of $\varepsilon_K(K_t, L_t, \cdot) + \varepsilon_L(K_t, L_t, \cdot) = 1$ under CRS. Since usually g_{Kt} exceeds g_{Lt} considerably, the difference between $g_{X,t}$ and $g_{\text{TFP}t}$ may be substantial.

Balanced growth at the aggregate level, hence Harrod neutrality, seems to characterize the growth experience of the UK and US over at least a century (Kongsamut et al., 2001; Attfield and Temple, 2010). At the same time the aggregate elasticity of factor substitution is generally estimated to be significantly less than one (cf. Chapter 2.7). This amounts to rejection of the Cobb-Douglas specification of the aggregate production function. So, at the aggregate level, Harrod neutrality rules out Hicks neutrality.

Since at least at the aggregate level Hicks-neutrality is empirically doubtful, the level of TFP can usually *not* be identified with the intuitive joint-productivity measure X_t, defined in (5.15) above. Then, to my knowledge there is no simple economic interpretation of what the TFP level actually measures.

A closer look at X_t vs. TFP_t

The fact that in the absence of Hicks-neutrality, TFP and the index X differ is the reason that we in Section 2.2 characterized the time-t TFP level relative to the time-0 level as the cumulative "*direct* contribution" on output growth since time 0 from cumulative technical change, thus excluding the possible indirect contribution coming about via the potential effect of technical change on the output elasticities w.r.t. capital and labor and thereby on the contribution to output from input growth.

Given that the joint-productivity index X is the more intuitive joint-productivity measure, why is TFP the more popular measure? There are at least two reasons for this. First, it can be shown that the TFP measure has more convenient balanced growth properties. Second, X is more difficult to measure. To see the reason for this, we substitute (5.3) into (5.16) to get

$$g_{Xt} = g_{Yt} - (\varepsilon_K(K_t, L_t, 0)g_{Kt} + \varepsilon_L(K_t, L_t, 0)g_{Lt}). \tag{5.17}$$

The relevant output elasticities, $\varepsilon_K(K_t, L_t, 0) \equiv \frac{K_t F_K(K_t,L_t,0)}{F(K_t,L_t,0)}$ and $\varepsilon_L(K_t, L_t, 0) \equiv \frac{L_t F_L(K_t,L_t,0)}{F(K_t,L_t,0)}$, are hypothetical constructs, referring to the technology as it was at time 0, but with the factor combination observed at time t, not at time 0. The nice thing about the Solow residual is that under the assumptions of perfect competition and absence of externalities, it allows measurement by using data on prices and quantities alone, that is, without knowledge of the production function. To evaluate g_X, however, we need estimates of the hypothetical output elasticities, $\varepsilon_K(K_t, L_t, 0)$ and $\varepsilon_L(K_t, L_t, 0)$. This requires knowledge about how the output elasticities depend on the factor combination and time, respectively, that is, knowledge about the production function.

5.5 A warning regarding cross-country TFP growth comparisons

When Harrod neutrality applies, relative TFP growth rates across sectors or countries can be quite deceptive. Consider a group of n countries that share some structural characteristics. Country i has the aggregate production function

$$Y_{it} = F^{(i)}(K_{it}, A_t L_{it}) \qquad i = 1, 2, ..., n,$$

where $F^{(i)}$ is a neoclassical production function with CRS, and A_t is the level of labor-augmenting technology which we assume shared by all the countries (these are open and "close" to each other). Technical progress is thus Harrod-neutral. Let the growth rate of A be a constant $g > 0$.

Define $\tilde{k}_{it} \equiv K_{it}/(A_t L_{it}) \equiv k_{it}/A_t$ and $\tilde{y}_{it} \equiv Y_{it}/(A_t L_{it}) \equiv y_{it}/A_t$. Suppose the countries feature (within-country) convergence, i.e.,

$$\tilde{k}_{it} \to \tilde{k}_i^* \quad \text{and} \quad \tilde{y}_{it} \to \tilde{y}_i^* = f^{(i)}(\tilde{k}_i^*) \quad \text{for} \quad t \to \infty,$$

where $f^{(i)}$ is the production function in intensive form. Since $k_{it} \equiv \tilde{k}_{it}A_t$ and $y_{it} \equiv \tilde{y}_{it}A_t$, we thus have

$$g_{k_i} \rightarrow g_A \ (= g) \quad \text{and} \quad g_{y_i} \rightarrow g_A \quad \text{for} \quad t \rightarrow \infty.$$

So in the long run g_{k_i} and g_{y_i} tend to the constant g.

Formula (5.6) then gives the TFP growth rate of country i in the long run as

$$g_{\text{TFP}_i} = g_{y_i} - \alpha_i^* g_{k_i} = (1 - \alpha_i^*)g, \tag{5.18}$$

where α_i^* is the output elasticity w.r.t. capital, $f^{(i)\prime}(\tilde{k}_i)\tilde{k}_i/f^{(i)}(\tilde{k}_i)$, evaluated at $\tilde{k}_i = \tilde{k}_i^*$. Under labor-augmenting technical progress, the TFP growth rate thus varies negatively with the output elasticity w.r.t. capital (the capital income share under perfect competition). Owing to differences in product and industry composition, the countries have different α_i^*'s. In view of (5.18), for two different countries, i and j, we therefore get

$$\frac{TFP_i}{TFP_j} \rightarrow \begin{cases} \infty \text{ if } \alpha_i^* < \alpha_j^*, \\ 1 \text{ if } \alpha_i^* = \alpha_j^*, \\ 0 \text{ if } \alpha_i^* > \alpha_j^*, \end{cases} \tag{5.19}$$

for $t \rightarrow \infty$.[9]

In spite of long-run growth in the essential variable, y, being the same across the countries, their TFP growth rates are very different. Countries with low α^* appear to be technologically very dynamic and countries with high α^* appear to be lagging behind. The explanation is simply that a higher α^* means that a larger fraction of $g_y = g_k = g$ becomes driven by ("explained by") g_k in the growth accounting (5.18), leaving a smaller residual. But it is the exogenous technology growth rate g that drives both g_k and g_{TFP}. The level of α^* is just the long-run output elasticity w.r.t. capital and reflects neither technological dynamism nor its opposite. Notwithstanding the countries' different α^*, their long-run growth rates of per capita consumption will be the same, namely g. Moreover, if the economies can be described, for instance, by a Solow model with the same s, δ, and n (standard notation) across the countries, and the ratio $s/(\delta + g + n)$ happens to equal 1, then even the *level* of per capita consumption in the countries will in the long run be the same growth rate. Nevertheless there will be persistent differences in their TFP growth rates, and (5.19) remains true.

We conclude that comparison of TFP growth rates across countries may misrepresent the intuitive meaning of productivity and technical progress

[9] If F is Cobb-Douglas with output elasticity w.r.t. capital equal to α_i, the key result, (5.18), can be derived more directly by first defining $B_t = A_t^{1-\alpha_i}$, then writing the production function in the Hicks-neutral form (5.12), and finally use (5.13).

when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral (even if technical progress were at the same time Hicks-neutral as is the case with a Cobb-Douglas specification).

On this background let us briefly consider a different decomposition than the one made in standard growth accounting. Under CRS, equation (5.10) holds. Subtracting $\varepsilon_{Kt}g_{yt}$ on both sides, dividing through by $1 - \varepsilon_{Kt}$, and rearranging gives

$$g_{yt} = \frac{\varepsilon_{Kt}}{1-\varepsilon_{Kt}} g_{\frac{k}{y}t} + \frac{1}{1-\varepsilon_{Kt}} g_{\text{TFP}t}. \tag{5.20}$$

This says that increases in the capital-output ratio as well as TFP contribute to growth in labor productivity via the "multipliers" $\varepsilon_{Kt}/(1-\varepsilon_{Kt})$ and $1/(1-\varepsilon_{Kt})$, respectively. This may speak for focusing on $g_{\text{TFP}t}/(1-\varepsilon_{Kt})$ rather than $g_{\text{TFP}t}$ it self. A growth path along which the capital-output ratio is constant (as it tends to be in the long run according to Kaldor's 'stylized facts") will feature labor productivity growth equal to the TFP growth rate multiplied by the inverse of the output elasticity w.r.t. labor (since, under CRS, $1 - \varepsilon_{Kt} = \varepsilon_{Lt}$).[10]

Thus, in the comparison of the n countries above, where in the long run the capital-output ratios are indeed constant ($\tilde{k}_i^*/f(\tilde{k}_i^*)$ is constant), it makes sense to focus on

$$\frac{g_{\text{TFP}_i}}{1-\alpha_{it}^*} = g_{yt} - \frac{\varepsilon_{Kt}}{1-\varepsilon_{Kt}} g_{\frac{k}{y}t} = \frac{1}{1-\varepsilon_{Kt}} g_{yt} - \frac{\varepsilon_{Kt}}{1-\varepsilon_{Kt}} g_{kt}. \tag{5.21}$$

This measure of the contribution of technical change ends up in the long run equal to the rate of Harrod-neutral technical progress, g, cf. (5.18).[11]

Since this "corrected TFP growth rate" in many models, including the one considered above, ultimately becomes identical to the labor productivity growth rate in the long run, it is not unreasonable in simple international comparisons to just compare levels and growth rates of Y/L across countries.

Remark on levels accounting

In growth accounting we consider productivity of a single country at different points in time. Another discipline is named *levels accounting*, where one compares productivity across different countries at a single point in time. See Caselli (2005), Acemoglu (2009, Chapter 3.5), and Jones and Vollrath (2013, Chapter 3).

[10] The labor productivity growth rate along a path along which the capital-output ratio is constant has occationally been called the *Harrod-productivity growth rate*.

[11] To focus on this, or a similar, measure was suggested by ....

5.6 Summing up

Growth accounting is – as the name indicates – a descriptive way of presenting growth data. So we should not confuse growth *accounting* with *causality* in growth analysis. To talk about causality we need a theoretical model supported by the data. On the basis of such a model we can say that this or that set of exogenous factors through the propagation mechanisms of the model cause this or that phenomenon, including economic growth.

In a complete model with exogenous technical progress, g_{kt} will be *induced* by this technical progress. If technical progress is endogenous through learning by investing, as in Arrow (1962), there is mutual causation between g_{kt} and technical progress. Yet other kinds of models explain both technical progress and capital accumulation through R&D, cf. Barro (1999) and Fernald and Jones (2014).

When technical change is not Hicks-neutral, the *level* of TFP can at best be *approximated* by the intuitive joint-productivity measure X_t, defined in (5.15) above. The approximation may not be good. And in absence of Hicks-neutrality, there seems to exist no simple economic interpretation of what the TFP *level* actually measures.

We also observed that relative TFP growth rates across sectors or countries can be quite deceptive when output elasticities w.r.t. capital differ. It may be more reasonable to compare the "corrected TFP growth rate" defined in (5.21) above or just compare levels and growth rates of Y/L across countries.

5.7 References

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Chapter 6

Transitional dynamics. Barro-style growth regressions

In this chapter we discuss three issues, all of which are related to the transitional dynamics of a growth model:

- Do poor countries necessarily tend to approach their steady state from below?
- How fast (or rather how slow) are the transitional dynamics in a growth model?
- What exactly is the theoretical foundation for a Barro-style growth regression analysis?

The Solow growth model may serve as the analytical point of departure for the first two issues and to some extent also for the third.

6.1 Point of departure: the Solow model

As is well-known, the fundamental differential equation for the Solow model is

$$\dot{\tilde{k}}(t) = sf(\tilde{k}(t)) - (\delta + g + n)\tilde{k}(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \tag{6.1}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $f(\tilde{k}(t)) \equiv F(\tilde{k}(t), 1)$, $A(t) = A_0 e^{gt}$, and $L(t) = L_0 e^{nt}$ (standard notation). The production function F is neoclassical with CRS and the parameters satisfy $0 < s < 1$ and $\delta + g + n > 0$. The production function on intensive form, f, therefore satisfies $f(0) \geq 0$, $f' > 0$, $f'' < 0$, and

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) > \frac{\delta + g + n}{s} > \lim_{\tilde{k} \to \infty} f'(\tilde{k}). \tag{A1}$$

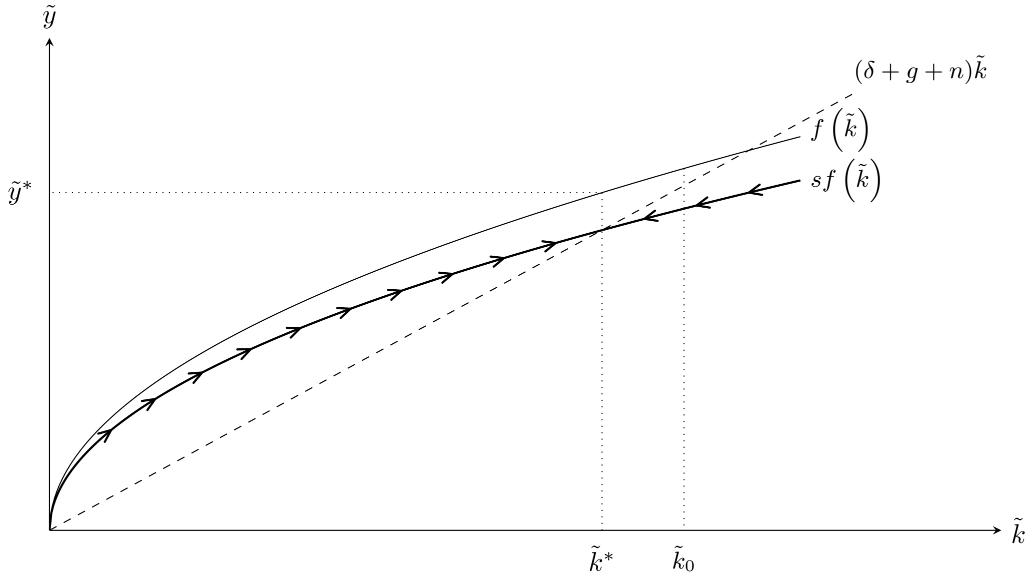

Figure 6.1: Phase diagram 1 (capital essential).

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, that is, a unique positive solution to the equation

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{6.2}$$

Furthermore, given an arbitrary $\tilde{k}_0 > 0$, we have for all $t \geq 0$,

$$\dot{\tilde{k}}(t) \gtreqless 0 \text{ for } \tilde{k}(t) \lesseqgtr \tilde{k}^*, \tag{6.3}$$

respectively. The steady state, $\tilde{k}^*$, is thus *globally asymptotically stable* in the sense that for all $\tilde{k}_0 > 0$, $\lim_{t\to\infty} \tilde{k}(t) = k^*$, and this convergence is *monotonic* (in the sense that $\tilde{k}(t) - \tilde{k}^*$ does not change sign during the adjustment process).

Figure 6.1 illustrates the dynamics as seen from the perspective of (6.1): $\tilde{k}$ is rising (falling) when saving per unit of effective labor, AL, is greater (less) than the amount needed to maintain the effective capital-labor ratio constant in spite of capital depreciation, more labor and better technology.

Figure 6.2 illustrates the dynamics emerging when we rewrite (6.1) this way:

$$\dot{\tilde{k}}(t) = s\left(f(\tilde{k}(t)) - \frac{\delta + g + n}{s}\tilde{k}(t)\right) \gtreqless 0 \text{ for } \tilde{k}(t) \lesseqgtr \tilde{k}^*.$$

In Figure 6.3 yet another illustration is exhibited, based on rewriting (6.1)

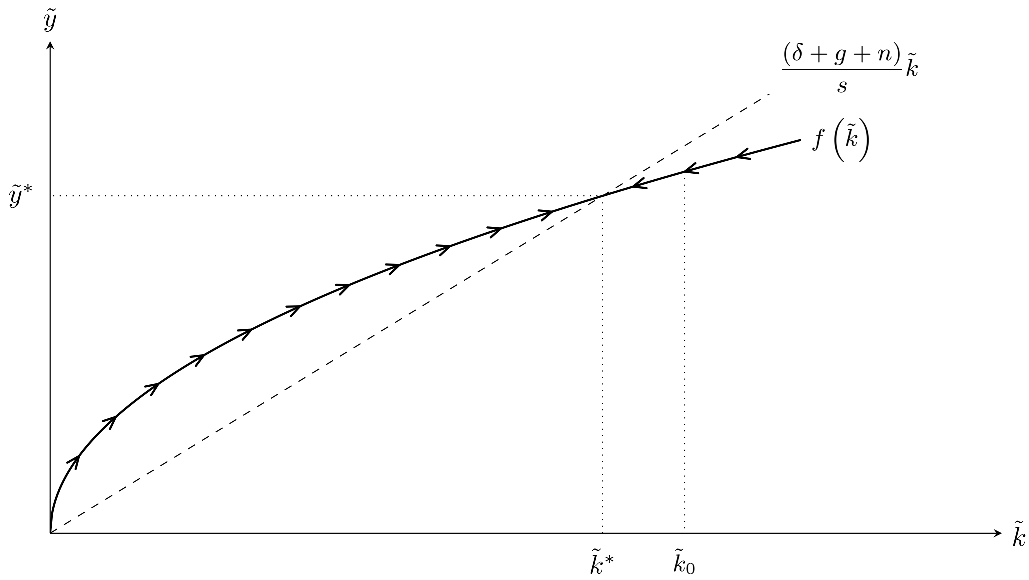

Figure 6.2: Phase diagram 2.

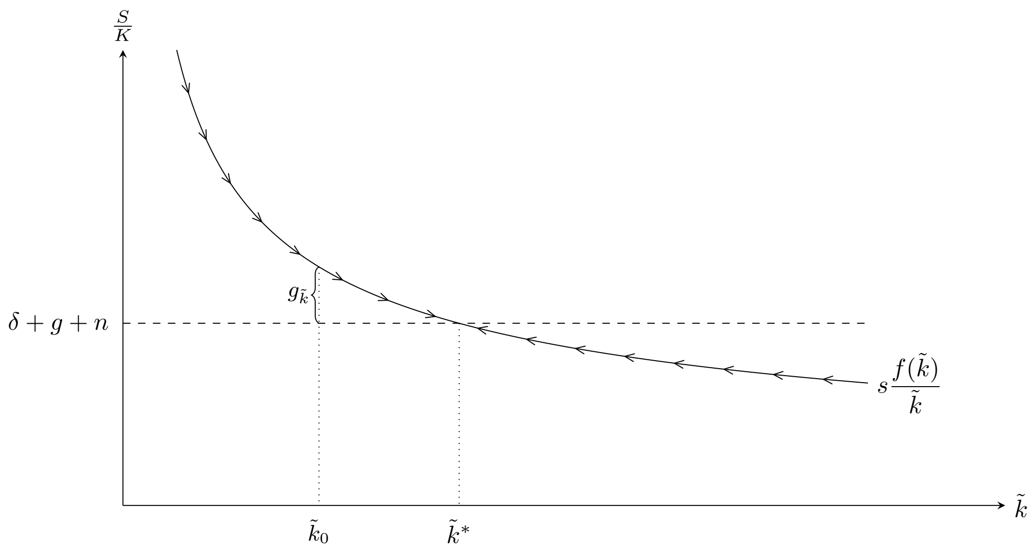

Figure 6.3: Phase diagram 3.

this way:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\frac{f(\tilde{k}(t))}{\tilde{k}(t)} - (\delta + g + n),$$

where $sf(\tilde{k}(t))/\tilde{k}(t)$ is gross saving per unit of capital, $S(t)/K(t) \equiv (Y(t) - C(t))/K(t)$.

From now on the dating of the variables is suppressed unless needed for clarity.

6.2 Do poor countries tend to approach their steady state from below?

From some textbooks (for instance Barro and Sala-i-Martin, 2004) one gets the impression that poor countries tend to approach their steady state *from below*. But this is *not* what the Penn World Table data seems to indicate. And from a theoretical point of view the size of $\tilde{k}_0$ relative to $\tilde{k}^*$ is certainly ambiguous, whether the country is rich or poor. To see this, consider a poor country with initial effective capital intensity

$$\tilde{k}_0 \equiv \frac{K_0}{A_0 L_0}.$$

Here K_0/L_0 will typically be small for a poor country (the country has not yet accumulated much capital relative to its fast-growing population). The technology level, A_0, however, *also* tends to be small for a poor country. Hence, whether we should expect $\tilde{k}_0 < \tilde{k}^*$ or $\tilde{k}_0 > \tilde{k}^*$ is not obvious *apriori*. Or equivalently: whether we should expect that a poor country's GDP at an arbitrary point in time grows at a rate higher or lower than the country's steady-state growth rate, $g + n$, is not obvious *apriori*.

While Figure 6.3 illustrates the case where the inequality $\tilde{k}_0 < \tilde{k}^*$ holds, Figure 6.1 and 6.2 illustrate the opposite case. There *exists* some empirical evidence indicating that poor countries tend to approach their steady state *from above*. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

The prejudice that poor countries *apriori* should tend to approach their steady state from below seems to come from a confusion of conditional and unconditional β convergence. The Solow model predicts - and data supports - that within a group of countries with similar structural characteristics (approximately the same f, A_0, g, s, n, and δ), the initially poorer countries will grow faster than the richer countries. This is because the poorer countries (small $y(0) = f(\tilde{k}_0)A_0$) will be the countries with relatively small initial capital-labor ratio, k_0. As all the countries in the group have approximately the same A_0, the poorer countries thus have $\tilde{k}_0 \equiv k_0/A_0$ relatively small, i.e., $\tilde{k}_0 < \tilde{k}^*$. From $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k})A$ follows that the growth rate in output per worker of these poor countries tends to exceed g. Indeed, we have generally (for instance in the Solow model as well as the Ramsey model)

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + g = \frac{f'(\tilde{k})\dot{\tilde{k}}}{f(\tilde{k})} + g \gtreqless g \text{ for } \dot{\tilde{k}} \gtreqless 0, \text{ i.e., for } \tilde{k} \lesseqgtr \tilde{k}^*.$$

So, *within* the group, the poor countries tend to approach the steady state, $\tilde{k}^*$, *from below.*

The countries in the world as a whole, however, differ a lot w.r.t. their structural characteristics, including their A_0. Unconditional β convergence is definitely rejected by the data. Then there is no reason to expect the poorer countries to have $\tilde{k}_0 < \tilde{k}^*$ rather than $\tilde{k}_0 > \tilde{k}^*$. Indeed, according to the mentioned study by Cho and Graham (1996), it turns out that the data for the relatively poor countries favors the latter inequality rather than the first.

6.3 Within-country convergence speed and adjustment time

Our next issue is: How fast (or rather how slow) are the transitional dynamics in a growth model? To put it another way: according to a given growth model with convergence, how fast does the economy approach its steady state? The answer turns out to be: not very fast - to say the least. This is a rather general conclusion and is confirmed by the empirics: adjustment processes in a growth context are quite time consuming.

In Acemoglu (2009) we meet the concept of speed of convergence at p. 54 (under an alternative name, rate of adjustment) and p. 81 (in connection with Barro-style growth regressions). Here we shall go more into detail with the issue of speed of convergence.

Again the Solow model is our frame of reference. We search for a formula for the *speed of convergence* of $\tilde{k}(t)$ and $y(t)/y^*(t)$ in a closed economy described by the Solow model. So our analysis is concerned with *within-country convergence*: how fast do variables such as $\tilde{k}$ and y approach their steady state paths in a closed economy? The key adjustment mechanism is linked to diminishing returns to capital (falling marginal productivity of capital) in the process of capital accumulation. The problem of *cross-country convergence* (which is what "β convergence" and "σ convergence" are about) is in principle more complex because also such mechanisms as technological catching-up and cross-country factor movements are involved.

6.3.1 Convergence speed for $\tilde{k}(t)$

The ratio of $\dot{\tilde{k}}(t)$ to $(\tilde{k}(t) - \tilde{k}^*) \neq 0$ can be written

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t) - \tilde{k}^*} = \frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*}, \tag{6.4}$$

since $d\tilde{k}^*/dt = 0$. We define the *instantaneous speed of convergence* at time t as the (proportionate) rate of *decline* of the distance $\left|\tilde{k}(t) - \tilde{k}^*\right|$ at time t and we denote it $\mathrm{SOC}_t(\tilde{k})$.[1] Thus,

$$\mathrm{SOC}_t(\tilde{k}) \equiv -\frac{d\left(\left|\tilde{k}(t) - \tilde{k}^*\right|\right)/dt}{\left|\tilde{k}(t) - \tilde{k}^*\right|} = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \tag{6.5}$$

per time unit, where the equality sign is valid for monotonic convergence.

Generally, $\mathrm{SOC}_t(\tilde{k})$ depends on both the absolute size of the difference $\tilde{k} - \tilde{k}^*$ at time t and its sign. But if the difference is already "small", $\mathrm{SOC}_t(\tilde{k})$ will be "almost" constant for increasing t and we can find an approximate measure for it. Let the function $\varphi(\tilde{k})$ be defined by $\dot{\tilde{k}} = sf(\tilde{k}) - m\tilde{k} \equiv \varphi(\tilde{k})$, where $m \equiv \delta + g + n$. A first-order Taylor approximation of $\varphi(\tilde{k})$ around $\tilde{k} = \tilde{k}^*$ gives

$$\varphi(\tilde{k}) \approx \varphi(\tilde{k}^*) + \varphi'(\tilde{k}^*)(\tilde{k} - \tilde{k}^*) = 0 + (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*).$$

For $\tilde{k}$ in a small neighborhood of the steady state, $\tilde{k}^*$, we thus have

$$\begin{aligned}
\dot{\tilde{k}} &= \varphi(\tilde{k}) \approx (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*) \\
&= (\frac{sf'(\tilde{k}^*)}{m} - 1)m(\tilde{k} - \tilde{k}^*) \\
&= (\frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (6.2))} \\
&\equiv (\varepsilon(\tilde{k}^*) - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (6.6))},
\end{aligned}$$

[1] Synonyms for speed of convergence are *rate of convergence, rate of adjustment* or *adjustment speed.*

where $\varepsilon(\tilde{k}^*)$ is the output elasticity w.r.t. capital, evaluated in the steady state. So

$$\frac{K}{Y}\frac{\partial Y}{\partial K} = \frac{\tilde{k}}{f(\tilde{k})}f'(\tilde{k}) \equiv \varepsilon(\tilde{k}), \tag{6.6}$$

where $0 < \varepsilon(\tilde{k}) < 1$ for all $\tilde{k} > 0$.

Applying the definition (6.5) and the identity $m \equiv \delta + g + n$, we now get

$$\mathrm{SOC}_t(\tilde{k}) = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} = \frac{-\dot{\tilde{k}}(t)}{\tilde{k}(t) - \tilde{k}^*} \approx (1-\varepsilon(\tilde{k}^*))(\delta+g+n) \equiv \beta(\tilde{k}^*) > 0. \tag{6.7}$$

This result tells us how fast, approximately, the economy approaches its steady state if it starts "close" to it. If, for example, $\beta(\tilde{k}^*) = 0.02$ per year, then 2 percent of the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ vanishes per year. We also see that everything else equal, a higher output elasticity w.r.t. capital implies a lower speed of convergence.

In the limit, for $\left|\tilde{k} - \tilde{k}^*\right| \to 0$, the instantaneous speed of convergence coincides with what is called the *asymptotic speed of convergence*, defined as

$$\mathrm{SOC}^*(\tilde{k}) \equiv \lim_{\left|\tilde{k}-\tilde{k}^*\right| \to 0} \mathrm{SOC}_t(\tilde{k}) = \beta(\tilde{k}^*). \tag{6.8}$$

Multiplying through by $-(\tilde{k}(t) - \tilde{k}^*)$, the equation (6.7) takes the form of a homogeneous linear differential equation (with constant coefficient), $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. With $x(t) = \tilde{k}(t) - \tilde{k}^*$ and "=" replaced by "≈", we get in the present case

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t} \to 0 \text{ for } t \to \infty. \tag{6.9}$$

This is the approximative time path for the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ and shows how the gap becomes smaller and smaller at the rate $\beta(\tilde{k}^*)$.

One of the reasons that the speed of convergence is important is that it indicates what weight should be placed on transitional dynamics of a growth model relative to the steady-state behavior. The speed of convergence matters for instance for the evaluation of growth-promoting policies. In growth models with diminishing marginal productivity of production factors, successful growth-promoting policies have transitory growth effects and permanent level effects. Slower convergence implies that the full benefits are slower to arrive.

6.3.2 Convergence speed for $\log \tilde{k}(t)$*

We have found an approximate expression for the convergence speed of $\tilde{k}$. Since models in empirical analysis and applied theory are often based on log-linearization, we might ask what the speed of convergence of $\log \tilde{k}$ is. The answer is: approximately the same and asymptotically exactly the same as that of $\tilde{k}$ itself! Let us see why.

A first-order Taylor approximation of $\log \tilde{k}(t)$ around $\tilde{k} = \tilde{k}^*$ gives

$$\log \tilde{k}(t) \approx \log \tilde{k}^* + \frac{1}{\tilde{k}^*}(\tilde{k}(t) - \tilde{k}^*). \tag{6.10}$$

By definition

$$\begin{aligned} \mathrm{SOC}_t(\log \tilde{k}) &= -\frac{d(\log \tilde{k}(t) - \log \tilde{k}^*)/dt}{\log \tilde{k}(t) - \log \tilde{k}^*} = -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)(\log \tilde{k}(t) - \log \tilde{k}^*)} \\ &\approx -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)\frac{\tilde{k}(t)-\tilde{k}^*}{\tilde{k}^*}} = \frac{\tilde{k}^*}{\tilde{k}(t)}\mathrm{SOC}_t(\tilde{k}) \rightarrow \mathrm{SOC}^*(\tilde{k}) = \beta(\tilde{k}^*) \qquad (6.11) \\ \text{for } \tilde{k}(t) &\rightarrow \tilde{k}^*, \end{aligned}$$

where in the second line we have used, first, the approximation (6.10), second, the definition in (6.7), and third, the definition in (6.8).

So, at least in a neighborhood of the steady state, the instantaneous rate of decline of the logarithmic distance of $\tilde{k}$ to the steady-state value of $\tilde{k}$ approximates the instantaneous rate of decline of the distance of $\tilde{k}$ itself to its steady-state value. The asymptotic speed of convergence of $\log \tilde{k}$ coincides with that of $\tilde{k}$ itself and is exactly $\beta(\tilde{k}^*)$.

In the Cobb-Douglas case (where $\varepsilon(\tilde{k}^*)$ is a constant, say α) it is possible to find an explicit solution to the Solow model, see Acemoglu (2009, p. 53) and Exercise II.2. It turns out that the instantaneous speed of convergence in a finite distance from the steady state is a constant and equals the asymptotic speed of convergence, $(1 - \alpha)(\delta + g + n)$.

6.3.3 Convergence speed for $y(t)/y^*(t)$*

The variable which we are interested in is usually not so much $\tilde{k}$ in itself, but rather labor productivity, $y(t) \equiv \tilde{y}(t)A(t)$. In the interesting case where $g > 0$, labor productivity does not converge towards a constant. We therefore focus on the ratio $y(t)/y^*(t)$, where $y^*(t)$ denotes the hypothetical value of labor productivity at time t, conditional on the economy being on its steady-state path, i.e.,

$$y^*(t) \equiv \tilde{y}^* A(t). \tag{6.12}$$

We have

$$\frac{y(t)}{y^*(t)} \equiv \frac{\tilde{y}(t)A(t)}{\tilde{y}^*A(t)} = \frac{\tilde{y}(t)}{\tilde{y}^*}. \tag{6.13}$$

As $\tilde{y}(t) \to \tilde{y}^*$ for $t \to \infty$, the ratio $y(t)/y^*(t)$ converges towards 1 for $t \to \infty$.

Taking logs on both sides of (6.13), we get

$$\begin{aligned}
\log \frac{y(t)}{y^*(t)} &= \log \frac{\tilde{y}(t)}{\tilde{y}^*} = \log \tilde{y}(t) - \log \tilde{y}^* \\
&\approx \log \tilde{y}^* + \frac{1}{\tilde{y}^*}(\tilde{y}(t) - y^*) - \log \tilde{y}^* \text{ (first-order Taylor approx. of } \log \tilde{y}\text{)} \\
&= \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}(t)) - f(\tilde{k}^*)) \\
&\approx \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}^*) + f'(\tilde{k}^*)(\tilde{k}(t) - \tilde{k}^*) - f(\tilde{k}^*)) \text{ (first-order approx. of } f(\tilde{k})) \\
&= \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \equiv \varepsilon(\tilde{k}^*) \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \\
&\approx \varepsilon(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \qquad \text{(by (6.10))}.
\end{aligned} \tag{6.14}$$

Multiplying through by $-(\log \tilde{k}(t) - \log \tilde{k}^*)$ in (6.11) and carrying out the differentiation w.r.t. time, we find an approximate expression for the growth rate of $\tilde{k}$,

$$\begin{aligned}
\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)} &\equiv g_{\tilde{k}}(t) \approx -\frac{\tilde{k}^*}{\tilde{k}(t)} \mathrm{SOC}_t(\tilde{k})(\log \tilde{k}(t) - \log \tilde{k}^*) \\
&\to -\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \quad \text{for } \tilde{k}(t) \to \tilde{k}^*,
\end{aligned} \tag{6.15}$$

where the convergence follows from the last part of (6.11). We now calculate the time derivative on both sides of (6.14) to get

$$\begin{aligned}
d(\log \frac{y(t)}{y^*(t)})/dt &= d(\log \frac{\tilde{y}(t)}{\tilde{y}^*})/dt = \frac{d\tilde{y}(t)/dt}{\tilde{y}(t)} \equiv g_{\tilde{y}}(t) \\
&\approx \varepsilon(\tilde{k}^*) g_{\tilde{k}}(t) \approx -\varepsilon(\tilde{k}^*)\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*).
\end{aligned} \tag{6.16}$$

from (6.15). Dividing through by $-\log(y(t)/y^*(t))$ in this expression, taking (6.14) into account, gives

$$-\frac{d(\log \frac{y(t)}{y^*(t)})/dt}{\log \frac{y(t)}{y^*(t)}} = -\frac{d(\log \frac{y(t)}{y^*(t)} - \log 1)/dt}{\log \frac{y(t)}{y^*(t)} - \log 1} \equiv \mathrm{SOC}_t(\log \frac{y}{y^*}) \approx \beta(\tilde{k}^*), \tag{6.17}$$

in view of $\log 1 = 0$. So the logarithmic distance of y from its value on the steady-state path at time t has approximately the same rate of decline as the logarithmic distance of $\tilde{k}$ from $\tilde{k}$'s value on the steady-state path at time t. The asymptotic speed of convergence for $\log y(t)/y^*(t)$ is exactly the same as that for $\tilde{k}$, namely $\beta(\tilde{k}^*)$.

What about the speed of convergence of $y(t)/y^*(t)$ itself? Here the same principle as in (6.11) applies. The asymptotic speed of convergence for $\log(y(t)/y^*(t))$ is the same as that for $y(t)/y^*(t)$ (and vice versa), namely $\beta(\tilde{k}^*)$.

With one year as our time unit, standard parameter values are: $g = 0.02$, $n = 0.01$, $\delta = 0.05$, and $\varepsilon(\tilde{k}^*) = 1/3$. We then get $\beta(\tilde{k}^*) = (1-\varepsilon(\tilde{k}^*))(\delta+g+n)$ $= 0.053$ per year. In the empirical Chapter 11 of Barro and Sala-i-Martin (2004), it is argued that a lower value of $\beta(\tilde{k}^*)$, say 0.02 per year, fits the data better. This requires $\varepsilon(\tilde{k}^*) = 0.75$. Such a high value of $\varepsilon(\tilde{k}^*)$ ($\approx$ the income share of capital) may seem difficult to defend. But if we reinterpret K in the Solow model so as to include *human* capital (skills embodied in human beings and acquired through education and learning by doing), a value of $\varepsilon(\tilde{k}^*)$ at that level may not be far out.

6.3.4 Adjustment time

Let τ_ω be the time that it takes for the fraction $\omega \in (0,1)$ of the initial gap between $\tilde{k}$ and $\tilde{k}^*$ to be eliminated, i.e., τ_ω satisfies the equation

$$\frac{\left|\tilde{k}(\tau_\omega) - \tilde{k}^*\right|}{\left|\tilde{k}(0) - \tilde{k}^*\right|} = \frac{\tilde{k}(\tau_\omega) - \tilde{k}^*}{\tilde{k}(0) - \tilde{k}^*} = 1 - \omega, \tag{6.18}$$

where $1-\omega$ is the fraction of the initial gap still remaining at time τ_ω. In (6.18) we have applied that $sign(\tilde{k}(t) - \tilde{k}^*) = sign(\tilde{k}(0) - \tilde{k}^*)$ in view of monotonic convergence.

By (6.9), we have

$$\tilde{k}(\tau_\omega) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

In view of (6.18), this implies

$$1 - \omega \approx e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

Taking logs on both sides and solving for τ_ω gives

$$\tau_\omega \approx -\frac{\log(1-\omega)}{\beta(\tilde{k}^*)}. \tag{6.19}$$

This is the approximate *adjustment time* required for $\tilde{k}$ to eliminate the fraction ω of the initial distance of $\tilde{k}$ to its steady-state value, $\tilde{k}^*$, when the adjustment speed (speed of convergence) is $\beta(\tilde{k}^*)$.

Often we consider the *half-life* of the adjustment, that is, the time it takes for half of the initial gap to be eliminated. To find the half-life of the adjustment of $\tilde{k}$, we put $\omega = \frac{1}{2}$ in (6.19). Again we use one year as our time unit. With the parameter values from Section 6.3.3, we have $\beta(\tilde{k}^*) = 0.053$ per year and thus

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.053} \approx \frac{0.69}{0.053} = 13,1 \text{ years.}$$

As noted above, Barro and Sala-i-Martin (2004) estimate the asymptotic speed of convergence to be $\beta(\tilde{k}^*) = 0.02$ per year. With this value, the half-life is approximately

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.02} \approx \frac{0.69}{0.02} = 34.7 \text{ years.}$$

And the time needed to eliminate three quarters of the initial distance to steady state, $\tau_{3/4}$, will then be about 70 years ($= 2 \cdot 35$ years, since $1 - 3/4 = \frac{1}{2} \cdot \frac{1}{2}$).

Among empirical analysts there is not general agreement about the size of $\beta(\tilde{k}^*)$. Some authors, for example Islam (1995), using a panel data approach, find speeds of convergence considerably larger, between 0.05 and 0.09. McQuinne and Whelan (2007) get similar results. There is a growing realization that the speed of convergence differs across periods and groups of countries. Perhaps an empirically reasonable range is $0.02 < \beta(\tilde{k}^*) < 0.09$. Correspondingly, a reasonable range for the half-life of the adjustment will be 7.6 years $< \tau_{\frac{1}{2}} < 34.7$ years.

Most of the empirical studies of convergence use a variety of cross-country regression analysis of the kind described in the next section. Yet the theoretical frame of reference is often the Solow model - or its extension with human capital (Mankiw et al., 1992). These models are closed economy models with exogenous technical progress and deal with "within-country" convergence. It is not obvious that they constitute an appropriate framework for studying cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and where some countries are pushing the technological frontier further out, while others try to imitate and catch up. At least one should be aware that the empirical estimates obtained may reflect mechanisms in addition to the falling marginal productivity of capital in the process of capital accumulation.

6.4 Barro-style growth regressions*

Barro-style growth regression analysis, which became very popular in the 1990s, draws upon transitional dynamics aspects (including the speed of convergence) as well as steady state aspects of neoclassical growth theory (for instance the Solow model or the Ramsey model).

Chapter 3.2 in Acemoglu (2009) presents Barro's growth regression equations in an unconventional form, see Acemoglu's equations (3.12), (3.13), and (3.14). The left-hand side appears as if it is just the growth rate of y (output per unit of labor) from one year to the next. But the true left-hand side of a Barro equation is the average compound annual growth rate of y over many years. Moreover, since Acemoglu's text is very brief about the formal links to the underlying neoclassical theory of transitional dynamics, we will spell the details out here.

Most of the preparatory work has already been done above. The point of departure is a neoclassical one-sector growth model for a closed economy:

$$\dot{\tilde{k}}(t) = s(\tilde{k}(t))f(\tilde{k}(t)) - (\delta + g + n)k(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \text{ given}, \tag{6.20}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $A(t) = A_0 e^{gt}$, and $L(t) = L_0 e^{nt}$ as above. The Solow model is the special case where the saving-income ratio, $s(\tilde{k}(t))$, is a constant $s \in (0, 1)$.

It is assumed that the model, (6.20), generates monotonic convergence, i.e., $\tilde{k}(t) \to \tilde{k}^* > 0$ for $t \to \infty$. Applying again a first-order Taylor approximation, as in Section 3.1, and taking into account that $s(\tilde{k})$ now may depend on $\tilde{k}$, as for instance it generally does in the Ramsey model, we find the asymptotic speed of convergence for $\tilde{k}$ to be

$$\text{SOC}^*(\tilde{k}) = (1 - \varepsilon(\tilde{k}^*) - \eta(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*) > 0, \tag{*}$$

where $\eta(\tilde{k}^*) \equiv \tilde{k}^* s'(\tilde{k}^*)/s(\tilde{k}^*)$ is the elasticity of the saving-income ratio w.r.t. the effective capital intensity, evaluated at $\tilde{k} = \tilde{k}^*$. (In case of the Ramsey model, one can alternatively use the fact that $\text{SOC}^*(\tilde{k})$ equals the absolute value of the negative eigenvalue of the Jacobian matrix associated with the dynamic system of the model, evaluated in the steady state. For a fully specified Ramsey model this eigenvalue can be numerically calculated by an appropriate computer algorithm; in the Cobb-Douglas case there exists even an explicit algebraic formula for the eigenvalue, see Barro and Sala-i-Martin, 2004). In a neighborhood of the steady state, the previous formulas remain valid with $\beta(\tilde{k}^*)$ defined as in (*). The asymptotic speed of convergence of for example $y(t)/y^*(t)$ is thus $\beta(\tilde{k}^*)$ as given in (*). For notational convenience, we will just denote it β, interpreted as a derived parameter, i.e.,

$$\beta = (1 - \varepsilon(\tilde{k}^*) - \eta(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*). \tag{6.21}$$

In case of the Solow model, $\eta(\tilde{k}^*) = 0$ and we are back in Section 3.

In view of $y(t) \equiv \tilde{y}(t)A(t)$, we have $g_y(t) = g_{\tilde{y}}(t) + g$. By (6.16) and the definition of β,

$$g_y(t) \approx g - \varepsilon(\tilde{k}^*)\beta(\log \tilde{k}(t) - \log \tilde{k}^*) \approx g - \beta(\log y(t) - \log y^*(t)), \tag{6.22}$$

where the last approximation comes from (6.14). This generalizes Acemoglu's Equation (3.10) (recall that Acemoglu concentrates on the Solow model and that his k^* is the same as our $\tilde{k}^*$).

With the horizontal axis representing time, Figure 6.4 gives an illustration of these transitional dynamics. As $g_y(t) = d \log y(t)/dt$ and $g = d \log y^*(t)/dt$, (6.22) is equivalent to

$$\frac{d(\log y(t) - \log y^*(t))}{dt} \approx -\beta(\log y(t) - \log y^*(t)). \tag{6.23}$$

So again we have a simple differential equation of the form $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. The solution of (6.23) is thus

$$\log y(t) - \log y^*(t) \approx (\log y(0) - \log y^*(0))e^{-\beta t}.$$

As $y^*(t) = y^*(0)e^{gt}$, this can be written

$$\log y(t) \approx \log y^*(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t}. \tag{6.24}$$

The solid curve in Figure 6.4 depicts the evolution of $\log y(t)$ in the case where $\tilde{k}_0 < \tilde{k}^*$ (note that $\log y^*(0) = \log f(\tilde{k}^*) + \log A_0$). The dotted curve exemplifies the case where $\tilde{k}_0 > \tilde{k}^*$. The figure illustrates per capita income convergence: low initial income is associated with a high subsequent growth rate which, however, diminishes along with the diminishing logarithmic distance of per capita income to its level on the steady state path.

For convenience, we will from now on treat (6.24) as an equality. Subtracting $\log y(0)$ on both sides, we get

$$\begin{aligned} \log y(t) - \log y(0) &= \log y^*(0) - \log y(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t} \\ &= gt - (1 - e^{-\beta t})(\log y(0) - \log y^*(0)). \end{aligned}$$

Dividing through by $t > 0$ gives

$$\frac{\log y(t) - \log y(0)}{t} = g - \frac{1 - e^{-\beta t}}{t}(\log y(0) - \log y^*(0)). \tag{6.25}$$

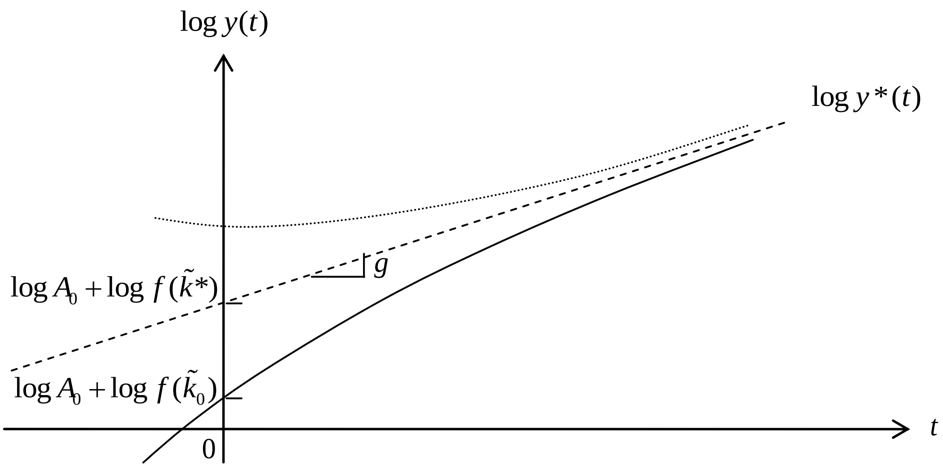

Figure 6.4: Evolution of $\log y(t)$. Solid curve: the case $\tilde{k}_0 < \tilde{k}^*$. Dotted curve: the case $\tilde{k}_0 > \tilde{k}^*$. Stippled line: the steady-state path.

On the left-hand side appears the average compound annual growth rate of y from period 0 to period t, which we will denote $\bar{g}_y(0,t)$. On the right-hand side appears the initial distance of $\log y$ to its hypothetical level along the steady state path. The coefficient, $-(1-e^{-\beta t})/t$, to this distance is negative and approaches zero for $t \rightarrow \infty$. Thus (6.25) is a translation into growth form of the convergence of $\log y_t$ towards the steady-state path, $\log y_t^*$, in the theoretical model without shocks. Rearranging the right-hand side, we get

$$\bar{g}_y(0,t) = g + \frac{1-e^{-\beta t}}{t}\log y^*(0) - \frac{1-e^{-\beta t}}{t}\log y(0) \equiv b^0 + b^1 \log y(0),$$

where both the constant $b^0 \equiv g + \left[(1-e^{-\beta t})/t\right]\log y^*(0)$ and the coefficient $b^1 \equiv -(1-e^{-\beta t})/t$ are determined by "structural characteristics". Indeed, β is determined by δ, g, n, $\varepsilon(\tilde{k}^*)$, and $\eta(\tilde{k}^*)$ through (6.21), and $y^*(0)$ is determined by A_0 and $f(\tilde{k}^*)$ through (6.12), where, in turn, $\tilde{k}^*$ is determined by the steady state condition $s(\tilde{k}^*)f(\tilde{k}^*) = (\delta+g+n)\tilde{k}^*$, $s(\tilde{k}^*)$ being the saving-income ratio in the steady state.

With data for N countries, $i = 1, 2, \ldots, N$, a test of the *unconditional convergence hypothesis* may be based on the regression equation

$$\bar{g}_{y_i}(0,t) = b^0 + b^1 \log y_i(0) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2), \tag{6.26}$$

where ϵ_i is the error term. This can be seen as a Barro growth regression equation in its simplest form. For countries in the entire world, the theoretical hypothesis $b^1 < 0$ is clearly not supported (or, to use the language of statistics, the null hypothesis, $b^1 = 0$, is not rejected).[2]

Allowing for the considered countries having different structural characteristics, the Barro growth regression equation takes the form

$$\bar{g}_{y_i}(0,t) = b_i^0 + b^1 \log y_i(0) + \epsilon_i, \qquad b^1 < 0, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2). \tag{6.27}$$

In this "fixed effects" form, the equation has been applied for a test of the *conditional convergence hypothesis,* $b^1 < 0$, often supporting this hypothesis. That is, within groups of countries with similar characteristics (like, e.g., the OECD countries), there is a tendency to convergence.

From the estimate of b^1 the implied estimate of the asymptotic speed of convergence, β, is readily obtained through the formula $b^1 \equiv (1 - e^{-\beta t})/t$. Even β, and therefore also the slope, b^1, does depend, theoretically, on country-specific structural characteristics. But the sensitivity on these do not generally seem large enough to blur the analysis based on (6.27) which abstracts from this dependency.

With the aim of testing hypotheses about growth determinants, Barro (1991) and Barro and Sala-i-Martin (1992, 2004) decompose b_i^0 so as to reflect the role of a set of potentially causal measurable variables,

$$b_i^0 = \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \ldots + \alpha_m x_{im},$$

where the α's are the coefficients and the x's are the potentially causal variables.[3] These variables could be measurable Solow-type parameters among those appearing in (6.20) or a broader set of determinants, including for instance the educational level in the labor force, and institutional variables like rule of law and democracy. Some studies include the initial within-country inequality in income or wealth among the x's and extend the theoretical framework correspondingly.[4]

From an econometric point of view there are several problematic features in regressions of Barro's form (also called the β convergence approach). These problems are discussed in Acemoglu pp. 82-85.

[2] Cf. Acemoglu, p. 16. For the OECD countries, however, b^1 is definitely found to be negative (cf. Acemoglu, p. 17).

[3] Note that our α vector is called β in Acemoglu, pp. 83-84. So Acemoglu's β is to be distinquished from our β which denotes the asymptotic speed of convergence.

[4] See, e.g., Alesina and Rodrik (1994) and Perotti (1996), who argue for a negative relationship between inequality and growth. Forbes (2000), however, rejects that there should be a robust negative correlation between the two.

6.5 References

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Chapter 7

Why the Malthusian era must come to an end

This chapter presents the *population-breeds-ideas model* by Michael Kremer (Kremer, 1993). The point of the model is to show that under certain conditions, the cumulative and nonrival character of technical knowledge makes it almost inevitable that the Malthusian regime of stagnating income per capita, close to subsistence minimum, will sooner or later in the historical evolution be surpassed.

This topic relates to Section 8.2 of Jones and Vollrath (2013). Section 4.2 of Acemoglu (2009) briefly discuss two special cases of the Kremer model.

7.1 The general model

Suppose a pre-industrial economy can be described by:

$$Y_t = A_t^{\sigma} L_t^{\alpha} Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, \tag{7.1}$$

$$\dot{A}_t = \lambda A_t^{\varepsilon} L_t, \qquad \lambda > 0, \quad A_0 > 0 \text{ given}, \tag{7.2}$$

$$L_t = \frac{Y_t}{\bar{y}}, \qquad \bar{y} > 0, \tag{7.3}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), Z the amount of land (fixed), and $\bar{y}$ subsistence minimum. By this is not meant some point almost at starvation, but an income level sufficient for food, clothing, shelter etc. to the worker, including family and offspring, thereby enabling reproduction of the labor force.

Both Z and $\bar{y}$ are considered as constant parameters. Time is continuous and it is understood that a kind of Malthusian population mechanism (see below) is operative behind the scene.

The exclusion of capital from the aggregate production function, (7.1), reflects the presumption that capital (tools etc.) is quantitatively of minor importance in a pre-industrial economy. In accordance with the replication argument, the production function has CRS w.r.t. the rival inputs, labor and land. The factor A_t^{σ} measures total factor productivity. As the right-hand side of (7.2) is positive, the technology level, A_t, is rising over time (although far back in time very very slowly). The increase in A_t per time unit is seen to be an increasing function of the size of the population. This reflects the hypothesis that population breeds ideas; these are *nonrival* and enter the pool of technical knowledge available for society as a whole. Indeed, the use of an idea by one agent does not preclude others' use of the same idea. Dividing through by L in (7.1) we see that $y \equiv Y_t/L_t = A_t^{\sigma}(Z/L_t)^{1-\alpha}$. The nonrival character is displayed by labor productivity being dependent on the total stock of knowledge, not on this stock per worker. In contrast, labor productivity depends on *land per worker.*

The rate per capita by which population breeds ideas is λA^{ε}. In case $\varepsilon > 0$, this rate is an increasing function of the already existing level of technical knowledge. This case reflects the hypothesis that the larger is the stock of ideas the easier do new ideas arise (perhaps by combination of existing ideas). The opposite case, $\varepsilon < 0$, is the one where "the easiest ideas are found first" or "the low-hanging fruits are picked first".

Equation (7.3) is a shortcut description of a Malthusian population mechanism. Suppose the true mechanism is

$$\dot{L}_t = \beta(y_t - \bar{y})L_t \gtreqless 0 \quad \text{for} \quad y_t \gtreqless \bar{y}, \tag{7.4}$$

where $\beta > 0$ is the speed of adjustment, y_t is per capita income, and $\bar{y} > 0$ is subsistence minimum. A rise in y_t above $\bar{y}$ will lead to increases in L_t through earlier marriage, higher fertility, and lower mortality. Thereby downward pressure on Y_t/L_t is generated, perhaps pushing y_t below $\bar{y}$. When this happens, population will be decreasing for a while and so return towards its sustainable level, $Y_t/\bar{y}$. Equation (7.3) treats this mechanism as if the population instantaneously adjusts to its sustainable level (i.e., as if $\beta \to \infty$). The model hereby gives a long-run picture, ignoring the Malthusian ups and downs in population and per capita income about the subsistence minimum. The important feature is that the technology level, and thereby Y_t, as well as the sustainable population will be *rising* over time. This speeds up the arrival of new ideas and so Y_t is raised even faster although per-capita income remains at its long-run level, $\bar{y}$.[1]

For simplicity, we now normalize the constant Z to be 1.

[1]Extending the model with the institution of private ownership and competitive markets, the absence of a growing standard of living corresponds to the doctrine from classical

7.2 Law of motion

The dynamics of the model can be reduced to one differential equation, the law of motion of technical knowledge. By (7.3) and (7.1), $L_t = Y_t/\bar{y} = A_t^\sigma L_t^\alpha/\bar{y}$. Consequently $L_t^{1-\alpha} = A_t^\sigma/\bar{y}$ so that

$$L_t = \bar{y}^{\frac{1}{\alpha-1}} A_t^{\frac{\sigma}{1-\alpha}}. \tag{7.5}$$

Substituting this into (7.2) gives the law of motion of technical knowledge:

$$\dot{A}_t = \lambda \bar{y}^{\frac{1}{\alpha-1}} A_t^{\varepsilon+\frac{\sigma}{1-\alpha}} \equiv \hat{\lambda} A_t^\mu, \tag{7.6}$$

where we have defined $\hat{\lambda} \equiv \lambda \bar{y}^{1/(\alpha-1)}$ and $\mu \equiv \varepsilon + \sigma/(1-\alpha)$. As will appear in the remainder, the "feedback parameter" μ is of key importance for the dynamics. We immediately see that if $\mu = 1$, the differential equation (7.6) is linear, while otherwise it is nonlinear.

The case $\mu = 1$: When $\mu = 1$, there will be a constant growth rate $g_A = \hat{\lambda}$ in technical knowledge. By (7.5), this results in a constant population growth rate $g_L = [\sigma/(1-\alpha)]\,\hat{\lambda}$, which is also the growth rate of output in view of (7.3). By the definition of $\hat{\lambda}$ in (7.6), we see that, as expected, the population and output growth rate is an increasing function of the creativity parameter λ and a decreasing function of the subsistence minimum.[2]

In this case the economy never leaves the Malthusian regime of a more or less constant standard of living close to existence minimum. Takeoff never occurs.

The case $\mu \neq 1$. Then (7.6) can be written

$$\dot{A}_t = \hat{\lambda} A_t^\mu, \tag{7.7}$$

which is a nonlinear differential equation in A.[3] Let $x \equiv A^{1-\mu}$. Then

$$\dot{x}_t = (1-\mu) A_t^{-\mu} \hat{\lambda} A_t^\mu = (1-\mu)\hat{\lambda}, \tag{7.8}$$

economics called the *iron law of wages.* This is the theory (from Malthus and Ricardo) that scarce natural resources and the pressure from population growth causes real wages to remain at subsistende level. There may occationally occur a technological improvement, which leads to a transitory real wage increase, triggering of an increase in population which ultimately brings down wages.

These classical economists did not recognize any tendency to sustained technical progress and therefore missed the immanent tendency to sustained population growth at the pre-industrial stage of economic development. Karl Marx was the first among the classical economists to really see and emphasize sustained technical progress.

[2]If $\sigma = 1-\alpha$ as in Acemoglu's analysis, $\mu = 1$ requires $\varepsilon = 0$, and in this case L and Y grow at the same rate as knowledge.

[3]The differential equation, (7.7), is a special case of what is known as the *Bernoulli equation.* In spite of being a non-linear differential equation, the Bernoulli equation always has an explicit solution.

a constant. To find x_t from this, we only need simple integration:

$$x_t = x_0 + \int_0^t \dot{x}_\tau d\tau = x_0 + (1-\mu)\hat{\lambda}t.$$

As $A = x^{\frac{1}{1-\mu}}$ and $x_0 = A_0^{1-\mu}$, this implies

$$A_t = x_t^{\frac{1}{1-\mu}} = \left[A_0^{1-\mu} + (1-\mu)\hat{\lambda}t\right]^{\frac{1}{1-\mu}} = \frac{1}{\left[A_0^{1-\mu} - (\mu-1)\hat{\lambda}t\right]^{\frac{1}{\mu-1}}}. \tag{7.9}$$

There are now two sub-cases, $\mu > 1$ and $\mu < 1$. The latter sub-case leads to permanent but decelerating growth in knowledge and population and the Malthusian regime is never transcended (see Exercise III.3). The former sub-case is the interesting one.

7.3 The inevitable ending of the Malthusian regime when $\mu > 1$

Assume $\mu > 1$. In this case the result (7.9) implies that the Malthusian regime *must* come to an end.

Although to begin with, A_t may grow extremely slowly, the growth in A_t will be *accelerating* because of the *positive feedback* (visible in (7.2)) from both rising population and rising A_t. Indeed, since $\mu > 1$, the denominator in (7.9) will be decreasing over time and approach zero in finite time, namely as t approaches the finite value $t^* = A_0^{1-\mu}/((\mu-1)\hat{\lambda})$. As an implication, according to (7.9), A_t goes towards *infinity* in *finite* time. The stylized graph in Fig. 7.1 illustrates. The evolution of technical knowledge becomes *explosive* as t approaches t^*.

It follows from (7.5) and (7.1) that explosive growth in A implies explosive growth in L and Y, respectively. The acceleration in the evolution of Y will sooner or later make Y rise fast enough so that the Malthusian population mechanism (which for biological reasons has to be slow) can not catch up. Then, what was in the Malthusian regime only a transitory excess of y_t over $\bar{y}$, will at some $t = \hat{t} < t^*$ become a permanent excess and take the form of sustained growth in y_t.

We may think of this post-Malthusian phase as describing pre-industrial Britain. Technological innovations speeded up, helped by market-friendly institutions, intellectual property rights, and deliberate and systematic application of science and engineering. This lead to the *takeoff* known as the *industrial revolution.*

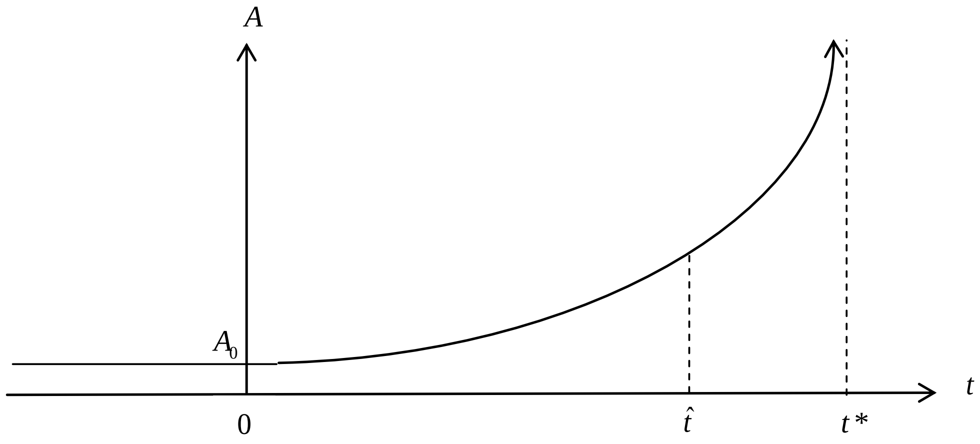

Figure 7.1: Accelerating growth in A when the feedback parameter μ exceeds one.

Note that Fig. 7.1 illustrates only what the process (7.7), with $\mu > 1$, implies *as long as it rules*, namely that knowledge goes towards infinity in *finite* time. The process necessarily ceases to rule long before time t^* is reached, however. This is because the process presupposes that the Malthusian population mechanism keeps track with output growth so as to maintain (7.3) which at some point before t^* becomes impossible because of the acceleration in the latter.

In a neighborhood of this point the takeoff will occur, featuring sustained growth in output per capita. According to equation (7.4), the takeoff should also feature a permanently rising population growth rate. As economic history has testified, however, along with the rising standard of living the demographics changed radically (in the U.K. during the 19th century). The *demographic transition* took place with fertility declining faster than mortality. This results in completely different dynamics, hence the model as it stands no longer fits.[4] As to the demographic transition as such, explanations suggested by economists include: higher real wages mean higher opportunity costs of raising children instead of producing; reduced use of child labor; the trade-off between "quality" (educational level) of the offspring and their "quantity" (Becker, Galor)[5]; skill-biased technical change; and improved contraception technology.

[4]Kremer (1993), however, also includes an extended model taking some of these changed dynamics into account.

[5]See Acemoglu, Section 21.2.

7.4 Closing remarks

The population-breeds-ideas model is about dynamics in the Malthusian regime of the pre-industrial epoch. The story told by the model is the following. When the feedback parameter, μ, is above one, the Malthusian regime has to come to an end because the battle between scarcity of land (or natural resources more generally) and technological progress (absent natural catastrophes) will inevitably be won by the latter. The reason is the cumulative and nonrival character of technical knowledge. This nonrivalry implies economies of scale. Moreover, the stock of knowledge is growing *endogenously.* This knowledge growth generates output growth and, through the demographic mechanism (7.3), growth in the stock of people, which implies a *positive feedback* to the growth of knowledge and so on. On top of this, if $\varepsilon > 0$, knowledge growth has a direct positive feedback on itself through (7.2). When the total positive feedback is strong enough ($\mu > 1$), it generates an explosive process.[6]

On the basis of demographers' estimates of the growth in global population over most of human history, Kremer (1993) finds empirical support for $\mu > 1$. Indeed, in the opposite case, $\mu \leq 1$, there would *not* have been a rising world population growth rate since one million years B.C. to the industrial revolution. The data in Kremer (1993, p. 682) indicates that the world population growth rate has been more or less proportional to the size of population until recently.

Final remark. Compared with Kremer's version of the model, we have allowed $\sigma \neq 1$, but at the same time introduced a simplification relative to Kremer's setup. Kremer starts from a slightly more general ideas-formation equation, namely $\dot{A}_t = \lambda A_t^{\varepsilon} L_t^{\psi}$ with $\psi > 0$, while in our (7.2) we have assumed $\psi = 1$. If $\psi > 1$, the ideas-creating brains reinforce one another. This only fortifies the acceleration in knowledge creation and thereby "supports" the case $\mu > 1$.[7] If on the other hand $0 < \psi < 1$, the idea-creating brains partly offset one another, for instance by simultaneously coming up with more or less the same ideas (the case of "overlap"). This generalization does not change the qualitative results. By assuming that the number of new ideas per time unit is proportional to the stock of brains, we have chosen to focus on an intermediate case in order to avoid secondary factors blurring the main mechanism.

[6] In the appendix the explosion result is considered in a general mathematical context.

[7] Kremer's calibration suggests $\psi \approx 6/5$.

7.5 Appendix

A. The mathematical background

Mathematically, the background for the explosion result is that the solution to a first-order differential equation of the form $\dot{x}(t) = \alpha + bx(t)^c$, $c > 1$, $b \neq 0$, $x(0) = x_0$ given, is always explosive. Indeed, the solution, $x = x(t)$, will have the property that $x(t) \to \pm\infty$ for $t \to t^*$ for some $t^* > 0$ where t^* depends on the initial conditions; and thereby the solution is defined only on a bounded time interval which depends on the initial condition.

Take the differential equation $\dot{x}(t) = 1 + x(t)^2$, $x(0) = 0$,as an example. As is well-known, the solution is $x(t) = \tan t = \sin t / \cos t$, defined for $t \in (-\pi/2, \pi/2)$.

B. Comparison with the two special cases considered in Acemoglu (2009)

At pp. 113-14 Acemoglu presents two versions of this framework, both of which assume $\sigma = 1 - \alpha$. This assumption is arbitrary; it is included as a special case in our formulation above. As to the other parameter relating to the role of knowledge, ε, Acemuglu assumes $\varepsilon = 0$ in his first version of the framework. This leads to constant population growth but a forever stagnating standard of living (Acemoglu, p. 113). In his second version, Acemoglu assumes $\varepsilon = 1$. This leads to many centuries of slow but (weakly) accelerating population growth and then ultimately a "takeoff" with sustained rise in the standard of living, to be followed by the "demographic transition" (outside the model). This latter outcome arises for a much larger set of parameter values than $\varepsilon = 1$ and is therefore theoretically more robust than appears in Acemoglu's exposition.

7.6 References

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Chapter 9

Human capital, learning technology, and the Mincer equation

We start with an overview of different approaches to the modeling of human capital formation in macroeconomics. Next we go into detail with one of these approaches, the life-cycle approach. In Section 9.3 a simple model of the choice of schooling length is considered. Finally, Section 9.4 presents the theory behind the empirical relationship named the Mincer equation.[1] In this connection it is emphasized that the Mincer equation should be seen as an equilibrium relationship for relative wages at a given point in time rather than as a production function for human capital.

9.1 Macroeconomic approaches to human capital

We define *human capital* as the stock of productive skills embodied in an individual. Human capital is thus a *production factor*, while by *human wealth* is meant the *present value* of expected future labor income (usually after tax).

Increases in the stock of human capital occurs through formal education and on-the-job-training. By contributing to the maintenance of life and well-being, also health care is of importance for the stock of human capital and the incentive to invest in human capital.

Since human capital is embodied in individuals and can only be used one place at a time, it is a *rival* and *excludable* good. Human capital is thus

[1] After Mincer (1958, 1974).

very different from technical knowledge. We think of *technical knowledge* as a list of instructions about how different inputs can be combined to produce a certain output. A principle of chemical engineering is an example of a piece of technical knowledge. In contrast to human capital, technical knowledge is a *non-rival* and only partially excludable good. Competence in *applying* technical knowledge is one of the skills that to a larger or smaller extent is part of human capital.

9.1.1 Modelling human capital

In the macroeconomic literature there are different theoretical approaches to the modelling of human capital. Broadly speaking we may distinguish these approaches along two "dimensions": 1) What characteristics of human capital are emphasized? 2) What characteristics of the decision maker investing in human capital are emphasized? Combining these two "dimensions", we get Table 1.

Table 1. Macroeconomic approaches to the modelling of human capital.

The character of the decision maker	*The character of human capital (hc):* Is hc treated as essentially different from physical capital? No	Yes
Solow-type rule-of-thumb households	Mankiw et al. (1992)	
Infinitely-lived family "dynasties" (the representative agent approach)	Barro&Sala-i-Martin (2004) Dalgaard&Kreiner (2001)	Lucas (1988)
Finitely-lived individuals going through a life cycle (the life cycle approach)		Ben-Porath (1967) Heijdra&Romp (2009)

My personal opinion is that for most issues the approach in the lower-right corner of Table 1 is preferable, that is, the approach treating human capital as a distinct capital good in a life cycle perspective. The viewpoint is:

First, by being *embodied* in a person and being lost upon death of this person, human capital is very different from physical capital. In addition, investment in human capital is *irreversible* (can not be recovered). Human capital is also distinct in view of the limited extend to which it can be used as a collateral, at least in non-slave societies. Financing an investment in physical capital, a house for example, by credit is comparatively easy because the house can serve as a collateral. A creditor can not gain title to a person, however. At most a creditor can gain title to a part of that person's future earnings in excess of a certain level required for a "normal" or "minimum" standard of living.

Second, educational investment is closely related to life expectancy and the life cycle of human beings: school - work - retirement. So a life cycle perspective seems the natural approach. Fortunately, convenient macroeconomic frameworks incorporating life cycle aspects exist in the form of overlapping generations models (for example Diamond's OLG model or Blanchard's continuous time OLG model).

9.1.2 Human capital and the efficiency of labor

Generally we tend to think of human capital as a combination of different skills. Macroeconomics, however, often tries (justified or not) to boil down the notion of human capital to a one-dimensional entity. So let us imagine that the current stock of human capital in society is measured by the one-dimensional index H. With L denoting the size of the labor force, we define $h \equiv H/L$. So, h is the average stock of human capital in the labor force. Further, let the "quality" (or "efficiency") of this stock in production be denoted q (under certain conditions this quality might be proxied by the average real wage per man-hour). Then it is reasonable to link q and h by some increasing *quality function*

$$q = q(h), \qquad \text{where } q(0) \geq 0, q' > 0. \tag{9.1}$$

Consider an aggregate production function, $\tilde{F}$, giving output per time unit at time t as

$$Y = \tilde{F}(K, q(h)L, t), \qquad \frac{\partial \tilde{F}}{\partial t} > 0, \tag{9.2}$$

where K is input of physical capital. The third argument of $\tilde{F}$ is time, t, indicating that the production function is time-dependent due to technical progress.

Generally the macroeconomic analyst would prefer a measure of human capital such that the quality of human capital is proportional to the stock of human capital, allowing us to write $q(h) = h$ by normalizing the factor of proportionality to be 1. The main reason is that an expedient variable representing human capital in a model requires that the analyst can decompose the real wage per working hour multiplicatively into two factors, the real wage per unit of human capital per working hour and the stock of human capital, h. That is, an expedient human capital concept requires that we can write

$$w = \hat{w} \cdot h, \tag{9.3}$$

where $\hat{w}$ is the real wage per unit of human capital per working hour. Indeed, if we have

$$Y = \tilde{F}(K, hL, t), \tag{9.4}$$

then, under perfect competition, we can write

$$w = \frac{\partial Y}{\partial L} = \tilde{F}_2(K, hL, t)h = \hat{w} \cdot h.$$

Under disembodied Harrod-neutral technical progress, (9.4) would take the form

$$Y = \tilde{F}(K, hL, t) = F(K, AhL) \equiv F(K, EL), \tag{9.5}$$

where $E \equiv A \cdot h$ is the "effective" labor input. The proportionality between E and h will under perfect competition allow us to write

$$w = \frac{\partial Y}{\partial L} = \tilde{F}_2(K, EL, t)E = w_E \cdot E = w_E \cdot A \cdot h = \hat{w} \cdot h.$$

So with the introduction of the technology level, A, an additional decomposition, $\hat{w} = w_E \cdot A$ comes in, while the original decomposition in (9.3) remains valid.

Whether or not the desired proportionality $q(h) = h$ can be obtained depends on how we model the formation of the "stuff" h. Empirically it turns out that treating the formation of human capital as similar to that of physical capital does *not* lead to the desired proportionality.

Treating the formation of human capital as similar to formation of physical capital

Consider a model where human capital is formed in a way similar to physical capital. The Mankiw-Romer-Weil (1992) extension of the Solow growth model with human capital is a case in point. Non-consumed aggregate output is split into one part generating additional physical capital one-to-one, while the other part is assumed to generate additional human capital one-to-one. Then for a closed economy in continuous time we can write:

$$\begin{aligned} Y &= C + I_K + I_H, \\ \dot{K} &= I_K - \delta_K K, \qquad \delta_K > 0, \\ \dot{H} &= I_H - \delta_H H, \qquad \delta_H > 0, \end{aligned} \tag{9.6}$$

where I_K and I_H denote gross investment in physical and human capital, respectively. This approach essentially assumes that human capital is produced by the same technology as consumption and investment goods.

Suppose the huge practical measurement problems concerning I_H have been somehow overcome. Then from long time series for I_H an index for H_t can be constructed by the *perpetual inventory method* in a way similar to the way an index for K_t is constructed from long time series for I_K. Indeed, in discrete time, with $0 < \delta_H < 1$, we get, by backward substitution,

$$\begin{aligned} H_{t+1} &= I_{H,t} + (1-\delta_H)H_t = I_{H,t} + (1-\delta_H)\left[I_{H,t-1} + (1-\delta_H)H_{t-1}\right] \\ &= \sum_{i=0}^{T}(1-\delta_H)^i I_{H,t-i} + (1-\delta_H)^{T+1}H_{t-T}. \end{aligned} \quad (9.7)$$

From the time series for I_H, an estimate of δ_H, and a rough conjecture about the initial value, H_{t-T}, we can calculate H_{t+1}. The result will not be very sensitive to the conjectured value of H_{t-T} since for large T the last term in (9.7) becomes very small.

In principle there need not be anything wrong with this approach. A snag arises, however, if, without further notice, the approach is combined with an explicit or implicit postulate that $q(h)$ is proportional to the "stuff", h, brought into being in the way described by (9.6). The snag is that the empirical evidence does *not* support this when the formation of human capital is modelled as in (9.6). This is an unintended by-product of the cross-country regression analysis by Mankiw, Romer, and Weil (1992), based on the approach in equation (9.6). One of their conclusions is that the following production function for a country's GDP is an acceptable approximation:

$$Y = BK^{1/3}H^{1/3}L^{1/3}, \quad (9.8)$$

where B stands for the total factor productivity of the country and is generally growing over time.[2] Applying that $H = hL$, we can write (9.8) this way:

$$Y = BK^{1/3}(hL)^{1/3}L^{1/3} = K^{1/3}(Ah^{1/2}L)^{2/3},$$

where $A = B^{3/2}$. That is, we end up with the form $Y = F(K, Aq(h)L)$ where $q(h) = h^{1/2}$, not $q(h) = h$. We should thus not expect the real wage to rise in proportion to h, when h is considered as some "stuff" formed in a way similar to the way physical capital is formed. (A further point is that writing a production function as in (9.8), i.e., with H and L as two separate inputs, may lead to confusion. The tangible input is L, and in this L, a certain

[2] The way Mankiw-Romer-Weil measure I_H is indirect and questionable. In addition, the way they let their measure enter the regression equation has been criticized for confounding the effects of the human capital *stock* and human capital *investment*, cf. Gemmel (1996) and Sianesi and Van Reenen (2003). It will take us too far to go into detail with these problems here.

"normal" or average h is *embodied.* In effect, varying L should immediately also imply variation of $H \equiv hL$.)

Before proceeding, a terminological point is in place. Why do we call $q(h)$ in (9.2) a "quality" function rather than simply a "productivity" function? The reason is the following. With perfect competition and *CRS*, in equilibrium the real wage per man-hour would be $w = \partial Y/\partial L = F_2'(K, Aq(h)L)Aq(h)$ $= \left[f(\tilde{k}) - \tilde{k}f'(\tilde{k})\right] Aq(h)$, where $\tilde{k} \equiv K/(Aq(h)L)$. So, with a converging $\tilde{k}$, the long-run growth rate of the real wage would in continuous time tend to be

$$g_w = g_A + g_q.$$

In this context we are inclined to identify "labor productivity" with $Aq(h)$ rather than just $q(h)$ and "growth in labor productivity" with $g_A + g_q$ rather than just g_q. So a distinct name for q seems appropriate and an often used name is "quality".

The conclusion so far is that specifying human capital formation as in (9.6) does not generally lead to a linear quality function. To obtain the desired linearity we have to specify the formation of human capital in a way different from the equation (9.6). This dissociation with the approach (9.6) applies, of course, also to its equivalent form on a per capita basis,

$$\dot{h} = (\frac{\dot{H}}{H} - n)h = \frac{I_H}{L} - (\delta_H + n)h. \tag{9.9}$$

(In the derivation of (9.9) we have first calculated the growth rate of $h \equiv H/L$, then inserted (9.6), and finally multiplied through by h.)

9.2 A life-cycle perspective on human capital

In the life-cycle approach to human capital formation we perceive h as the human capital embodied in a single individual and lost upon death of this individual. We study how h evolves over the lifetime of the individual as a result of both educational investment (say time spent in school) and work experience. In this way the life-cycle approach recognizes that human capital is different from physical capital. By seeing human capital formation as the result of individual learning, the life-cycle approach opens up for distinguishing between the production technologies for human and physical capital. Thereby the life-cycle approach offers a better chance for obtaining the linear relationship, $q(h) = h$.

Let the human capital of an individual of "age" τ (beyond childhood) be denoted h_τ. Let the total time available per time unit for study, work, and leisure be normalized to 1. Let s_τ denote the fraction of time the individual spends in school at age τ. This allows the individual to go to school only part-time and spend the remainder of non-leisure time working. If ℓ_τ denotes the fraction of time spent at work, we have

$$0 \leq s_\tau + \ell_\tau \leq 1.$$

The fraction of time used as leisure (or child rearing, say) at age τ is $1-s_\tau-\ell_\tau$. If full retirement occurs at age $\bar{\tau}$, we have $s_\tau = \ell_\tau = 0$ for $\tau \geq \bar{\tau}$.

We measure age in the same time units as calendar time. It seems natural to assume that the increase in h_τ per unit of time (age) generally depends on four variables: current time in school, current time at work (resulting in work experience), human capital already obtained, and current calendar time itself, that is,

$$\dot{h}_\tau \equiv \frac{dh_\tau}{d\tau} = G(s_\tau, \ell_\tau, h_\tau, t), \qquad h_0 \geq 0 \text{ given.} \tag{9.10}$$

The function G can be seen as a production function for human capital − in brief a *learning technology.* The first argument of G reflects the role of formal education. Empirically, the primary input in formal education is the time spent by the students studying; this time is not used in work or leisure and it thereby gives rise to an opportunity cost of studying.[3] The second argument of G takes learning through work experience into account and the third argument allows for the already obtained level of human capital to affect the strength of the influence from given s_τ and ℓ_τ (the sign of this effect is theoretically ambiguous). Finally, the fourth argument, current calendar time allows for changes over time in the learning technology (organization of the learning process).

Consider an individual "born" (as a youngster) at date $v \leq t$ (v for vintage). If still alive at time t, the age of this individual is $\tau \equiv t - v$. The obtained stock of human capital at age τ will be

$$h_\tau = h_0 + \int_0^\tau G(s_x, \ell_x, h_x, v + x)dx.$$

A basic supposition in the life-cycle approach is that it is possible to specify the function G such that a person's time-t human capital embodies a time-t labor productivity proportional to this amount of human capital and thereby, under perfect competition, a real wage proportional to this human capital.

[3] We may perceive the costs associated with teachers' time and educational buildings and equipment as being either quantitatively negligible or implicit in the function symbol G.

Below we consider four specifications of the learning technology that one may encounter in the literature.

EXAMPLE 1 In a path-breaking model by the Israeli economist Ben-Porath (1967) the learning technology is specified this way:

$$\dot{h}_\tau = g(s_\tau h_\tau) - \delta h_\tau, \qquad g' > 0, g'' < 0,\ \delta > 0, \qquad h_0 > 0. \tag{9.11}$$

Here time spent in school is more efficient in building human capital the more human capital the individual has already. Work experience does not add to human capital formation. The parameter δ enters to reflect obsolescence (due to technical change) of skills learnt in school. □

EXAMPLE 2 Growiec (2010) and Growiec and Groth (2015) consider the aggregate implications of a learning technology specified this way:

$$\dot{h}_\tau = (\lambda s_\tau + \xi \ell_\tau) h_\tau, \qquad \lambda > 0, \xi \geq 0, \qquad h_0 > 0. \tag{9.12}$$

Here λ measures the efficiency of schooling and ξ the efficiency of work experience. The effects of schooling and (if $\xi > 0$) work experience are here assumed proportional to the level of human capital already obtained by the individual (a strong assumption which may be questioned).[4] The linear differential equation (9.12) allows an explicit solution,

$$h_\tau = h_0 e^{\int_0^\tau (\lambda s_x + \xi \ell_x) dx}, \tag{9.13}$$

a formula valid as long as the person is alive. This result has some affinity with the familiar "Mincer equation", to be considered below.[5] □

EXAMPLE 3 Here we consider an individual with exogenous and constant leisure. Hence time available for study and work is constant and conveniently normalized to 1 (as if there were no leisure at all). Moreover, in the beginning of life beyond childhood the individual goes to school full-time in S time units (years) and thereafter works full-time until death (no retirement). Thus

$$s_\tau = \begin{cases} 1 & \text{for } 0 \leq \tau < S, \\ 0 & \text{for } \tau \geq S. \end{cases} \tag{9.14}$$

We further simplify by ignoring the effect of work experience (or we may say that work experience just offsets obsolescence of skills learnt in school). The learning technology is specified as

$$\dot{h}_\tau = \eta \tau^{\eta - 1} s_\tau, \qquad \eta > 0, \quad h_0 \geq 0, \tag{9.15}$$

[4] Lucas (1988) builds on the case $\xi = 0$.

[5] In case $\xi = 0$ and s_x = constant = 1, while $\tau = S$ = schooling length, (9.13) reduces to $h = h_0 e^{\lambda S}$. This looks like a simple version of the "Mincer equation".

If $\eta < 1$, it becomes more difficult to learn more the longer you have already been to school. If $\eta > 1$, it becomes easier to learn more the longer you have already been under education.

The specification (9.14) implies that throughout working life the individual has constant human capital equal to $h_0 + S^\eta$. Indeed, integrating (9.15), we have for $t \geq S$ and until time of death,

$$h_\tau = h_0 + \int_0^\tau \dot{h}_x dx = h_0 + \int_0^S \eta x^{\eta-1} dx = h_0 + x^\eta|_0^S = h_0 + S^\eta. \tag{9.16}$$

So the parameter η measures the elasticity of human capital w.r.t. the number of years in school. As briefly commented on in the concluding section, there is some empirical support for the power function specification in (9.16) and even the hypothesis $\eta = 1$ may not be rejected. □

In Example 1 there is no explicit solution for the level of human capital. Then the solution can be characterized by phase diagram analysis (as in Acemoglu, §10.3). In the examples 2 and 3 we can find an explicit solution for the level of human capital. In this case the term "learning technology" is used not only in connection with the original differential form as in (9.10), but also for the integrated form, as in (9.13) and (9.16), respectively. Sometimes the integrated form, like (9.16), is called a *schooling technology.*

EXAMPLE 4 Here we still assume the setup in (9.14) of Example 3, including the absence of both after-school learning and gradual depreciation. But the right-hand side of (9.15) is generalized to $\varphi(\tau)s_\tau$, where $\varphi(\tau)$ is some positively valued function of age. Then we end up with human capital after leaving school equal to some increasing function of S :

$$h = h(S), \qquad \text{where } h(0) \geq 0,\ h' > 0. \tag{9.17}$$

In cross-section or time series analysis it may be relevant to extend this by writing $h = ah(S)$, $a > 0$. The parameter a could then reflect quality of schooling. In the next section we shall focus on the form (9.17) where the quality-of-schooling parameter a can be seen as implicit in the function h. □

Before proceeding, let us briefly comment on the problem of aggregation over the different members of the labor force at a given point in time. In the aggregate framework of Section 9.1 multiplicity of skill types and job types is ignored. Human capital is treated as a one-dimensional and additive production factor. In production functions like (9.4) only aggregate human capital, H, matters. So output is thought to be the same whether the input is 2 million workers, each with one unit of human capital, or 1 million workers, each with 2 units of human capital. In human capital theory this questionable assumption is called the *perfect substitutability assumption* or the *efficiency unit assumption* (Sattinger, 1980). If we are willing to impose this assumption, going from micro to macro is conceptually simple. With h denoting individual human capital and $f(h)$ being the density function at a given point in time (so that $\int_0^\infty f(h)dh = 1$), we find average human capital in the labor force at that point in time to be $\bar{h} = \int_0^\infty hf(h)dh$ and aggregate human capital as $H = \bar{h}L$, where L is the size of the labor force. To build a theory of the evolution over time of the density function, $f(h)$, is, however, a complicated matter. Within as well as across the different cohorts there is heterogeneity regarding both schooling and retirement. And the fertility and mortality patterns are changing over time.

If we want to open up for a distinction between different types of jobs and different types of labor, say, skilled and unskilled labor, we may replace the production function (9.4) with

$$Y = \tilde{F}(K, h_1L_1, h_2L_2, t), \tag{9.18}$$

where L_1 and L_2 indicate man-hours delivered by the two types of workers, respectively, and h_1 and h_2 are the given embodied human capital levels (measured in efficiency units for each of the two kinds of jobs), respectively. This could be the basis for studying skill-biased technical change.

Whether or not the aggregate human capital, H, is a useful concept or not in connection with production can be seen as a question about whether or not we can rewrite a production function like (9.18) as $Y = F(K, H, t)$, where $H = h_1L_1 + h_2L_2$. We *can* if the two types of labor are *perfectly substitutable*, otherwise not. *Perfect substitutability* in this context means that the marginal rate of substitution between the two kinds of labor in (9.18) is a constant, i.e.,

$$MRS \equiv -\frac{dL_1}{dL_2}_{|Y=\bar{Y},K=\bar{K}} = \frac{\partial Y/\partial L_2}{\partial Y/\partial L_1} = \text{ a constant.} \tag{9.19}$$

This is satisfied if we can rewrite the production function such that $Y = F(K, H, t)$, where $H = h_1L_1 + h_2L_2$. Indeed, in this case we get $MRS = F_Hh_2/(F_Hh_1) = h_2/h_1$, a constant.

9.3 Choosing length of education

First some simplifying demographic assumptions. We assume, realistically, that expected lifetime of an individual is finite while the age at death is stochastic (uncertain) ex ante. We further assume, unrealistically, that *independently* of the already obtained age, the probability of surviving x more time units (years) is

$$P(X > x) = e^{-mx},$$

where X is remaining lifetime, a stochastic variable, while $m > 0$ is the *mortality rate* which is thus taken to be independent of age (and also independent of calendar time). Under this assumption, the "crude death rate", that is, the number of deaths per year divided by the size of population at the beginning of the year, will be approximately equal to m. Moreover, the mortality rate, m, will for an arbitrary person indicate the approximate probability of dying within one year "from now".[6]

Consider an individual's educational planning as seen from time of "birth" (entering life beyond childhood). Let the time of birth be denoted v. Suppose schooling is a full-time activity and that the individual plans to attend school in the first S years of life and after that work "full time" until death ("no retirement"). Let $\ell_{t-v}(S)$ denote the planned supply of labor (hours per year) to the labor market at age $t-v$ in the future. As $\ell_{t-v}(S)$ depends on the *stochastic* age, T, at death, $\ell_{t-v}(S)$ is itself a stochastic variable with two possible outcomes:

$$\ell_{t-v}(S) = \begin{cases} 0 & \text{when } t \leq v+S \text{ or } t > v+T, \\ \ell & \text{when } v+S < t \leq v+T, \end{cases}$$

where $\ell > 0$ is an exogenous constant ("full-time" working).

The combination of age-independent mortality rate and no retirement is sometimes called the "perpetual youth" assumption.

9.3.1 Human wealth

Let $w_t(S)$ denote the real wage received *per working hour* delivered at time t by a person who after S years in school works ℓ hours per year until death. This allows us to write the present value as seen from time v of expected lifetime earnings, i.e., the human wealth, for a person "born" at time v as

[6] If T denotes the uncertain age at death (a stochastic variable), the mortality rate (or "hazard rate" of death) at the age τ, denoted $m(\tau)$, is defined as $m(\tau) = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} P\left(T \leq \tau + \Delta\tau | \, T > \tau\right)$.

In the present model this is assumed equal to a constant, m. The unconditional probability of not reaching age τ is $P(T \leq \tau) = 1 - e^{-m\tau} \equiv F(\tau)$. Hence the density function is $f(\tau) = F'(\tau) = me^{-m\tau}$ and $P(\tau < T \leq \tau + \Delta\tau) \approx me^{-m\tau}\Delta\tau$. So, for $\tau = 0$, $P(0 < T \leq \Delta\tau) \approx m\Delta\tau = m$ if $\Delta\tau = 1$. Life expectancy is $E(T) = \int_0^\infty \tau me^{-m\tau} d\tau = 1/m$. All this is like in the "perpetual-youth" overlapping generations model by Blanchard (1985).

$$
\begin{aligned}
HW(v,S) =& \\
& 0 + E_v\left(\int_{v+S}^{v+T} w_t(S)\ell e^{-r(t-v)}dt\right) \\
=& E_v\left(\int_{v+S}^{\infty} w_t(S)\ell_{t-v}(S)e^{-r(t-v)}dt\right) \\
=& \int_{v+S}^{\infty} E_v(w_t(S)\ell_{t-v}(S)e^{-r(t-v)})dt = \int_{v+S}^{\infty} w_t(S)e^{-r(t-v)}E_v(\ell_{t-v}(S))dt,
\end{aligned} \tag{9.20}
$$

as in this context the integration operator $\int_{v+S}^{\infty}(\cdot)dt$ acts like a discrete-time summation operator, $\sum_{t=v}^{\infty}$. The rate of discount for potential future labor income conditional on being alive at the moment concerned is denoted r.[7] We get

$$
\begin{aligned}
HW(v,S) &= \int_{v+S}^{\infty} w_t(S)e^{-r(t-v)}\left(\ell \cdot P(T > t-v) + 0 \cdot P(T \leq t-v)\right)dt \\
&= \int_{v+S}^{\infty} w_t(S)e^{-r(t-v)}\ell e^{-m(t-v)}dt \\
&= \int_{v+S}^{\infty} w_t(S)\ell e^{-(r+m)(t-v)}dt.
\end{aligned} \tag{9.21}
$$

In writing the present value of the expected stream of labor income this way, we have assumed that:

A1 The discount rate, r, is constant over time.

A2 There is no educational fee.

We now introduce two additional assumptions:

A3 Labor efficiency (human capital) of a person with S years of schooling is $h(S)$, so that

$$w_t(S) = \hat{w}_t h(S), \qquad h' > 0,$$

where $\hat{w}_t$ is the real wage *per unit of human capital* per working hour at time t.[8]

[7] This rate is related to the opportunity cost of going to school instead of working and depends on conditions in the credit market. Under the idealized assumption A5 below, $r =$ the risk-free interest rate.

[8] Cf. Example 4 of Section 9.2.

A4 Owing to Harrod-neutral technical progress at a constant rate $g \in [0, r+m) \geq 0$, the evolution of $\hat{w}_t$ is given by $\hat{w}_t = \hat{w}_0 e^{gt}$. So technical progress makes a given h more and more productive (there is complementarity between the technology level and human capital as in (9.5) above).

Given A3 and A4, we get from (9.21) the expected "lifetime earnings" conditional on a schooling level S :

$$\begin{aligned} HW(v,S) &= h(S)\ell \int_{v+S}^{\infty} \hat{w}_0 e^{gt} e^{-(r+m)(t-v)} dt \qquad (9.22) \\ &= \hat{w}_0 e^{gv} h(S)\ell \int_{v+S}^{\infty} e^{[g-(r+m)](t-v)} dt \quad (\text{since } e^{gt} = e^{gv} e^{g(t-v)}) \\ &= \hat{w}_0 e^{gv} h(S)\ell \left(\left. \frac{e^{[g-(r+m)](t-\nu)}}{g-(r+m)} \right|_{\nu+S}^{\infty} \right) = \hat{w}_0 e^{g\nu} h(S)\ell \frac{e^{[g-(r+m)]S}}{r+m-g}. \end{aligned}$$

Below we chose measurement units such that the "normal" working time per year is 1 rather than ℓ.

The result in (9.22) provides a convenient formula for human wealth as seen from time of "birth", v. To say something reasonable about the *choice* of S, we need to specify the set of possibilities for the individual. These possibilities depend on the market environment. In particular, we need to specify how students make a living while studying.

9.3.2 Financing education

Assuming the students are born with no financial wealth and themselves have to finance their costs of living, they have to borrow while studying. Later in life, when they receive an income, they repay the loans with interest.

In this context we shall introduce the simplifying assumption:

A5 There is a *perfect credit and life annuity market.*

Financial intermediaries will be unwilling to offer the students loans at the going risk-free interest rate. Indeed, a creditor faces the risk that the student fails in the studies, never achieves the hoped job, or dies before having paid off the debt including the compound interest. The financial intermediaries may, however, be willing to offer student loans in the form of contracts stipulating later repayment with an interest rate above the risk-free rate and with the agreement that if the debtor dies before the principal has been paid back with interest, the debtor's estate is held free of any obligations.

Given the described constant mortality rate and given existence of a perfect credit and life insurance market, it can be shown that the equilibrium interest rate on this kind of student loans is what is known as the "actuarial rate". This rate equals the risk-free interest rate plus the mortality rate, m.[9] The relevant discount rate, r, in (9.20) will under these circumstances coincide with the risk-free interest rate. So we let this rate be denoted r and write the actuarial rate as $r + m$.

If the individual later in life, after having paid off the debt and obtained a positive net financial position, places the savings on life annuity accounts in life insurance companies, the *actuarial rate*, $r + m$, will also be the equilibrium rate of return received (until death) on these deposits. At death the liability of the insurance company is cancelled which means that the deposit is transferred to the insurance company in return for the high annuity payouts while the depositor was alive. The advantage of saving in life annuities (at least for people without a bequest motive) is that life annuities imply a transfer of income from after time of death to before time of death by offering a higher rate of return than risk-free bonds, but only until the depositor dies.

9.3.3 Maximizing human wealth

Suppose that neither the educational process itself nor the resulting stock of human capital enter the utility function. That is, assume

A6 There is no "joy of going to school" and no "joy of being a learned person".

In the perspective of this assumption, human capital is only an investment good, not also a durable consumption good.[10] If moreover there is no utility from leisure, the educational decision can be separated from whatever plan for the time path of consumption and saving through life the individual may decide; this is known as the *Separation Theorem*.[11] Under the described circumstances, the only incentive for acquiring human capital is to increase the human wealth $HW(\nu, S)$ given in (9.22).

[9] See Yaari (1965). This result presupposes that the insurance companies have negligible administration costs.

Owing to asymmetric information and related credit market imperfections, in real world situations such loan contracts are rare; this is one of the reasons for public sector intervention in the provision of loans to students. These credit market imperfections are ignored by the present model, but are briefly dealt with in for instance Acemoglu (2009), pp. 761-764.

[10] For a broader conception of human capital, see for instance Sen (1997).

[11] See, e.g., Acemoglu (2009), Ch. 10.1.

By the assumptions A1, A2, ..., A6, we have hereby reduced the problem of choosing schooling length to the unconstrained static problem of maximizing $HW(\nu, S)$ with respect to S. An interior solution to this problem satisfies the first-order condition:

$$\begin{aligned}\frac{\partial HW}{\partial S}(v,S) &= \frac{\hat{w}_0}{r+m-g}\left[h'(S)e^{[g-(r+m)]S} - h(S)e^{[g-(r+m)]S}(r+m-g)\right] \\ &= HW(v,S)\left[\frac{h'(S)}{h(S)} - (r+m-g)\right] = 0, \qquad (9.23)\end{aligned}$$

from which follows

$$\frac{h'(S)}{h(S)} = r+m-g \equiv \tilde{r}. \qquad (9.24)$$

This may be called the *schooling first-order condition*, and $\tilde{r}$ can be seen as the "required rate of return" in units of human capital. In the optimal plan the actual rate of return in units of human capital equals $\tilde{r}$, which in turn equals the risk-free interest rate adjusted for (a) the approximate probability of dying within a year from "now", $1 - e^{-m} \approx m$; and (b) wage growth due to technical progress. The trade-off faced by the individual is the following: increasing S by one year results in a higher level of human capital (higher future earning power) but postpones by one year the time when earning an income begins. The effective interest cost (opportunity cost) is diminished by g, reflecting the fact that next year the real wage per unit of human capital is $100 \cdot g$ percent higher than in the current year.

The intuition behind the first-order condition (9.24) may be easier to grasp if we put g on the left-hand-side and multiply by $\hat{w}_t$ in the numerator as well as the denominator. Then the condition looks like a standard no-arbitrage condition:

$$\frac{\hat{w}_t h'(S) + \hat{w}_t g h(S)}{\hat{w}_t h(S)} = r+m. \qquad (9.25)$$

On the left-hand side we have the rate of return (in units of consumption) obtained by "investing" one more year in education. In the numerator we have the direct increase in wage income by increasing S by one unit plus the gain arising from the fact that human capital, $h(S)$, has higher earnings capacity one year later due to technical progress. In the denominator we have the educational investment made by letting the obtained human capital, $h(S)$, "stay" one more year in school instead of at the labor market. Indeed, $\hat{w}_t h(S)$ is the size of that investment in the sense of the opportunity cost of staying in school one more year.

On the right-hand side of (9.25) appears the rate of return, $r+m$, that could be obtained by the alternative strategy, which is to leave school already after S years and then use next years's labor income to pay off study loans. This alternative would give the rate of return $r + m$.

The first-order condition (9.24) has thus similarity with a no-arbitrage equation in financial markets. (As is usual, our interpretation treats marginal changes as if they were discrete.)

Now, suppose $S = S^* > 0$ is a unique value of S satisfying (9.24). Then a sufficient (but not necessary) condition for S^* to be the unique optimal length of education for the individual is that $h'' \leq 0$ at $S = S^*$ (see Appendix A). If individuals are alike in the sense of having the same innate abilities and facing the same schooling technology $h(\cdot)$, they will all choose S^*.

EXAMPLE 5 Suppose $h(S) = S^\eta$, $\eta > 0$, as in Example 3, but with $h_0 = 0$. Then the first-order condition (9.24) gives a unique solution $S^* = \eta/(r+m-g)$; and the second-order condition (9.32) holds for all $\eta > 0$. More sharply decreasing returns to schooling (smaller η) shortens the optimal time spent in school as does of course a higher effective discount rate, $r + m - g$.

Consider two countries, one rich (industrialized) and one poor (agricultural). With one year as the time unit, let the parameter values be as in the first four columns in the table below. The resulting optimal S for each of the countries is given in the last column.

	η	r	m	g	S^*
rich country	0.6	0.06	0.01	0.02	12.0
poor country	0.6	0.12	0.02	0.00	4.3

The difference in S^* is due to r and m being higher and g lower in the poor country. □

9.4 What the Mincer equation is and is not

In this section we consider the issue whether the *exponential* form,

$$h(S) = h(0)e^{\lambda S}, \qquad \lambda > 0, \tag{9.26}$$

is a plausible specification of the production function for human capital. This specification is quite popular in the literature, and in Acemoglu (2009) it is used in connection with "levels accounting" in his Chapter 3, pp. 96-99 and is treated also theoretically in his Chapter 10.2. The form (9.26) can be seen as a special case of equation (9.13) from Example 2 above, namely the case $\xi = 0$ in combination with equation (9.14) from Example 3.

There exists a presumption in the macroeconomic literature that the famous *Mincer equation* provides an empirical foundation for the exponential form (9.26). The Mincer equation is the following semi-loglinear *cross-sectional* relationship at a given point in time, t:

$$\log w_t(S) = \log w_t(0) + \lambda S, \qquad \lambda > 0, \tag{9.27}$$

where, as in Section 9.3, $w_t(S)$ is the real wage *per working hour* delivered at time t by a person with S years' schooling level, cf. Figure 9.1. Such a semi-loglinear relationship is well documented in the empirical literature and was first discovered by the American economist Jacob Mincer (Mincer 1958, 1974).

But does it provide evidence for any particular form for the *production function* for human capital? No! *First*, as briefly commented in the concluding section, there seems to be little empirical support for an exponential production function. *Second*, as we shall now see, the microeconomic theory, proposed by Mincer (1958) as an explanation of the observed semi-loglinear relationship (9.27), has nothing to do with a specific production function for human capital.[12]

Explaining the Mincer equation

In Mincer's theory behind the observed exponential relationship called the Mincer equation, there is no role at all for any specific schooling technology, $h(\cdot)$, leading to a unique solution, S^*. The point of departure is that there is *heterogeneity in the jobs* offered to people (different educational levels not being perfectly substitutable). Assuming people are ex ante alike, they end up ex post choosing different educational levels. This outcome arises through the competitive equilibrium forces of *supply and demand in the job markets*.

Imagine, first, a case where all individuals have in fact chosen the same educational level, S^*, because they are ex ante alike and all face the same arbitrary human capital production function, $h(S)$, satisfying (9.32). Then jobs that require other educational levels will go unfilled and so the job markets will not clear. The forces of excess demand and excess supply will then tend to generate an educational wage profile different from the one presumed in (9.22), that is, different from $\hat{w}_t h(S)$. Sooner or later an *equilibrium* educational wage profile tends to arise such that people are *indifferent* as to how much schooling they choose, thereby allowing market clearing. This requires

[12] The above Example 5 follows a short note by Jones (2007) entitled "A simple Mincerian approach to endogenizing schooling". The term "Mincerian approach" should here be interpreted in a very broad sense as more or less synonymous with "life-cycle approach" rather than be associated with a particular choice regarding the form of $h(S)$.

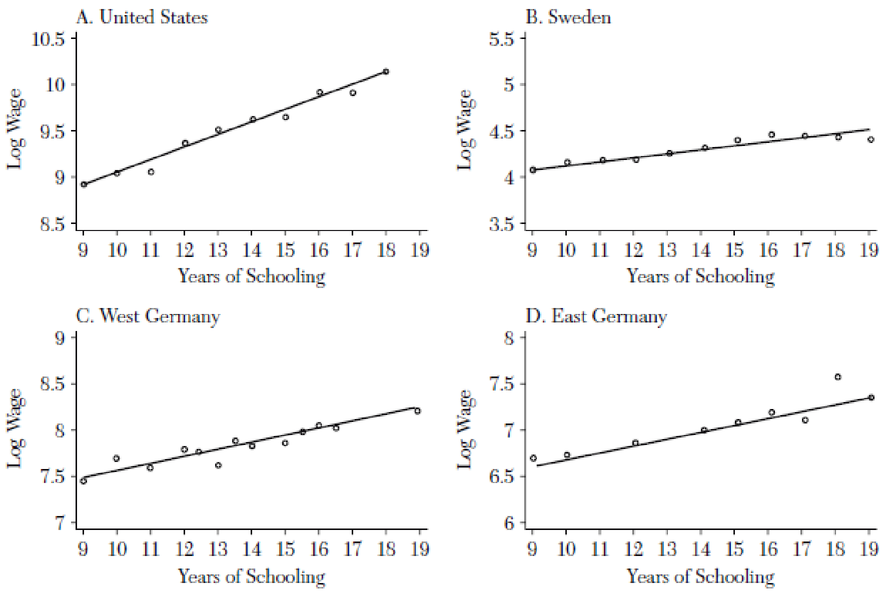

Figure 9.1: The semi-log schooling-wage relationship for fixed t. Different countries. Source: Krueger and Lindahl (2001).

a wage profile, $w_t(S)$, such that a marginal condition analogue to (9.24) holds for *all* S for which there is a positive amount of labor traded in equilibrium, say all $S \in [0, \bar{S}]$:

$$\frac{dw_t(S)/dS}{w_t(S)} = r + m - g \equiv \tilde{r} \qquad \text{for all } S \in [0, \bar{S}]. \tag{9.28}$$

It is here assumed, in the spirit of assumption A4 above, that technical progress implies that $w_t(S)$ *for fixed* S grows at the rate g, i.e., $w_t(S) = w_0(S)e^{gt}$, for all $S \in [0, \bar{S}]$. The equation (9.28) is a linear *differential equation* for w_t w.r.t. S, defined in the interval $0 \leq S \leq \bar{S}$, while t is fixed. And the function $w_t(S)$ is the so far unknown solution to this differential equation. That is, we have a differential equation of the form $dx(S)/dS = \tilde{r}x(S)$, where the unknown function, $x(S)$, is a function of schooling length rather than calendar time. The solution is $x(S) = x(0)e^{\tilde{r}S}$. Replacing the function $x(\cdot)$ with the function $w_t(\cdot)$, we thus have the solution

$$w_t(S) = w_t(0)e^{\tilde{r}S}. \tag{9.29}$$

Note that in the previous section, in the context of (9.24), we required the proportionate marginal return to schooling to equal $\tilde{r}$ only for a specific S, i.e.,

$$\frac{d(\hat{w}_t h(S))/dS}{\hat{w}_t h(S)} = \frac{h'(S)}{h(S)} = r + m - g \equiv \tilde{r} \qquad \text{for } S = S^*. \tag{9.30}$$

This is no more than a first-order condition assumed to hold at some point, S^*. It will generally not be a differential equation the solution of which gives a Mincerian exponential relationship. A differential equation requires a derivative relationship to hold not only at one point, but in an interval for the independent variable (S in (9.28)). Indeed, in (9.28) we require the proportionate marginal return to schooling to equal $\tilde{r}$ in a whole interval of schooling levels. Otherwise, with heterogeneity in the jobs offered, there could not be equilibrium.[13]

Returning to (9.29), by taking logs on both sides and substituting $\tilde{r}$ by λ, we get (9.27), which is the *Mincer equation* in semi-loglinear form.

As mentioned, empirically, the Mincer equation does surprisingly well in cross-section regression analysis, cf. Figure 9.1.[14] Note that (9.29) also yields a theory of how the "Mincerian slope", λ, in (9.26) is *determined*, namely as

[13] It seems that Acemoglu (2009, p. 362) makes the logical error of identifying a first-order condition, (9.30), with a differential equation, (9.28).

[14] The slopes are in the interval $(0.05, 0.15)$.

the mortality- and growth-corrected real interest rate, $\tilde{r}$. The evidence for this part of the theory is more scarce.

Given the equilibrium educational wage profile, $w_t(S)$, the human wealth of an individual "born" at time 0 can be written

$$\begin{aligned} HW_0 &= \int_S^\infty w_t(0)e^{\tilde{r}S}e^{-(r+m)t}dt = e^{\tilde{r}S}\int_S^\infty w_0(0)e^{\bar{g}t}e^{-(r+m)t}dt \\ &= w_0(0)e^{\tilde{r}S}\int_S^\infty e^{[g-(r+m)]t}dt = w_0(0)e^{\tilde{r}S}\left[\frac{e^{[g-(r+m)]t}}{g-(r+m)}\right]_S^\infty \\ &= \frac{w_0(0)}{r+m-g}, \qquad (9.31) \end{aligned}$$

since $\tilde{r} \equiv r+m-g$. In equilibrium the human wealth of the individual is thus *independent of* S (within an interval) according to the Mincerian theory. This is due to "compensating wage differentials", that is, the adjustment of the S-dependent wage levels so as to compensate for the S-dependent differences in length of work life after schooling. Indeed, the essence of Mincer's theory is that if one level of schooling implies a higher human wealth than the other levels of schooling, the number of individuals choosing that level of schooling will rise until the associated wage has been brought down so as to be in line with the human wealth associated with the other levels of schooling. Of course, such adjustment processes must in practice be quite time consuming and can only be approximative. Moreover, who among the ex ante similar individuals ends up with what schooling level is indeterminate in this setup.

In this context, the original schooling technology, $h(\cdot)$, for human capital formation has lost any importance. It does not enter human wealth in a long-run equilibrium in this disaggregate model where human wealth is simply given by (9.31). In this equilibrium people have different S's and the received wage of an individual per unit of work has no relationship with the human capital production function, $h(\cdot)$, by which we started in this section.

Although there thus exists a microeconomic theory behind a Mincerian relationship, this theory gives us a relationship for relative wages in a cross-section at a given point in time. It leaves open what an intertemporal production function for human capital, relating educational investment, S, to a resulting level, h, of labor efficiency, looks like. Besides, the Mincerian slope, $\tilde{r}$, is a market price, not an aspect of schooling technology.

We have up to now been silent about the fact that our simple framework in Section 9.3 does not fully embrace the case of strong convexity implied by an exponential specification of $h(S)$. Appendix B briefly comments.

9.5 Empirics relating to $h(S)$

The empirical macroeconomic literature typically measures S as the average number of years of schooling in the working-age population, taken for instance from the Barro and Lee (2001) data set.[15]

In their cross-country regression analysis de la Fuente and Domenech (2006) find a relationship essentially like that in Example 3 with $\eta = 1$. The authors find that the elasticity of GDP w.r.t. average years in school in the labor force is at least 0.60.

Similarly, the cross-country study, based on calibration, by Bills and Klenow (2000) as well as the time series study by Cervelatti and Sunde (2010) favor the hypothesis of diminishing returns to schooling. According to this, the linear term, $\tilde{r}S$, in the exponent in (9.26) should be replaced by a strictly concave function of S. These findings are in accordance with the results by Psacharopoulus (1994). They give empirical reasons for scepticism towards the linearity in h assumed in Example 2 of Section 9.2.

For $S > 0$, the power function in Example 5 can be written $h = S^{\eta} = e^{\eta \ln S}$ and is thus in better harmony with the data than the exponential function (9.26). A parameter indicating the quality of schooling may be added: $h = ae^{\eta \ln S}$, where $a > 0$ may be a function of the teacher-pupil ratio, teaching materials per student etc. See Caselli (2005).

9.6 Concluding remarks

Our formulation of the schooling length decision problem in Section 9.3 contained several simplifications such that we ended up with a static maximization problem in Section 9.3.3. More general setups lead to truly dynamic human capital accumulation problems.

This chapter considered human capital as a productivity-enhancing factor. There is a partly complementary perspective on human capital, often named the Nelson-Phelps hypothesis about the key role of human capital for technology *adoption* and technological *catching up.* An increase in human capital leads to an increase in the technology absorption capability of a nation.

A simple way of formalizing this idea is obtained by recognizing that it is not obvious that technical knowledge and human capital should enter the production function in the simple *multiplicative* way, $Y = F(K, AhL)$, as

[15] This means that complicated aggregation issues, arising from cohort heterogeneity and from the fact that individual human capital is lost upon death, are bypassed. For discussion, see Growiec and Groth (2015).

assumed in (9.5) above. The complementarity between A and h may take another form, perhaps better reflecting that workers with high skill level can use more advanced technology than workers with low skill level:

$$Y = \tilde{F}(K, hL, t) = F(K, \min(h/\eta(A), 1)AL), \qquad \eta'(A) > 0,$$

where $\eta(A)$ is the level of human capital *required* to fully exploit the current technology level, A. If actual $h < \eta(A)$, only the fraction $h/\eta(A)$ of A is utilized. Similar ideas are sketched in Jones and Vollrath (2013, Ch. 6.1) and Acemoglu (2009, Ch. 10.8). See also Exercise V.3.

Models based on the life-cycle approach to human capital typically conclude that education is productivity enhancing, i.e., more education has a positive *level* effect on income per capita but can only temporarily raise the per capita growth rate. Education is not a factor which in itself can explain sustained per capita growth. A more plausible main driving factor behind growth rather seems to be technological innovations. A higher level of per capita human capital may raise the speed of innovations, however. These themes are taken up in the next chapter (and in Exercise V.7 and V.8).

9.7 Appendix

Appendix A

Suppose $S = S^* > 0$ satisfies the first-order condition (9.24). To check the second-order condition, we consider

$$\begin{aligned}
&\frac{\partial^2 HW}{\partial S^2}(v, S^*) \\
= \;& \frac{\partial HW}{\partial S}(v, S^*)\left[\frac{h'(S^*)}{h(S^*)} - (r + m - g)\right] + HW(v, S^*)\frac{h(S^*)h''(S^*) - h'(S^*)^2}{h(S^*)^2} \\
= \;& HW(v, S^*)\frac{\frac{S^*}{h'(S^*)}h''(S^*) - \frac{S^*}{h(S^*)}h'(S^*)}{S^*h(S^*)}h'(S^*), \qquad (9.32)
\end{aligned}$$

since the first term on the right-hand side in the second row vanishes due to (9.24) being satisfied at $S = S^*$. The second-order condition, $\partial^2 HW/\partial S^2 < 0$ at $S = S^*$ holds if and only if the elasticity of h w.r.t. S exceeds that of h' w.r.t. S at $S = S^*$. A sufficient but not necessary condition for this is that $h'' \leq 0$. Anyway, since $HW(v, S)$ is a continuous function of S, if there is a unique $S^* > 0$ satisfying (9.24), and if $\partial^2 HW/\partial S^2 < 0$ holds for this S^*, then this S^* is the unique optimal length of education for the individual.

Appendix B

As alluded to at the end of Section 9.4, the strong convexity implied by the exponential specification $h(S) = h(0)e^{\lambda S}$ does not fit entirely well with the model in Section 9.3 based on the "perpetual youth" assumption of age-independent mortality and no retirement. The problem is that when $h(S) = h(0)e^{\lambda S}$, the "perpetual youth" setup implies that the first-order condition (9.24) holds for all S; moreover, we get $\partial^2 HW/\partial S^2 = 0$ for all S.

This problem reflects a limitation of the "perpetual youth" setup, where there is no conclusive upper bound for anyone's lifetime. It is not an argument for *apriori* rejection of the exponential specification.

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Chapter 11

AK and reduced-form AK models. Consumption taxation

The simplest model featuring "fully-endogenous" exponential per capita growth is what is known as the *AK model*. Jones and Vollrath (2013, Chapter 9, introduce a "Solow-style" AK model. Acemoglu (2009, Chapter 11) discusses *AK* and *reduced-form AK models* within a framework with Ramsey-style households (i.e., a representative agent approach). The name "AK" refers to a special feature of the aggregate production function, namely the absence of diminishing returns to capital. A characteristic result from AK models is that they have *no* transitional dynamics.

With the aim of synthesizing the formal characteristics of Ramsey-style AK models, this lecture note gives a brief account of the common features of AK models and reduced-form AK models (Section 11.1 and 11.2, respectively). Finally, for later application we discuss in Section 11.3 conditions under which consumption taxation is not distortionary. Notation is standard.

11.1 General equilibrium dynamics in the simple AK model

In the simple AK model we imagine a fully automatized economy where the aggregate production function is

$$Y(t) = AK(t), \qquad A > 0 \text{ and constant,} \tag{11.1}$$

so that marginal productivity of capital is

$$\frac{\partial Y(t)}{\partial K(t)} = A.$$

There are thus constant returns to capital, not diminishing returns. And labor is no longer a production factor. The model should be considered a thought experiment, not a model of reality.

This section provides a detailed proof that when we embed the AK technology (11.1) in a Ramsey framework with perfect competition, the model generates balanced growth *from the beginning*. So there will be no transitional dynamics.

We consider a closed economy with perfect competition and no government sector. The dynamic resource constraint for the economy is

$$\dot{K}(t) = Y(t) - c(t)L(t) - \delta K(t) = AK(t) - c(t)L(t) - \delta K(t), \qquad K(0) > 0 \text{ given}, \tag{11.2}$$

where $L(t)$ is population size. With the aim of maximizing profit, the representative firm demands capital services according to

$$K^d(t) = \begin{cases} \infty \text{ if } r(t) + \delta < A, \\ \text{undetermined if } r + \delta = A, \\ 0 \text{ if } r(t) + \delta > A. \end{cases} \tag{11.3}$$

So, in equilibrium ($K^d(t) = K(t)$), the interest rate is $r(t) = A - \delta \equiv r$. With Ramsey households with rate of time preference, ρ, and constant elasticity of marginal utility of consumption, θ, we thus find the equilibrium growth rate of per capita consumption to be

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho) \equiv \frac{1}{\theta}(A - \delta - \rho) \equiv g_c, \tag{11.4}$$

a constant. To ensure positive growth we impose the parameter restriction

$$A - \delta > \rho. \tag{A1}$$

And to ensure boundedness of discounted utility (and thereby a possibility of satisfying the transversality condition of the representative household), we impose the additional parameter restriction:

$$\rho - n > (1 - \theta)g_c. \tag{A2}$$

Reordering gives

$$r = \theta g_c + \rho > g_c + n, \tag{11.5}$$

where the equality is due to (16.27).

Solving the linear differential equation (16.27) gives

$$c(t) = c(0)e^{g_c t}, \tag{11.6}$$

where $c(0)$ is unknown so far (because c is not a predetermined variable). We shall find $c(0)$ by appealing to the household's transversality condition,

$$\lim_{t\to\infty} a(t)e^{-(r-n)t} = 0, \tag{TVC}$$

where $a(t)$ is per capita financial wealth at time t. Recalling the No-Ponzi-Game condition,

$$\lim_{t\to\infty} a(t)e^{-(r-n)t} \geq 0, \tag{NPG}$$

we see that the transversality condition is equivalent to the No-Ponzi-Game condition being not over-satisfied.

Defining $k(t) \equiv K(t)/L(t)$, the dynamic resource constraint, (11.2), is in per-capita terms

$$\dot{k}(t) = (A - \delta - n)k(t) - c(0)e^{g_c t}, \qquad k(0) > 0 \text{ given}, \tag{11.7}$$

where we have inserted (12.24). The solution to this linear differential equation is (cf. Appendix to Chapter 3)

$$k(t) = \left(k(0) - \frac{c(0)}{r - n - g_c}\right)e^{(r-n)t} + \frac{c(0)}{r - n - g_c}e^{g_c t}, \qquad r \equiv A - \delta. \tag{11.8}$$

In our closed-economy framework with no public debt, $a(t) = k(t)$. So the question is: When will the time path (11.8) satisfy (TVC) with $a(t) = k(t)$? To find out, we multiply by the discount factor $e^{-(r-n)t}$ on both sides of (11.8) to get

$$k(t)e^{-(r-n)t} = k(0) - \frac{c(0)}{r - n - g_c} + \frac{c(0)}{r - n - g_c}e^{-(r-g_c-n)t}.$$

Thus, in view of the assumption (A2), (11.5) holds and thereby the last term on the right-hand side vanishes for $t \to \infty$. Hence

$$\lim_{t\to\infty} k(t)e^{-(r-n)t} = k(0) - \frac{c(0)}{r - n - g_c}.$$

From this we see that the representative household satisfies (TVC) if and only if it chooses

$$c(0) = (r - n - g_c)k(0). \tag{11.9}$$

This is the equilibrium solution for the household's chosen per capita consumption at time $t = 0$. If the household instead had chosen $c(0) < (r - n - g_c)k(0)$, then $\lim_{t\to\infty} k(t)e^{-(r-n)t} > 0$ and so the household would not satisfy (TVC) but instead be over-saving. And if it had chosen $c(0) > (r-n-g_c)k(0)$, then $\lim_{t\to\infty} k(t)e^{-(r-n)t} < 0$ and so the household would be over-consuming and violate (NPG) (hence also (TVC)).

Substituting the solution for $c(0)$ into (11.8) gives the evolution of $k(t)$ in equilibrium,

$$k(t) = \frac{c(0)}{r - n - g_c} e^{g_c t} = k(0)e^{g_c t}.$$

So from the beginning k grows at the same constant rate as c. Since per capita output is $y \equiv Y/L = Ak$, the same is true for per capita output. Hence, from start the system is in balanced growth (there is no transitional dynamics).

The AK model features one of the simplest kinds of *endogenous growth* one can think of. Exponential growth is endogenous in the model in the sense that there is positive per capita growth in the long run, generated by an internal mechanism in the model (not by exogenous technology growth). The endogenously determined capital accumulation constitutes the mechanism through which sustained per capita growth is generated and sustained. It is because the net marginal productivity of capital is assumed constant and, according to (A1), higher than the rate of impatience, ρ, that capital accumulation itself is so powerful.

11.2 Reduced-form AK models

The models known as reduced-form AK models are a generalization of the simple AK model considered above. In contrast to the simple AK model, where only physical capital is an input, a reduced-form AK model assumes a technology involving at least two different inputs. Yet it is possible that in general equilibrium the aggregate production function ends up implying proportionality between output and some measure of "broad capital", i.e.,

$$Y(t) = B\tilde{K}(t),$$

where B is some endogenously determined positive constant, and $\tilde{K}(t)$ is "broad capital". If in addition the real interest rate in general equilibrium ends up being a constant, the model is called a *reduced-form AK model.* In the simple AK model constancy of average productivity of capital is postulated from the beginning. In the reduced-form AK models the average productivity of capital becomes and remains *endogenously* constant over time.

An example is the "AK model with physical and human capital" in Acemoglu, Chapter 11.2. With Ramsey-style households and the formation of human capital treated as similar to that of physical capital, Acemoglu finds that along the balanced growth path (obtained after an initial phase with full specialization in either physical or human capital accumulation), we have[1]

$$Y(t) = F(K(t), h(t)L(t)) = f(\hat{k}^*)h(t)L(t) = f(\hat{k}^*)H(t), \tag{11.10}$$

where we have defined

$$\hat{k} \equiv \frac{K}{H} \equiv \frac{K}{hL} \equiv \frac{k}{h}.$$

We further define

$$\tilde{K}(t) \equiv K(t) + H(t) = \text{ "broad capital".}$$

Then

$$\tilde{K}(t) \equiv (\frac{K(t)}{H(t)} + 1)H(t) = (\hat{k}^* + 1)H(t),$$

along the BGP. Isolating $H(t)$ and inserting into (11.10) gives

$$Y(t) = f(\hat{k}^*)\frac{1}{\hat{k}^* + 1}\tilde{K}(t) \equiv B\tilde{K}(t).$$

At an abstract level it is thus conceivable that "broad capital", defined as the sum of physical and human capital, can be meaningful. Empirically, however, there exists no basis for believing *this* concept of "broad capital" to be useful, cf. Exercises V.4 and V.5.

Anyway, a reduced-form AK model ends up with quite similar aggregate relations as those in the simple AK model. Hence the solution procedure to find the equilibrium path is quite similar to that in the simple AK model above. Again there will be no transitional dynamics.[2]

The nice feature of AK models is that they provide very simple theoretical examples of endogenous growth. The problematic feature is that they tend to simplify the technology description *too much.* They constitute extreme knife-edge cases, not something intermediate. This weakness also characterizes so-called asymptotic AK models or asymptotic reduced-form AK models (Exercise I.7 provides simple examples).

[1] The mentioned initial phase is left unnoticed in Acemoglu. In our notation $k \equiv K/L$ and $\hat{k} \equiv K/H$, while Acemoglu's text has $k \equiv K/H$.

[2] See Chapter 12.3. See also Exercise VI.4.

11.3 On consumption taxation

As a preparation for the discussion shortly in this course of fiscal policy in relation to economic growth, we shall here try to clarify an aspect of consumption taxation. This is the question: is a consumption tax distortionary - always? never? sometimes?

The answer is the following.

1. Suppose labor supply is *elastic* (due to leisure entering the utility function). Then a consumption tax (whether constant or time-dependent) is generally distortionary, creating a wedge between the MRS between consumption and leisure and labor's marginal productivity. The tax reduces the effective opportunity cost of leisure by reducing the amount of consumption forgone by working one hour less. Indeed, the tax makes consumption goods more expensive and so the amount of consumption that the agent can buy for the hourly wage becomes smaller. The substitution effect on leisure of a consumption tax is thus positive, while the income and wealth effects will be negative. Generally, the net effect will not be zero, but it can be of any sign; it may be small in absolute terms.

2. Suppose labor supply is *inelastic* (no trade-off between consumption and leisure). Then, at least in the type of growth models we consider in this course, a constant (time-independent) consumption tax acts as a lump-sum tax and is thus non-distortionary. If the consumption tax is *time-dependent*, however, a distortion of the *intertemporal* aspect of household decisions tends to arise.

To understand answer 2, consider a Ramsey household with inelastic labor supply. Suppose the household faces a time-varying consumption tax rate $\tau_t > 0$. To obtain a consumption level per time unit equal to c_t per capita, the household has to spend

$$\bar{c}_t = (1 + \tau_t)c_t$$

units of account (in real terms) per capita. Thus, spending $\bar{c}_t$ per capita per time unit results in the per capita consumption level

$$c_t = (1 + \tau_t)^{-1}\bar{c}_t. \tag{11.11}$$

In order to concentrate on the consumption tax as such, we assume the tax revenue is simply given back as lump-sum transfers and that there are no other government activities. Then, with a balanced government budget, we have

$$x_t L_t = \tau_t c_t L_t,$$

where x_t is the per capita lump-sum transfer, exogenous to the household, and L_t is the size of the representative household.

Assuming CRRA utility with parameter $\theta > 0$, the instantaneous per capita utility can be written

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} = \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta}.$$

In our standard notation the household's intertemporal optimization problem, in continuous time, is then to choose $(\bar{c}_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned}
U_0 &= \int_0^\infty \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\
\bar{c}_t &\geq 0, \\
\dot{a}_t &= (r_t - n)a_t + w_t + x_t - \bar{c}_t, \quad a_0 \text{ given,} \\
\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s - n)ds} &\geq 0.
\end{aligned}$$

From now, we let the timing of the variables be implicit unless needed for clarity. The current-value Hamiltonian is

$$H = \frac{(1+\tau)^{\theta-1}\bar{c}^{1-\theta} - 1}{1-\theta} + \lambda\left[(r-n)a + w + x - \bar{c}\right],$$

where λ is the co-state variable associated with financial per capita wealth, a. An interior optimal solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial \bar{c}} = (1+\tau)^{\theta-1}\bar{c}^{-\theta} - \lambda = 0, \text{ so that } (1+\tau)^{\theta-1}\bar{c}^{-\theta} = \lambda, \quad (11.12)$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} + (\rho - n)\lambda, \quad (11.13)$$

and a transversality condition which amounts to

$$\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s - n)ds} = 0. \quad (11.14)$$

We take logs in (11.12) to get

$$(\theta - 1)\log(1+\tau) - \theta \log \bar{c} = \log \lambda.$$

Differentiating w.r.t. time, taking into account that $\tau = \tau_t$, gives

$$(\theta - 1)\frac{\dot{\tau}}{1+\tau} - \theta\frac{\dot{\bar{c}}}{\bar{c}} = \frac{\dot{\lambda}}{\lambda} = \rho - r.$$

By ordering, we find the growth rate of consumption spending,

$$\frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right].$$

Using (11.11), this gives the growth rate of consumption,

$$\frac{\dot{c}}{c} = \frac{\dot{\bar{c}}}{\bar{c}} - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right] - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}(r - \frac{\dot{\tau}}{1+\tau} - \rho).$$

Assuming firms maximize profit under perfect competition, in equilibrium the real interest rate will satisfy

$$r = \frac{\partial Y}{\partial K} - \delta. \tag{11.15}$$

But the *effective* real interest rate, $\hat{r}$, faced by the consuming household, is

$$\hat{r} = r - \frac{\dot{\tau}}{1+\tau} \lesseqgtr r \text{ for } \dot{\tau} \gtreqless 0,$$

respectively. If for example the consumption tax is increasing, then the effective real interest rate faced by the consumer is smaller than the market real interest rate, given in (11.15), because saving implies postponing consumption and future consumption is more expensive due to the higher consumption tax rate.

The conclusion is that a time-varying consumption tax rate is distortionary. It implies a wedge between the intertemporal rate of transformation faced by the consumer, reflected by $\hat{r}$, and the intertemporal rate of transformation available in the technology of society, indicated by r in (11.15). On the other hand, *if* the consumption tax rate is constant, the consumption tax is non-distortionary when there is no utility from leisure.

A remark on tax smoothing

In models with transitional dynamics it is often so that maintaining constant tax rates is inconsistent with maintaining a balanced government budget. Is the implication of this that we should recommend the government to let tax rates be continually adjusted so as to maintain a forever balanced budget? No! As the above example as well as business cycle theory suggest, maintaining tax rates constant ("tax smoothing"), and thereby allowing government deficits and surpluses to arise, will generally make more sense. In itself, a budget deficit is not worrisome. It only becomes worrisome if it is not accompanied later by sufficient budget surpluses to avoid an exploding government debt/GDP ratio to arise. This requires that the tax rates taken together have a *level* which in the long run matches the level of government expenses.

Chapter 12

Learning by investing: two versions

The *learning-by-investing model,* sometimes called the *learning-by-doing model,* is one of the basic endogenous growth models. By basic is meant that the model specifies not only the technological aspects of the economy but also the market structure and the household sector, including household preferences. As in much other endogenous growth theory, the modeling of the household sector follows Ramsey and assumes the existence of a representative infinitely-lived household. Since this results in a simple determination of the long-run interest rate (the modified golden rule), the analyst can in a first approach concentrate on the main issue, technological change, without being detracted by aspects secondary to this issue.

In the present model learning from investment experience and diffusion across firms of the resulting new technical knowledge (positive externalities) play a key role.

There are two popular alternative versions of the model. The distinguishing feature is whether the learning parameter (see below) is less than one or equal to one. The first case corresponds to (a simplified version of) a model by Nobel laureate Kenneth Arrow (1962). The second case has been drawn attention to by Paul Romer (1986) who assumes that the learning parameter equals one. These two contributions start out from a common framework which we now present.[1]

[1] This lecture note also contains, in the appendix, a refresher on the concepts of a saddle point and saddle point stability.

12.1 The common framework

We consider a closed economy with firms and households interacting under conditions of perfect competition. Later, a government attempting to internalize the positive investment externality is introduced.

Let there be N firms in the economy (N "large"). Suppose they all have the same neoclassical production function, F, with CRS. Firm no. i faces the technology

$$Y_{it} = F(K_{it}, A_t L_{it}), \qquad i = 1, 2, ..., N, \tag{12.1}$$

where the economy-wide technology level A_t is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$A_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \qquad 0 < \lambda \leq 1, \tag{12.2}$$

where I_s^n is aggregate net investment and $K_t = \sum_i K_{it}$.[2]

The idea is that investment − the production of capital goods − as an unintended *by-product* results in *experience* or what we may call on-the-job *learning*. Experience allows producers to recognize opportunities for process and quality improvements. In this way knowledge is achieved about how to use the new capital goods efficiently and how to produce them in a cost-efficient way. This includes learning how to improve their design so that in combination with labor they are more productive and better satisfy the needs of the users. As formulated by Arrow:

> "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).[3]

The learning is assumed to benefit producers in many lines of production in the economy: learning by doing, learning by watching, learning by using. There are knowledge spillovers across firms and these spillovers are

[2] With arbitrary units of measurement for labor and output, the hypothesis is $A_t = BK_t^{\lambda}$, $B > 0$. In (12.2) measurement units are chosen such that $B = 1$.

[3] Concerning empirical evidence of learning-by-doing and learning-by-investing, see Chapter 13. The citation of Arrow indicates that it was rather experience from cumulative *gross* investment he had in mind as the basis for learning. Yet the hypothesis in (12.2) is the more popular one - seemingly for no better reason than that it leads to simpler dynamics. Another way in which (12.2) deviates from Arrow's original ideas is by assuming that technical progress is disembodied rather than embodied, an important distinction we defined in Chapter 2.2.

reasonably fast relative to the time horizon relevant for growth theory. In our macroeconomic approach both F and A are in fact assumed to be exactly the same for all firms in the economy. That is, in this specification the firms producing consumption-goods benefit from the learning just as much as the firms producing capital-goods.

The parameter λ indicates the elasticity of the general technology level, A, with respect to cumulative aggregate net investment and is named the "learning parameter". Whereas Arrow assumes $\lambda < 1$, Romer focuses on the case $\lambda = 1$. The case of $\lambda > 1$ is ruled out since it would lead to explosive growth (infinite output in finite time) and is therefore not plausible.

12.1.1 The individual firm

In the simple Ramsey model we assumed that households directly own the capital goods in the economy and rent them out to the firms. When discussing learning-by-investment, it somehow fits the intuition better if we (realistically) assume that the firms generally own the capital goods they use. They then finance their capital investment by issuing shares and bonds. Households' financial wealth then consists of these shares and bonds.

Consider firm i. There is perfect competition in all markets. So the firm is a price taker. Its problem is to choose a production and investment plan which maximizes the present value, V_i, of expected future cash-flows. Thus the firm chooses $(L_{it}, I_{it})_{t=0}^{\infty}$ to maximize

$$V_{i0} = \int_0^{\infty} \left[F(K_{it}, A_t L_{it}) - w_t L_{it} - I_{it}\right] e^{-\int_0^t r_s ds} dt$$

subject to $\dot{K}_{it} = I_{it} - \delta K_{it}$. Here w_t and I_t are the real wage and gross investment, respectively, at time t, r_s is the real interest rate at time s, and $\delta \geq 0$ is the capital depreciation rate. Rising marginal capital installation costs and other kinds of adjustment costs are assumed minor and can be ignored. It can be shown that in this case the firm's problem is equivalent to maximization of current pure profits in every short time interval. So, as hitherto, we can describe the firm as just solving a series of static profit maximization problems.

We suppress the time index when not needed for clarity. At any date firm i maximizes current pure profits, $\Pi_i = F(K_i, AL_i) - (r + \delta)K_i - wL_i$. This leads to the first-order conditions for an interior solution:

$$\begin{aligned} \partial \Pi_i / \partial K_i &= F_1(K_i, AL_i) - (r + \delta) = 0, \\ \partial \Pi_i / \partial L_i &= F_2(K_i, AL_i)A - w = 0. \end{aligned} \tag{12.3}$$

Behind (12.3) is the presumption that each firm is small relative to the economy as a whole, so that each firm's investment has a negligible effect on the economy-wide technology level A_t. Since F is homogeneous of degree one, by Euler's theorem,[4] the first-order partial derivatives, F_1 and F_2, are homogeneous of degree 0. Thus, we can write (12.3) as

$$F_1(k_i, A) = r + \delta, \tag{12.4}$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (12.4) determines k_i uniquely. From (12.4) follows that the chosen capital-labor ratio, k_i, will be the same for all firms, say $\bar{k}$.

12.1.2 The household

The representative household (or family dynasty) has $L_t = L_0 e^{nt}$ members each of which supplies one unit of labor inelastically per time unit, $n \geq 0$. The household has CRRA instantaneous utility with parameter $\theta > 0$. The pure rate of time preference is a constant, ρ. The flow budget identity in per head terms is

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given},$$

where a is per head financial wealth. The NPG condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0.$$

The resulting consumption-saving plan implies that per head consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho),$$

and the transversality condition that the NPG condition is satisfied with strict equality. In general equilibrium of our closed economy without natural resources and government debt, a_t will equal K_t/L_t.

[4]Recall that a function $f(x, y)$ defined in a domain D is homogeneous of degree h if for all (x, y) in D, $f(\lambda x, \lambda y) = \lambda^h f(x, y)$ for all $\lambda > 0$. If a differentiable function $f(x, y)$ is homogeneous of degree h, then (i) $xf_1'(x, y) + yf_2'(x, y) = hf(x, y)$, and (ii) the first-order partial derivatives, $f_1'(x, y)$ and $f_2'(x, y)$, are homogeneous of degree $h - 1$.

12.1.3 Equilibrium in factor markets

In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $\sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital-labor ratio, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{12.5}$$

As a consequence we can use (12.4) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, A_t) - \delta. \tag{12.6}$$

That is, whereas in the firm's first-order condition (12.4) causality goes from r_t to k_{it}, in (12.6) causality goes from k_t to r_t. Note also that in our closed economy with no natural resources and no government debt, a_t will equal k_t.

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, A) L_i = \sum_i F(k, A) L_i \quad \text{(by (12.1) and (12.5))} \\ &= F(k, A) \sum_i L_i = F(k, A)L = F(K, AL) = F(K, K^\lambda L) \quad \text{(by (12.2))}, \end{aligned} \tag{12.7}$$

where we have several times used that F is homogeneous of degree one.

12.2 The arrow case: $\lambda < 1$

The Arrow case is the robust case where the learning parameter satisfies $0 < \lambda < 1$. The method for analyzing the Arrow case is analogue to that used in the study of the Ramsey model with exogenous technical progress. In particular, aggregate capital per unit of effective labor, $\tilde{k} \equiv K/(AL)$, is a key variable. Let $\tilde{y} \equiv Y/(AL)$. Then

$$\tilde{y} = \frac{F(K, AL)}{AL} = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{12.8}$$

We can now write (12.6) as

$$r_t = f'(\tilde{k}_t) - \delta, \tag{12.9}$$

where $\tilde{k}_t$ is pre-determined.

12.2.1 Dynamics

From the definition $\tilde{k} \equiv K/(AL)$ follows

$$\begin{aligned}\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - \lambda\frac{\dot{K}}{K} - n \quad \text{(by (12.2))} \\ &= (1-\lambda)\frac{Y - C - \delta K}{K} - n = (1-\lambda)\frac{\tilde{y} - \tilde{c} - \delta\tilde{k}}{\tilde{k}} - n, \quad \text{where } \tilde{c} \equiv \frac{C}{AL} \equiv \frac{c}{A}.\end{aligned}$$

Multiplying through by $\tilde{k}$ we have

$$\dot{\tilde{k}} = (1-\lambda)(f(\tilde{k}) - \tilde{c}) - [(1-\lambda)\delta + n]\,\tilde{k}. \tag{12.10}$$

In view of (12.9), the Keynes-Ramsey rule implies

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left(f'(\tilde{k}) - \delta - \rho\right). \tag{12.11}$$

Defining $\tilde{c} \equiv c/A$, now follows

$$\begin{aligned}\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{A}}{A} = \frac{\dot{c}}{c} - \lambda\frac{\dot{K}}{K} = \frac{\dot{c}}{c} - \lambda\frac{Y - cL - \delta K}{K} = \frac{\dot{c}}{c} - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}) \\ &= \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}).\end{aligned}$$

Multiplying through by $\tilde{c}$ we have

$$\dot{\tilde{c}} = \left[\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(f(\tilde{k}) - \tilde{c} - \delta\tilde{k})\right]\tilde{c}. \tag{12.12}$$

The two coupled differential equations, (12.10) and (12.12), determine the evolution over time of the economy.

Phase diagram

Figure 12.1 depicts the phase diagram. The $\dot{\tilde{k}} = 0$ locus comes from (12.10), which gives

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}, \tag{12.13}$$

where we realistically may assume that $\delta + n/(1-\lambda) > 0$. As to the $\dot{\tilde{c}} = 0$ locus, we have

$$\begin{aligned}\dot{\tilde{c}} &= 0 \text{ for } \tilde{c} = f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda\theta}(f'(\tilde{k}) - \delta - \rho) \\ &= f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda}g_c \equiv c(\tilde{k}) \quad \text{(from (12.11))}.\end{aligned} \tag{12.14}$$

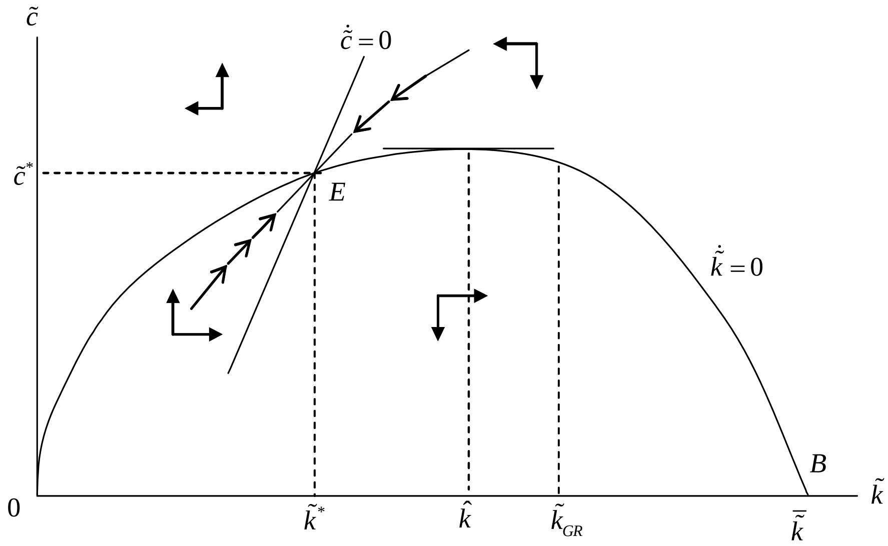

Figure 12.1: Phase diagram for the Arrow model.

Before determining the slope of the $\dot{\tilde{c}} = 0$ locus, it is convenient to consider the steady state, $(\tilde{k}^*, \tilde{c}^*)$.

Steady state

In a steady state $\tilde{c}$ and $\tilde{k}$ are constant so that the growth rate of C as well as K equals $\dot{A}/A + n$, i.e.,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{A}}{A} + n = \lambda\frac{\dot{K}}{K} + n.$$

Solving gives

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda}.$$

Thence, in a steady state

$$g_c = \frac{\dot{C}}{C} - n = \frac{n}{1-\lambda} - n = \frac{\lambda n}{1-\lambda} \equiv g_c^*, \quad \text{and} \tag{12.15}$$

$$\frac{\dot{A}}{A} = \lambda\frac{\dot{K}}{K} = \frac{\lambda n}{1-\lambda} = g_c^*. \tag{12.16}$$

The steady-state values of r and $\tilde{k}$, respectively, will therefore satisfy, by (12.11),

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda}. \tag{12.17}$$

To ensure existence of a steady state we assume that the private marginal product of capital is sufficiently sensitive to capital per unit of effective labor, from now called the "capital intensity":

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \delta + \rho + \theta\frac{\lambda n}{1-\lambda} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

The transversality condition of the representative household is that $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$, where a_t is per capita financial wealth. In general equilibrium $a_t = k_t \equiv \tilde{k}_t A_t$, where A_t in steady state grows according to (12.16). Thus, in steady state the transversality condition can be written

$$\lim_{t\to\infty} \tilde{k}^* e^{(g_c^* - r^* + n)t} = 0. \tag{TVC}$$

For this to hold, we need

$$r^* > g_c^* + n = \frac{n}{1-\lambda}, \tag{12.18}$$

by (12.15). In view of (12.17), this is equivalent to

$$\rho - n > (1-\theta)\frac{\lambda n}{1-\lambda}, \tag{A2}$$

which we assume satisfied.

As to the slope of the $\dot{\tilde{c}} = 0$ locus we have, from (12.14),

$$c'(\tilde{k}) = f'(\tilde{k}) - \delta - \frac{1}{\lambda}(\tilde{k}\frac{f''(\tilde{k})}{\theta} + g_c) > f'(\tilde{k}) - \delta - \frac{1}{\lambda}g_c, \tag{12.19}$$

since $f'' < 0$. At least in a small neighborhood of the steady state we can sign the right-hand side of this expression. Indeed,

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* = \rho + \theta g_c^* - \frac{1}{\lambda}g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} = \rho - n - (1-\theta)\frac{\lambda n}{1-\lambda} > 0, \tag{12.20}$$

by (12.15) and (A2). So, combining with (12.19), we conclude that $c'(\tilde{k}^*) > 0$. By continuity, in a small neighborhood of the steady state, $c'(\tilde{k}) \approx c'(\tilde{k}^*) > 0$. Therefore, close to the steady state, the $\dot{\tilde{c}} = 0$ locus is positively sloped, as indicated in Figure 12.1.

Still, we have to check the following question: In a neighborhood of the steady state, which is steeper, the $\dot{\tilde{c}} = 0$ locus or the $\dot{\tilde{k}} = 0$ locus? The slope of the latter is $f'(\tilde{k}) - \delta - n/(1-\lambda)$, from (12.13). At the steady state this slope is

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* \in (0, c'(\tilde{k}^*)),$$

in view of (12.20) and (12.19). The $\dot{\tilde{c}} = 0$ locus is thus steeper. So, the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus from below and can only cross once.

The assumption (A1) ensures existence of a $\tilde{k}^* > 0$ satisfying (12.17). As Figure 12.1 is drawn, a little more is implicitly assumed namely that there exists a $\hat{k} > 0$ such that the *private* net marginal product of capital equals the steady-state growth rate of output, i.e.,

$$f'(\hat{k}) - \delta = (\frac{\dot{Y}}{Y})^* = (\frac{\dot{A}}{A})^* + \frac{\dot{L}}{L} = \frac{\lambda n}{1-\lambda} + n = \frac{n}{1-\lambda}, \tag{12.21}$$

where we have used (12.16). Thus, the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \hat{k}$ is horizontal and $\hat{k} > \tilde{k}^*$ as indicated in the figure.

Note, however, that $\hat{k}$ is not the golden-rule capital intensity. The latter is the capital intensity, $\tilde{k}_{GR}$, at which the *social* net marginal product of capital equals the steady-state growth rate of output (see Appendix). If $\tilde{k}_{GR}$ exists, it will be larger than $\hat{k}$ as indicated in Figure 12.1. To see this, we now derive a convenient expression for the social marginal product of capital. From (12.7) we have

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= F_1(\cdot) + F_2(\cdot)\lambda K^{\lambda-1}L = f'(\tilde{k}) + F_2(\cdot)K^{\lambda}L(\lambda K^{-1}) \quad \text{(by (12.8))} \\
&= f'(\tilde{k}) + (F(\cdot) - F_1(\cdot)K)\,\lambda K^{-1} \quad \text{(by Euler's theorem)} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda}L - f'(\tilde{k})K)\lambda K^{-1} \quad \text{(by (12.8) and (12.2))} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda-1}L - f'(\tilde{k}))\lambda = f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} > f'(\tilde{k}).
\end{aligned}$$

in view of $\tilde{k} = K/(K^{\lambda}L) = K^{1-\lambda}L^{-1}$ and $f(\tilde{k})/\tilde{k} - f'(\tilde{k}) > 0$. As expected, the positive externality makes the social marginal product of capital larger than the private one. Since we can also write $\partial Y/\partial K = (1-\lambda)f'(\tilde{k}) + \lambda f(\tilde{k})/\tilde{k}$, we see that $\partial Y/\partial K$ is (still) a decreasing function of $\tilde{k}$ since both $f'(\tilde{k})$ and $f(\tilde{k})/\tilde{k}$ are decreasing in $\tilde{k}$. So the golden rule capital intensity, $\tilde{k}_{GR}$, will be that capital intensity which satisfies

$$f'(\tilde{k}_{GR}) + \lambda\frac{f(\tilde{k}_{GR}) - \tilde{k}_{GR}f'(\tilde{k}_{GR})}{\tilde{k}_{GR}} - \delta = \left(\frac{\dot{Y}}{Y}\right)^* = \frac{n}{1-\lambda}.$$

To ensure there exists such a $\tilde{k}_{GR}$, we strengthen the right-hand side inequality in (A1) by the assumption

$$\lim_{\tilde{k}\to\infty}\left(f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}}\right) < \delta + \frac{n}{1-\lambda}. \tag{A3}$$

This, together with (A1) and $f'' < 0$, implies existence of a unique $\tilde{k}_{GR}$, and in view of our additional assumption (A2), we have $0 < \tilde{k}^* < \hat{k} < \tilde{k}_{GR}$, as displayed in Figure 12.1.

Stability

The arrows in Figure 12.1 indicate the direction of movement, as determined by (12.10) and (12.12)). We see that the steady state is a *saddle point.* Moreover, the dynamic system is *saddle-point stable.*[5] The dynamic system has one pre-determined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$. The saddle path is not parallel to the jump variable axis. We claim that for a given $\tilde{k}_0 > 0$, (i) the initial value of $\tilde{c}_0$ will be the ordinate to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path; (ii) over time the economy will move along the saddle path towards the steady state. Indeed, this time path is consistent with all conditions of general equilibrium, including the transversality condition (TVC). And the path is the *only* technically feasible path with this property. Indeed, all the divergent paths in Figure 12.1 can be ruled out as equilibrium paths because they can be shown to violate the transversality condition of the household.

In the long run c and $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k}^*)A$ grow at the rate $\lambda n/(1-\lambda)$, which is positive if and only if $n > 0$. This is an example of *endogenous growth* in the sense that the positive long-run per capita growth rate is generated through an internal mechanism (learning) in the model (in contrast to exogenous technology growth as in the Ramsey model with exogenous technical progress).

12.2.2 Two types of endogenous growth

As also touched upon elsewhere in these lecture notes, it is useful to distinguish between two types of endogenous exponential growth. *Fully endogenous* exponential growth occurs when the long-run growth rate of c is positive without support from growth in any exogenous factor; the Romer case, to be considered in the next section, provides an example. *Semi-endogenous* exponential growth occurs if growth is endogenous but a positive per capita growth rate can not be maintained in the long run without the support from growth in some exogenous factor (for example exogenous growth in the labor force). Clearly, in the Arrow version of learning by investing, exponential growth is "only" semi-endogenous. The technical reason for this is the assumption that the learning parameter, λ, is below 1, which implies diminishing marginal returns to capital at the aggregate level. As a consequence,

[5] A formal definition is given in Appendix B.

if and only if $n > 0$, do we have $\dot{c}/c > 0$ in the long run. In line with this, $\partial g_y^*/\partial n > 0$.[6]

The key role of population growth derives from the fact that although there are diminishing marginal returns to capital at the aggregate level, there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. To put it differently: when there are increasing returns to K and L together, growth in the labor force not only counterbalances the falling marginal productivity of aggregate capital (this counter-balancing role reflects the direct complementarity between K and L), but also upholds sustained productivity growth via the learning mechanism.

Note that in the semi-endogenous growth case, $\partial g_y^*/\partial\lambda = n/(1-\lambda)^2 > 0$ for $n > 0$. That is, a higher value of the learning parameter implies higher per capita growth in the long run, when $n > 0$. Note also that $\partial g_y^*/\partial\rho = 0$ $= \partial g_y^*/\partial\theta$, that is, in the semi-endogenous growth case, preference parameters do not matter for the long-run per capita growth rate. As indicated by (12.15), the long-run growth rate is tied down by the learning parameter, λ, and the rate of population growth, n. Like in the simple Ramsey model, however, it can be shown that preference parameters matter for the *level* of the growth path. For instance (12.17) shows that $\partial\tilde{k}^*/\partial\rho < 0$ so that more patience (lower ρ) imply a higher $\tilde{k}^*$ and thereby a higher $y_t = f(\tilde{k}^*)A_t$.

This suggests that although taxes and subsidies do not have long-run growth effects, they can have *level* effects.

In this model there is clearly a motivation for government intervention due to the positive externality of private investment. But details about the design of government policy vis-a-vis this externality will in this lecture note only be discussed in relation to the Romer case of $\lambda = 1$, which is simpler and to which we now return.

12.3 Romer's limiting case: $\lambda = 1$, $n = 0$

We now consider the limiting case $\lambda = 1$. We should think of it as a thought experiment because, by most observers, the value 1 is considered an unrealistically high value for the learning parameter. Moreover, in combination with $n > 0$, the value 1 will lead to a forever rising per capita growth rate which does not accord the economic history of the industrialized world over more

[6] Note, however, that the model, and therefore (12.15), presupposes $n \geq 0$. If $n < 0$, the steady-state formulas in Section 12.2 are no longer valid. The formula in (12.16) would for instance imply a *decreasing* level of technical knowledge, which, at least in a modern economy, is implausible.

than a century. To avoid a forever rising growth rate, we therefore introduce the parameter restriction $n = 0$.

The resulting model turns out to be extremely simple and at the same time it gives striking results (both circumstances have probably contributed to its popularity).

First, with $\lambda = 1$ we get $A = K$ and so the equilibrium interest rate is, by (12.6),

$$r = F_1(k, K) - \delta = F_1(1, L) - \delta \equiv \bar{r},$$

where we have divided the two arguments of $F_1(k, K)$ by $k \equiv K/L$ and again used Euler's theorem. Note that the interest rate is constant "from the beginning" and independent of the historically given initial value of K, K_0. The aggregate production function is now

$$Y = F(K, KL) = F(1, L)K, \qquad L \text{ constant}, \tag{12.22}$$

and is thus *linear* in the aggregate capital stock.[7] In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. Thereby the Romer model belongs to the class of *reduced-form AK models*, that is, models where in general equilibrium the interest rate and the aggregate output-capital ratio are necessarily constant over time whatever the initial conditions.

The method for analyzing an AK model is different from the one used for a diminishing returns model as above.

12.3.1 Dynamics

The Keynes-Ramsey rule now takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv \gamma, \tag{12.23}$$

which is also constant "from the beginning". To ensure positive growth, we assume

$$F_1(1, L) - \delta > \rho. \tag{A1'}$$

And to ensure bounded intertemporal utility (and thereby a possibility of satisfying the transversality condition of the representative household), it is assumed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A2'}$$

[7] Acemoglu, p. 400, writes this as $Y = \tilde{f}(L)K$.

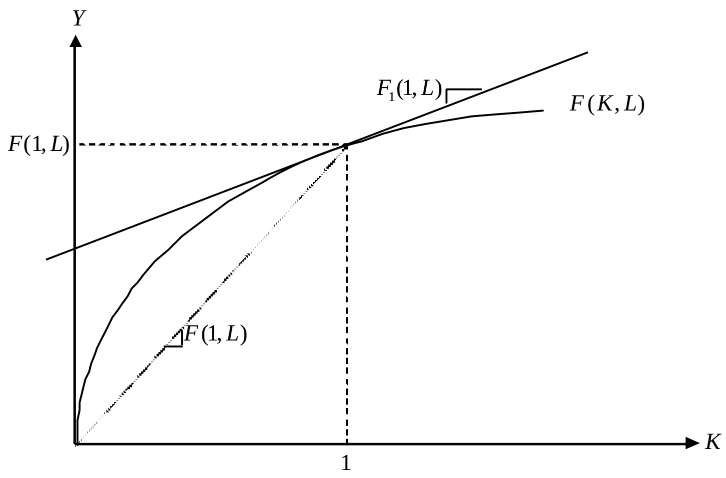

Figure 12.2: Illustration of the fact that for L given, $F(1, L) > F_1(1, L)$.

Solving the linear differential equation (12.23) gives

$$c_t = c_0 e^{\gamma t}, \tag{12.24}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by applying the households' transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

First, note that the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1, L)K - cL - \delta K,$$

or, in per-capita terms,

$$\dot{k} = [F(1, L) - \delta]\, k - c_0 e^{\gamma t}. \tag{12.25}$$

In this equation it is important that $F(1, L) - \delta - \gamma > 0$. To understand this inequality, note that, by (A2'), $F(1, L) - \delta - \gamma > F(1, L) - \delta - \bar{r} = F(1, L) - F_1(1, L) = F_2(1, L)L > 0$, where the first equality is due to $\bar{r} = F_1(1, L) - \delta$ and the second is due to the fact that since F is homogeneous of degree 1, we have, by Euler's theorem, $F(1, L) = F_1(1, L) \cdot 1 + F_2(1, L)L > F_1(1, L) > \delta$, in view of (A1'). The key property $F(1, L) - F_1(1, L) > 0$ is illustrated in Figure 12.2.

The solution of a general linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$, is

$$x(t) = (x(0) - \frac{c}{a+h})e^{-at} + \frac{c}{a+h}e^{ht}. \tag{12.26}$$

Thus the solution to (12.25) is

$$k_t = (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1,L) - \delta - \gamma}e^{\gamma t}. \tag{12.27}$$

To check whether (TVC) is satisfied we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L) - \delta - \gamma}e^{(\gamma-\bar{r})t} \\ &\to (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \to \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A2'). But $\bar{r} = F_1(1,L) - \delta < F(1,L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma)k_0. \tag{12.28}$$

If c_0 is less than this, there will be over-saving and (TVC) is violated ($a_t e^{-\bar{r}t} \to \infty$ for $t \to \infty$, since $a_t = k_t$). If c_0 is higher than this, both the NPG and (TVC) are violated ($a_t e^{-\bar{r}t} \to -\infty$ for $t \to \infty$).

Inserting the solution for c_0 into (12.27), we get

$$k_t = \frac{c_0}{F(1,L) - \delta - \gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional dynamics).

This is a case of *fully endogenous growth* in the sense that the long-run growth rate of c is positive without the support by growth in any exogenous factor. This outcome is due to the absence of diminishing returns to aggregate capital, which is implied by the assumed high value of the learning parameter. But the empirical foundation for this high value is weak, to say the least, cf. Chapter 13. A further drawback of this special version of the learning model is that the results are *non-robust*. With λ slightly less than 1, we are back in the Arrow case and growth peters out, since $n = 0$. With λ slightly above 1, it can be shown that growth becomes explosive: infinite output in finite time![8]

[8] See Appendix B in Chapter 13.

The Romer case, $\lambda = 1$, is thus a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which *apriori* can take any value within an interval. Second, the imposed value leads to non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system.

Note that the *causal structure* in the long run in the diminishing returns case is different than in the AK-case of Romer. In the diminishing returns case the steady-state growth rate is determined first, as g_c^* in (12.15), then r^* is determined through the Keynes-Ramsey rule and, finally, Y/K is determined by the technology, given r^*. In contrast, the Romer case has Y/K and r directly given as $F(1, L)$ and $\bar{r}$, respectively. In turn, $\bar{r}$ determines the (constant) equilibrium growth rate through the Keynes-Ramsey rule.

12.3.2 Economic policy in the Romer case

In the AK case, that is, the fully endogenous growth case, we have $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$. Thus, preference parameters *matter* for the long-run growth rate and not "only" for the *level* of the upward-sloping time path of per capita output. This suggests that taxes and subsidies can have *long-run* growth effects. In any case, in this model there is a motivation for government intervention due to the positive externality of private investment. This motivation is present whether $\lambda < 1$ or $\lambda = 1$. Here we concentrate on the latter case, for no better reason than that it is simpler. We first find the social planner's solution.

The social planner

Recall that by a *social planner* we mean a fictional "all-knowing and all-powerful" decision maker who maximizes an objective function under no other constraints than what follows from technology and initial resources. The social planner faces the aggregate production function (12.22) or, in per capita terms, $y_t = F(1, L)k_t$. The social planner's problem is to choose $(c_t)_{t=0}^{\infty}$ to maximize

$$
\begin{aligned}
U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\
c_t &\geq 0, \\
\dot{k}_t &= F(1, L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, && (12.29) \\
k_t &\geq 0 \text{ for all } t > 0. && (12.30)
\end{aligned}
$$

The current-value Hamiltonian is

$$H(k, c, \eta, t) = \frac{c^{1-\theta}}{1-\theta} + \eta \left(F(1, L)k - c - \delta k \right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \tag{12.31}$$

$$\frac{\partial H}{\partial k} = \eta(F(1, L) - \delta) = -\dot{\eta} + \rho\eta. \tag{12.32}$$

We guess that also the transversality condition,

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \tag{12.33}$$

must be satisfied by an optimal solution.[9] This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really an optimal solution.

Log-differentiating w.r.t. t in (12.31) and combining with (12.32) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1, L) - \delta - \rho) \equiv \gamma_{SP}. \tag{12.34}$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal product of capital is $\partial y_t/\partial k_t = F(1, L) > F_1(1, L)$. To ensure bounded intertemporal utility we sharpen (A2') to

$$\rho > (1 - \theta)\gamma_{SP}. \tag{A2"}$$

To find the time path of k_t, note that the dynamic resource constraint (12.29) can be written

$$\dot{k}_t = (F(1, L) - \delta)k_t - c_0 e^{\gamma_{SP} t},$$

in view of (12.34). By the general solution formula (12.26) this has the solution

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t}. \tag{12.35}$$

[9] The proviso implied by saying "guess" is due to the fact that optimal control theory does not guarantee that this "standard" transversality condition is necessary for optimality in *all* infinite horizon optimization problems.

In view of (12.32), in an interior optimal solution the time path of the adjoint variable η is

$$\eta_t = \eta_0 e^{-[(F(1,L)-\delta-\rho]t},$$

where $\eta_0 = c_0^{-\theta} > 0$, by (12.31). Thus, the conjectured transversality condition (12.33) implies

$$\lim_{t\to\infty} k_t e^{-(F(1,L)-\delta)t} = 0, \tag{12.36}$$

where we have eliminated η_0. To ensure that this is satisfied, we multiply k_t from (12.35) by $e^{-(F(1,L)-\delta)t}$ to get

$$\begin{aligned} k_t e^{-(F(1,L)-\delta)t} &= k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} + \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{[\gamma_{SP}-(F(1,L)-\delta)]t} \\ &\to k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} \text{ for } t \to \infty, \end{aligned}$$

since, by (A2"), $\gamma_{SP} < \rho + \theta\gamma_{SP} = F(1,L) - \delta$ in view of (12.34). Thus, (12.36) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma_{SP})k_0. \tag{12.37}$$

Inserting this solution for c_0 into (12.35), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{\gamma_{SP}t} = k_0 e^{\gamma_{SP}t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, our candidate for the social planner's solution is from start in balanced growth (there is no transitional dynamics).

The next step is to check whether our candidate solution satisfies a set of *sufficient* conditions for an optimal solution. Here we can use *Mangasarian's theorem* which, applied to a problem like this, with one control variable and one state variable, says that the following conditions are sufficient:

(a) Concavity: The Hamiltonian is jointly concave in the control and state variables, here c and k.

(b) Non-negativity: There is for all $t \geq 0$ a non-negativity constraint on the state variable; and the co-state variable, η, is non-negative for all $t \geq 0$.

(c) TVC: The candidate solution satisfies the transversality condition $\lim_{t\to\infty} k_t\eta_t e^{-\rho t} = 0$, where $\eta_t e^{-\rho t}$ is the discounted co-state variable.

In the present case we see that the Hamiltonian is a sum of concave functions and therefore is itself concave in (k, c). Further, from (12.30) we see that condition (b) is satisfied. Finally, our candidate solution is constructed so as to satisfy condition (c). The conclusion is that our candidate solution *is* an optimal solution. We call it the SP allocation.

Implementing the SP allocation in the market economy

Returning to the market economy, we assume there is a policy maker, say the government, with only two activities. These are (i) paying an investment subsidy, s, to the firms so that their capital costs are reduced to

$$(1-s)(r+\delta)$$

per unit of capital per time unit; (ii) financing this subsidy by a constant consumption tax rate τ.

Let us first find the size of s needed to establish the SP allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-s)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-s)(r+\delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-s)(r+\delta),$$

where $k \equiv K/L$, which is pre-determined from the supply side. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k, K)}{1-s} - \delta = \frac{F_1(1, L)}{1-s} - \delta, \tag{12.38}$$

again using Euler's theorem.

It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1, L) - \delta$. If we can obtain $r = F(1, L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required s thus satisfies

$$r = \frac{F_1(1, L)}{1-s} - \delta = F(1, L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1,L)}{F(1,L)} = \frac{F(1,L) - F_1(1,L)}{F(1,L)} = \frac{F_2(1,L)L}{F(1,L)}.$$

In case $Y_i = K_i^{\alpha}(AL_i)^{1-\alpha}$, $0 < \alpha < 1$, $i = 1, \ldots, N$, this gives $s = 1 - \alpha$.

It remains to find the required consumption tax rate τ. The tax revenue will be τcL, and if the government budget should be balanced at every instant,[10] the *required* tax revenue is

$$\mathcal{T} = s(r+\delta)K = (F(1,L) - F_1(1,L))\,K = \tau cL.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{\mathcal{T}}{cL} = \frac{F(1,L) - F_1(1,L)}{c/k} = \frac{F(1,L) - F_1(1,L)}{F(1,L) - \delta - \gamma_{SP}} > 0, \qquad (12.39)$$

where we have used that the proportionality in (12.37) between c and k holds for all $t \geq 0$. Substituting (12.34) into (12.39), the solution for τ can be written

$$\tau = \frac{\theta\,[F(1,L) - F_1(1,L)]}{(\theta-1)(F(1,L)-\delta)+\rho} = \frac{\theta F_2(1,L)L}{(\theta-1)(F(1,L)-\delta)+\rho}.$$

The required tax rate on consumption is thus a constant. It therefore does not distort the consumption/saving decision on the margin, cf. Chapter 11.

It follows that the allocation obtained by this subsidy-tax policy *is* the SP allocation. A policy, here the policy (s, τ), which in a decentralized system induces the SP allocation, is called a *first-best policy.*

12.4 Appendix

A. The golden-rule capital intensity in the Arrow case

In our discussion of the Arrow model in Section 12.2 (where $0 < \lambda < 1$), we claimed that the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that effective capital-labor ratio at which the social net marginal productivity of capital equals the steady-state growth rate of output. In this respect the Arrow model with endogenous technical progress is similar to the standard neoclassical growth model with exogenous technical progress.

The claim corresponds to a very general theorem, valid also for models with many capital goods and non-existence of an aggregate production function. This theorem says that the highest sustainable path for consumption

[10] We say "if" because in a growing economy there is scope for some persistent deficit financing without threatening fiscal sustainability.

per unit of labor in an economy will be that path which results from those techniques which profit maximizing firms choose under perfect competition when the real interest rate equals the steady-state growth rate of GNP (see Gale and Rockwell, 1975).

To prove our claim, note that in steady state, (12.14) holds whereby consumption per unit of labor (here the same as per capita consumption in view of L = labor force = population) can be written

$$\begin{aligned} c_t &\equiv \tilde{c}_t A_t = \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] K_t^{\lambda} \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \left(K_0 e^{\frac{n}{1-\lambda}t} \right)^{\lambda} \quad (\text{by } g_K^* = \frac{n}{1-\lambda}) \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \left((\tilde{k}L_0)^{\frac{1}{1-\lambda}} e^{\frac{n}{1-\lambda}t} \right)^{\lambda} \quad (\text{from } \tilde{k} = \frac{K_t}{K_t^{\lambda} L_t} = \frac{K_t^{1-\lambda}}{L_t} \text{ also for } t = 0) \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \tilde{k}^{\frac{\lambda}{1-\lambda}} L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t} \equiv \varphi(\tilde{k}) L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}, \end{aligned}$$

defining $\varphi(\tilde{k})$ in the obvious way.

We look for that value of $\tilde{k}$ at which this steady-state path for c_t is at the highest technically feasible level. The positive coefficient, $L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}$, is the only time-dependent factor and can be ignored since it is exogenous. The problem is thereby reduced to the static problem of maximizing $\varphi(\tilde{k})$ with respect to $\tilde{k} > 0$. We find

$$\begin{aligned} \varphi'(\tilde{k}) &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda}) \right] \tilde{k}^{\frac{\lambda}{1-\lambda}} + \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \frac{\lambda}{1-\lambda} \tilde{k}^{\frac{\lambda}{1-\lambda}-1} \\ &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda}) + \left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \frac{n}{1-\lambda}) \right) \frac{\lambda}{1-\lambda} \right] \tilde{k}^{\frac{\lambda}{1-\lambda}} \\ &= \left[(1-\lambda) f'(\tilde{k}) - (1-\lambda)\delta - n + \lambda \frac{f(\tilde{k})}{\tilde{k}} - \lambda(\delta + \frac{n}{1-\lambda}) \right] \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \\ &= \left[(1-\lambda) f'(\tilde{k}) - \delta + \lambda \frac{f(\tilde{k})}{\tilde{k}} - \frac{n}{1-\lambda} \right] \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \equiv \psi(\tilde{k}) \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda}, \quad (12.40) \end{aligned}$$

defining $\psi(\tilde{k})$ in the obvious way. The first-order condition for the problem, $\varphi'(\tilde{k}) = 0$, is equivalent to $\psi(\tilde{k}) = 0$. After ordering this gives

$$f'(\tilde{k}) + \lambda \frac{f(\tilde{k}) - \tilde{k} f'(\tilde{k})}{\tilde{k}} - \delta = \frac{n}{1-\lambda}. \quad (12.41)$$

We see that

$$\varphi'(\tilde{k}) \gtreqless 0 \quad \text{for} \quad \psi(\tilde{k}) \gtreqless 0,$$

respectively. Moreover,

$$\psi'(\tilde{k}) = (1-\lambda)f''(\tilde{k}) - \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}^2} < 0,$$

in view of $f'' < 0$ and $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$. So a $\tilde{k} > 0$ satisfying $\psi(\tilde{k}) = 0$ is the unique maximizer of $\varphi(\tilde{k})$. By (A1) and (A3) in Section 12.2 such a $\tilde{k}$ exists and is thereby the same as the $\tilde{k}_{GR}$ we were looking for.

The left-hand side of (12.41) equals the social marginal productivity of capital and the right-hand side equals the steady-state growth rate of output. At $\tilde{k} = \tilde{k}_{GR}$ it therefore holds that

$$\frac{\partial Y}{\partial K} - \delta = \left(\frac{\dot{Y}}{Y}\right)^{*}.$$

This confirms our claim in Section 12.2 about $\tilde{k}_{GR}$.

Remark about the absence of a golden rule in the Romer model. In the Romer model, which has constant L, the golden rule is not a well-defined concept for the following reason. Along any balanced growth path we have from (12.29),

$$g_k \equiv \frac{\dot{k}_t}{k_t} = F(1,L) - \delta - \frac{c_t}{k_t} = F(1,L) - \delta - \frac{c_0}{k_0},$$

because g_k $(= g_K)$ is by definition constant along a balanced growth path, whereby also c_t/k_t must be constant. We see that g_k is decreasing linearly from $F(1,L) - \delta$ to $-\delta$ when c_0/k_0 rises from nil to $F(1,L)$. So choosing among alternative technically feasible balanced growth paths is inevitably a choice between starting with low consumption to get high growth forever or starting with high consumption to get low growth forever. Given any $k_0 > 0$, the alternative possible balanced growth paths will therefore sooner or later cross each other in the $(t, \ln c)$ plane. Hence, there exists no balanced growth path which for all $t \geq 0$ has c_t higher than along any other technically feasible balanced growth path. So no golden rule path exists. This is a general property of AK and reduced-form AK models.

B. Saddle-point stability

This appendix is a refresher on the concept of saddle-point stability, a concept which perplexes many people.

Consider a *two-dimensional* dynamic system (two coupled first-order differential equations). Suppose the system has a steady state which is a *saddle point* (which is the case if and only if the two eigenvalues of the associated Jacobian matrix, evaluated at the steady state, are of opposite sign). Then, so far, either presence or absence of saddle-point stability is possible. And which of the two cases occur can not be diagnosed from the two differential equations in isolation. One has to consider the boundary conditions. Here is a complete definition of (local) saddle-point stability.

DEFINITION. A steady state of a two-dimensional dynamic system is (locally) *saddle-point stable* if:

1. the steady state is a saddle point;
2. one of the two endogenous variables is predetermined while the other is a jump variable;
3. the saddle path is not parallel to the jump variable axis; and
4. there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

Thus, to establish saddle-point stability, all four properties must be verified. If for instance point 1 and 2 hold, but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \to \infty$.

To say that the steady state is saddle-point stable is synonymous with saying that the *dynamic system* is saddle-point stable.

For an *n-dimensional* dynamic system (n coupled first-order differential equations, $n \geq 2$) the concepts of a saddle point and saddle-point stability are defined via a generalization of point 1 to 4. As to point 1: A steady state of an n-dimensional dynamic system is called a *saddle point* if all eigenvalues of the associated Jacobian matrix have non-zero real parts and at least two of the eigenvalues, have real parts of opposite sign. As to point 2: The number of predetermined variables in the system equals the number of eigenvalues with negative real part (and the number of jump variables in the system consequently equals the number of eigenvalues with positive real part). The generalization of points 3 and 4 is more nerdy, so we refer to, for instance, Hirsch and Smale, *Differential equations, dynamic systems, and linear algebra*, Academic Press, 1974.

Chapter 13

Perspectives on learning by doing and learning by investing

By adding some theoretical and empirical perspectives to learning-by-doing and learning-by-investing models of endogenous growth, this chapter is a follow-up on Chapter 12. The contents are:

1. Learning by doing, learning by using, learning by watching
2. Empirics on learning by investing
3. Disembodied vs. embodied technical change*
4. Static comparative advantage vs. dynamics of learning by doing*

The growth rate of any time-dependent variable $z > 0$ is written $g_z \equiv \dot{z}/z$. In this chapter the economy-wide technology level at time t is denoted T_t rather than A_t.

13.1 Learning by doing, learning by using, learning by watching

The term *learning by doing* refers to the hypothesis that accumulated work experience, including repetition of the same type of action, improves workers' productivity and adds to technical knowledge. In connection with training in applying new production equipment, sometimes the related term *learning by using* is appropriate. In a broader context, the literature sometimes refers to spillover effects as *learning by watching.*

A specific form of learning by doing is called learning by investing and is treated separately in Section 13. 2 and 13.3.

A learning-by-doing model typically combines an aggregate CRS production function,

$$Y_t = F(K_t, T_t L_t), \tag{13.1}$$

with a learning function, for example,

$$\dot{T}_t = BY_t^{\lambda}, \qquad B > 0, 0 < \lambda \leq 1, \tag{13.2}$$

where λ is a learning parameter and B is a constant that, depending on the value of λ and the complete model in which (13.2) is embedded, is either an unimportant constant that depends only on measuring units or a parameter of importance for the productivity level or even the productivity growth rate. In Section 13.4 below, on the resource curse problem, we consider a two-sector model where each sector's productivity growth is governed by such a relationship.[1]

Another learning hypothesis is of the form

$$\dot{T}_t = BT_t^{\lambda} L_t^{\mu}, \qquad T_0 > 0 \text{ given}, B > 0, \lambda \leq 1, \mu > 0. \tag{13.3}$$

Here both λ and μ are learning parameters, reflecting the elasticities of learning w.r.t. the technology level and labor hours, respectively. The higher the number of human beings involved in production and the more time they spend in production, the more experience is accumulated. Sub-optimal ingredients in the production processes are identified and eliminated. The experience and knowledge arising in one firm or one sector is speedily diffused to other firms and other sectors in the economy (knowledge spillovers or *learning by watching*), and as a result the aggregate productivity level is increased.[2]

Since hours spent, L_t, is perhaps a better indicator for "new experience" than output, Y_t, specification (13.3) may seem more appealing than specification (13.2). So this section concentrates on (13.3).

If the labor force is growing, λ should be assumed strictly less than one, because with $\lambda = 1$ there would be a built-in tendency to forever faster growth, which does not seem plausible. In fact, $\lambda < 0$ can not be ruled out; that would reflect that learning becomes more and more difficult ("the easiest ideas are found first"). On the other hand, the case of "standing on

[1] In his Chapter 20, Section 20.4, on industrialization and structural change Acemoglu considers a model with two sectors, an agrarian and a manufacturing sector, where in the latter learning by doing in the form (13.2) with $\lambda = 1$ plays an important role.

[2] Diffusion of proficiency also occurs via apprentice-master relationships.

the shoulders" is also possible, that is, the case $0 < \lambda \leq 1$, which is the case where new learning becomes easier, the more is learnt already.

In "very-long-run" growth theory concerned with human development in an economic history perspective, the L in (13.3) has been replaced simply by the size of population in the relevant region (which may be considerably larger than a single country). This is the "population breeds ideas" view, cf. Kremer (1993). Anyway, many simple models consider the labor force to be proportional to population size, and then it does not matter whether we use the learning-by-doing interpretation or the population-breeds-ideas interpretation.

The so-called Horndal effect (reported by Lundberg, 1961) was one of the empirical observations motivating the learning-by-doing idea in growth theory:

> "The Horndal-iron works in Sweden had no new investment (and therefore presumably no significant change in its methods of production) for a period of 15 years, yet productivity (output per man-hour) rose on the average close to 2 % per annum. We find again steadily increasing performance which can only be imputed to learning from experience" (here cited after Arrow, 1962).

Similar patterns of on-the-job productivity improvements have been observed in ship-building, airframe construction, and chemical industries. On the other hand, within a single production line there seems to be a tendency for this kind of productivity increases to gradually peter out, which suggests $\lambda < 0$ in (13.3). We may call this phenomenon "diminishing returns in the learning process": the potential for new learning gradually evens out as more and more learning has already taken place. But new products are continuously invented and the accumulated knowledge is transmitted, more or less, to the production of these new products that start on a "new learning curve", along which there is initially "a large amount to be learned".[3] This combination of qualitative innovation and continuous productivity improvement through learning *may* at the aggregate level end up in a $\lambda \geq 0$ in (13.3).

In any case, whatever the sign of λ at the aggregate level, with $\lambda < 1$, this model is capable of generating sustained endogenous per capita growth (without "growth explosion") if the labor force is growing at a rate $n > 0$. Indeed, as in Chapter 12, there are two cases that are consistent with a balanced growth path (BGP for short) with positive per capita growth,

[3] A *learning curve* is a graph of estimated productivity (or its inverse, cf. Fig. 13.1 or Fig. 13.2 below) as a function of cumulative output or of time passed since production of the new product began at some plant.

namely the case $\lambda < 1$ combined with $n > 0$, and the case $\lambda = 1$ combined with $n = 0$.

We will show this for a closed economy with $L_t = L_0 e^{nt}$, $n \geq 0$, and with capital accumulation according to

$$\dot{K}_t = I_t - \delta K_t = Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ given.} \tag{13.4}$$

13.1.1 The case: $\lambda < 1$ in (13.3)

Let us first consider the growth rate of $y \equiv Y/L$ along a BGP. There are two steps in the calculation of this growth rate.

Step 1. Given (13.4), from basic balanced growth theory (Chapter 4) we know that along a BGP with positive gross saving, not only are, by definition, g_Y and g_K constant, but they are also the same, so that Y_t/K_t is constant over time. Owing to the CRS assumption, (13.1) implies that

$$1 = F(\frac{K_t}{Y_t}, \frac{T_t L_t}{Y_t}). \tag{13.5}$$

When Y_t/K_t is constant, $T_t L_t / Y_t \equiv T_t / y_t$ must be constant, whereby

$$g_T = g_y = g_Y - n, \tag{13.6}$$

a constant.

Step 2. Dividing through by T_t in (13.3), we get

$$g_T \equiv \frac{\dot{T}_t}{T_t} = A T_t^{\lambda-1} L_t^{\mu}.$$

Taking logs gives $\log g_T = \log A + (\lambda - 1)\log T + \mu \log L$. And taking the time derivative on both sides of this equation leads to

$$\frac{\dot{g}_T}{g_T} = (\lambda - 1) g_T + \mu n. \tag{13.7}$$

In view of g_T being constant along a BGP, we have $\dot{g}_T = 0$, and so (13.7) gives

$$g_T = \frac{\mu n}{1 - \lambda},$$

presupposing $\lambda < 1$. Hence, by (13.6),

$$g_y = \frac{\mu n}{1 - \lambda}.$$

Under the assumption that $n > 0$, this per capita growth rate is positive, whatever the sign of λ. Given n, the growth rate is an increasing function of *both* learning parameters. Since a positive per capita growth rate can in the long run be maintained only if supported by $n > 0$, this is an example of *semi-endogenous exponential growth* (as long as n is exogenous).

This model thus gives growth results somewhat similar to the results in Arrow's learning-by-investing model, cf. Chapter 12. In both models the learning is an unintended by-product of the work process and construction of investment goods, respectively. And both models assume that knowledge is non-appropriable (non-exclusive) and that knowledge spillovers across firms are fast (in the time perspective of growth theory). So there are positive externalities which may motivate government intervention.

Methodological remark: Different approaches to the calculation of long-run growth rates Within this semi-endogenous growth case, depending on the situation, different approaches to the calculation of long-run growth rates may be available. In Chapter 12, in the analysis of the Arrow case $\lambda < 1$, the point of departure in the calculation was the steady state property of Arrow's model that $\tilde{k} \equiv K/(TL)$ is a constant. But this point of departure presupposes that we have established a well-defined steady state in the sense of a stationary point of a complete dynamic system (which in the Arrow model consists of two first-order differential equations in $\tilde{k}$ and $\tilde{c}$, respectively), usually involving also a description of the household sector.

In the present case we are not in this situation because we have not specified how the saving in (13.4) is determined. This explains why above (as well as in Chapter 10) we have taken another approach to the calculation of the long-run growth rate. We simply assume balanced growth and ask what the growth rate must then be. If the technologies in the economy are such that per capita growth in the long run can only be due to either exogenous productivity growth or semi-endogenous productivity growth, this approach is usually sufficient to determine a unique growth rate.

Note also, however, that this latter feature is in itself an interesting and useful result (as exemplified in Chapter 10). It tells us what the growth rate *must* be in the long run provided that the system converges to balanced growth. The growth rate will be the same, independently of the market structure and the specification of the household sector, that is, it will be the same whether, for example, there is a Ramsey-style household sector or an overlapping generations set-up.[4] And at least in the first case the growth

[4] Specification of these things is needed if we want to study the transitional dynamics: the adjustment processes outside balanced growth/steady state, including the question of

rate will be the same whatever the size of the preference parameters (the rate of time preference and the elasticity of marginal utility of consumption). Moreover, only if economic policy affects the learning parameters or the population growth rate (two things that are often ruled out inherently by the setup), will the long-run growth rate be affected. Still, economic policy can *temporarily* affect economic growth and in this way affect the *level* of the long-run growth path.

13.1.2 The case $\lambda = 1$ in (13.3)

With $\lambda = 1$ in (13.3), the above growth rate formulas are no longer valid. But returning to (13.3), we have $g_T = BL_t^{\mu}$. Then, unless $n = 0$, the growth rate of y will tend to rise forever, since we have $g_T = BL_0^{\mu}e^{\mu nt} \rightarrow \infty$ for $n > 0$.

So we will assume $n = 0$. Then $L_t = L_0$ for all t, implying $g_T = BL_0^{\mu}$ for all t. Since both B and L_0 are exogenous, it is *as if* the rate of technical progress, g_T, were exogenous. Yet, technical progress is generated by an internal mechanism. If the government by economic policy could affect B or L_0, also g_T would be affected. In any case, under balanced growth, (13.5) holds again and so $T_tL_t/Y_t = T_t/y_t$ must be constant. This implies $g_y = g_T = BL_0^{\mu} > 0$. Consequently, positive per capita growth can be maintained forever without support of growth in any exogenous factor. So we consider *fully endogenous exponential growth*.

As in the semi-endogenous growth case we can here determine the growth rate along a BGP independently of how the household sector is described. And preference parameters do *not* affect the growth rate. The fact that this is so even in the fully-endogenous growth case is due to the "law of motion" of technology making up a subsystem that is independent of the remainder of the economic system. This is a special feature of the "growth engine" (13.3). Although it is not a typical ingredient of endogenous growth models, this growth engine can not be ruled out *apriori*. The simple alternative, (13.2), is very different in that the endogenous aggregate output, Y_t, is involved. We return to (13.2) in Section 13.4 below.

Before proceeding, a brief remark on the explosive case $\lambda > 1$ in (13.2) or (13.3) is in place. If we imagine $\lambda > 1$, growth becomes explosive in the extreme sense that output as well as productivity, hence also per capita consumption, will tend to *infinity in finite time.* This is so even if $n = 0$. The argument is based on the mathematical fact that, given a differential equation $\dot{x} = x^a$, where $a > 1$ and $x_0 > 0$, the solution x_t has the property

convergence to balanced growth/steady state.

that there exists a $t_1 > 0$ such that $x_t \to \infty$ for $t \to t_1$. For details, see Appendix B.

13.2 Disembodied learning by investing

In the above framework the work process is a source of learning whether it takes place in the consumption or capital goods sector. This is *learning by doing* in a broad sense. If the source of learning is specifically associated with the construction of capital goods, the learning by doing is often said to be of the form of *learning by investing.* Why in the headline of this section we have added the qualification "disembodied", will be made clear in Section 13.3. Another name for learning by investing is *investment-specific learning by doing.*

The prevalent view in the empirical literature seems to be that learning by investing is the most important form of learning by doing; ship-building and airframe construction are prominent examples. To the extent that the construction of capital equipment is based on more complex and involved technologies than is the production of consumer goods, we are also, intuitively, inclined to expect that the greatest potential for productivity increases through learning is in the investment goods sector.[5]

In the simplest version of the learning-by-investment hypothesis, (13.3) above is replaced by

$$T_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{13.8}$$

where I_s^n is aggregate *net* investment. This is the hypothesis that the economy-wide technology level T_t is an increasing function of society's previous experience, proxied by cumulative aggregate net investment.[6] The Arrow and Romer models, as described in Chapter 12, correspond to the cases $0 < \lambda < 1$ and $\lambda = 1$, respectively.

In this framework, where the "growth engine" depends on capital accumulation, it is only in the Arrow case that we can calculate the per-capita growth rate along a BGP without specifying anything about the household sector.

[5] After the information-and-communication technology (ICT) revolution, where a lot of technically advanced consumer goods have entered the scene, this traditional presumption may be less compelling.

[6] Contrary to the dynamic learning-by-doing specification (13.3), there is here no good reason for allowing $\lambda < 0$.

13.2.1 The Arrow case: $\lambda < 1$ and $n \geq 0$

We may apply the same two steps as in Section 13.1.1. Step 1 is then an exact replication of step 1 above. Step 2 turns out to be even simpler than above, because (13.8) immediately gives $\log T = \lambda \log K$ so that $g_T = \lambda g_K$, which substituted into (13.6) yields

$$g_T = \lambda g_K = g_y = g_Y - n = g_K - n.$$

From this follows, first,

$$g_K = \frac{n}{1-\lambda}, \tag{13.9}$$

and, second,

$$g_y = \frac{\lambda n}{1-\lambda}.$$

Alternatively, we may in this case condense the two steps into one by rewriting (13.5) in the form

$$\frac{Y_t}{K_t} = F(1, \frac{T_t L_t}{K_t}) = F(1, K_t^{\lambda-1} L_t),$$

by (13.8). Along the BGP, since Y/K is constant, so must the second argument, $K_t^{\lambda-1} L_t$, be. It follows that

$$(\lambda - 1)g_K + n = 0,$$

thus confirming (13.9).

Whatever the approach to the calculation, the per capita growth rate is here tied down by the size of the learning parameter and the growth rate of the labor force.

13.2.2 The Romer case: $\lambda = 1$ and $n = 0$

In the Romer case, however, the growth rate along a BGP cannot be determined until the saving behavior in the economy is modeled. Indeed, the knife-edge case $\lambda = 1$ opens up for many different per capita growth rates under balanced growth. Which one is "selected" by the economy depends on how the household sector is described.

For a Ramsey setup with $n = 0$ the last part of Chapter 12 showed how the growth rate generated by the economy depends on the rate of time preference and the elasticity of marginal utility of consumption of the representative household. Growth is here *fully-endogenous* in the sense that a positive per capita growth rate can be maintained forever without the support by growth in any exogenous factor. Moreover, according to this model, economic policy that internalizes the positive externality in the system can raise not only the productivity level, but also the long-run productivity growth rate.

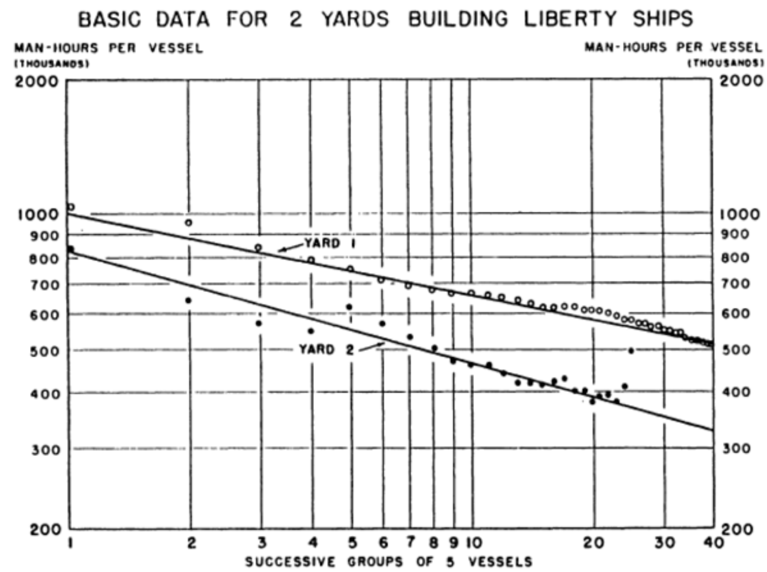

FIGURE 1.—Reductions in man-hours per vessel with increasing production. Merchant shipyards.

Figure 13.1: Man-hours per vessel against cumulative number of vessels completed to date in shipyard 1 and shipyard 2, respectively. Log-log paper. Source: Searle (1945).

13.2.3 The size of the learning parameter

What is from an empirical point of view a plausible value for the learning parameter, λ? This question is important because quite different results emerge depending on whether λ is close to 1 or considerably lower (fully-endogenous growth versus semi-endogenous growth). At the same time the question is not easy to answer because λ in the models is a parameter that is meant to reflect the aggregate effect of the learning going on in single firms and spreading across firms and industries.

Like Lucas (1993), we will consider the empirical studies of on-the-job productivity increases in ship-building by Searle (1945) and Rapping (1965). Both studies used data on the production of different types of cargo vessels during the second world war. Figures 1 and 2 are taken from Lucas' review article, Lucas (1993), but the original source is Searle (1945). For the vessel type called "Liberty Ships" Lucas cites the observation by Searle (1945):

> "the reduction in man-hours per ship with each doubling of cumulative output ranged from 12 to 24 percent."

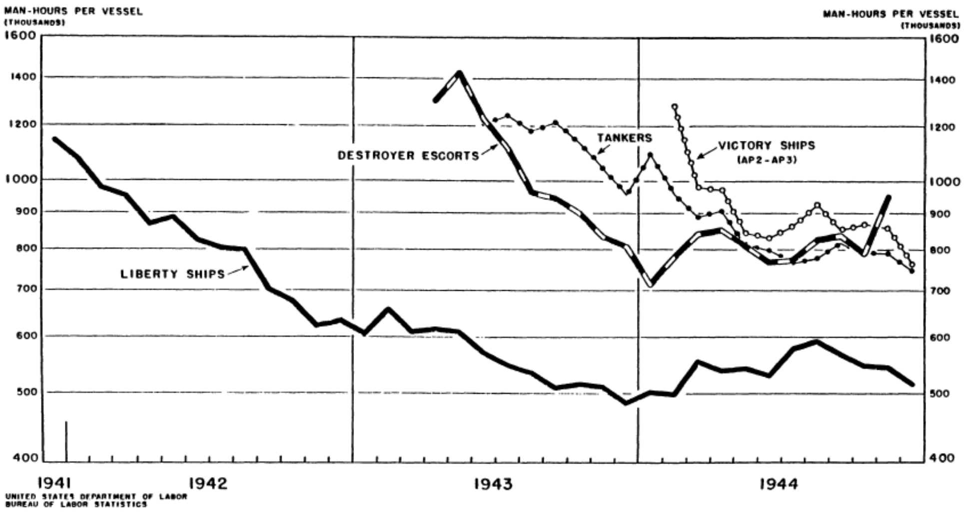

FIGURE 2.—Unit man-hour requirements for selected shipbuilding programs. Vessels delivered December 1941–December 1944.

Figure 13.2: Average man-hours (over ten shipyards) per vessel against calendar time. Four different vessel types. Source: Searle (1945).

Let us try to connect this observation to the learning parameter λ in Arrow's and Romer's framework. We begin by considering firm i which operates in the investment goods sector. We imagine that firm i's equipment is unchanged during the observation period (as is understood in the above citation as well as the citation from Arrow (1962) in Section 13.1). Let firm i's current output and employment be Y_{it} and L_{it}, respectively. The current labor productivity is then $a_{it} = Y_{it}/L_{it}$. Let the firm's *cumulative* output be denoted Q_{it}. This cumulative output is a part of cumulative investment in society. At the micro-level the learning-by-investing hypothesis is the hypothesis that labor productivity is an increasing function of the firm's cumulative output, Q_{it}.

In figures 1 and 2 the dependent variable is not directly labor productivity, but its inverse, namely the required man-hours per unit of output, $m_{it} = L_{it}/Y_{it} = 1/a_{it}$. Figure 13.1 suggests a log-linear relationship between this variable and the cumulative output:

$$\log m_{it} = \alpha - \beta \log Q_{it}. \tag{13.10}$$

That is, as cumulative output rises, the required man-hours per unit of output declines over time in this way:

$$m_{it} = \frac{e^{\alpha}}{Q_{it}^{\beta}}.$$

Equivalently, labor productivity rises over time in this way:

$$a_{it} = \frac{1}{m_{it}} = e^{-\alpha} Q_{it}^{\beta}.$$

So, specifying the relationship by a power function, as in (13.8), makes sense.

Now, let $t = t_1$ be a fixed point in time. Then, (13.10) becomes

$$\log m_{it_1} = \alpha - \beta \log Q_{it_1}.$$

Let t_2 be the later point in time where cumulative output has been doubled. Then at time t_2 the required man-hours per unit of output has declined to

$$\log m_{it_2} = \alpha - \beta \log Q_{it_2} = \alpha - \beta \log(2Q_{it_1}).$$

Hence,

$$\log m_{it_1} - \log m_{it_2} = -\beta \log Q_{it_1} + \beta \log(2Q_{it_1}) = \beta \log 2. \tag{13.11}$$

Lucas' citation above from Searle amounts to a claim that

$$0.12 < \frac{m_{it_1} - m_{it_2}}{m_{it_1}} < 0.24. \tag{13.12}$$

By a first-order Taylor approximation we have $\log m_{it_2} \approx \log m_{it_1} + (m_{it_2} - m_{it_1})/m_{it_1}$. Hence, $(m_{it_1} - m_{it_2})/m_{it_1} \approx \log m_{it_1} - \log m_{it_2}$. Substituting this into (13.12) gives, approximately,

$$0.12 < \log m_{it_1} - \log m_{it_2} < 0.24.$$

Combining this with (13.11) gives $0.12 < \beta \log 2 < 0.24$ so that

$$0.17 = \frac{0.12}{\log 2} < \beta < \frac{0.24}{\log 2} = 0.35.$$

Rapping (1965) finds by a more rigorous econometric approach β to be in the vicinity of 0.26 (still ship building). Arrow (1962) and Solow (1997) refer to data on airframe building. This data roughly suggests $\beta = 1/3$.

How can this be translated into a guess about the size of the "aggregate" learning parameter λ in (13.8)? This is a complicated question and the subsequent remarks are very tentative. First of all, the potential for both internal and external learning seems to vary a lot across different industries. Second, the amount of spillovers can not simply be added to the β above, since they are already partly included in the estimate of β. Even theoretically, the role of experience in different industries cannot simply be added up because to some extent there is redundancy due to *overlapping* experience and sometimes the learning in other industries is of limited relevance. Given that we are interested in an upper bound for λ, a "guestimate" is that the spillovers matter for the final λ at most the same as β from ship building so that $\lambda \leq 2\beta$.[7]

On the basis of these casual considerations we claim that a λ higher than about 2/3 may be considered fairly implausible. This speaks for the Arrow case of semi-endogenous exponential growth rather than the Romer case of fully-endogenous exponential growth, at least as long as we think of learning by investing as the sole source of productivity growth. Another point is that to the extent learning is internal and at least temporarily appropriable, we should expect at least some firms to internalize the phenomenon in its optimizing behavior (Thornton and Thompson, 2001). Although the learning is far from fully excludable, it takes time for others to discover and imitate technical and organizational improvements. Many simple growth models ignore this and treat all learning by doing and learning by investing as a 100 percent externality, which seems an exaggeration.

A further issue is to what extent learning by investing takes the form of *disembodied* versus *embodied* technical change. This is the topic of the next section.

13.3 Disembodied vs. embodied technical change

Arrow's and Romer's models build on the idea that the *source* of learning is primarily experience in the investment goods sector. Both models assume that the learning, via knowledge spillovers across firms, provides an engine of productivity growth in essentially *all* sectors of the economy. And both models (Arrow's, however, only in its simplified version, which we considered

[7] For more elaborate studies of empirical aspects of learning by doing and learning by investing, see Irwin and Klenow (1994), Jovanovic and Nyarko (1995), and Greenwood and Jovanovic (2001). Caballero and Lyons (1992) find clear evidence of positive externalities across US manufacturing industries. Studies finding that the quantitative importance of spillovers is significantly smaller than required by the Romer case include Englander and Mittelstadt (1988) and Benhabib and Jovanovic (1991). See also the surveys by Syverson (2011) and Thompson (2012).

Although in this lecture note we focus on learning as an externality, there exists studies focusing on *internal* learning by doing, see, e.g., Gunn and Johri, 2011.

in Chapter 12, not in its original version) assume that a firm can benefit from recent technical advances irrespective of whether it buys new equipment or just uses old equipment. That is, the models assume that technical change is *disembodied*.

13.3.1 Disembodied technical change

Disembodied technical change occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again (13.1) and (13.3). When the K_t appearing in (13.1) refers to the total, historically accumulated capital stock, then the interpretation is that the higher technology level generated in (13.3) or (13.8) results in higher productivity of *all* labor, independently of the vintage of the capital equipment with which this labor is combined. Thus also firms with old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.

Examples of this kind of productivity increases include improvement in management and work practices/organization and improvement in accounting.

13.3.2 Embodied technical change

In contrast, we say that technical change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technical progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.[8]

One way to formally represent embodied technical progress is to write

[8]The concept of embodied technical change was introduced by Johansen (1959) and Solow (1960). The notion of Solow-neutral technical change is related to embodied technical change and capital of different vintages.

capital accumulation in the following way,

$$\dot{K}_t = q_t I_t - \delta K_t, \tag{13.13}$$

where I_t is gross investment at time t and q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising q_t so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in efficiency units. Even if technical change does not directly appear in the production function, that is, even if for instance (13.1) is replaced by $Y_t = F(K_t, L_t)$, the economy may in this manner still experience a rising standard of living.

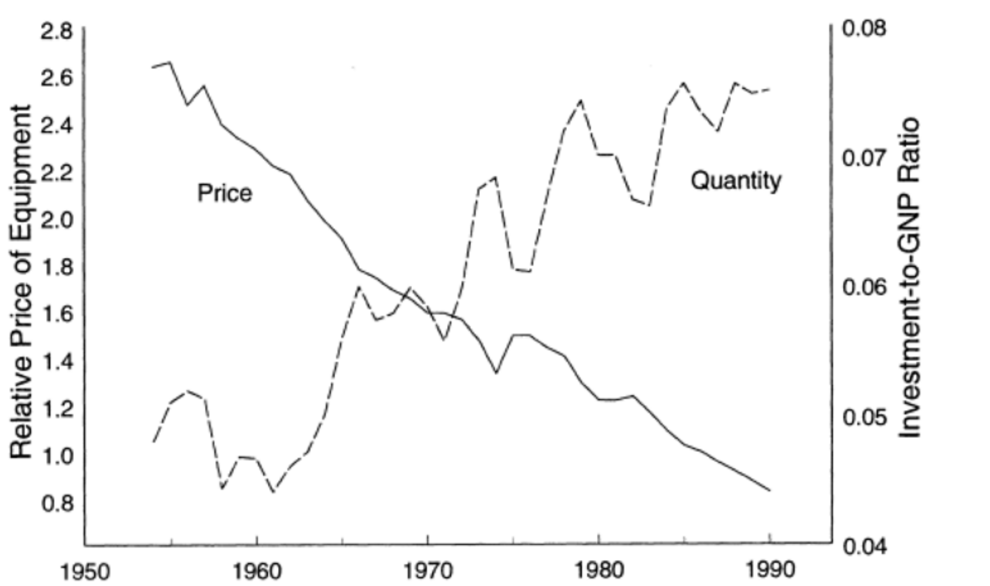

Figure 13.3: Relative price of equipment and quality-adjusted equipment investment-to-GNP ratio. Source: Greenwood, Hercowitz, and Krusell (1997).

Embodied technical progress is likely to result in a steady decline in the price of capital equipment relative to the price of consumption goods. This prediction is confirmed by the data. Greenwood et al. (1997) find for the U.S. that the relative price, p, of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990, cf. the "Price" curve in Figure 13.3.[9] As the "Quantity" curve in Figure 13.3 shows, over

[9] The relative price index in Fig. 13.3 is based on the book by R. Gordon (1990), which is an attempt to correct previous price indices for equipment by better taking into account quality improvements in new equipment.

the same period there has been a secular rise in the ratio of new equipment investment (in efficiency units) to GNP; note that what in the figure is called the "investment-to-GNP Ratio" is really "quality-adjusted investment-to-GNP Ratio", qI/GNP, not the usual investment-income ratio, I/GNP.

Moreover, the correlation between de-trended p and de-trended qI/GNP is -0.46. Greenwood et al. interpret this as evidence that technical advances have made equipment less expensive, triggering increases in the accumulation of equipment both in the short and the long run. The authors also estimate that embodied technical change explains 60% of the growth in output per man hour.

13.3.3 Embodied technical change and learning by investing

Whether technological progress is disembodied or embodied says nothing about whether its *source* is exogenous or endogenous. Indeed, the increases of q in (13.13) may be modeled as exogenous or endogenous. In the latter case, a popular hypothesis is that the source is learning by investing. This learning may take the form (13.8) above. In that case the experience that matter for learning is cumulative *net* investment.

An alternative hypothesis is:

$$q_t = \left(\int_{-\infty}^{t} I_s ds \right)^{\lambda}, \qquad 0 < \lambda \leq \bar{\lambda}, \tag{13.14}$$

where I_s is *gross* investment at time s. Here the experience that matter has its basis in cumulative *gross* investment. An upper bound, $\bar{\lambda}$, for the learning parameter is introduced to avoid explosive growth. The hypothesis (13.14) seems closer to both intuition and the original ideas of Arrow:

> "Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).

Contrary to the integral based on net investment in (13.8), the integral in the learning hypothesis (13.14) does not allow an immediate translation into an expression in terms of the accumulated capital stock. Instead a new state variable, cumulative gross investment, enter the system and opens up for richer dynamics.

We may combine (13.14) with an aggregate Cobb-Douglas production function,

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}. \tag{13.15}$$

Then the upper bound for the learning parameter in (13.14) is $\bar{\lambda} = (1-\alpha)/\alpha$.[10]

The case $\lambda < (1-\alpha)/\alpha$

Suppose $\lambda < (1-\alpha)/\alpha$. Using (13.14) together with (13.13), (13.15), and $I = Y - C$, one finds, under balanced growth with $s = I/Y$ constant and $0 < s < 1$,

$$\begin{aligned} g_K &= \frac{(1-\alpha)(1+\lambda)n}{1-\alpha(1+\lambda)}, & (13.16)\\ g_q &= \frac{\lambda}{1+\lambda} g_K, & (13.17)\\ g_Y &= \frac{1}{1+\lambda} g_K, & (13.18)\\ g_c &= g_y = g_Y - n = \frac{\alpha\lambda n}{1-\alpha(1+\lambda)}, & (13.19) \end{aligned}$$

cf. Appendix A. We see that $g_y > 0$ if and only if $n > 0$. So exponential growth is here semi-endogenous.

Let us assume there is perfect competition in all markets. Since q capital goods can be produced at the same minimum cost as one consumption good, the equilibrium price, p, of capital goods in terms of the consumption good must equal the inverse of q, that is, $p = 1/q$. With the consumption good being the numeraire, let the rental rate in the market for capital services be denoted R and the real interest rate in the market for loans be denoted r. Ignoring uncertainty, we have the no-arbitrage condition

$$\frac{R_t - (\delta p_t - \dot{p}_t)}{p_t} = r_t, \tag{13.20}$$

[10] An alternative to the specification of embodied learning by gross investment in (13.14) is

$$q_t = \left(\int_{-\infty}^{t} q_s I_s ds \right)^{\tilde{\lambda}}, \qquad 0 < \tilde{\lambda} \leq \bar{\tilde{\lambda}},$$

implying that it is cumulative quality-adjusted gross investment that matters, cf. Greenwood and Jovanovic (2001). If combined with the production function (13.15) the appropriate upper bound on the learning parameter, $\tilde{\lambda}$, is $\bar{\tilde{\lambda}} = 1 - a$.

where $\delta p_t - \dot{p}_t$ is the true economic depreciation of the capital good per time unit. Since $p = 1/q$, (13.17) and (13.16) indicate that along a BGP the relative price of capital goods will be declining according to

$$g_p = -\frac{(1-\alpha)\lambda n}{1-\alpha(1+\lambda)} < 0.$$

Note that $g_K > g_Y$ along the BGP. Is this a violation of Proposition 1 of Chapter 4? No, that proposition presupposes that capital accumulation occurs according to the standard equation (13.4), not (13.13). And although g_K differs from g_Y, the output-capital ratio in *value* terms, $Y/(pK)$, is constant along the BGP. In fact, the BGP complies entirely with Kaldor's stylized facts if we interpret "capital" as the value of capital, pK.

The formulas (13.16) and (13.19) display that $\alpha(1+\lambda) < 1$ is needed to avoid a forever rising growth rate if $n > 0$. This inequality is equivalent to $\lambda < (1-\alpha)/\alpha$ and confirms that the upper bound, $\bar{\lambda}$, in (13.14) equals $(1-\alpha)/\alpha$. With $\alpha = 1/3$, this upper bound is 2. The bound is thus no longer 1 as in the simple learning-by-investing model of Section 13.2. The reason is twofold, namely partly that now q is formed via cumulative gross investment instead of net investment, partly that the role of q is to strengthen capital formation rather than the efficiency of production factors in aggregate final goods produce.

When $n = 0$, the system can no longer generate a constant positive per capita growth rate (exponential growth). Groth et al. (2010) show, however, that the system is capable of generating *quasi-arithmetic growth.* This class of growth processes, which fill the whole range between exponential growth and complete stagnation, was briefly commented on in Section 10.5 of Chapter 10.

The case $\lambda = (1-\alpha)/\alpha$ and $n = 0$*

When $\lambda = (1-\alpha)/\alpha$, we have $\alpha(1+\lambda) = 1$ and so the growth formulas (13.16) and (13.19) no longer hold. But the way that (13.17) and (13.18) are derived (see Appendix A) ensures that these two equations remain valid along a BGP. Given $\lambda = (1-\alpha)/\alpha$, (13.17) can be written $g_q = (1-\alpha)g_K$, which is equivalent to

$$q_t = BK_t^{1-\alpha}$$

along a BGP (B is some positive constant to be determined).

To see whether a BGP exists, note that (13.14) implies

$$g_q = \frac{\dot{q}_t}{q_t} = \lambda q_t^{-1/\lambda} I_t = \lambda q_t^{-\alpha/(1-\alpha)} I_t = \lambda B^{-\alpha/(1-\alpha)} K_t^{-\alpha} I_t = \lambda B^{-\alpha/(1-\alpha)} K_t^{-\alpha} s Y_t,$$

considering a BGP with $s = I/Y$ constant. Substituting (13.15) into this, we get

$$g_q = \lambda B^{-\alpha/(1-\alpha)} K_t^{-\alpha} s K_t^{\alpha} L^{1-\alpha} = \lambda B^{-\alpha/(1-\alpha)} s L^{1-\alpha}. \quad (13.21)$$

If $n = 0$, the right-hand side of (13.21) is constant and so is $g_K = g_q/(1-\alpha)$, by (13.17), and $g_Y = \alpha g_K = \alpha g_q/(1-\alpha)$, by (13.19).

If $n > 0$ at the same time as $\lambda = (1-\alpha)/\alpha$, however, there is a tendency to a forever rising growth rate in q, hence also in K and Y. No BGP exists in this case.

Returning to the case where a BGP exists, a striking feature revealed by (13.21) is that the saving rate, s, matters for the growth rate of q, hence also for the growth rate of K and Y, respectively, along a BGP. As in the Romer case of the disembodied learning-by-investing model, the growth rates along a BGP cannot be determined until the saving behavior in the economy is modeled.

So the considered knife-edge case, $\lambda = (1-\alpha)/\alpha$ combined with $n = 0$, opens up for many different per capita growth rates under balanced growth. Which one is "selected" by the economy depends on how the household sector is described. In a Ramsey setup with $n = 0$ one can show that the growth rate under balanced growth depends negatively on the rate of time preference and the elasticity of marginal utility of consumption of the representative household. And not only is growth in this case *fully endogenous* in the sense that a positive per capita growth rate can be maintained forever without the support by growth in any exogenous factor. An economic policy that subsidizes investment can generate not only a transitory rise in the productivity growth rate, but also a permanently higher productivity growth rate.

In contrast to the Romer (1986) model, cf. Section 13.2.2 above, we do not here end up with a reduced-form AK model. Indeed, we end up with a model with transitional dynamics, as a consequence of the presence of *two* state variables, K and q.

If instead $\alpha > 1/(1+\lambda)$, we get a tendency to explosive growth − infinite output in finite time − a not plausible scenario, cf. Appendix B.

13.4 Static comparative advantage vs. dynamics of learning by doing*

In this section we will briefly discuss a development economics perspective of the above learning-based growth models.

More specifically we will take a look at the possible "conflict" between static comparative advantage and economic growth. The background to this possible "conflict" is the dynamic externalities inherent in learning by doing and learning by investing.[11]

13.4.1 A simple two-sector learning-by-doing model

We consider an isolated economy with two production sectors, *sector 1* and *sector 2*, each producing its specific consumption good. Labor is the only input and aggregate labor supply L is constant. There are many small firms in the two sectors. Aggregate output in the sectors are:

$$Y_{1t} = T_{1t}L_{1t}, \tag{13.22}$$

$$Y_{2t} = T_{2t}L_{2t}, \tag{13.23}$$

where

$$L_{1t} + L_{2t} = L.$$

There are *sector-specific* learning-by-doing externalities in the following form:

$$\dot{T}_{1t} = B_1 Y_{1t}, \qquad B_1 \geq 0, \tag{13.24}$$

$$\dot{T}_{2t} = B_2 Y_{2t}, \qquad B_2 \geq 0. \tag{13.25}$$

Although not visible in our aggregate formulation, there are substantial knowledge spillovers across firms within the sectors. Across sectors, spillovers are assumed negligible.

Assume firms maximize profits and that there is perfect competition in the goods and labor markets. Then, prices are equal to the (constant) marginal costs. Let the relative price of sector 2-goods in terms of sector-1 goods be called p_t (i.e., we use sector-1 goods as numeraire). Let the hourly wage in terms of sector-1 goods be w_t. In general equilibrium with production in both sectors we then have

$$T_{1t} = p_t T_{2t} = w_t,$$

saying that the value of the (constant) marginal productivity of labor in each sector equals the wage. Hence,

$$p_t \frac{T_{2t}}{T_{1t}} = 1 \quad \text{or} \quad p_t = \frac{T_{1t}}{T_{2t}}, \tag{13.26}$$

saying that the relative price of the two goods is inversely proportional to the relative labor productivities in the two sectors. The demand side, which

[11] Krugman (1987), Lucas (1988, Section 5).

is not modelled here, will of course play a role for the final allocation of labor to the two sectors.

Taking logs in (13.26) and differentiating w.r.t. t gives

$$\frac{\dot{p}_t}{p_t} = \frac{\dot{T}_{1t}}{T_{1t}} - \frac{\dot{T}_{2t}}{T_{2t}} = \frac{B_1 Y_{1t}}{T_{1t}} - \frac{B_2 Y_{2t}}{T_{2t}} = B_1 L_{1t} - B_2 L_{2t},$$

using (13.24) and (13.25). Thus,

$$\dot{p}_t = (B_1 L_{1t} - B_2 L_{2t}) p_t.$$

Assume sector 2 (say some industrial activity) is more disposed to learning-by-doing than sector 1 (say mining) so that $B_2 > B_1$. Consider for simplicity the case where at time 0 there is symmetry in the sense that $L_{10} = L_{20}$. Then, the relative price p_t of sector-2 goods in terms of sector-1 goods will, at least initially, tend to diminish over time. The resulting substitution effect is likely to stimulate demand for sector-2 goods. Suppose this effect is large enough to ensure that $L_2 = Y_2/T_2$ never becomes lower than $B_1 L_1/B_2$, that is, $B_2 L_2 \geq B_1 L_1$ for all t. Then the scenario with $\dot{p} \leq 0$ is sustained over time and the sector with highest growth potential remains a substantial constituent of the economy. This implies sustained economic growth in the aggregate economy.

Now, suppose the country considered is a rather backward, developing country which until time t_0 has been a closed economy (very high tariffs etc.). Then the country decides to open up for free foreign trade. Let the relative world market price of sector 2-goods be $\bar{p}$, which we for simplicity assume is constant. At time t_0 there are two alternative possibilities to consider:

Case 1: $\bar{p} > \frac{T_{1t_0}}{T_{2t_0}}$ (world-market price of good 2 higher than the opportunity cost of producing good 2). Then the country specializes fully in sector-2 goods. Since this is the sector with a high growth potential, economic growth is stimulated. The relative productivity level T_{1t}/T_{2t} decreases so that the scenario with $\bar{p} > T_{1t}/T_{2t}$ remains. A virtuous circle of dynamics of learning by doing is unfolded and high economic growth is sustained.

Case 2: $\bar{p} < \frac{T_{1t_0}}{T_{2t_0}}$ (world-market price of good 2 lower than the opportunity cost of producing good 2). Then the country specializes fully in sector-1 goods. Since this is the sector with a low growth potential, economic growth is impeded or completely halted. The relative productivity level T_{1t}/T_{2t} does not decrease. Hence, the scenario with $\bar{p} < T_{1t}/T_{2t}$ sustains itself and persists. Low or zero economic growth is sustained. The static comparative advantage in sector-1 goods remains and the country is locked in low growth.

If instead $\bar{p}$ is time-dependent, suppose $\dot{\bar{p}}_t < 0$ (by similar arguments as for the closed economy). Then the case 2 scenario is again self-sustaining.

The point is that there may be circumstances (like in case 2), where temporary protection for a backward country is growth promoting (this is a specific kind of "infant industry" argument).

13.4.2 A more robust specification

The way (13.24) and (13.25) are formulated, we have

$$\frac{\dot{T}_{1t}}{T_{1t}} = B_1 L_{1t}, \tag{13.27}$$

$$\frac{\dot{T}_{2t}}{T_{2t}} = B_2 L_{2t}, \tag{13.28}$$

by (13.22) and (13.23). Thus, the model implies scale effects on growth, that is, *strong* scale effects.

An alternative specification introduces limits to learning-by-doing in the following way:

$$\dot{T}_{1t} = B_1 Y_{1t}^{\lambda_1}, \qquad \lambda_1 < 1,$$

$$\dot{T}_{2t} = B_2 Y_{2t}^{\lambda_2}, \qquad \lambda_2 < 1.$$

Then (13.27) and (13.28) are replaced by

$$\frac{\dot{T}_{1t}}{T_{1t}} = B_1 T_{1t}^{\lambda_1 - 1} L_{1t}^{\lambda_1}, \tag{13.29}$$

$$\frac{\dot{T}_{2t}}{T_{2t}} = B_2 T_{2t}^{\lambda_2 - 1} L_{2t}^{\lambda_2}. \tag{13.30}$$

Now the problematic strong scale effect has disappeared. At the same time, since $\lambda_1 - 1 < 0$ and $\lambda_2 - 1 < 0$, (13.29) and (13.30) show that growth peters out as long as the "diminishing returns" to learning-by-doing are not offset by an increasing labor force or an additional source (outside the model) of technical progress. If $n > 0$, we get sustained growth of the semi-endogenous type as in the Arrow model of learning-by-investing.

Yet the analysis may still be a basis for an "infant industry" argument. If the circumstances are like in case 2, temporary protection may help a backward country to enter a higher long-run path of evolution. Stiglitz underlines South Korea as an example:

> What matters is *dynamic* comparative advantage, or comparative advantage in the long run, which can be shaped. Forty years ago, South Korea had a comparative advantage in growing rice. Had

> it stuck to that strength, it would not be the industrial giant that it is today. It might be the world's most efficient rice grower, but it would still be poor (Stiglitz, 2012, p. 2).

This point is related to two different aspects of technical knowledge. On the one hand, technical knowledge is a nonrival good and this non-rivalness speaks for *openness,* thereby improving conditions for knowledge spillovers and learning from other countries. On the other hand, the potential for knowledge accumulation and internal learning by doing is different in different production sectors. And some sectors with a lot of internal learning potential and economies of scale never gets started unless to begin with they are protected from foreign competition.

13.4.3 Resource curse?

The analysis also suggests a mechanism that, along with others, may help explaining what is known as the *resource curse* problem. This problem refers to the paradox that being abundant in natural resources may sometimes seem a curse for a country rather than a blessing. At least quite many empirical studies have shown a negative correlation between resource abundance and economic growth (see, e.g., Sachs and Warner 1995, Gylfason et al., 1999).

The mechanism behind this phenomenon could be the following. Consider a mining country with an abundance of natural resources in the ground. Empirically, growth in total factor productivity in mining activity is relatively low. Interpreting this as reflecting a relatively low learning potential, the mining sector may be represented by sector 1 above. Given the abundance of natural resources, T_{1t_0} is likely to be high relative to the productivity in the manufacturing sector, T_{2t_0}. So the country is likely to be in the situation described as case 2. As a result, economic growth may never get started.

The basic problem here is, however, not of an economic nature in a narrow sense, but rather of an institutional character. Taxation on the natural resource and use of the tax revenue for public investment in growth promoting factors (infrastructure, health care, education, R&D) or directly in the sector with high learning potential can from an economic point of view circumvent the curse to a blessing. It is not the natural resources as such, but rather barriers of a political character, conflicts of interest among groups and social classes, even civil war over the right to exploit the resources, or dominance by foreign superpowers, that may be the obstacles to a sound economic development (Mehlum et al., 2006). An additional potential obstacle is related to the possible response of a country's real exchange rate, and therefore its competitiveness, to a new discovery of natural resources in a country.[12]

Summing up: Discovery of a valuable mineral in the ground in a country with weak institutions may, through corruption etc. have adverse effects on resource allocation and economic growth in the country. But: "Resources should be a blessing, not a curse. They can be, but it will not happen on its own. And it will not happen easily" (Stiglitz, 2012, p. 2).

13.5 Appendix

A. Balanced growth in the embodied technical change model with investment-specific learning

In this appendix the results (13.16), (13.17), (13.18), and (13.19) are derived. The model is:

$$\begin{aligned} Y &= K^{\alpha}L^{1-\alpha}, \qquad 0<\alpha<1, && (13.31)\\ I &= Y-C, && (13.32)\\ \dot{K} &= qI-\delta K, && (13.33)\\ q_t &= \left(\int_{-\infty}^{t} I_s ds\right)^{\lambda}, \qquad 0<\lambda\leq\bar{\lambda}, && (13.34)\\ L &= L_0e^{nt}, \qquad n\geq 0. && (13.35) \end{aligned}$$

Consider a BGP. By definition, Y, K, and C then grow at constant rates, not necessarily positive. With $s=I/Y$ constant and $0<s<1$, (13.31) gives

$$g_I = g_Y = \alpha g_K + (1-\alpha)n, \tag{13.36}$$

a constant. By (13.33), $g_K = q\frac{I}{K} - \delta$, showing that qI/K is constant along a BGP. Hence,

$$g_q + g_I = g_K, \tag{13.37}$$

and so also g_q must be constant. From (13.34) follows that $g_q = \lambda q^{-1/\lambda}I$. Taking logs in this equation and differentiating w.r.t. t gives

$$\frac{\dot{g}_q}{g_q} = -\frac{1}{\lambda}g_q + g_I = 0,$$

in view of constancy of g_q. Substituting into (13.37) yields $(1+\lambda)g_I = g_K$, which combined with (13.36) gives

$$g_K = \frac{(1-\alpha)(1+\lambda)n}{1-\alpha(1+\lambda)},$$

[12] Ploeg (2011) provides a survey over different theories related to the resource curse problem. See also Ploeg and Venables (2012) and Stiglitz (2012).

which is (13.16). In view of $g_q = \lambda g_I = \lambda g_Y = \lambda(g_y + n) = \lambda g_K/(1+\lambda)$, the results (13.17), (13.18), and (13.19) immediately follow.

B. Big bang a hair's breadth from the AK

Here we shall prove the statement in Section 13.5.2: a hair's breadth from the AK assumption the technology is so productive as to generate infinite output in finite time.

The simple AK model as well as reduced-form AK models end up in an aggregate production function

$$Y = AK.$$

We ask the question: what happens if the exponent on K is not exactly 1, but slightly above. For simplicity, let $A = 1$ and consider

$$Y = K^{\alpha}, \qquad \alpha = 1 + \varepsilon,\ \varepsilon \gtrapprox 0.$$

Our claim is that *if* $\alpha > 1$, a constant saving rate, s, will generate infinite Y and C in finite time.

We embed the technology in a Solow-style model with $\delta = n = 0$ and get:

$$\dot{K} \equiv \frac{dK}{dt} = sK^{\alpha}, \qquad 0 < s < 1, \qquad K(0) = K_0 > 0 \text{ given.} \tag{13.38}$$

We see that not only is $\dot{K} > 0$ for all $t \geq 0$, but $\dot{K}$ is increasing over time since K is increasing. So, for sure, $K \to \infty$, but how fast?

One way of answering this question exploits the fact that $\dot{x} = x^a$ is a Bernouilli equation and can be solved by considering the transformation $z = x^{1-\alpha}$ as we do in Chapter 7 and Exercise III.3. Closely related to that method is the approach below, which may have the advantage of being somewhat more transparent and intuitive.

To find out, note that (13.38) is a separable differential equation which implies

$$K^{-\alpha}dK = sdt.$$

By integration,

$$\begin{aligned} \int K^{-\alpha}dK &= \int sdt + \mathcal{C} \ \Rightarrow \\ \frac{K^{-\alpha+1}}{1-\alpha} &= st + \mathcal{C}, \end{aligned} \tag{13.39}$$

where $\mathcal{C}$ is some constant, determined by the initial condition $K(0) = K_0$. For $t = 0$ (13.39) gives $\mathcal{C} = K_0^{-\alpha+1}/(1-\alpha)$. Consequently, the solution $K = K(t)$ satisfies

$$\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} = st. \tag{13.40}$$

As t increases, the left-hand side of this equation follows suit since $K(t)$ increases and $\alpha > 1$. There is a $\bar{t} < \infty$ such that when $t \to \bar{t}$ from below, $K(t) \to \infty$. Indeed, by (13.40) we see that such a $\bar{t}$ must be the solution to the equation

$$\lim_{K(t)\to\infty} \left(\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} \right) = s\bar{t}.$$

Since

$$\lim_{K(t)\to\infty} \left(\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} \right) = \frac{{K_0}^{1-\alpha}}{\alpha - 1},$$

we find

$$\bar{t} = \frac{1}{s}\frac{{K_0}^{1-\alpha}}{\alpha - 1}.$$

To get an idea about the implied order of magnitude, let the time unit be one year and $s = 0.1$, $K_0/Y_0 = K_0^{1-\alpha} = 2$, and $\alpha = 1.05$. Then $\bar{t} = 400$ years. So the Big Bang ($Y = \infty$) would occur in 400 years from now if $\alpha = 1.05$.

As Solow remarks (Solow 1994), this arrival to the Land of Cockaigne would imply the "end of scarcity", a very optimistic perspective.

In a discrete time setup we get an analogue conclusion. With airframe construction in mind let us imagine that the learning parameter λ is slightly above 1. Then we must accept the implication that it takes only a finite number of labor hours to produce an infinite number of airframes. This is because, given the (direct) labor input required to produce the q'th in a sequence of identical airframes is proportional to $q^{-\lambda}$, the total labor input required to produce the first q airframes is proportional to $1/1 + 1/2^{\lambda} + 1/3^{\lambda} + ... + 1/k^{\lambda}$. Now, the infinite series $\sum_{k=1}^{\infty} 1/q^{\lambda}$ converges if $\lambda > 1$. As a consequence only a finite amount of labor is needed to produce an infinite number of airframes. "This seems to contradict the whole idea of scarcity", Solow observes (Solow 1997, p. 8).

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Chapter 16

Natural resources and economic growth

In these lecture notes, up to now, the relationship between economic growth and the earth's finite natural resources has been briefly touched upon in connection with: the discussion of returns to scale (Chapter 2), the transition from a pre-industrial to an industrial economy (in Chapter 7), the environmental problem of global warming (Chapter 8), and the resource curse (in Chapter 13.4.3). In a more systematic way the present chapter reviews how natural resources, including the environment, relate to economic growth.

The contents are:

- Classification of means of production.
- The notion of sustainable development.
- Renewable natural resources.
- Non-renewable natural resources and exogenous technology growth.
- Non-renewable natural resources and endogenous technology growth.
- Natural resources and the issue of limits to economic growth.

The first two sections aim at establishing a common terminology for the discussion.

16.1 Classification of means of production

We distinguish between different categories of production factors. First two broad categories:

1. *Producible* means of production, also called man-made inputs.

2. *Non-producible* means of production.

The first category includes:

1.1 *Physical inputs* like processed raw materials, other intermediate goods, machines, and buildings.

1.2 *Human inputs* of a produced character in the form of technical knowledge (available in books, USB sticks etc.) and human capital.

The second category includes:

2.1 Human inputs of a non-produced character, sometimes called "raw labor".[1]

2.2 Natural resources. By definition in limited supply on this earth.

Natural resources can be sub-divided into:

2.2.1 *Renewable resources*, that is, natural resources the stock of which can be replenished by a natural self-regeneration process. Hence, if the resource is not over-exploited, it can in production as well as consumption be sustained in a more or less constant amount per time unit. Examples: ground water, fertile soil, fish in the sea, clean air, national parks.

2.2.2 *Non-renewable resources*, that is, natural resources which have no natural regeneration process (at least not within a relevant time scale). The stock of a non-renewable resource is thus depletable. Examples: fossil fuels, many non-energy minerals, virgin wilderness and endangered species.

The climate change problem due to "greenhouse gasses" can be seen as belonging to somewhere between category 2.2.1 or 2.2.2 in that the quality of the atmosphere *has* a natural self-regeneration ability, but the speed of regeneration is very low. A very important facet of natural resources is that they function as direct or indirect sources of energy. Think of animal power, waterfalls, coal, oil, natural gas, biomass, wind, and geothermic energy in modern times

Given the scarcity of natural resources and the pollution problems caused by economic activity, key issues are:

[1]Outside a slave society, biological reproduction is usually not considered as part of the economic sphere of society even though formation and maintainance of raw labor requires child rearing, health, food etc. and is thus conditioned on economic circumstances.

a. Is sustainable development possible?

b. Is sustainable economic growth (in a per capita welfare sense) possible?

c. How should a better "thermometer" for the evolution of the economy than measurement of GNP be designed?

But first: what does "sustainable" and "sustainability" really mean"?

16.2 The notion of sustainable development

The basic idea in the notion of sustainable development is to emphasize intergenerational responsibility. The Brundtland Commission (1987) defined sustainable development as "development that meets the needs of present generations without compromising the ability of future generations to meet theirs".

In more standard economic terms we may define *sustainable economic development* as a time path along which per capita "welfare" (somehow measured) remains non-decreasing across generations forever. An aspect of this is that current economic activities should not impose significant economic risks on future generations. The "forever" in the definition can not, of course, be taken literally, but as equivalent to "for a very long time horizon". We know that the sun will eventually (in some billion years) burn out and consequently life on earth will become extinct.

Our definition emphasizes *welfare,* which should be understood in a broad sense, that is, as more or less synonymous with "quality of life", "living conditions", or "well-being" (the term used in Smulders, 1995). What may matter is thus not only the per capita amount of marketable consumption goods, but also fundamental aspects like health, life expectancy, and enjoyment of services from the ecological system. In summary: capability to lead a worthwhile life.

To make this more specific, consider preferences as represented by the period utility function of a "typical individual". Suppose two variables enter as arguments, namely consumption, c, of a marketable produced good and some measure, q, of the quality of services from the eco-system. Suppose further that the period utility function is of constant-elasticity-of-substitution (CES) form:

$$u(c,q) = \left[\alpha c^{\beta} + (1-\alpha)q^{\beta}\right]^{1/\beta}, \quad 0 < \alpha < 1, \beta < 1. \tag{16.1}$$

The parameter β is called the *substitution parameter*. The elasticity of substitution between the two goods is $\sigma = 1/(1-\beta) > 0$, a constant. When $\beta \to 1$

(from below), the two goods become perfect substitutes (in that $\sigma \to \infty$). The smaller is β, the less substitutable are the two goods. When $\beta < 0$, we have $\sigma < 1$, and as $\beta \to -\infty$, the indifference curves become near to right angled.[2] According to many environmental economists, there are good reasons to believe that $\sigma < 1$, since water, basic foodstuff, clean air, absence of catastrophic climate change, etc. are difficult to replace by produced goods and services. In this case there is a limit to the extent to which a rising c, obtainable through a rising per capita income, can compensate for falling q.

At the same time the techniques by which the consumption good is currently produced may be "dirty" and thereby *cause* a falling q. An obvious policy response is the introduction of pollution taxes that give an incentive for firms (or households) to replace these techniques (or goods) with cleaner ones. For certain forms of pollution (e.g., sulfur dioxide, SO_2, in the air) there is evidence of an inverted U-curve relationship between the degree of pollution and the level of economic development measured by GDP per capita – the *environmental Kuznets curve*.[3]

So an important element in *sustainable* economic development is that the economic activity of current generations does not spoil the environmental conditions for future generations. Living up to this requirement necessitates economic and environmental strategies consistent with the planet's endowments. This means recognizing the role of environmental constraints for economic development. A complicating factor is that specific abatement policies vis-a-vis particular environmental problems may face resistance from interest groups, thus raising political-economics issues.

As defined, a criterion for sustainable economic *development* to be present is that per capita welfare remains *non-decreasing* across generations. A subcategory of this is *sustainable* economic *growth* which is present if per capita welfare is *growing* across generations. Here we speak of growth in a *welfare* sense, not in a physical sense. Permanent exponential per capita output

[2]By L'Hôpital's rule for "0/0" it follows that, for fixed c and q,

$$\lim_{\beta \to 0, \beta \neq 0} \left[\alpha c^{\beta} + (1-\alpha) q^{\beta} \right]^{1/\beta} = c^{\alpha} q^{1-\alpha}.$$

So the Cobb-Douglas function, which has elasticity of substitution between the goods equal to 1, is an intermediate case, corresponding to $\beta = 0$. More technical details in Chapter 2, albeit from the perspective of production rather than preferences.

[3]See, e.g., Grossman and Krueger (1995). Others (e.g., Perman and Stern, 2003) claim that when paying more serious attention to the statistical properties of the data, the environmental Kuznets curve is generally rejected. Important examples of pollutants accompanied by *absence* of an environmental Kuznets curve include waste storage, reduction of biodiversity, and emission of CO_2 to the atmosphere. A very serious problem with the latter is that emissions from a single country is spread all over the globe.

growth in a *physical* sense is of course not possible with limited natural resources (matter or energy). The issue about sustainable *growth* is whether, by combining the natural resources with man-made inputs (knowledge, human capital, and physical capital), an output stream of *increasing quality*, and therefore increasing *economic value*, can be maintained. In modern times capabilities of many digital electronic devices provide conspicuous examples of exponential growth in *quality* (or *efficiency*). Think of processing speed, memory capacity, and efficiency of electronic sensors. What is known as "Moore's Law" is the rule of thumb that there is a doubling of the efficiency of microprocessors within every two years. The evolution of the internet has provided much faster and widened dissemination of information and fine arts.

Of course there are intrinsic difficulties associated with measuring sustainability in terms of well-being. There now exists a large theoretical and applied literature dealing with these issues. A variety of extensions and modifications of the standard national income accounting GNP has been developed under the heading *Green NNP* (green net national product). An essential feature in the measurement of Green NNP is that from the conventional GDP (which essentially just measures the level of economic activity) is subtracted the depreciation of not only the physical capital but also the environmental assets. The latter depreciate due to pollution, overburdening of renewable natural resources, and depletion of reserves of non-renewable natural resources.[4] In some approaches the focus is on whether a comprehensive measure of *wealth* is maintained over time. Along with reproducible assets and natural assets (including the damage to the atmosphere from "greenhouse gasses"), Arrow et al. (2012) include health, human capital, and "knowledge capital" in their measure of "wealth". They apply this measure in a study of the United States, China, Brazil, India, and Venezuela over the period 1995-2000. They find that all five countries over this period satisfy the sustainability criterion of non-decreasing wealth in this broad sense. Indeed the wealth measure referred to is found to be growing in all five countries – in the terminology of the field positive "genuine saving" has taken place.[5] Note that it is sustainability that is claimed, not optimality.

In the next two sections we will go more into detail with the challenge

[4] The depreciation of these environmental and natural assets is evaluated in terms of the social planner's shadow prices. See, e.g., Heal (1998), Weitzman (2001, 2003), and Stiglitz et al. (2010).

[5] Of course, many measurement uncertainties and disputable issues of weighting are involved; brief discussions, and questioning, of the study are contained in Solow (2012), Hamilton (2012), and Smulders (2012). Regarding Denmark 1990-2009, a study by Lind and Schou (2013), along lines somewhat similar to those of Arrow et al. (2012), also suggests sustainability to hold.

to sustainability and growth coming from renewable and non-renewable resources, respectively. We shall primarily deal with the issues from the point of view of technical feasibility of non-decreasing, and possibly rising, per-capita consumption. Concerning questions about appropriate institutional regulation the reader is referred to the specialized literature.

We begin with renewable resources.

16.3 Renewable resources

A useful analytical tool is the following simple model of the stock dynamics associated with a renewable resource.

Let $S_t \geq 0$ denote the *stock* of the renewable resource at time t (so in this chapter S is not our symbol for saving). Then we may write

$$\dot{S}_t \equiv \frac{dS_t}{dt} = G_t - R_t = G(S_t) - R_t, \qquad S_0 > 0 \text{ given}, \tag{16.2}$$

where G_t is the self-regeneration of the resource per time unit and $R_t \geq 0$ is the resource extraction (and use) per time unit at time t. If for instance the stock refers to the number of fish in the sea, the flow R_t represents the number of fish caught per time unit. And if, in a pollution context, the stock refers to "cleanness" of the air in cities, R_t measures, say, the emission of sulfur dioxide, SO_2, per time unit. The self-regenerated amount per time unit depends on the available stock through the function $G(S_t)$, known as a *self-regeneration function.*[6]

Let us briefly consider the example where S stands for the size of a fish population in the sea. Then the self-regeneration function will have a bell-shape as illustrated in the upper panel of Figure 16.1. Essentially, the self-regeneration ability is due to the flow of solar energy continuously entering the the eco-system of the earth. This flow of solar energy is constant and beyond human control.

The size of the stock at the lower intersection of the $G(S)$ curve with the horizontal axis is $\underline{S}(0) \geq 0$. Below this level, even with $R = 0$ there are too few female fish to generate offspring, and the population necessarily shrinks and eventually reaches zero. We may call $\underline{S}(0)$ the minimum sustainable stock.

At the other intersection of the $G(S)$ curve with the horizontal axis, $\bar{S}(0)$ represents the maximum sustainable stock. The eco-system cannot support further growth in the fish population. The reason may be food scarcity,

[6] The equation (16.2) also covers the case where S represents the stock of a *non-renewable* resource if we impose $G(S) \equiv 0$, i.e., there is no self-regeneration.

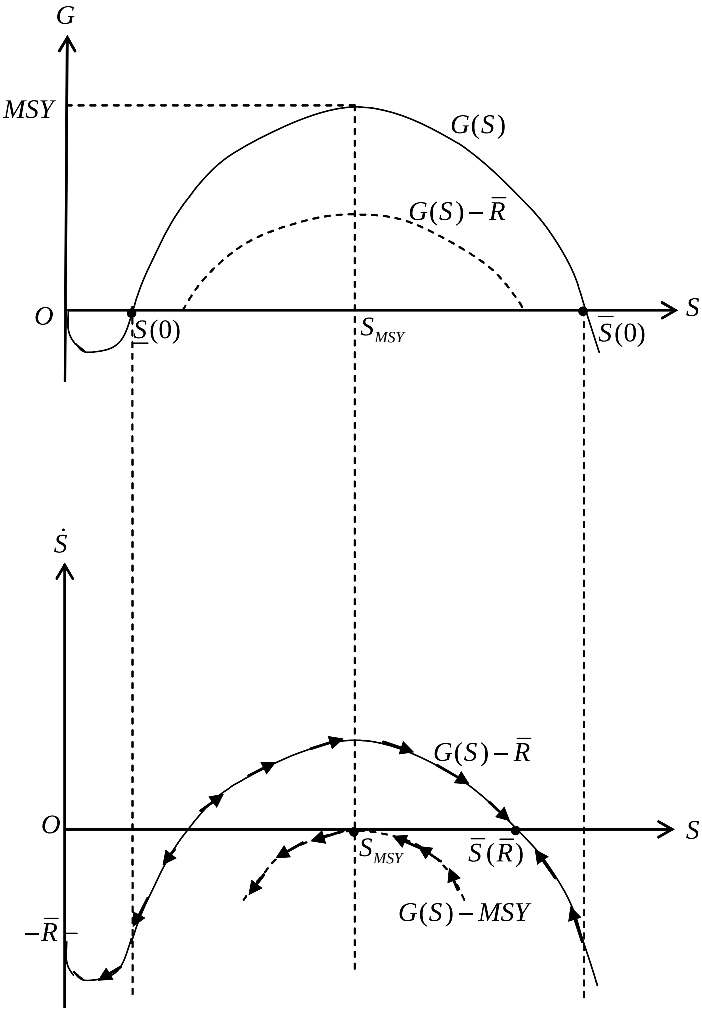

Figure 16.1

spreading of diseases because of high population density, and easiness for predators to catch the considered fish species and themselves expand. Popular mathematical specifications of $G(\cdot)$ include the logistic function $G(S) = \alpha S(1 - S/\beta)$, where $\alpha > 0$, $\beta > 0$, and the quasi-logistic function $G(S) = \alpha S(1 - S/\beta)(S/\gamma - 1)$, where also $\gamma > 0$. In both cases $\bar{S}(0) = \beta$, but $\underline{S}(0)$ equals 0 in the first case and γ in the second.

The value MSY, indicated on the vertical axis in the upper panel, equals $\max_S g(S)$. This value is thus the *maximum sustainable yield* per time unit. This yield is sustainable from time t_0, provided the fish population is at time t_0 at least of size $S_{MSY} = \arg\max_S g(S)$ which is that value of S where $G(S) = MSY$. The size, S_{MSY}, of the fish population is consistent with maintaining the harvest MSY per time unit forever in a steady state.

The lower panel in Figure 16.1 illustrates the dynamics in the $(S, \dot{S})$ plane, given a fixed rate of resource extraction $R = \bar{R} \in (0, MSY]$. The arrows indicate the direction of movement in this plane. In the long run, if $R = \bar{R}$ for all t, the stock will settle down at the size $\bar{S}(\bar{R})$. The stippled curve in the upper panel indicates $G(S) - \bar{R}$, which is the same as $\dot{S}$ in the lower panel when $R = \bar{R}$. The stippled curve in the lower panel indicates the dynamics in case $R = MSY$. In this case the steady-state stock, $\bar{S}(MSY) = S_{MSY}$, is unstable. Indeed, a small negative shock to the stock will not lead to a gradual return but to a self-reinforcing reduction of the stock as long as the extraction $R = MSY$ is maintained.

Note that MSY is an *ecological maximum* and not necessarily in any sense an economic optimum. Indeed, since the search and extraction costs may be a decreasing function of the fish density in the sea, hence of the stock of fish, it may be worthwhile to increase the stock beyond S_{MSY}, thus settling for a smaller harvest per time unit. Moreover, a fishing industry cost-benefit analysis may consider maximization of the discounted expected aggregate profits per time unit, taking into account the expected evolution of the market price of fish, the cost function, and the dynamic relationship (16.2).

In addition to its importance for regeneration, the stock, S, may have amenity value and thus enter the instantaneous utility function. Then again some conservation of the stock over and above S_{MSY} may be motivated.

A dynamic model with a renewable resource and focus on technical feasibility Consider a simple model consisting of (16.2) together with

$$\begin{aligned} Y_t &= F(K_t, L_t, R_t, t), \qquad \partial F/\partial t \geq 0, \\ \dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ given}, \\ L_t &= L_0 e^{nt}, \qquad n \geq 0, \quad L_0 > 0 \text{ given}, \end{aligned} \tag{16.3}$$

where Y_t is aggregate output and K_t, L_t, and R_t are inputs of capital, labor, and a renewable resource, respectively, per time unit at time t. Let the aggregate production function, F, be neoclassical[7] with constant returns to scale w.r.t. K, L, and R. The assumption $\partial F/\partial t \geq 0$ represents exogenous technical progress. Further, C_t is aggregate consumption ($\equiv c_t L_t$, where c_t is per capita consumption) and δ denotes a constant rate of capital depreciation. There is no distinction between employment and population, L_t. The population growth rate, n, is assumed constant.

Is sustainable economic development in this setting technically feasible? By definition, the answer will be yes if non-decreasing per capita consumption can be sustained forever. From economic history we know of examples of "tragedy of the commons", like over-grazing of unregulated common land. As our discussion is about technical feasibility, we assume this kind of problem is avoided by appropriate institutions.

Suppose the use of the renewable resource is kept constant at a sustainable level $\bar{R} \in (0, MSY)$. To begin with, suppose $n = 0$ so that $L_t = L$ for all $t \geq 0$. Assume that at $R = \bar{R}$, the system is "productive" in the sense that

$$\lim_{K\to 0} F_K(K, L, \bar{R}, 0) > \delta > \lim_{K\to\infty} F_K(K, L, \bar{R}, 0). \tag{A1}$$

This condition is satisfied in Figure 16.2 where the value $\bar{K}$ has the property $F(\bar{K}, L, \bar{R}, 0) = \bar{K}$. Given the circumstances, this value is the least upper bound for a sustainable capital stock in the sense that

$$\begin{aligned} \text{if } K &\geq \bar{K}, \text{ we have } \dot{K} < 0 \text{ for any } C > 0; \\ \text{if } 0 &< K < \bar{K}, \text{ we have } \dot{K} = 0 \text{ for } C = F(K, L, \bar{R}, 0) - \delta K > 0. \end{aligned}$$

For such a C, illustrated in Figure 16.2, a constant $Y = F(K, L, \bar{R}, 0)$ is maintained forever which implies non-decreasing per-capita income, $y \equiv Y/L$, forever. So, in spite of the limited availability of the natural resource, a *non-decreasing* level of consumption is technically feasible even without technical progress. A forever *growing* level of consumption will, of course, require sufficient technical progress capable of substituting for the natural resource.

Now consider the case $n > 0$ and assume CRS w.r.t. K, L, and R. In view of CRS, we have

$$1 = F(\frac{K_t}{Y_t}, \frac{L_t}{Y_t}, \frac{\bar{R}}{Y_t}, t), \tag{16.4}$$

[7] That is, marginal productivities of the production factors are positive, but diminishing, and the upper contour sets are strictly convex.

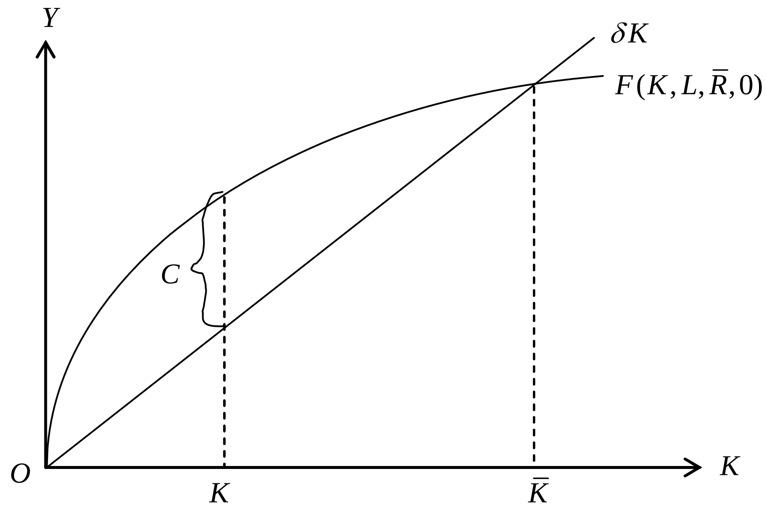

Figure 16.2: Sustainable consumption in the case of $n = 0$ and no technical progress (L and $\bar{R}$ fixed).

Along a balanced growth path with positive gross saving (if it exists) we know that K_t/Y_t and C_t/Y_t must be constant, cf. the balanced growth equivalence theorem of Chapter 4. Maintaining C_t/L_t $(= (C_t/Y_t)/(L_t/Y_t))$ constant along such a path, requires that L_t/Y_t is constant and thereby that Y_t grows at the rate n. But then $\bar{R}/Y_t$ will be declining over time. To compensate for this in (16.4), sufficient technical progress is necessary. This necessity of course is present, a fortiori, for sustained *growth* in per-capita consumption to occur.

As technical progress in the far future is by its very nature uncertain and unpredictable, there can be no guarantee for sustained per capita growth if there is sustained population growth.

Pollution As hinted at above, the concern that certain production methods involve pollution is commonly incorporated into economic analysis by subsuming environmental quality into the general notion of renewable resources. In that context S in (16.2) and Figure 16.1 will represent the "level of environmental quality" and R_t will be the amount of dirty emissions per time unit. Since the level of the environmental quality is likely to be an argument in both the utility function and the production function, again some limitation of the "extraction" (the pollution flow) is motivated. Pollution taxes may help to encourage abatement activities and make technical innovations towards cleaner production methods more profitable.

16.4 Non-renewable resources: The DHSS model

Whereas extraction and use of renewable resources can be sustained at a more or less constant level (if not too high), the situation is different with non-renewable resources. They have no natural regeneration process (at least not within a relevant time scale) and so continued extraction per time unit of these resources will inevitably have to decline and approach zero in the long run.

To get an idea of the implications, we will consider the Dasgupta-Heal-Solow-Stiglitz model (DHSS model) from the 1970s.[8] The production side of the model is described by:

$$\begin{aligned} Y_t &= F(K_t, L_t, R_t, t), \qquad \partial F/\partial t \geq 0, && (16.5)\\ \dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ given}, && (16.6)\\ \dot{S}_t &= -R_t \equiv -u_t S_t, \qquad S_0 > 0 \text{ given}, && (16.7)\\ L_t &= L_0 e^{nt}, \qquad n \geq 0. && (16.8) \end{aligned}$$

The new element is the replacement of (16.2) with (16.7), where S_t is the *stock of the non-renewable* resource (e.g., oil reserves), and u_t is the depletion rate. Since we must have $S_t \geq 0$ for all t, there is a finite upper bound on cumulative resource extraction:

$$\int_0^\infty R_t dt \leq S_0. \tag{16.9}$$

Since the resource is non-renewable, no re-generation function appears in (16.7). Uncertainty is ignored and the extraction activity involves no costs.[9] As before, there is no distinction between employment and population, L_t.

The model was formulated as a response to the pessimistic Malthusian views expressed in the book *The Limits to Growth* written by MIT ecologists Meadows et al. (1972).[10] Stiglitz, and fellow economists, asked the question: what are the technological conditions needed to avoid falling per capita consumption in the long run in spite of the inevitable decline in the use of non-renewable resources? The answer is that there are three ways in which this decline in resource use may be counterbalanced:

[8] See, e.g., Stiglitz, 1974.

[9] This simplified description of resource extraction is the reason that it is common to classify the model as a *one-sector* model, notwithstanding there are two productive activities in the economy, manufacturing and resource extraction.

[10] An up-date came in 2004, see Meadows at al. (2004).

1. input substitution;
2. resource-augmenting technical progress;
3. increasing returns to scale.

Let us consider each of them in turn (although in practice the three mechanisms tend to be intertwined).

16.4.1 Input substitution

By input substitution is here meant the gradual replacement of the input of the exhaustible natural resource by man-made input, capital. Substitution of fossil fuel energy by solar, wind, tidal and wave energy resources is an example. Similarly, more abundant lower-grade non-renewable resources can substitute for scarce higher-grade non-renewable resources - and this *will* happen when the scarcity price of these has become sufficiently high. A rise in the price of a mineral makes a synthetic substitute cost-efficient or lead to increased recycling of the mineral. Finally, the composition of final output can change toward goods with less material content. Overall, capital accumulation can be seen as the key background factor for such substitution processes (though also the arrival of new technical knowledge may be involved - we come back to this).

Whether capital accumulation can do the job depends crucially on the degree of substitutability between K and R. To see this, let the production function F be a three-factor CES production function. Suppressing the explicit dating of the variables when not needed for clarity, we have.

$$Y = \left(\alpha_1 K^\beta + \alpha_2 L^\beta + \alpha_3 R^\beta\right)^{1/\beta}, \ \alpha_1, \alpha_2, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 1, \beta < 1, \beta \neq 0. \tag{16.10}$$

We omit the the time index on Y, K, L, and R, when not needed for clarity. The important parameter is β, the *substitution parameter*. Let p_R denote the cost to the firm per unit of the resource flow and let $\hat{r}$ be the cost per unit of capital (generally, $\hat{r} = r + \delta$, where r is the real rate of interest). Then $p_R/\hat{r}$ is the relative factor price, which may be expected to increase as the resource becomes more scarce. The *elasticity of substitution* between K and R can be measured by $[d(K/R)/d(p_R/\hat{r})]\,(p_R/\hat{r}\ )/(K/R)$ evaluated along an isoquant curve, i.e., the percentage rise in the K-R ratio that a cost-minimizing firm will choose in response to a one-percent rise in the relative factor price, $p_R/\hat{r}$. Since we consider a CES production function, this elasticity is a constant $\sigma = 1/(1-\beta) > 0$. Indeed, the three-factor CES production function has the property that the elasticity of substitution between any pair of the three production factors is the same.

First, suppose $\sigma > 1$, i.e., $0 < \beta < 1$. Then, for fixed K and L, $Y \to \left(\alpha_1 K^\beta + \alpha_2 L^\beta\right)^{1/\beta} > 0$ when $R \to 0$. In this case of high substitutability the resource is seen to be *inessential* in the sense that it is not necessary for a positive output. That is, from a production perspective, conservation of the resource is not vital.

Suppose instead $\sigma < 1$, i.e., $\beta < 0$. Although increasing when R decreases, output per unit of the resource flow is then bounded from above. Consequently, the finiteness of the resource inevitably implies doomsday sooner or later if input substitution is the only salvage mechanism. To see this, keeping K and L fixed, we get

$$\frac{Y}{R} = Y(R^{-\beta})^{1/\beta} = \left[\alpha_1(\frac{K}{R})^\beta + \alpha_2(\frac{L}{R})^\beta + \alpha_3\right]^{1/\beta} \to {\alpha_3}^{1/\beta} \text{ for } R \to 0, \tag{16.11}$$

since $\beta < 0$. Even if K and L are increasing, $\lim_{R\to 0} Y = \lim_{R\to 0}(Y/R)R = \alpha_3^{1/\beta} \cdot 0 = 0$. Thus, when substitutability is low, the resource is *essential* in the sense that output is nil in the absence of the resource.

What about the intermediate case $\sigma = 1$? Although (16.10) is not defined for $\beta = 0$, using L'Hôpital's rule (as for the two-factor function, cf. Chapter 2), it can be shown that $\left(\alpha_1 K^\beta + \alpha_2 L^\beta + \alpha_3 R^\beta\right)^{1/\beta} \to K^{\alpha_1} L^{\alpha_2} R^{\alpha_3}$ for $\beta \to 0$. In the limit a three-factor Cobb-Douglas function thus appears. This function has $\sigma = 1$, corresponding to $\beta = 0$ in the formula $\sigma = 1/(1-\beta)$. The circumstances giving rise to the resource being essential thus include the Cobb-Douglas case $\sigma = 1$.

The interesting aspect of the Cobb-Douglas case is that it is the only case where the resource is essential while at the same time output per unit of the resource is unbounded from above (since $Y/R = K^{\alpha_1} L^{\alpha_2} R^{\alpha_3 - 1} \to \infty$ for $R \to 0$).[11] Under these circumstances it was considered an open question whether non-decreasing per capita consumption could be sustained. Therefore the Cobb-Douglas case was studied intensively. For example, Solow (1974) showed that if $n = \delta = 0$, then a necessary and sufficient condition that a constant positive level of consumption can be sustained is that $\alpha_1 > \alpha_3$. This condition in itself seems fairly realistic, since, empirically, α_1 is many times the size of α_3 (Nordhaus and Tobin, 1972, Neumayer 2000). Solow added the observation that under competitive conditions, the *highest* sustainable level of consumption is obtained when investment in capital exactly

[11] To avoid misunderstanding: by "Cobb-Douglas case" we refer to any function where R enters in a "Cobb-Douglas fashion", i.e., any function like $Y = \tilde{F}(K, L)^{1-\alpha_3} R^{\alpha_3}$.

equals the resource rent, $R \cdot \partial Y/\partial R$. This result was generalized in Hartwick (1977) and became known as *Hartwick's rule*. If there is population growth ($n > 0$), however, not even the Cobb-Douglas case allows sustainable per capita consumption unless there is sufficient technical progress, as equation (16.15) below will tell us.

Neumayer (2000) reports that the empirical evidence on the elasticity of substitution between capital and energy is inconclusive. Ecological economists tend to claim the poor substitution case to be much more realistic than the optimistic Cobb-Douglas case, not to speak of the case $\sigma > 1$. This invites considering the role of technical progress.

16.4.2 Technical progress

Solow (1974) and Stiglitz (1974) analyzed the theoretical possibility that resource-augmenting technological change can overcome the declining use of non-renewable resources that must be expected in the future. The focus is not only on whether a non-decreasing consumption level can be maintained, but also on the possibility of sustained per capita *growth* in consumption.

New production techniques may raise the efficiency of resource use. For example, Dasgupta (1993) reports that during the period 1900 to the 1960s, the quantity of coal required to generate a kilowatt-hour of electricity fell from nearly seven pounds to less than one pound.[12] Further, technological developments make extraction of lower quality ores cost-effective and make more durable forms of energy economical. On this background we incorporate resource-augmenting technical progress at the rate γ_3 and also allow labor-augmenting technical progress at the rate γ_2. So the CES production function now reads

$$Y_t = \left(\alpha_1 K_t^\beta + \alpha_2 (A_{2t} L_t)^\beta + \alpha_3 (A_{3t} R_t)^\beta\right)^{1/\beta}, \qquad (16.12)$$

where $A_{2t} = e^{\gamma_2 t}$ and $A_{3t} = e^{\gamma_3 t}$, considering $\gamma_2 \geq 0$ and $\gamma_3 > 0$ as exogenous constants. If the (proportionate) rate of decline of R_t is kept smaller than γ_3, then the "effective resource" input is no longer decreasing over time. As a consequence, even if $\sigma < 1$ (the poor substitution case), the finiteness of nature need not be an insurmountable obstacle to non-decreasing per capita consumption.

Actually, a technology with $\sigma < 1$ *needs* a considerable amount of resource-augmenting technical progress to obtain compliance with the empirical fact that the income share of natural resources has *not* been rising (Jones, 2002). When $\sigma < 1$, market forces tend to increase the income share of the factor

[12] For a historical account of energy technology, see Smil (1994).

that is becoming relatively more scarce. Empirically, K/R and Y/R have increased systematically. However, with a sufficiently increasing A_3, the income share $p_R R/Y$ need not increase in spite of $\sigma < 1$. Compliance with Kaldor's "stylized facts" (more or less constant growth rates of K/L and Y/L and stationarity of the output-capital ratio, the income share of labor, and the rate of return on capital) can be maintained with moderate labor-augmenting technical change (A_2 growing over time). The motivation for not allowing a rising A_1 and replacing K in (16.12) by $A_1 K$, is that this would be at odds with Kaldor's "stylized facts", in particular the absence of a trend in the rate of return to capital.

With $\gamma_3 > \gamma_2 + n$, we end up with conditions allowing a *balanced growth path* (BGP for short), which we in the present context, with an essential resource, define as a path along which the quantities Y, C, K, R, and S are positive and change at constant proportionate rates (some or all of which may be negative). Given (16.12), it can be shown that along a BGP with positive gross saving, $Y/(A_2 L)$ is constant and so $g_y = \gamma_2$ (hence also $g_c = \gamma_2$).[13] There is thus scope for a positive g_y if $0 < \gamma_2 < \gamma_3 - n$.

Of course, one thing is that such a combination of assumptions allows for an upward trend in per capita consumption - which is what we have seen since the industrial revolution. Another thing is: will the needed assumptions be satisfied for a long time in the future? Since we have considered *exogenous* technical change, there is so far no hint from theory. But, even taking endogenous technical change into account, there will be many uncertainties about what kind of technological changes will come through in the future and how fast.

Balanced growth in the Cobb-Douglas case

The described results go through in a more straightforward way in the Cobb-Douglas case. So let us consider this. A convenience is that capital-augmenting, labor-augmenting, and resource-augmenting technical progress become indistinguishable and can thus be merged into one technology variable, the total factor productivity A_t :

$$Y_t = A_t K_t^{\alpha_1} L_t^{\alpha_2} R_t^{\alpha_3}, \qquad \alpha_1, \alpha_2, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 1, \tag{16.13}$$

where we assume that A_t is growing at some constant rate $\tau > 0$. This, together with (16.6) - (16.8), is now the model under examination.

Log-differentiating w.r.t. time in (16.13) yields the growth-accounting relation

$$g_Y = \tau + \alpha_1 g_K + \alpha_2 n + \alpha_3 g_R. \tag{16.14}$$

[13] See Appendix.

In Appendix it is shown that along a BGP with positive gross saving the following holds:

(i) $g_K = g_Y = g_C \equiv g_c + n$;
(ii) $g_R = g_S \equiv \dot{S}_t/S_t \equiv -R_t/S_t \equiv -\bar{u} =$ constant < 0;
(iii) nothing of the resource is left unutilized forever.

With constant depletion rate, R_t/S_t, denoted $\bar{u}$, along the BGP, (16.14) thus implies

$$g_c = g_y = \frac{1}{1-\alpha_1}(\tau - \alpha_3 n - \alpha_3 \bar{u}), \tag{16.15}$$

since $\alpha_1 + \alpha_2 - 1 = -\alpha_3$.

Absent the need for input of limited natural resources, we would have $\alpha_3 = 0$ and so $g_c = \tau/(1-\alpha_1)$. But with $\alpha_3 > 0$, the non-renewable resource is essential and implies a *drag on per capita growth* equal to $\alpha_3(n+\bar{u})/(1-\alpha_1)$. We get $g_c > 0$ if and only if $\tau > \alpha_3(n+\bar{u})$ (where, the constant depletion rate, $\bar{u}$, can in principle, from a social point of view, be chosen very small if we want a strict conservation policy).

It is noteworthy that in spite of per-capita growth being due to exogenous technical progress, (16.15) shows that there is scope for policy affecting the long-run per-capita growth rate. Indeed, a policy affecting the depletion rate u in one direction will affect the growth rate in the opposite direction.

"Sustained growth" in K_t and c_t should not be understood in a narrow physical sense. As alluded to earlier, we have to understand K_t broadly as "produced means of production" of rising quality and falling material intensity; similarly, c_t must be seen as a composite of consumer "goods" with declining material intensity over time (see, e.g., Fagnart and Germain, 2011). This accords with the empirical fact that as income rises, the share of consumption expenditures devoted to agricultural and industrial products declines and the share devoted to services, hobbies, sports, and amusement increases. Although "economic development" is perhaps a more appropriate term (suggesting qualitative and structural change), we retain standard terminology and speak of "economic growth".

In any event, simple aggregate models like the present one should be seen as no more than a frame of reference, a tool for thought experiments. At best such models might have some validity as an approximate summary description of a certain period of time. One should be aware that an economy in which the ratio of capital to resource input grows without limit might well enter a phase where technological relations (including the elasticity of factor substitution) will be very different from now. For example, along *any* economic development path, the aggregate input of non-renewable resources must in the long run asymptotically approach zero. From a physical point of view, however, there must be some minimum amount of the resource below which it can not fulfil its role as a productive input. Thus, strictly speaking, sustainability requires that in the "very long run", non-renewable resources become inessential.

A backstop technology We end this sub-section by a remark on a rather different way of modeling resource-augmenting technical change. Dasgupta and Heal (1974) present a model of resource-augmenting technical change, considering it not as a smooth gradual process, but as something arriving in a discrete once-for-all manner with economy-wide consequences. The authors envision a future major discovery of, say, how to harness a lasting energy source such that a hitherto essential resource like fossil fuel becomes inessential. The contour of such a *backstop technology* might be currently known, but its practical applicability still awaits a technological breakthrough. The time until the arrival of this breakthrough is uncertain and may well be long. In Dasgupta, Heal and Majumdar (1977) and Dasgupta, Heal and Pand (1980) the idea is pursued further, by incorporating costly R&D. The likelihood of the technological breakthrough to appear in a given time interval depends positively on the accumulated R&D as well as the current R&D. It is shown that under certain conditions an index reflecting the probability that the resource becomes unimportant acts like an addition to the utility discount rate and that R&D expenditure begins to decline after some time. This is an interesting example of an early study of *endogenous* technological change.[14]

16.4.3 Increasing returns to scale

The third circumstance that might help overcoming the finiteness of nature is increasing returns to scale. For the CES function with poor substitution ($\sigma < 1$), however, increasing returns to scale, though helping, are not by themselves sufficient to avoid doomsday. For details, see, e.g., Groth (2007).

16.4.4 Summary on the DHSS model

Apart from a few remarks by Stiglitz, the focus of the fathers of the DHSS model is on constant returns to scale; and, as in the simple Solow and Ramsey growth models, only *exogenous* technical progress is considered. For our purposes we may summarize the DHSS results in the following way. Non-renewable resources do not really matter seriously if the elasticity of substi-

[14] A similar problem has been investigated by Kamien and Schwartz (1978) and Just et al. (2005), using somewhat different approaches.

tution between them and man-made inputs is above one. If not, however, then:

(a) absent technical progress, if $\sigma = 1$, sustainable per capita consumption requires $\alpha_1 > \alpha_3$ and $n = 0 = \delta$; otherwise, declining per capita consumption is inevitable and this is definitely the prospect, if $\sigma < 1$;

(b) on the other hand, if there is enough resource-augmenting and labor-augmenting technical progress, non-decreasing per capita consumption and even growing per capita consumption may be sustained;

(c) population growth, implying more mouths to feed from limited natural resources, exacerbates the drag on growth implied by a declining resource input; indeed, as seen from (16.15), the drag on growth is $\alpha_3(n + u)/(1 - \alpha_1)$ along a BGP.

The obvious next step is to examine how *endogenizing* technical change may throw new light on the issues relating to non-renewable resources, in particular the visions (b) and (c). Because of the non-rival character of technical knowledge, endogenizing knowledge creation may have profound implications, in particular concerning point (c). Indeed, the relationship between population growth and economic growth may be circumvented when endogenous creation of ideas (implying a form of increasing returns to scale) is considered. This is taken up in Section 16.5.

16.4.5 An extended DHSS model

The above discussion of sustainable economic development in the presence of non-renewable resources was carried out on the basis of the original DHSS model with only capital, labor, and a non-renewable resource as inputs. In practice the issues of input substitution and technological change are to a large extent interweaved into the question of substitutability of non-renewable with renewable resources. A more natural point of departure for the discussion may therefore be an extended DHSS model of the form:

$$\begin{aligned}
Y_t &= F(K_t, L_t, R_{rt}, R_{nt}, t), \qquad \partial F/\partial t \geq 0, \\
\dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ given}, \\
\dot{S}_{rt} &= G(S_{rt}) - R_{rt}, \qquad S_{r0} > 0 \text{ given}, \\
\dot{S}_{nt} &= -R_{nt}, \qquad S_{n0} > 0 \text{ given}, \\
\int_0^\infty R_{nt}dt &\leq S_{n0},
\end{aligned}$$

where R_{rt} is input of the renewable resource and S_{rt} the corresponding stock, while R_{nt} is input of the non-renewable resource to which corresponds the stock S_{nt}. Only the non-renewable resource is subject to the constraint of a finite upper bound on cumulative resource extraction.

Within such a framework a more or less gradual transition from use of non-renewable energy forms to renewable energy forms (hydro-power, wind energy, solar energy, biomass, and geothermal energy), likely speeded up learning by doing as well as R&D, can be studied (see for instance Tahvonen and Salo, 2001).

16.5 A two-sector R&D-based model

We shall look at the economy from the perspective of a fictional social planner who cares about finding a resource allocation so as to maximize the intertemporal utility function of a representative household subject to technical feasibility as given from the initial technology and initial resources.

16.5.1 The model

In addition to cost-free resource extraction, there are two "production" sectors, the manufacturing sector and the R&D sector. In the manufacturing sector the aggregate production function is

$$Y_t = A_t^{\varepsilon} K_t^{\alpha} L_t^{\beta} R_t^{\gamma}, \qquad \varepsilon, \alpha, \beta, \gamma > 0, \quad \alpha + \beta + \gamma + \lambda = 1, \tag{16.16}$$

whereY_t is output of manufacturing goods, while K_t, L_t, and R_t are inputs of capital, labor, and a non-renewable resource, respectively, per time unit at time t. Total factor productivity is A_t^{ε} where the variable A_t is assumed proportional to the stock of technical knowledge accumulated through R&D investment. Due to this proportionality we can simply identify A_t with the stock of knowledge at time t.

Aggregate manufacturing output is used for consumption, C_t, investment, I_{Kt}, in physical capital, and investment, I_{At}, in R&D,

$$C_t + I_{Kt} + I_{At} = Y_t.$$

Accumulation of capital occurs according to

$$\dot{K}_t = I_{Kt} - \delta_K K_t = Y_t - C_t - I_{At} - \delta_K K_t, \qquad \delta_K \geq 0, \qquad K_0 > 0 \text{ given}, \tag{16.17}$$

where δ_K is the (exogenous) rate of depreciation (decay) of capital.

In the R&D sector additions to the stock of technical knowledge are created through R&D investment, I_A :

$$\dot{A}_t = I_{At} - \delta_A A_t, \qquad \delta_A \geq 0 \qquad A_0 > 0 \text{ given.} \tag{16.18}$$

We allow for a positive depreciation rate, δ_A, to take into account the possibility that as technology advances, old knowledge becomes obsolete and then over time gradually becomes useless in production.

Extraction of the non-renewable resource is again given by

$$\dot{S}_t = -R_t \equiv -u_t S_t, \qquad S_0 > 0 \text{ given,} \tag{16.19}$$

where S_t is the stock of the non-renewable resource (e.g., oil reserves) and u_t is the depletion rate. Since we must have $S_t \geq 0$ for all t, there is a finite upper bound on cumulative resource extraction:

$$\int_0^\infty R_t dt \leq S_0. \tag{16.20}$$

Finally, population (= labor force) grows according to

$$L_t = L_0 e^{nt}, \qquad n \geq 0, \qquad L_0 > 0 \text{ given.}$$

Uncertainty is ignored and the extraction activity involves no costs.

This setup is elementarily related to what is known as "lab-equipment models". By investing a part of the manufacturing output, new knowledge is directly generated without intervention by researchers and similar.[15] Note also that there are no intertemporal knowledge-spillovers.

16.5.2 Analysis

We now skip the explicit dating of the variables where not needed for clarity. The model has *three* state variables, the stock, K, of physical capital, the stock, S, of non-renewable resources, and the stock, A, of technical knowledge. To simplify the dynamics, we will concentrate on the special case $\delta_A = \delta_K = \delta$. In this case, as we shall see, after an initial adjustment period, the economy behaves in many respects similarly to a reduced-form AK model.

Let us first consider *efficient paths*, i.e., paths such that aggregate consumption can not be increased in some time interval without being decreased

[15] An interpretation is that part of the activity in the manufacturing sector is directly R&D activity using the same technology (production function) as is used in the production of consumption goods and capital goods.

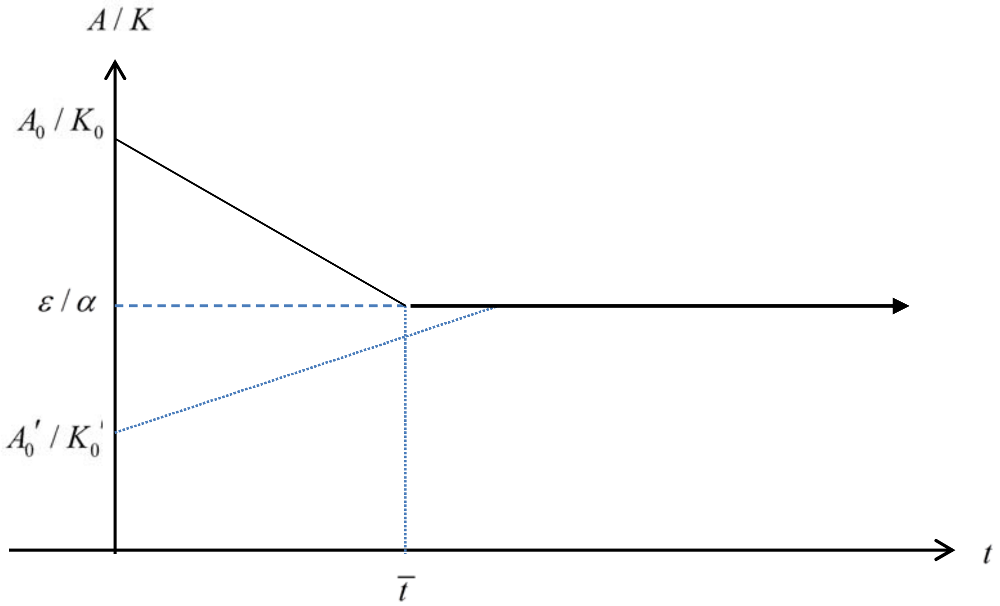

Figure 16.3: Initial complete specialization followed by balanced growth.

in another time interval. The net marginal productivities of A and K are equal if and only if $\varepsilon Y/A - \delta = \alpha Y/K - \delta$, i.e., if and only if

$$A/K = \varepsilon/\alpha.$$

The initial stocks, A_0 and K_0 are historically given. Suppose $A_0/K_0 > \varepsilon/\alpha$ as in Figure 16.3. Then, initially, the net marginal productivity of capital is larger than that of knowledge, i.e., capital is relatively scarce. An efficient economy will therefore for a while invest only in capital, i.e., there will be a phase where $I_A = 0$. This phase of complete specialization lasts until $A/K = \varepsilon/\alpha$, a state reached in finite time, say at time $\bar{t}$, cf. the figure. Hereafter, there is investment in both assets so that their ratio remains equal to the efficient ratio ε/α forever. Similarly, if initially $A_0/K_0 < \varepsilon/\alpha$, then there will be a phase of complete specialization in R&D, and after a finite time interval the efficient ratio $A/K = \varepsilon/\alpha$ is achieved and maintained forever.

For $t > \bar{t}$, at the aggregate level it is thus as if there were only one kind of capital, which we may call "broad capital" and define as $\tilde{K} = K + A = (\alpha+\varepsilon)K/\alpha$. Indeed, substitution of $A = \varepsilon K/\alpha$ and $K = \alpha\tilde{K}/(\varepsilon+\alpha)$ into (16.16) gives

$$Y = \frac{\varepsilon^{\varepsilon}\alpha^{\alpha}}{(\varepsilon+\alpha)^{\varepsilon+\alpha}}\tilde{K}^{\varepsilon+\alpha}L^{\beta}R^{\gamma} \equiv B\tilde{K}^{\tilde{\alpha}}L^{\beta}R^{\gamma}, \qquad \tilde{\alpha} \equiv \alpha+\varepsilon, \qquad (16.21)$$

so that $\tilde{\alpha} + \beta + \gamma > 1$. Further, adding (16.18) and (16.17) gives

$$\dot{\tilde{K}} = \dot{A} + \dot{K} = Y - cL - \delta\tilde{K}, \tag{16.22}$$

where c is per capita consumption.

We now proceed with a model based on broad capital, using (16.21), (16.22) and the usual resource depletion equation (16.19). Essentially, this model amounts to an extended DHSS model allowing increasing returns to scale, thereby offering a simple framework for studying *endogenous* growth with essential non-renewable resources.

We shall focus on questions like:

1 Is sustainable development (possibly even growth) possible within the model?

2 Can the utilitarian principle of discounted utility maximizing possibly clash with a requirement of sustainability? If so, under what conditions?

3 How can environmental policy be designed so as to enhance the prospects of sustainable development or even sustainable economic growth?

Balanced growth

Log-differentiating (16.21) w.r.t. t gives the "growth-accounting equation"

$$g_Y = \tilde{\alpha} g_{\tilde{K}} + \beta n + \gamma g_R. \tag{16.23}$$

Hence, along a BGP we get

$$(1 - \tilde{\alpha})g_c + \gamma u = (\tilde{\alpha} + \beta - 1)n. \tag{16.24}$$

Since $u > 0$, it follows immediately that:

Result (i) A BGP has $g_c > 0$ if and only if

$$(\tilde{\alpha} + \beta - 1)n > 0 \qquad \text{or} \qquad \tilde{\alpha} > 1. \tag{16.25}$$

Proof. Since $\gamma u > 0$, (*) implies $(1 - \tilde{\alpha})g_c < (\tilde{\alpha} + \beta - 1)n$. Hence, if $g_c > 0$, either $\tilde{\alpha} > 1$ or ($\tilde{\alpha} \leq 1$ and $(\tilde{\alpha} + \beta - 1)n > 0$). This proves "only if". The "if" part is more involved but follows from Proposition 2 in Groth (2004). □

Result (i) tells us that endogenous growth is theoretically possible, if there are either increasing returns to the capital-*cum*-labor input combined with population growth *or* increasing returns to capital (broad capital) itself. At least one of these conditions is required in order for capital accumulation to offset the effects of the inescapable waning of resource use over time. Based on Nordhaus (1992), $\alpha \approx 0.2, \beta \approx 0.6, \gamma \approx 0.1$, and $\lambda \approx 0.1$ seem reasonable. Given these numbers,

(i) semi-endogenous growth requires $(\varepsilon + \alpha + \beta - 1)n > 0$, hence $\varepsilon > 0.20$;

(ii) fully endogenous growth requires $\varepsilon + \alpha > 1$, hence $\varepsilon > 0.80$.

We have defined *fully endogenous growth* to be present if the long-run growth rate in per capita output is positive without the support of growth in any exogenous factor. Result (i) shows that only if $\tilde{\alpha} > 1$, is *fully* endogenous growth possible. Although the case $\tilde{\alpha} > 1$ has potentially explosive effects on the economy, if $\tilde{\alpha}$ is not too much above 1, these effects can be held back by the strain on the economy imposed by the declining resource input.

In some sense this is "good news": fully endogenous steady growth is theoretically possible and no knife-edge assumption is needed. As we have seen in earlier chapters, in the conventional framework without non-renewable resources, fully endogenous growth requires constant returns to the producible input(s) in the growth engine. In our one-sector model the growth engine is the manufacturing sector itself, and without the essential non-renewable resource, fully endogenous growth would require the knife-edge condition $\tilde{\alpha} = 1$ ($\tilde{\alpha}$ being above 1 is excluded in this case, because it would lead to explosive growth in a setting without some countervailing factor). When non-renewable resources are an essential input in the growth engine, a drag on the growth potential is imposed. To be able to offset this drag, fully endogenous growth requires *increasing* returns to capital.

The "bad news" is, however, that even in combination with essential non-renewable resources, an assumption of increasing returns to capital seems too strong and too optimistic. A technology having $\tilde{\alpha}$ just slightly above 1 can sustain *any* per capita growth rate − there is no upper bound on g_c.[16] This appears overly optimistic.

This leaves us with *semi-endogenous* growth as the only plausible form of endogenous growth (as long as n is not endogenous). Indeed, Result (i) indicates that semi-endogenous growth corresponds to the case $1-\beta < \tilde{\alpha} \leq 1$. In this case sustained positive per capita growth driven by some internal mechanism is possible, but only if supported by $n > 0$, that is, by growth in an exogenous factor, here population size.

[16] This is shown in Groth (2004).

Growth policy and conservation

Result (i) is only about whether the technology as such allows a positive per capita growth rate or not. What about the *size* of the growth rate? Can the growth rate temporarily or perhaps permanently be affected by economic policy? The simple growth-accounting relation (16.24) immediately shows:

Result (ii) Along a BGP, policies that decrease (increase) the depletion rate u (and only such policies) will increase (decrease) the per capita growth rate (here we presuppose $\tilde{a} < 1$, the plausible case).

This observation is of particular interest in view of the fact that changing the perspective from exogenous to endogenous technical progress implies bringing a source of numerous market failures to light. On the face of it, the result seems to run against common sense. Does high growth not imply *fast* depletion (high u)? Indeed, the answer is affirmative, but with the addition that exactly because of the fast depletion such high growth will only be temporary − it carries the seeds to its own obliteration. For faster sustained growth there must be sustained slower depletion. The reason for this is that with protracted depletion, the rate of decline in resource input becomes smaller. Hence, so does the drag on growth caused by this decline.

As a statement about policy and long-run growth, (ii) is a surprisingly succinct conclusion. It can be clarified in the following way. For policy to affect long-run growth, it must affect a linear differential equation linked to the basic goods sector in the model. In the present framework the resource depletion relation,

$$\dot{S} = -uS,$$

is such an equation. In balanced growth $g_S = -R/S \equiv -u$ is constant, so that the proportionate rate of decline in R must comply with, indeed be equal to, that of S. Through the growth accounting relation (16.23), given u, this fixes g_Y and $g_{\tilde{K}}$ (equal in balanced growth), hence also $g_c = g_Y - n$.

The conventional wisdom in the endogenous growth literature is that interest income taxes impede economic growth and investment subsidies promote economic growth. Interestingly, this need not be so when non-renewable resources are an essential input in the growth engine (which is here the manufacturing sector itself). At least, starting from a Cobb-Douglas aggregate production function as in (16.16), it can be shown that only those policies that interfere with the depletion rate u in the long run (like a profits tax on resource-extracting companies or a time-dependent tax on resource use) can affect long-run growth. It is noteworthy that this long-run policy result holds whether $g_c > 0$ or not and whether growth is exogenous, semi-endogenous or fully endogenous.[17] The general conclusion is that with non-renewable resources entering the growth-generating sector in an essential way, conventional policy tools receive a different role and there is a role for new tools (affecting long-run growth through affecting the depletion rate).[18]

Introducing preferences

To be more specific we introduce household preferences and a social planner. The resulting resource allocation will coincide with that of a decentralized competitive economy if agents have perfect foresight and the government has introduced appropriate subsidies and taxes. As in Stiglitz (1974a), let the utilitarian social planner choose a path $(c_t, R_t)_{t=0}^{\infty}$ so as to optimize

$$U_0 = \int_0^{\infty} \frac{{c_t}^{1-\theta}}{1-\theta} L_t e^{-\rho t} dt, \qquad \theta > 0, \tag{16.26}$$

subject to the constraints given by technology, i.e., (16.21), (16.22), and (16.19), and initial conditions. The parameter $\theta > 0$ is the (absolute) elasticity of marginal utility of consumption (reflecting the strength of the desire for consumption smoothing) and ρ is a constant rate of time preference.[19]

Using the Maximum Principle, the first-order conditions for this problem lead to, first, the social planner's *Keynes-Ramsey* rule,

$$g_c = \frac{1}{\theta}(\frac{\partial Y}{\partial \tilde{K}} - \delta - \rho) = \frac{1}{\theta}(\tilde{\alpha}\frac{Y}{\tilde{K}} - \delta - \rho), \tag{16.27}$$

second, the social planner's *Hotelling rule,*

$$\frac{d(\partial Y/\partial R)}{dt} = \frac{\partial Y}{\partial R}(\frac{\partial Y}{\partial \tilde{K}} - \delta) = \gamma\frac{Y}{R}(\tilde{\alpha}\frac{Y}{\tilde{K}} - \delta). \tag{16.28}$$

The Keynes-Ramsey rule says: as long as the net return on investment in capital is higher than the rate of time preference, one should let current c be low enough to allow positive net saving (investment) and thereby higher consumption in the future. The Hotelling rule is a no-arbitrage condition saying that the return ("capital gain") on leaving the marginal unit of the resource

[17]This is a reminder that the distinction between fully endogenous growth and semi-endogenous growth is not the same as the distinction between policy-dependent and policy-invariant growth.

[18]These aspects are further explored in Groth and Schou (2006).

[19]For simplicity we have here ignored (as does Stiglitz) that also environmental quality should enter the utility function.

in the ground must equal the return on extracting and using it in production and then investing the proceeds in the alternative asset (reproducible capital).[20]

After the initial phase of complete specialization described above, we have, due to the proportionality between K, A and $\tilde{K}$, that $\partial Y/\partial K = \partial Y/\partial A = \partial Y/\partial \tilde{K} = \tilde{\alpha} Y/\tilde{K}$. Notice that the Hotelling rule is independent of preferences; *any* path that is *efficient* must satisfy the Hotelling rule (as well as the exhaustion condition $\lim_{t\to\infty} S(t) = 0$).

Using the Cobb-Douglas specification, we may rewrite the Hotelling rule as $g_Y - g_R = \tilde{\alpha}Y/\tilde{K} - \delta$. Along a BGP $g_Y = g_C = g_c + n$ and $g_R = -u$, so that the Hotelling rule combined with the Ramsey rule gives

$$(1-\theta)g_c + u = \rho - n. \tag{16.29}$$

This linear equation in g_c and u combined with the growth-accounting relationship (16.24) constitutes a linear two-equation system in the growth rate and the depletion rate. The determinant of this system is $D \equiv 1 - \tilde{\alpha} - \gamma + \theta\gamma$. We assume $D > 0$, which seems realistic and is in any case necessary (and sufficient) for stability.[21] Then

$$g_c = \frac{(\tilde{\alpha} + \beta + \gamma - 1)n - \gamma\rho}{D}, \quad \text{and} \tag{16.30}$$

$$u = \frac{\left[(\tilde{\alpha} + \beta - 1)\theta - \beta\right] n + (1-\tilde{\alpha})\rho}{D}. \tag{16.31}$$

To ensure boundedness from above of the utility integral (16.26) we need the parameter restriction $\rho - n > (1-\theta)g_c$, which we assume satisfied for g_c as given in (16.30).

Interesting implications are:

Result (iii) If there is impatience ($\rho > 0$), then even when a non-negative g_c is technically feasible (i.e., (16.25) satisfied), a negative g_c can be optimal and stable.

[20] After Hotelling (1931), who considered the "rule" in a market economy. Assuming perfect competition, the real resource price is $p_R = \partial Y/\partial R$ and the real rate of interest is $r = \partial Y/\partial K - \delta$. Then the rule takes the form $\dot{p}_R/p_R = r$. If there are extraction costs at rate $C(R, S, t)$, then the rule takes the form $\dot{p}_S - \partial C/\partial S = rp_S$, where p_S is the price of the unextracted resource (whereas $p_R = p_S + \partial C/\partial R$).

It is another matter that the rise in resource prices and the predicted decline in resource use have not yet shown up in the data (Krautkraemer 1998, Smil 2003); this may be due to better extraction technology and discovery of new deposits. But in the long run, if non-renewable resources *are* essential, this tendency inevitably will be reversed.

[21] As argued above, $\tilde{\alpha} < 1$ seems plausible. Generally, θ is estimated to be greater than one (see, e.g., Attanasio and Weber 1995); hence $D > 0$. The stability result as well as other findings reported here are documented in Groth and Schou (2002).

Result (iv) Population growth is *good* for economic growth. In its absence, when $\rho > 0$, we get $g_c < 0$ along an optimal BGP; if $\rho = 0$, $g_c = 0$ when $n = 0$.

Result (v) There is never a scale effect on the growth rate.

Result (iii) reflects that utility discounting and consumption smoothing weaken the "growth incentive".

Result (iv) is completely contrary to the conventional (Malthusian) view and the learning from the DHSS model. The point is that two offsetting forces are in play. On the one hand, higher n means more mouths to feed and thus implies a drag on per capita growth (Malthus). On the other hand, a growing labour force is exactly what is needed in order to exploit the benefits of increasing returns to scale (anti-Malthus). And at least in the present framework this dominates the first effect. This feature might seem to be contradicted by the empirical finding that there is no robust correlation between g_c and population growth in cross-country regressions (Barro and Sala-i-Martin 2004, Ch. 12). However, the proper unit of observation in this context is not the individual country. Indeed, in an internationalized world with technology diffusion a positive association between n and g_c as in (16.30) should not be seen as a prediction about individual countries, but rather as pertaining to larger regions, perhaps the global economy. In any event, the second part of Result (iv) is a dismal part − in view of the projected long-run stationarity of world population (United Nations 2005).

A somewhat surprising result appears if we imagine (unrealistically) that $\tilde{\alpha}$ is sufficiently above one to make D a negative number. If population growth is absent, $D < 0$ is in fact needed for $g_c > 0$ along a BGP. However, $D < 0$ implies instability. Hence this would be a case of an instable BGP with fully endogenous growth.[22]

As to Result (v), it is noteworthy that the absence of a scale effect on growth holds for *any* value of $\tilde{\alpha}$, including $\tilde{\alpha} = 1$.[23]

A pertinent question is: are the above results just an artifact of the very simplified reduced-form AK-style set-up applied here? The answer turns out

[22]Thus, if we do not require $D > 0$ in the first place, (iv) could be reformulated as: existence of a *stable* optimal BGP with $g_c > 0$ *requires* $n > 0$. This is not to say that reducing n from positive to zero renders an otherwise stable BGP instable. Stability-instability is governed solely by the sign of D. Given $D > 0$, letting n decrease from a level above the critical value, $\gamma\rho/(\tilde{\alpha} + \beta + \gamma - 1)$, given from (16.30), to a level below, changes g_c from positive to negative, i.e., growth comes to an end.

[23]More commonplace observations are that increased impatience leads to faster depletion and lower growth (in the plausible case $\tilde{\alpha} < 1$). Further, in the log-utility case ($\theta = 1$) the depletion rate u equals the effective rate of impatience, $\rho - n$.

to be no. For models with a distinct research technology and intertemporal knowledge spillovers, this is shown in Groth (2007).

16.6 Natural resources and the issue of limits to economic growth

Two distinguished professors were asked by a journalist: Are there limits to economic growth?
The answers received were:[24]

Clearly YES:

- A finite planet!
- The amount of cement, oil, steel, and water that we can use is limited!

Clearly NO:

- Human creativity has no bounds!
- The quality of wine, TV transmission of concerts, computer games, and medical treatment knows no limits!

An aim of this chapter has been to bring to mind that it would be strange if there were no limits to growth. So a better question is:

What determines the limits to economic growth?

The answer suggested is that these limits are determined by the capability of the economic system to substitute limited natural resources by man-made goods the variety and quality of which are expanded by creation of new ideas. In this endeavour frontier countries, first the U.K. and Europe, next the United States, have succeeded at a high rate for two and a half century. To what extent this will continue in the future nobody knows. Some economists, e.g. Gordon (2012), argue there is an enduring tendency to slowing down of innovation and economic growth (the low-hanging fruits have been taken). Others, e.g. Brynjolfsson and McAfee (2012, 2014), disagree. They reason that the potentials of information technology and digital communication are on the verge of the point of ubiquity and flexible application. For these authors the prospect is "The Second Machine Age" (the title of their recent book), by which they mean a new innovative epoch where smart machines and new ideas are combined and recombined - with pervasive influence on society.

[24] Inspired by Sterner (2008).

16.7 Bibliographic notes concerning Section 16.5

It is not always recognized that the research of the 1970s on macro implications of essential natural resources in limited supply already laid the groundwork for a theory of endogenous and policy-dependent growth with natural resources. Actually, by extending the DHSS model, Suzuki (1976), Robson (1980) and Takayama (1980) studied how endogenous innovation may affect the prospect of overcoming the finiteness of natural resources.

Suzuki's (1976) article contains an additional model, involving a resource externality. Interpreting the externality as a "greenhouse effect", Sinclair (1992, 1994) and Groth and Schou (2006) pursue this issue further. In the latter paper a configuration somewhat similar to the model in Section 16.5 is studied. The source of increasing returns to scale is not intentional creation of knowledge, however, but learning as a by-product of investing as in Arrow (1962a) and Romer (1986). Empirically, the evidence furnished by, e.g., Hall (1990) and Caballero and Lyons (1992) suggests that there are quantitatively significant increasing returns to scale w.r.t. capital and labour or external effects in US and European manufacturing. Similarly, Antweiler and Trefler (2002) examine trade data for goods-producing sectors and find evidence for increasing returns to scale.

Concerning Result (i) in Section 16.5, note that if some irreducibly exogenous element in the technological development is allowed in the model by replacing the constant B in (16.21) by $e^{\tau t}$, where $\tau \geq 0$, then (16.25) is replaced by $\tau + (\tilde{\alpha} + \beta - 1)n > 0$ or $\tilde{\alpha} > 1$. Both Stiglitz (1974a, p. 131) and Withagen (1990, p. 391) ignore implicitly the possibility $\tilde{\alpha} > 1$. Hence, from the outset they preclude fully endogenous growth.

16.8 Appendix: Balanced growth with an essential non-renewable resource

The production side of the DHSS model with CES production function is described by:

$$\begin{aligned} Y_t &= \tilde{F}(K_t, L_t, R_t, t), \qquad \partial\tilde{F}/\partial t \geq 0, & (16.32) \\ \dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ given}, & (16.33) \\ \dot{S}_t &= -R_t \equiv -u_t S_t, \qquad S_0 > 0 \text{ given}, & (16.34) \\ L_t &= L_0 e^{nt}, \qquad n \geq 0, & (16.35) \\ \int_0^\infty R_t dt &\leq S_0. & (16.36) \end{aligned}$$

We will assume that the non-renewable resource is essential, i.e.,

$$R_t = 0 \text{ implies } Y_t = 0. \tag{16.37}$$

From now we omit the dating of the time-dependent variables where not needed for clarity. Recall that in the context of an essential non-renewable resource, we define a *balanced growth path* (BGP for short) as a path along which the quantities Y, C, K, R, and S are positive and change at constant proportionate rates (some or all of which may be negative).

Lemma 1 Along a BGP the following holds: (a) $g_S = g_R < 0$; (b) $R(0) = -g_R S(0)$, and

$$\lim_{t\to\infty} S = 0. \tag{16.38}$$

Proof Consider a BGP. (a) From (16.34), $g_S = -R/S$; differentiating with respect to time gives

$$\dot{g}_S = -(g_R - g_S)R/S = 0,$$

by definition of a BGP. Hence, $g_S = g_R$ since $R > 0$ by definition. For any constant g_R we have $\int_0^\infty R_t dt = \int_0^\infty R_0 e^{g_R t} dt$. If $g_R \geq 0$, (16.36) would thus be violated. Hence, $g_R < 0$. (b) With $t = 0$ in (16.34), we get $\dot{S}_0/S_0 = -R_0/S_0 = g_R$, the last equality following from (a). Hence, $R_0 = -g_R S_0$. Finally, the solution to (16.34) can be written $S_t = S_0 e^{g_S t}$. Then, since g_S is a negative constant, $S_t \to 0$ for $t \to \infty$. □

Define

$$z \equiv \frac{Y}{K}, \qquad x \equiv \frac{C}{K}, \qquad \text{and} \qquad u \equiv \frac{R}{S}. \tag{16.39}$$

We may write (16.34) as

$$g_K = z - x - \delta. \tag{16.40}$$

Similarly, by (16.34),

$$g_S \equiv -u. \tag{16.41}$$

Lemma 2 Along a BGP, $g_R = g_S = -u < 0$ is constant and $g_Y = g_C$. If gross saving is positive in some time interval, we have along the BGP in addition that $g_K = g_Y$, both constant, and that z and x are constant.

Proof Consider a BGP. Since g_S is constant by definition of a BGP, u must also be constant in view of (16.41). Then, by Lemma 1, $g_R = g_S = -u$ is constant and $u > 0$. Differentiating in (16.40) with respect to t gives $\dot{g}_K = \dot{z} - \dot{x} = (g_Y - g_K)z - (g_C - g_K)x = 0$ since g_K is constant along a BGP. Dividing through by z, which is positive along a BGP, and reordering gives

$$g_Y - g_K = (g_C - g_K)\frac{x}{z}. \tag{16.42}$$

But this is a contradiction unless $g_Y = g_C$; indeed, if $g_Y \neq g_C$, then $g_Y - g_K \neq g_C - g_K$ at the same time as $x/z = C/Y \to 0$ if $g_Y > g_C$, and $x/z = C/Y \to \infty$ if $g_Y < g_C$, both cases being incompatible with (16.42) and the presumed constancy of g_Y, g_K, and g_C, hence constancy of both $g_Y - g_K$, and $g_C - g_K$. So $g_Y = g_C$ along a BGP. Suppose gross saving is positive in some time interval and that at the same time $g_K \neq g_Y = g_C$, then (16.42) implies $x/z \equiv 1$, i.e., $C = Y$ for all t or gross saving $= 0$ for all t, a contradiction. Hence, $g_K = g_Y = g_C$. It follows by (16.39) that z and x are constant. □

Consider the case where the production function is neoclassical with CRS, and technical progress is labor- and resource-augmenting:

$$\begin{aligned} Y_t &= F(K_t, A_{2t}L_t, A_{3t}R_t), \\ A_{2t} &= e^{\gamma_2 t}, \quad \gamma_2 \geq 0, \qquad A_{3t} = e^{\gamma_3 t}, \quad \gamma_3 \geq 0. \end{aligned} \tag{16.43}$$

Let $\hat{L} \equiv A_2 L$ and $\hat{R} \equiv A_3 R$. Let ε_K, $\varepsilon_{\hat{L}}$, and $\varepsilon_{\hat{R}}$ denote the output elasticities w.r.t. K, $\hat{L}$, and $\hat{R}$, i.e.,

$$\varepsilon_K \equiv \frac{K}{Y}\frac{\partial Y}{\partial K}, \qquad \varepsilon_{\hat{L}} \equiv \frac{A_2 L}{Y}\frac{\partial Y}{\partial (A_2 L)}, \qquad \varepsilon_{\hat{R}} \equiv \frac{A_3 R}{Y}\frac{\partial Y}{\partial (A_3 R)}.$$

Differentiating in (16.43) w.r.t. t and dividing through by Y (as in growth-accounting), we then have

$$\begin{aligned} g_Y &\equiv \frac{\dot{Y}}{Y} = \varepsilon_K g_K + \varepsilon_{\hat{L}}(\gamma_2 + n) + \varepsilon_{\hat{R}}(\gamma_3 + g_R) \\ &= \varepsilon_K g_K + \varepsilon_{\hat{L}}(\gamma_2 + n) + (1 - \varepsilon_K - \varepsilon_{\hat{L}})(\gamma_3 + g_R), \end{aligned} \tag{16.44}$$

the last equality being implied by the CRS property in (16.43).

Suppose the economy follows a BGP with positive gross saving. Then, by Lemma 2, $g_K = g_Y$ and $g_R = -u < 0$. Hence, (16.44) can be written

$$(1 - \varepsilon_K)(g_Y - (\gamma_3 - u)) = \varepsilon_{\hat{L}}(\gamma_2 + n - (\gamma_3 - u)). \tag{16.45}$$

Consider the special case where F is CES:

$$Y_t = \left(\alpha_1 K_t^\beta + \alpha_2 (A_{2t} L_t)^\beta + \alpha_3 (A_{3t} R_t)^\beta\right)^{1/\beta}, \ \alpha_1, \alpha_2, \alpha_3 > 0, \sum_i \alpha_i = 1, \beta < 1. \tag{16.46}$$

As we know from Chapter 2 that, for $\beta = 0$, the CES formula can be interpreted as the Cobb-Douglas formula (16.13). Applying (**??**) from Chapter 2, the output elasticities w.r.t. K, $\hat{L}$, and $\hat{R}$ are

$$\varepsilon_K = \alpha_1 \left(\frac{Y}{K}\right)^{-\beta}, \quad \varepsilon_{\hat{L}} = \alpha_2 \left(\frac{Y}{A_2 L}\right)^{-\beta}, \quad \text{and} \quad \varepsilon_{\hat{R}} = \alpha_3 \left(\frac{Y}{A_3 R}\right)^{-\beta}, \tag{16.47}$$

respectively.

Lemma 3 Let $y \equiv Y/L$ and $c \equiv C/L$. Given (16.35) and (16.46), along a BGP with positive gross saving, ε_K and $\varepsilon_{\hat{L}}$ are constant, and $g_c = g_y = \gamma_2$. In turn, such a BGP exists if and only if

$$u = \gamma_3 - (\gamma_2 + n) > 0. \tag{16.48}$$

Proof Consider a BGP with positive gross saving. By Lemma 2, $0 > g_R = -u$ is constant and $Y/K \equiv z$ is constant, hence so is ε_K. The left hand side of (16.45) is thus constant and so must the right-hand side therefore be. Suppose that, contrary to (16.48), $u \neq \gamma_3 - (\gamma_2 + n)$. Then constancy of the right-hand of (16.45) requires that $\varepsilon_{\hat{L}}$ is constant. In turn, by (16.47), this requires that $Y/(A_2 L) \equiv y/A_2$ is constant. Consequently, $g_y = \gamma_2 = g_c$, where the second equality is implied by the claim in Lemma 2 that along a BGP with positive gross saving in some time interval, $g_C = g_Y$. As Y/K is constant, it follows that $g_K = g_Y \equiv g_y + n = \gamma_2 + n$. Inserting this into (16.44) and rearranging, we get

$$(1 - \varepsilon_K - \varepsilon_{\hat{L}})(\gamma_2 + n) = (1 - \varepsilon_K - \varepsilon_{\hat{L}})(\gamma_3 - u)$$

where the last equality follows from $g_R = -u$. Isolating u gives the equality in (16.48). Thereby, our assumption $u \neq \gamma_3 - (\gamma_2 + n)$ leads to a contradiction. Hence, given (16.35) and (16.46), if a BGP with positive gross saving exists, then $u = \gamma_3 - (\gamma_2 + n) > 0$. This shows the necessity of (16.48). The sufficiency of (16.48) follows by construction, starting by fixing u, and thereby $-g_R$, in accordance with (16.48) and moving "backward", showing consistency with (16.44) for $g_K = g_Y = \gamma_2 + n$. □

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Short Note 1. Economic Growth.

April 11, 2016. Christian Groth

Robustness issues and scale effects

This note adds some conceptual and empirical perspectives to the discussion in Chapter 5 and 9 in Jones and Vollrath (2013).

1 Different growth patterns

Notation: $Y = GDP$, $y \equiv Y/L$, and $g_y \equiv \dot{y}/y$; time is continuous.

Economic growth can take different forms. It can be *exponential*:

$$y_t = y_0 e^{gt}, \qquad g > 0. \tag{1}$$

Ignoring business cycle fluctuations this describes quite well what we have seen in industrialized economies since the industrial revolution (with $g \in (0.01, 0.02)$ on annual basis). When growth is exponential, the *growth rate,* $\dot{y}/y$, is a positive constant, here equal to g.

Growth can alternatively take the form of *arithmetic growth:*

$$y_t = y_0 + \alpha t, \qquad \alpha > 0. \tag{2}$$

Here $\dot{y} = \alpha$, the *momentum,* is a positive constant. So, in spite of the growth rate, $\dot{y}/y$, approaching zero for t going to infinity, we have $y_t \to \infty$ for $t \to \infty$.

More generally, growth can take the form of *quasi-arithmetic growth*:

$$y_t = y_0(1 + \alpha\beta t)^{1/\beta}, \qquad \alpha > 0,\ \beta > 0. \tag{3}$$

In the special case $\beta = 1$ and $y_0 = 1$, this is arithmetic growth. The parameter β is the *damping coefficient.* The case of *strict stagnation*, $y_t = y_0$ for all $t \geq 0$, can be interpreted as the limiting case $\beta \to \infty$.[1] On the other hand, in the limit, when $\beta \to 0$ (no damping), the growth path (3) becomes exponential growth, $y_t = y_0 e^{\alpha t}$.[2]

[1] To see this, use L'Hôpital's rule for "∞/∞" on $\ln y_t = \ln y_0 + \frac{1}{\beta}\ln(1 + \alpha\beta t)$. The term *stagnation* also covers the case of *asymptotic stagnation* where in spite of $\dot{y} > 0$ for all $t \geq 0$, $\dot{y}$ goes towards zero fast enough so that there is an upper bound, $\bar{y}$, for y_t, i.e., $y_t < \bar{y}$ for all $t \geq 0$. For instance *logistic growth* has this property. (Logistic growth is the growth path generated by the differential equation $\dot{y}_t = \alpha y_t(1 - y_t/\bar{y})$, $\alpha > 0$, $0 < y_0 < \bar{y}$.)

[2] To see this, use L'Hôpital's rule for "0/0" on $\ln y_t = \ln y_0 + \frac{1}{\beta}\ln(1 + \alpha\beta t)$.

These alternative growth patterns can be generated for alternative parameter values of essentially the same model, namely a model that leads to the differential equation

$$\dot{y}_t = \alpha y_0^{\beta} {y_t}^{1-\beta}, \qquad \alpha > 0, \beta \geq 0. \tag{4}$$

In case $\beta = 0$, (4) is a linear differential equation that has the solution (1) with $g = \alpha$, which is exponential growth. In case $\beta > 0$, (4) is an autonomous Bernoulli equation that has the solution (3), which is quasi-arithmetic growth.[3] For alternative values of β between 0 and infinity, quasi-arithmetic growth covers the whole range between exponential growth and strict stagnation. We rule out the case of $\beta < 0$ which would imply that the model could only temporarily describe reality, because $\beta < 0$ leads to *explosive* growth: y_t approaching infinity in *finite* time (the "end of scarcity").

Several prominent macroeconomists, e.g., Lawrence Summers, Robert Gordon, and our own Charles Jones, predict that economic growth in the future will be lower than what we have seen in the 20th century. One of the reasons emphasized by Jones and others is the slowdown of population growth and thereby, everything else equal, dampening of growth of the source of new ideas. Along this line, in a coming exercise you will be asked to show what long-run growth pattern the horizontal innovations model with $\varphi < 1$ and $n = 0$ implies.

2 The term "endogenous growth" and all that

How terms like "endogenous growth" and "semi-endogenous growth" are defined varies in the literature. In this course we use the following definitions. A model features:

endogenous growth if $y_t \to \infty$ for $\to \infty$, and the source of this evolution is some *internal* mechanism in the model (rather than exogenous technology growth);

fully-endogenous growth if growth is endogenous in such way that $y_t \to \infty$ for $\to \infty$ occurs even if there is no support by growth in any exogenous factor;[4]

[3] It is clear that with $0 < \beta < \infty$, the solution formula (3) can not be extended, without bound, *backward* in time. For $t = -(\alpha\beta)^{-1} \equiv \bar{t}$, we get $y_t = 0$, and thus, according to (4), $y_t = 0$ for all $t \leq \bar{t}$. This should not, however, be considered a necessarily problematic feature. A certain growth regularity need not be applicable to all periods in history. It may apply only to specific historical epochs characterized by a particular institutional environment.

[4] An alternative name for this case is *strictly endogenous growth.*

semi-endogenous growth if growth is endogenous in such way that $y_t \to \infty$ for $\to \infty$ occurs only if the growth path is supported by growth in *some* exogenous factor (for example exogenous growth in the labor force).

If in the above three cases, the weak growth criterion "$y_t \to \infty$ for $\to \infty$" is replaced by exponential growth, then we speak of *endogenous, fully-endogenous,* and *semi-endogenous exponential growth,* respectively. If instead the weak growth criterion is replaced by, for instance, arithmetic growth, we speak of *endogenous, fully-endogenous,* and *semi-endogenous arithmetic growth,* respectively,

An example of fully endogenous exponential growth is the endogenous growth generated in the Romer case ($\varphi = 1, n = 0$) of the horizontal innovations model. An example of semi-endogenous exponential growth is the Jones case ($\varphi < 1, n > 0$) of the horizontal innovations model.

When Romer's case is combined with Ramsey households, we get steady-state results of the following kind: $\partial g_y^*/\partial \rho < 0$ and $\partial g_y^*/\partial \theta < 0$ (standard notation). That is, preference parameters matter for long-run growth. This suggests, at least at the theoretical level, that taxes and subsidies, by affecting incentives, may have effects on long-run growth.

In any case, fully-endogenous exponential growth is technologically possible if and only if there are *non-diminishing returns* (at least asymptotically) *to the producible inputs* in the growth-generating sector(s), also called the *growth engine.* The growth engine in an endogenous growth model is defined as the set of input-producing sectors or activities using their own output as input. This set may consist of only one element, for instance the R&D sector in the horizontal innovations model, the manufacturing sector in the simple AK model, and the educational sector in the Lucas (1988) model. A model is capable of generating fully-endogenous exponential growth if the growth engine has *CRS w.r.t. producible inputs.*

No argument, however, like the replication argument for CRS w.r.t. the *rival* inputs exists regarding CRS w.r.t. the *producible inputs.* This theoretical limitation, combined with strong empirically founded skepticism, motivated Jones to introduce his *semi-endogenous* version of the horizontal innovations model (Jones1995a, 1995b), where $\varphi < 1, n > 0$. In that version, in the long run

$$g_y = g_k = g_c = \frac{n}{1-\varphi} \equiv g_y^*. \tag{5}$$

If a certain degree, ξ, of R&D overlap is added, $0 \leq \xi < 1$, we instead get $g_y^* = (1 - \xi)n/(1-\varphi)$.[5]

So, in this case, if and only if $n > 0$, can a positive constant per capita growth rate be maintained forever. Only when the R&D outcome is *assisted* by growth in the exogenous *source* of ideas, population, is the growth engine strong enough to maintain exponential growth. The key role of population growth derives from the fact that at the aggregate level there are increasing returns to scale w.r.t. capital, labor, *and* knowledge. For the increasing returns to be sufficiently exploited to generate exponential growth, population growth is needed. Note that if the Jones case is combined with Ramsey households, we get $\partial g_y^*/\partial\rho = 0 = \partial g_y^*/\partial\theta$, that is, preference parameters do not matter for *long-run* growth (only for the *level* of the growth path, see Section 4 below). This suggests that taxes and subsidies do not have *long-run* growth effects. Yet, in the Jones model and similar semi-endogenous growth models, economic policy can have important permanent *level* effects. Moreover, the only temporary growth effects can be quite durable because the speed of convergence is low (see Jones, 1995a).

Strangely enough, some textbooks (for example Barro and Sala-i-Martin, 2004) do not call much attention to the distinction between fully-endogenous growth and semi-endogenous growth (and even less attention to the distinction between exponential growth and weaker forms of growth). Rather, they tend to use the term "endogenous growth" as synonymous with what we here call "fully-endogenous exponential growth". But there is certainly no reason to rule out *apriori* the parameter cases corresponding to semi-endogenous growth.

In the Acemoglu textbook (Acemoglu, 2009, p. 448), "semi-endogenous growth" is defined or characterized as endogenous growth where the long-run per capita growth rate of the economy "does not respond to taxes or other policies". As an implication, endogenous growth which is not semi-endogenous is in Acemoglu's text implicitly defined as endogenous growth where the long-run per capita growth rate of the economy *does* respond to taxes or other policies.

We have defined the distinction between "semi-endogenous growth" and "fully-endogenous growth" in a different way. In our terminology, this distinction does not coincide with the distinction between policy-dependent and policy-invariant growth. Indeed, in our ter-

[5] Of course the model shifts from featuring "semi-" to featuring "fully-endogenous" exponential growth if the model is extended with an internal mechanism *determining* the population growth rate. Jones (2003) takes steps towards such a model.

minology positive per capita growth may rest on an "exogenous source" in the sense of deriving from exogenous technical progress and yet the long-run per capita growth rate may be policy-dependent. In Chapter 16 of the lecture notes we will see an example in connection with the Dasgupta-Heal-Solow-Stiglitz model, also known as the DHSS model.

There also exist models that according to our definition feature *semi*-endogenous growth and yet the long-run per capita growth rate is policy-*dependent* (Cozzi, 1997; Sorger, 2010). Similarly, there exist models that according to our definition feature fully-endogenous exponential growth and yet the long-run per capita growth rate is policy-*invariant* (some learning-by-doing models have this property).

Before proceeding, a word of warning. The distinction between "exogenous" and "endogenous" growth is only meaningful within a *given meta-theoretical framework*. It is always possible to make the meta-theoretical framework so broad that the per capita growth rate *must* be considered endogenous within that framework. From the perspective of society as a whole we can imagine many different political and institutional structures − as witnessed by long-run historical evolution − some of which clearly are less conducive to economic growth than others. From this broad point of view, growth is always endogenous.

3 Robustness of endogenous growth models

The horizontal innovations model illustrates the fact that endogenous growth models with exogenous population typically exist in two varieties or cases. One is the fully-endogenous growth case where a particular value is imposed on a key parameter in the growth engine. This value is such that there are constant returns (at least asymptotically) to *producible* inputs in the growth engine of the economy. In the "corresponding" semi-endogenous growth case, the key parameter is allowed to take any value in an open interval. The endpoint of this interval appears as the "knife-edge" value assumed in the fully-endogenous growth case.

Although the two varieties build on qualitatively the same mathematical model of a certain growth mechanism (say, research and development or learning by doing, to be considered later in the course), the long-run results turn out to be very sensitive to which of the two cases is assumed. In the fully-endogenous growth case a positive per-capita growth rate is maintained forever without support of growth in any exogenous factor. In the semi-endogenous growth case, the growth process needs "support" by some growing exogenous factor in order for sustained growth to be possible. The established terminology is somewhat seductive here. "Fully endogenous" sounds as something going much deeper than "semi-endogenous". But nothing of that sort should be implied. It is just a matter of different parameter values.

As Solow (1997, pp. 7-8) emphasizes in connection with learning-by-investing models (with constant population), the knife-edge case assumed in the fully-endogenous growth versions is a very special case, indeed an "extreme case, not something intermediate". A value slightly above the knife-edge value leads to explosive growth: infinite output in finite time even when $n = 0$. And a value slightly below the knife-edge value leads to growth petering out in the long run when $n = 0$.

Whereas the strength of the semi-endogenous growth case is its theoretical and empirical robustness, the convenience of the fully-endogenous growth case is that it has much simpler dynamics. Then the question arises to what extent a fully-endogenous growth model can be seen as a useful approximation to its semi-endogenous growth "counterpart". Imagine that we contemplate applying the fully-endogenous growth case as a basis for making forecasts or for policy evaluation in a situation where the "true" case is the semi-endogenous growth case. Then we would like to know: Are the impulse-response functions generated by a shock in the fully-endogenous growth case an *acceptable approximation* to those generated by the same shock in the corresponding semi-endogenous growth case for *a sufficiently long time horizon to be of interest*?[6] The answer is "yes" if the critical parameter has a value "close" to the knife edge value and "no" otherwise. How close it need be, depends on circumstances. My own tentative impression is that usually it is "closer" than what the empirical evidence warrants.

Even if a single growth-generating mechanism, like learning by doing, does not in itself seem strong enough to generate a reduced-form AK model (the fully-endogenous growth case), there might exist complementary factors and mechanisms that in total could generate something close to a reduced-form AK model. The time-series test by, for instance, Jones (1995b) and Romero-Avila (2006), however, reject this.[7]

Comment on "growth petering out" when $n = 0$ The above-mentioned "petering out" of long-run growth in the semi-endogenous case when $n = 0$ takes different forms in

[6] Obviously, the ultimate effects of the shock tend to be very different in the two models.

[7] For an opposite view, see Kocherlakota and Yi (1997). There is a longstanding discussion about these time-series econometric issues. See the course website under Supplementary Material.

different models. When exponential growth cannot be sustained in a model, sometimes it remains true that nevertheless $y \to \infty$ for $t \to \infty$, for instance in the form of quasi-arithmetic growth, and sometimes instead asymptotic stagnation results.[8]

Another issue is whether there exist factors that in spite of $n = 0$ (or, to be more precise, in spite of n decreasing, possibly to zero as projected by the United Nations (2013) to happen within a century from now) may *replace* the growth-supporting role of population growth under semi-endogenous parameter conditions like $\varphi < 1$. Both urbanization and the evolution of digital information and communication technologies seem likely for a long time to at least help in that direction.

4 Weak and strong scale effects

The distinction between weak and strong scale effects is important. In the Romer case ($\varphi = 1, n = 0$) of the horizontal innovations model there a *strong scale effect:*

$$\frac{\partial g_y^*}{\partial L} > 0. \tag{6}$$

Interpreting the size ("scale") of the economy as measured by the size, L, of the labor force, we call such an effect a *strong scale effect,* that is, "scale" has an effect on the long-run *growth rate*. This kind of scale effect has clearly been rejected by the empirics, cf. Jones and Vollrath (2013, p. 106).

Scale effects can be of a less dramatic form. In this case we speak of a *weak scale effect* or a *scale effect on levels.* This form arises when φ is less than 1. We see from (5) that in the Jones case ($\varphi < 1, n > 0$) of the horizontal innovations model, the steady state growth rate is independent of the *size* of the economy. Consequently, in Jones' version there is no strong scale effect. Yet there is a scale effect on *levels* unless $\varphi = 0$. If $\varphi > 0$, the scale effect is positive in the sense that along a steady state growth path, $(y_t^*)_{t=0}^{\infty}$,

$$\frac{\partial y_t^*}{\partial L_0} > 0, \tag{7}$$

cf. Exercise VII.7.

The result (7) says the following. Suppose we consider two closed economies characterized by the same parameters, including the same $n > 0$ and the same $\varphi \in (0, 1)$. The economies differ only w.r.t. initial size of the labor force. Suppose both economies are

[8] See Groth et al., 2010.

in steady state. Then, according to (7), the economy with the larger labor force has, for all t, larger output per unit of labor. The background is the increasing returns to scale w.r.t. capital, labor, *and* technical knowledge, which in turn is due to *technical knowledge being a non-rival good* – its use by one firm does not (in itself) limit the amount of knowledge available to other firms.[9] In a large economic system, say an integrated set of open economies, *more* people benefit from a given increase in knowledge than in a small economic system. At the same time the per capita cost of creating the increase in knowledge is less in the large system than in the small system.

The scale effect on levels displayed by (7) can be shown to be increasing in the parameter φ, which measures the elasticity of the economy-wide R&D productivity w.r.t. the stock of knowledge. When $\varphi \to 1$, the scale effect becomes more and more powerful. In the limit it ends up as a scale effect on the growth rate, as in the Romer case.

5 Discussion

Are there good theoretical and/or empirical reasons to believe in the existence of (positive) scale effects on levels or perhaps even on growth in the long run?

Let us start with some theoretical considerations.

5.1 Theoretical aspects

From the point of view of theory, we should recognize the likelihood that offsetting forces are in play. On the one hand, there is the problem of *limited natural resources.* For a given level of technology, if there are CRS w.r.t. capital, labor, *and* land (or other natural resources), there are diminishing returns to capital and labor taken together. In this *Malthusian* perspective, an increased scale (increased population) results, everything else equal, in lower rather than higher per capita output, that is, a negative scale effect should be expected.

On the other hand, there is the *anti-Mathusian* view that repeated improvements in technology tend to overcome, or rather *more* than overcome, this Malthusian force, if appropriate socio-economic conditions are present. Here the theory of endogenous technical change comes in by telling us that a large population may be good for technical

[9] By patent protection, secrecy, and copyright some aspects of technical knowledge are sometimes *partially* and *temporarily excludable,* but that is another matter.

progress if the institutions in society are growth-friendly. A larger population breeds more ideas, the more so the better its education is; a larger population also promotes division of labor and larger markets. This helps the creation of new technologies or, from the perspective of an open economy, it helps the local adoption of already existing technologies outside the country. In a less spectacular way it helps by furthering day-by-day productivity increases due to learning by doing and learning by watching. The non-rival character of technical knowledge is an important feature behind all this. It implies that output per capita depends on the *total* stock of ideas, not on the stock per person. This implies – everything else equal – an advantage of scale.

In the models considered so far in this course, natural resources and the environment have been more or less ignored. Here only a few remarks about this limitation. The approach we have followed is intended to clarify certain *mechanisms* – in abstraction from numerous things. The models in focus have primarily been about aspects of an industrialized economy. Yet the natural environment is always a precondition. A tendency to positive scale effects on levels *may* be more or less counteracted by *congestion* and aggravated *environmental problems* ultimately caused by increased population and a population density above some threshold.

What can we say from an *empirical* point of view?

5.2 Empirical aspects

First of all we should remember that in view of cross-border diffusion of ideas and technology, a positive scale effect (whether weak or strong) should not be seen as a prediction about individual countries, but rather as pertaining to larger regions, nowadays probably the total industrialized part of the world. So cross-country regression analysis is not the right framework for testing for scale effects, whether on levels or the growth rate. The relevant scale variable is not the size of the country, but the size of a larger region to which the country belongs, perhaps the whole world; and multivariate time series analysis seems the most relevant approach.

Since in the last century there has been no clear upward trend in per capita growth rates in spite of a growing world population (and also a growing population in the industrialized part of the world separately), most economists do not believe in *strong* scale effects. But on the issue of *weak* scale effects the opinion is definitely more divided.

Considering the *very*-long run *history* of population and per capita income of different regions of the world, there clearly exists evidence in favour of scale effects (Kremer, 1993). Whether advantages of scale are present also in a contemporary context is more debated. Recent econometric studies supporting the hypothesis of positive scale effects on levels include Antweiler and Trefler (2002) and Alcalá and Ciccone (2004). Finally, considering the economic growth in China and India since the 1980s, we must acknowledge that this impressive performance at least does *not* speak *against* the existence of positive scale effects on levels.

Acemoglu seems to find positive scale effects on levels plausible at the theoretical level (pp. 113-114). At the same time, however, later in his book he seems somewhat skeptical as to the existence of empirical support for this. Indeed, with regard to the fact that R&D-based theoretical growth models tend to generate at least weak scale effects, Acemoglu claims: "It is not clear whether data support these types of scale effects" (Acemoglu, 2009, p. 448).

My personal view on the matter is that we should, of course, recognize that offsetting forces, coming from our finite natural environment, are in play and that a lot of uncertainty is involved. Nevertheless it seems likely that at least up to a certain point there are positive scale effects on levels.

5.3 Policy implications

If this holds true, it supports the view that international economic integration is generally a good idea. The concern about congestion and environmental problems, in particular global warming, should probably, however, preclude recommending governments and the United Nations to try to *promote* population growth.

Moreover, it is important to remember the distinction between the global and the local level. The n in the formula (5) refers to a much larger region than a single country; we may refer to this region as "the set of knowledge-producing countries in the world". No recommendation of higher population growth in a single country is implied by this theoretical formula. When discussing economic policy from the perspective of a single country, all aspects of relevance in the given local context should be incorporated. For a developing country with limited infrastructure and weak educational system, family-planning programs and similar may in many cases make sense from both a social and a productivity point of view (cf. Dasgupta, 1995).

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Short Note 2. Economic Growth.

May 6, 2016. Christian Groth

The Romer-Jones horizontal innovations model

Below is a compact version of the Romer-Jones model of horizontal innovations in a closed industrialized economy. In contrast to Jones and Vollrath, Ch. 5.1-2, we specify the household sector to be of Ramsey type. There is no uncertainty and households have perfect foresight. The text is not meant to be a substitute to Jones and Vollrath's Ch. 5.1-2, but a complement to be read *after* Jones and Vollrath's introduction has been read. The aim is to give a systematic overview and to clarify some of the more technical issues. Our notation is as in exercises VII.10 - VII.14, thereby only in a few respects deviating from that in Jones and Vollrath.

1 The household sector

There is a fixed number of infinitely-lived households, all alike. Each household has $L(t) = L(0)e^{nt}$ members, $n \geq 0$, and each member supplies inelastically one unit of labor per time unit. We normalize the number of households to be one. Given $\theta > 0$ and $\rho > 0$, the representative household's problem is to choose a plan $(c(t))_{t=0}^{\infty}$ so as to

$$\begin{aligned} \max U_0 &= \int_0^{\infty} \frac{c(t)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} && (*) \\ c(t) &\geq 0, \\ \dot{a}(t) &= (r(t)-n)a(t) + w(t) - c(t), \qquad a(0) \text{ given}, \\ \lim_{t\to\infty} a(t)e^{-\int_0^t (r(s)-n)ds} &\geq 0. && \text{(NPG)} \end{aligned}$$

Here $r(t)$ is the risk-free interest rate, and $a(t)$ is per capita financial wealth, which can be placed in "raw capital" or perpetual patents, as described below.

The solution to the problem (*) is given by the Keynes-Ramsey rule,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{1}$$

and the transversality condition,

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (r(s)-n)ds} = 0. \tag{2}$$

This follows from applying Pontryagin's Maximum Principle to the problem.

2 The production side of the economy

There are three production sectors:

Firms in *Sector 1* produce *final goods* (consumption goods and "raw capital" goods) in the amount $Y(t)$ per time unit, under perfect competition. The final good is the numeraire.

Firms in *Sector 2* supply *specialized capital goods,* indexed by $j = 1, 2, \ldots, A(t)$. These specialized capital goods are *rented out* to firms in Sector 1, under conditions of monopolistic competition and barriers to entry. Like Jones and Vollrath, we sometimes refer to these specialized capital good services as "intermediate goods".[1]

Firms in *Sector 3* perform R&D to develop *technical designs* ("blueprints") for new specialized capital goods under conditions of perfect competition and free entry.

Labor is homogeneous, and also the labor market has perfect competition.

From now on, the explicit timing of the time-dependent variables is omitted unless needed for clarity; $\forall j$ means $j = 1, 2, \ldots, A$. The basic assumptions and conditions at the production side (technologies, behavior, use of Sector-1 output, no-arbitrage condition) can be presented the following way.

Sector 1: Final goods. The representative firm:

$$Y = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha}, \qquad 0 < \alpha < 1, \tag{3}$$

$$\frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L_Y} = w, \tag{FOC1}$$

$$\frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} = p_j, \quad \forall j, \tag{FOC2}$$

[1] By definition, "intermediate goods" are non-human inputs that cannot be stored. Rental services cannot be stored.

Uses of Y:

$$Y = C + I_K = cL + \dot{K} + \delta K, \qquad \delta \geq 0, \quad K(0) > 0 \text{ given.} \tag{4}$$

Sector 2: Specialized capital goods. Given the technical design j, firm j in Sector 2 can effortless transform x_j units of "raw capital" into x_j units of the specialized capital good j simply by pressing a button on a computer. Price-setting and accounting profit:

$$p_j = \frac{1}{\alpha}(r+\delta) \equiv p, \qquad \forall j, \tag{5}$$

$$\pi_j = (\frac{1}{\alpha} - 1)(r+\delta)x_j \equiv (\frac{1}{\alpha} - 1)(r+\delta)x \equiv \pi, \quad \forall j, \tag{6}$$

Sector 3: All R&D labs in Sector 3 face the same linear "research technology":

$$\# \text{ viable inventions per time unit } = \bar{\eta}\ell_A,$$

where ℓ_A is input of research labor, and $\bar{\eta}$ is productivity in R&D, which the individual R&D lab takes as given.[2] Let P_A denote the market value of the license to commercial utilization of a patent, j, forever. In brief, we may refer to P_A as the "market value of a patent", which in equilibrium turns out to be the same for all j, see below. Then the single lab's demand for research labor is

$$\ell_A = \begin{cases} \infty \text{ if } w < P_A\bar{\eta}, \\ \text{undetermined if } w = P_A\bar{\eta}, \\ 0 \text{ if } w > P_A\bar{\eta}. \end{cases} \tag{7}$$

This reflects that the value of the marginal product of research labor is $P_A\bar{\eta}$.

At the economy-wide level the accumulated stock of viable inventions, measured by the level of A, is treated as a continuous and differentiable function of time so that we can write the increase in A per time unit as

$$\dot{A} \equiv \frac{dA(t)}{dt} = \bar{\eta}L_A \equiv \eta A^{\varphi} L_A^{1-\xi}, \qquad \eta > 0, \varphi \leq 1, 0 \leq \xi < 1, \quad A(0) > 0 \text{ given,} \tag{8}$$

where $L_A \equiv \sum \ell_A$ is aggregate employment in Sector 3. Each R&D lab is "small" and therefore perceives, correctly, its contribution to aggregate $\dot{A}$, hence to $\bar{\eta}$, to be negligible.

While in (3) we consider j as a discrete variable taking values in $\{1, 2, \ldots, A\}$, at the aggregate level in (8) we "smooth out" the time path of A. This approximation seems acceptable when A is "large", and the increases in A per time unit are "small" relative to the size of A.

[2] By "viable" we mean non-duplicated.

3 General equilibrium

In general equilibrium with $L_A > 0$ we have:

$$
\begin{aligned}
(K^d &=) \ Ax \ = K \ (= K^s), && (9)\\
L_Y + L_A &= L, && (10)\\
Y &= K^\alpha (AL_Y)^{1-\alpha}, \qquad \text{(by (3) and (9))} && (11)\\
\frac{1}{\alpha}(r+\delta) &= \frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha}(\frac{K}{A})^{\alpha-1} = \alpha \frac{Y}{K} = \frac{\partial Y}{\partial K}, \qquad \text{(by (5), (FOC2), (9))} && (12)\\
\pi &= (1-\alpha)\alpha \frac{Y}{A}, \qquad \text{(by (6), (12), and (9))} && (13)\\
w &= (1-\alpha)\frac{Y}{L_Y} = P_A \bar{\eta} = P_A \eta A^\varphi L_A^{-\xi}, \quad \text{(by (FOC1), (7), and (8))} && (14)\\
P_A r &= \pi + \dot{P}_A. && (15)
\end{aligned}
$$

The equation (15) is the no-arbitrage condition which the market value, P_A, of a patent must satisfy in equilibrium. Assuming absence of asset price bubbles, this condition is equivalent to a statement saying that the market value of the patent equals the *fundamental value* of the patent.[3] By fundamental value is meant the present value of the expected future accounting profits from commercial utilization of the technical design in question. That is,

$$P_A(t) = \int_t^\infty \pi(s) e^{-\int_t^s r(u)du} ds. \tag{16}$$

Indeed, in view of no uncertainty and perfect foresight, we may consider the no-arbitrage condition (15) as a differential equation for the function $P_A(t)$. The solution to this differential equation, presupposing that there are no bubbles, is given in (16) (as derived in Appendix A). The convenience of (16) is that, given the expected future profits and interest rates, the formula directly tells us the market value of a patent. If, for instance, π grows at a constant rate n, and r is constant, then (16) reduces to

$$P_A(t) = \int_t^\infty \pi(t) e^{n(s-t)} e^{-r(s-t)} ds = \pi(t) \int_t^\infty e^{-(r-n)(s-t)} ds = \pi(t) \frac{1}{r-n}. \tag{17}$$

This present-value formula is, among other things, useful for intuitive interpretation of the effects of a change in the interest rate in the economy (everything else equal: higher r implies lower present value).

[3] Because accounting profits, π, per time unit is the same for all j, so is the market value P_A.

The size of per capita financial wealth is now given as

$$a(t) \equiv \frac{K(t) + P_A(t)A(t)}{L(t)}. \tag{18}$$

4 National income accounting

At this stage some national income accounting may be useful. From the final use side we have:

$$GNP = C + I_K + I_A = C + I_K + wL_A = C + \dot{K} + \delta K + P_A\dot{A} = Y + P_A\dot{A},$$

where we have applied (4) and the fact that, from (8) and (14), we have, in equilibrium, $P_A\dot{A} = P_A\bar{\eta}L_A = wL_A$ (no pure profits in R&D).

From the production (value added) side we have:

$$\begin{aligned} \text{value added in Sector 1} &= Y - pAx, \\ \text{value added in Sector 2} &= pAx, \\ \text{value added in Sector 3} &= P_A\dot{A}. \end{aligned}$$

So, total value added $= GNP = Y + P_A\dot{A}$.

From the income side:

$$\begin{aligned} GNP &= wL_Y + (r+\delta)K + A\pi + wL_A = wL_Y + (r+\delta)K + (1-\alpha)\alpha Y + wL_A \\ &= (1-\alpha)Y + \alpha^2 Y + (1-a)\alpha Y + wL_A = \left(1 - \alpha + \alpha^2 + \alpha - \alpha^2\right) Y + wL_A \\ &= Y + wL_A, \end{aligned}$$

where, as noted above, $wL_A = P_A\dot{A}$.

5 Balanced growth

Taking logs and then time derivatives in (11), we get

$$g_Y = \alpha g_K + (1-\alpha)(g_A + g_{L_Y}). \tag{19}$$

Now assume balanced growth. Since we have here two endogenous state variables, the capital stock, K, and the knowledge stock, A, we extend our definition from Lecture

Notes, Chapter 4, of a balanced growth path, BGP, to be a path along which $g_Y, g_C, g_K,$ *and* g_A are constant.[4] From the balanced growth equivalence theorem of Lecture Notes, Chapter 4, we know that, given the capital accumulation equation (4) and given that $I_K > 0$, a BGP will satisfy that

$$g_Y = g_K = g_C.$$

In view of $g_Y = g_K$, (19) implies that along a BGP

$$g_Y = g_A + g_{L_Y} = \text{ constant.} \tag{20}$$

Since g_A is constant along a BGP, so is g_{L_Y}.

In addition to $c \equiv C/L$, we define $y \equiv Y/L$ and $k \equiv K/L$. From now on we have to distinguish between two alternative cases, the Romer case and the Jones case.

5.1 The Romer case: $\varphi = 1$, $n = 0$, and $\xi = 0$

Since here $\varphi = 1$, we have $g_A = \eta L_A$. So, along the BGP, L_A must be constant and so must $L_Y = L - L_A$ since L is constant. Along the BGP, therefore,

$$g_Y = g_y = g_k = g_c = g_A = \eta L_A. \tag{21}$$

To determine L_A we need to take the household behavior, described in the Keynes-Ramsey rule (1) and the transversality condition (2), into account. Isolating r in (1) along a BGP immediately gives

$$r^* = \rho + \theta g_A^*, \tag{22}$$

using that $g_c = g_A$ by (21); an asterisk signifies that a value in steady state or balanced growth is considered. With this in mind, it can be shown (Exercise VII.14) that an equilibrium path featuring balanced growth with active R&D has

$$0 \;<\; L_A = \frac{\alpha\eta L - \rho}{(\theta+\alpha)\eta} \equiv L_A^*, \quad \text{and} \tag{23}$$

$$0 \;<\; g_A = \frac{\alpha\eta L - \rho}{\theta+\alpha} \equiv g_A^* \equiv g_c^*. \tag{24}$$

This is the "fully-endogenous growth" case.

[4] Recall that, on the one hand, the immediate interpretation of our symbol A is that it makes up an index for the most recently invented capital good type. On the other hand, we may also see A as an index of the stock of technical knowledge in society. In that context we treat A as a continuous and differentiable function of time.

The result is derived under the pre-condition that the transversality condition of the representative household is satisfied along the BGP and that L_A *is* positive along the path. Let us check what the necessary and sufficient parameter conditions are for this to hold.

It can be shown (Exercise VII.14) that the transversality condition (2) with $n = 0$, in combination with (18), holds if and only if $\rho > (1-\theta)g_A^*$. By inserting (24) and isolating ρ, this inequality is equivalent to

$$\rho > \frac{(1-\theta)\alpha\eta L}{1+\alpha}. \tag{A1-R}$$

From (24) follows immediately that $L_A^* > 0$ if and only if

$$\rho < \alpha\eta L. \tag{A2}$$

Note that the right-hand side of (A1-R) is always smaller than the right-hand side of (A2) (since both θ and α are positive). Hence, (A1-R) and (A2) can hold at the same time. To assume both (A1-R) and (A2) is equivalent to assuming

$$\frac{(1-\theta)\alpha\eta L}{1+\alpha} < \rho < \alpha\eta L. \tag{**}$$

So for a BGP to be an equilibrium path in the Romer case, it is needed both that households are *not too patient* (in which case (A1-R) would be violated), *and* that they are *not too impatient* (in which case (A2) would be violated). On the one hand, being "too patient" means that households tend to save so much that the interest rate in the economy (implied by combining the result (24) with the Keynes-Ramsey rule along the BGP) would be larger than the growth rate of labor income. Because of the infinite time horizon of the households, this would imply that they had *infinite* human wealth, in which case it is a paradox that they do not consume much more than they do. When (A1-R) is violated, this paradox in unavoidable with Ramsey households. So general equilibrium within the Ramsey framework is in that case impossible.[5]

On the other hand, the meaning of being "too impatient", and thus violating (A2), is more straightforward. It simply means that households are not willing to deliver the saving needed to finance capital accumulation and R&D. Indeed, when $\rho \geq \alpha\eta L$, the willingness to save is so low that in the long run the economy will be in a *stationary state*

[5] This reflects one of the *limitations* of the Ramsey framework.

with just enough saving to maintain the capital stock and no saving left to finance R&D and net capital investment.[6]

It can be shown that the transitional dynamics of the model in the Romer case can be reduced to a three-dimensional dynamic system in $z_1 \equiv Y/K$, $z_2 \equiv C/K$, and $z_3 \equiv L_Y$. Under the assumptions (A1-R) and (A2), the system has a unique steady state, $z_1^* = (\rho+\theta g_A^*+\delta)/\alpha^2$, $z_2^* = z_1^*-g_A^*-\delta$, and $z_3^* = L_Y^*$, given in (23). In the steady state, y, k, c, and A follow the BGP described above. At least under realistic parameter values, the dynamic system can be shown to be saddle-point stable so that $(z_1(t), z_2(t), z_3(t)) \rightarrow (z_1^*, z_2^*, z_3^*)$ for $t \rightarrow \infty$ (Arnold, 2000). The transitional dynamics thus imply convergence towards the steady state which also means convergence towards balanced growth. Assuming (**), we thus know that, without recurrent disturbances, the system will in the long run be in balanced growth with a per capita growth rate equal to g_A^* given in (24).

Comments on the BGP solution in the Romer case Imposing both (A1-R) and (A2), in brief (**), there is in the Romer case a meaningful solution to the model. The solution features "fully endogenous" exponential growth. Exponential per capita growth is generated by an internal mechanism, through which labor is allocated to R&D; and this exponential per capita growth is maintained without support of growth in any exogenous factor.

Among other things, one can make comparative static analysis on the result in (24). For instance, we see that $\partial g_A^*/\partial L = \alpha\eta/(\theta+\alpha) > 0$. The Romer case thus implies a *scale effect on growth,* which is an empirically problematic feature.[7] In Exercise VII.14 the reader is asked to do further comparative static analysis on the result for g_A^*.

5.2 The Jones case: $\varphi < 1$, $n > 0$, and $\xi \in [0, 1)$

In this case, the "semi-endogenous growth" case, we can immediately determine g_A along a BGP with $L_A > 0$. We have

$$g_A \equiv \frac{\dot{A}}{A} = \eta A^{\varphi-1} L_A^{1-\xi}.$$

[6] This stationary state is in a sense still a BGP but with $g_y = g_k = g_c = g_A = 0$. Note that when $\rho > \alpha\eta L$, the formulas (23) and (24) cease to hold. This should be no surprise. Indeed, a path with $L_A < 0$ is obviously *impossible*; moreover, in the derivation of the two formulas we *relied* on the assumption that $L_A > 0$.

[7] If $n > 0$, the Romer case leads to a forever rising per capita growth rate, an implausible scenario.

Since, by assumption, $L_A > 0$, also $g_A > 0$, and so we can take logs on both sides and thereafter time derivatives, using the chain rule to get:

$$\frac{\dot{g}_A}{g_A} = (\varphi - 1)g_A + (1 - \xi)g_{L_A} = 0$$

along a BGP where, by definition, g_A must be constant. Hence

$$g_A = \frac{1 - \xi}{1 - \varphi} g_{L_A} = \frac{1 - \xi}{1 - \varphi} n \equiv g_A^*. \tag{25}$$

The last equality comes from the fact that since along a BGP with $L_A > 0$, g_{L_A} must be a positive constant at the same time as we know from (20) that g_{L_Y} is a constant along a BGP. Then, if either g_{L_A} or g_{L_Y} were smaller than n, the other would be larger than n and sooner or later violate $L_Y + L_A = L$. Hence, $g_{L_Y} = g_{L_A} = n$. From (20) then also follows that along a BGP,

$$g_Y = g_K = g_C = g_A^* + n. \tag{26}$$

The method of solving the model for $s_R \equiv L_A/L$ along the BGP is somewhat different from the method in the Romer case. To be able to pin down P_A under balanced growth, we first note that the no-arbitrage condition (15) can be written

$$P_A = \frac{\pi}{r - g_{P_A}}. \tag{27}$$

From (14) follows

$$g_{P_A} + \varphi g_A - \xi g_{L_A} = g_Y - g_{L_Y}. \tag{28}$$

We know that along the BGP, $g_{L_Y} = n = g_{L_A}$, so that (28) implies

$$g_{P_A} = g_Y - \varphi g_A - (1 - \xi)n = g_A^* + n - \varphi g_A^* - (1 - \xi)n = n, \tag{29}$$

where the second and third equalities build on (26) and (25). From the no-arbitrage condition (27) then follows that under balanced growth,

$$P_A = \frac{\pi}{r - n} = (1 - \alpha)\alpha \frac{Y}{(r - n)A}, \tag{30}$$

the last equality following from (13).

By (12), $r = \alpha^2 Y/K - \delta$. Since under balanced growth, Y/K is a constant, so is r. Hence, (30) shows that $g_\pi = g_{P_A} = n$ under balanced growth. That is, the monopolies' accounting profit grow at the rate of population growth, n. This relationship reflects that a larger population growth rate means that the markets for the specialized intermediate goods grow faster, which in view of increasing returns makes R&D more profitable.

Now (14) gives

$$(1-\alpha)\frac{Y}{L_Y} = P_A\bar{\eta} = (1-\alpha)\alpha\frac{Y}{(r-n)A}\bar{\eta}.$$

Cancelling out $(1-\alpha)Y$ and multiplying through by L_A gives

$$\frac{L_A}{L_Y} = \alpha\frac{\bar{\eta}L_A}{(r-n)A} = \alpha\frac{g_A}{r-n},$$

where the last equality follows from (8). As $L_A/L_Y = s_R/(1-s_R)$, we get from this,

$$s_R = \frac{1}{1+\frac{r-n}{\alpha g_A^*}} \tag{31}$$

along a BGP with $L_A > 0$.

This is not the final solution for s_R since r is endogenous. But again, reordering the Keynes-Ramsey rule gives, under balanced growth, $r = \rho + \theta g_A^* = r^*$ as in (22). Substituting this into (31) yields the solution for s_R along the BGP:

$$s_R = \frac{1}{1+\frac{1}{\alpha}(\frac{\rho-n}{g_A^*}+\theta)} = \frac{1}{1+\frac{1}{\alpha}(\frac{\rho-n}{\frac{1-\xi}{1-\varphi}n}+\theta)} \equiv s_R^*, \tag{32}$$

the second equality coming from (25).

Like the Romer results, the Jones results are derived under the pre-condition that the transversality condition of the representative household is satisfied along the BGP and that L_A (hence also g_A) *is* positive. Let us check what the necessary and sufficient parameter conditions (over and above the basic conditions $\varphi < 1$, $n > 0$, and $\xi \in [0,1)$) are for these conditions to hold.

First, as to the transversality condition (2), note that under balanced growth,

$$a(t) \equiv \frac{K(t)+P_A(t)A(t)}{L(t)} = \frac{K(0)e^{(g_A^*+n)t}+P_A(0)A(0)e^{(n+g_A^*)t}}{L(0)e^{nt}} = a(0)e^{g_A^* t},$$

where the second equality follows from (26) and (29). Consequently, along a BGP

$$a(t)e^{-(r^*-n)t} = a(0)e^{-(r^*-n-g_A^*)t} = a(0)e^{-(r^*-n-g_A^*)t} \to 0 \text{ if and only if } r^* > g_A^* + n,$$

where $r^* = \rho + \theta g_A^*$ by (22) which also holds here. So (2) holds along the BGP if and only if $\rho + \theta g_A^* > g_A^* + n$, that is, if and only if $\rho - n > (1-\theta)g_A^*$. By (25), this inequality is equivalent to

$$\rho > (1-\theta)\left(\frac{1-\xi}{1-\varphi}+1\right)n. \tag{A1-J}$$

Second, for $g_c = g_A > 0$ to be an outcome in balanced growth, we need $r^* > \rho$. In view of $r^* = \rho + \theta g_A^*$, this condition is equivalent to $\rho + \theta g_A^* > \rho$, which is automatically satisfied when $n > 0$, see (25).

We conclude that for a BGP to be an equilibrium path in the Jones case, it is just needed that households are *not too patient,* in the sense of violating the parameter condition (A1-J). For $0 < \theta < 1$, the right-hand side of (A1-J) defines a positive lower bound for the rate of impatience. For $\theta \geq 1$, the condition (A1-J) imposes only a mild constraint in that it is satisfied whenever just $\rho > 0$ ($\theta > 1$ even allows a negative ρ, although not "too large" in absolute value).

So, given the basic conditions $\varphi < 1$, $n > 0$, and $\xi \in [0, 1)$, we need only to add the assumption (A1-J) to ensure that in the Jones case there is a meaningful solution to the model. It can be shown that the transitional dynamics in the Jones case can be reduced to a *four*-dimensional dynamic system, that there is a unique steady state, equivalent to a balanced growth path, and that the dynamic system is saddle-point stable. Assuming (A1-J) we thus know that, without recurrent disturbances, the system will in the long run end up in balanced growth with per capita growth rate equal to g_A^*, given in (25).

Comments on the BGP solution in the Jones case From the result (25) we see that exponential growth is in the Jones case not "fully endogenous" since it can only be sustained if $n > 0$. In other words, exponential growth can only be sustained if the growth engine receives an inflow of "energy" from growth in the labor force, an exogenous source. In this sense the exponential growth in the Jones case is often referred to as "semi-endogenous". As mentioned in Short Note 1, p. 6, this terminology is somewhat seductive. The "semi-endogenous" Jones model sounds as something less deep than the "fully endogenous" Romer model. But nothing of that sort should be implied. It is just a matter of different parameter values (in fact, a matter of a "knife-edge" case versus a robust parameter case).

Before proceeding, note the striking simplicity of the result (25). The growth rate in income per capita under balanced growth depends only on three parameters: the growth rate of the labor force, n, the elasticity of research productivity with respect to the stock of knowledge, A, and the degree of duplication, ξ, in economy-wide research. Neither household preferences, represented by the parameters ρ and θ, nor for instance an R&D subsidy that raises the share of labor allocated to R&D, affect g_A^*. There will be a temporarily higher growth rate of A, but in the long run g_A will return to the same g_A^*

as before, namely that given in (25), cf. Jones and Vollrath, p. 109-110.

On the basis of the formula (32), long-run level effects on s_R of different parameter shifts can be studied (exercises VII.12 and VII.13). While for instance the preference parameters ρ and θ do not here have long-run growth effects, they affect the share of labor allocated to R&D. They thus have *level* effects on $L_A^*(t) = s_R^* L(t)$ along a BGP. As expected, both a rise in ρ and a rise in θ affect $L_A^*(t)$ *negatively.* The intuition is as follows. A rise in impatience, ρ, implies reduced saving, hence less R&D can be financed by the saving. We could also say that a rise in impatience means greater scarcity of finance, which in turn tends to raise the interest rate. This implies lower present value of expected future accounting profits to be obtained by an invention, cf. (16). In turn, this means that R&D is less rewarding.

Likewise, a rise in θ (the desire for consumption smoothing) implies reduced saving in the normal case where $r > \rho$, cf. the Keynes-Ramsey rule. The level effect on $L_A^*(t)$ of a rise in θ has thus similarity with that of a rise in ρ.

The level effects on $L_A^*(t)$ will not affect g_A in the long run, since (25) shows that g_A^* only depends on n and φ, not on s_R. A higher s_R will temporarily increase both the growth rate of A and that of y. But the fact that $\varphi < 1$ ("diminishing returns to knowledge" in the growth engine) makes it impossible to maintain the higher growth rate in A forever. The growth rate will, after a possibly quite *durable* adjustment process[8] return to the same g_A^* as before. But the level of the growth path will generally be permanently affected. This is like in the Solow model or the original Ramsey model, where an increase in the propensity to save raises the growth rate only temporarily due to the falling marginal productivity of capital.

While level effects of shifts in s_R on $L_A^*(t)$ are straightforward to analyze, level effects on $y^*(t)$ and $c^*(t)$ are a bit more complicated. Indeed, a shift in s_R has ambiguous effects on both $y^*(t)$ and $c^*(t)$ along a BGP. If s_R initially is "low", a "small" increase in s_R will have a positive level effect on y via the productivity-enhancing effect of more knowledge creation. But if s_R is already quite large initially, L_Y will be small, which implies that $\partial Y / \partial L_Y$ is large. This large marginal productivity constitutes the opportunity cost of increasing s_R further and dominates the benefit of a higher s_R, when $s_R > 1/(2 - \varphi)$ (in the case $\xi = 0$), cf. Exercise VII.7e).

[8]See Jones (1995).

6 Economic policy

The presented version of the Romer-Jones model implies in the Romer case that under laissez-faire, the decentralized market equilibrium unambiguously leads to too little R&D. This is due to three circumstances: (a) the positive externality generated by the intertemporal knowledge spillover, represented by $\varphi = 1$; (b) the "surplus appropriability problem" illustrated in Jones and Vollrath, p. 134; and (c) the demand-reducing monopoly pricing over and above marginal cost of intermediates. All three circumstances contribute to too little R&D. And there are no externalities going in the opposite direction. It can be shown that in combination with a subsidy to R&D, a subsidy to purchases of specialized capital good services can solve the problem, if these subsidies are financed by lump-sum taxes or lump-sum-equivalent taxes like, in the present framework, a labor income tax (recall that the model's labor supply is inelastic).

In the Jones case, the "stepping-on-toes" effect ($\xi > 0$) is a negative externality pointing in the opposite direction. And so is the intertemporal knowledge spillover *if* $\varphi < 0$. The different calibrations made by Jones and coauthors use a positive value of both φ and ξ. Even taking to some extent creative destruction into account, Jones and Williams (1998) estimate the resource allocation to R&D in USA to be only a fourth of the social optimum, given that discounted utility of the representative household is the optimality criterion.

7 Concluding remarks

A weakness of the presented Romer-Jones model is the unrealistic feature that obsolescence of specialized capital goods never occurs. The mentioned Jones and Williams (1998) paper attempts to surmount that problem.

Another weakness is that there are two hidden arbitrary parameter links in the specification of the production function for final goods. One is related to the way the variety index A enters the production function. The parameter reflecting "gains to variety", sometimes called the "gains to specialization" parameter, below denoted μ, is arbitrarily identified with the output elasticity w.r.t. labor, $1 - \alpha$. Another arbitrary parameter link is that the elasticity of substitution between the different capital good types calculated from the production function (3) is $1/(1 - \alpha)$ and thus implies market power equal to $1/\alpha$, the monopoly markup. Thereby, effects of a rise in monopoly power can not be studied independently of a fall in the output elasticity w.r.t. capital, α. This arbitrary parameter link has in the Romer case the implication that a rise in market power *reduces* g_c^*, an effect arising solely because the positive effect on growth of the rise in the markup is *blurred* by a negative effect coming from a diminished output elasticity w.r.t. capital.

A specification of the production function free of these two arbitrary parameter links, but maintaining power functions throughout, is the following:

$$Y = A^{\mu} X^{\alpha} {N_Y}^{1-\alpha}, \qquad 0 < \alpha < 1, \quad \mu > 0,$$

where X is a CES aggregate (with constant returns to scale) of quantities, x_j, of specialized capital goods:

$$X = A(\frac{1}{A}\sum_{j=1}^{A} {x_j}^{\varepsilon})^{\frac{1}{\varepsilon}}, \qquad 0 < \varepsilon < 1.$$

Here the existing specialized capital goods exhibit an elasticity of substitution equal to $1/(1-\varepsilon)$, implying that the market power, or the monopoly markup, is given by $1/\varepsilon > 1$. Now g_{c^*} generally differs from g_A^* and the formulas become more complicated. But a rise in market power, $1/\varepsilon$, can be shown to unambiguously *raise* g_c^*. This is the opposite of what we got above, where market power was arbitrarily linked to the output elasticity w.r.t. capital, α. For details, see Alvarez-Pelaez and Groth (2005).

8 Appendix: Solving the no-arbitrage equation for P_A(t) in the absence of asset price bubbles

In Section 3 we claimed that in the absence of bubbles, the differential equation implied by the no-arbitrage equation (15) has the solution

$$P_A(t) = \int_t^{\infty} \pi(s) e^{-\int_t^s r(u)du} ds. \tag{*}$$

To prove this, we write the no-arbitrage equation on the standard form for a linear differential equation:

$$\dot{P}_A(t) - r(t)P_A(t) = -\pi(t).$$

The general solution to this (see Appendix B to Chapter 3 of Lecture Notes) is

$$P_A(t) = P_A(t_0)e^{\int_{t_0}^t r(u)du} - e^{\int_{t_0}^t r(u)du} \int_{t_0}^t \pi(s) e^{-\int_{t_0}^s r(u)du} ds.$$

Multiplying through by $e^{-\int_{t_0}^{t} r(u)du}$ gives

$$P_A(t)e^{-\int_{t_0}^{t} r(u)du} = P_A(t_0) - \int_{t_0}^{t} \pi(s)e^{-\int_{t_0}^{s} r(u)du} ds.$$

Rearranging and letting $t \to \infty$, we get

$$P_A(t_0) = \int_{t_0}^{\infty} \pi(s)e^{-\int_{t_0}^{s} r(u)du} ds + \lim_{t\to\infty} P_A(t)e^{-\int_{t_0}^{t} r(u)du}. \quad (33)$$

The first term on the right-hand side is the fundamental value of the patent, i.e., the present value of the expected future accounting profits on using the patent commercially. The second term on the right-hand side thus amounts to the difference between the market value, $P_A(t_0)$, of the patent and its fundamental value. By definition, this difference represents a bubble. In the absence of bubbles, the difference is nil, and the market price, $P_A(t_0)$, coincides with the fundamental value. So (*) holds (in (33) replace t by T and t_0 by t), as was to be shown.

9 References

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Short Note 3. Economic Growth.

May 9, 2016. Christian Groth

A Schumpeterian model of vertical innovations

This note presents the Schumpeterian model of vertical innovations from Chapter 5.3-4 of Jones and Vollrath (2013). The aim is to give a systematic presentation of the model and to clarify some of the technical issues.[1] The focus is on the *core* of the model, namely the production, inventions, and financing aspects. This core can be combined with alternative models of the household sector. In Section 5 we use the Ramsey-style representative agent description of the household sector.

The new element in the Schumpeterian model compared with the horizontal innovations model is the implication that innovations imply "creative destruction" − the process through which existing businesses and technologies are competed out of the market by new technologies.

We start with an overview of the production sectors.

1 Overview of the production sectors

The economy is closed and has population $L = L_0 e^{nt}$, $n \geq 0$. Labor is homogeneous. Each member of the population supplies one unit of labor per time unit. In contrast to the horizontal innovations model, there is only one type of capital good. But over time, better and better qualities − or "versions" in the terminology of Jones and Vollrath − are invented.

There are three production sectors:

Firms in *Sector 1* produce *final goods* (consumption goods and "raw capital" goods) in the amount $Y(t)$ per time unit, under perfect competition. The final good is the *numeraire.*

[1] The model, as presented in Jones and Vollrath as well as here, is in some respects a simplified version of the contribution by Aghion and Howitt (1992), e.g., by considering only one type of intermediate good. In other respects it is an extension of that contribution, e.g., by considering *durable* capital goods and allowing $\varphi < 1$ and $n > 0$.

In *Sector 2* there is at any point in time only one active firm, the incumbent monopolist. This firm supplies the leading edge quality of the economy's single kind of capital good *on a leasing basis* to firms in Sector 1 under conditions of monopoly and barriers to entry.

Firms in *Sector 3,* the R&D labs, perform R&D to develop *technical designs* ("blueprints") for improved qualities of the capital good under conditions of perfect competition and free entry.

The reason that the inputs from Sector 2 to Sector 1 are by Jones and Vollrath called "intermediate goods" is that they are delivered on a leasing basis. In national-income terminology this makes them "intermediate goods" (in the sense of non-human inputs that cannot be stored). As to the raw capital goods produced in sector 1, it is easiest to imagine that they are *sold* at the price 1 to either the incumbent sector-2 monopolist or to households that then *rent* them out to the incumbent sector-2 monopolist at the capital cost $r + \delta$ per unit of raw capital. To fix ideas, we choose the former interpretation.

There is a labor market and a market for risk-free loans. Both markets have perfect competition. We denote the real wage w_t and the risk-free real interest rate r_t. There is "ideosyncratic" uncertainty (to be defined below). The risk associated with R&D and "creative destruction" can be diversified via the equity-share market because the economy is "large", and there are "many" R&D labs in the economy. All firms are profit maximizers. Time is continuous.

2 The interaction between Sector 1 and Sector 2

2.1 Sector 1: Final goods

The representative firm in Sector 1 has the production function

$$Y(t) = x_i(t)^\alpha (A_i L_Y(t))^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{1}$$

where $Y(t)$ is the produced quantity of final goods per time unit at time t, $L_Y(t)$ is labor input, and $x_i(t)$ is input of the currently superior version of the capital good, version i. The version of the capital good that was in use from time $t = 0$ until a new innovation occurred is indexed 0, the version associated with that new innovation is indexed 1, the subsequent version is indexed 2 and so on up to the current version, i. Labor working with version i has efficiency A_i. It is assumed that A_i evolves stepwise from innovation to innovation:

$$A_i = (1+\gamma)A_{i-1}, \qquad \gamma > 0.$$

Observe that γ is the relative increase in A *per step*, not the growth rate of A per time unit. Not only may the number of steps per time unit be generally below one or generally above one, but this number is stochastic (uncertain, governed by a probability distribution). This reflects that the length of the time interval between successive innovations is stochastic.

The output of final goods is used partly for consumption, $C(t) \equiv c(t)L(t)$, partly for investment in raw capital, $I_K(t)$:

$$Y(t) = C(t) + I_K(t) = c(t)L(t) + \dot{K}(t) + \delta K(t), \qquad \delta \geq 0, \quad K(0) > 0 \text{ given}, \tag{2}$$

where $K(t)$ is the stock of raw capital goods in the economy at time t and δ is the capital depreciation rate.

From now on, the explicit dating of the time-dependent variables is omitted unless needed for clarity. With the final good as numeraire we let p_i denote the rental rate per time unit for using one unit of the capital good in its current version i.

Maximizing profit under perfect competition leads to the FOCs:

$$\frac{\partial Y}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y} = w, \tag{3}$$

$$\frac{\partial Y}{\partial x_i} = \alpha x_i^{\alpha-1} A_i^{1-\alpha} L_Y^{1-\alpha} = p_i. \tag{4}$$

2.2 Sector 2: The currently superior version of the capital good

Let the owner of the exclusive and perpetual[2] right to use technical design i commercially be called firm i. Given the technical design i, firm i can effortless transform raw capital goods into the specific version i simply by pressing a button on a computer, thereby activating a computer code. The following linear transformation rule applies:

it takes $x_i > 0$ units of raw capital to supply x_i units of capital of version i.

The reason that the model assumes that the capital good in version i is *rented out* to the users in Sector 1 is related to the IO problem known as the "durable-goods-monopoly

[2] Recall, patents are assumed to be perpetual – or at least durable enough so that they have not expired before the next innovation turns up.

problem". Selling the capital good implies a transfer of ownership to a durable good and thereby a risk that a second-hand market for the good arises. This could threaten the market power of the monopolist.

The *pure profit* per time unit of firm i obtained by renting out x_i units of the capital good in version i can be written:

$$p_i(x_i)x_i - (r+\delta)x_i - rP_A \equiv \pi_i - rP_A,$$

where $p_i(x_i)$ denotes the maximum price at which the amount x_i can be rented out, r is the risk-free real interest rate, δ is the capital depreciation rate (and so $r+\delta$ is the capital cost per unit of raw capital held),[3] P_A is the market value of the right to use the technical design i, and π_i is the *accounting profit* in the sense of net revenue before subtraction of the imputed interest cost, rP_A. The latter is the opportunity cost of being in this business rather than for instance offering loans in the loan market. This interest cost is a *fixed* cost as long as the entrepreneur remains in the business. So, being in the business, maximizing pure profit is equivalent to maximizing the accounting profit π_i. The quantity x_i (or the price p_i) is thus set so as to maximize

$$\pi_i = p_i(x_i)x_i - (r+\delta)x_i.$$

The profit maximizing p_i $(= p_i(x_i))$ is such that marginal revenue, MR, equals marginal cost, MC :

$$\begin{aligned} MR &= \frac{dTR}{dx_i} = p_i(x_i) + x_i p_i'(x_i) = p_i\left(1 + \mathrm{E}\ell_{x_i} p_i\right) = p_i(1+\alpha-1) = p_i\alpha = MC = r+\delta \\ &\Rightarrow \;\; p_i = \frac{1}{\alpha}(r+\delta) \equiv p, \qquad (5) \end{aligned}$$

where the third equality comes from (4). We observe that the profit maximizing price, p_i, is independent of what rung, i, on the quality ladder has been reached. Hence, we can just denote it p.

Can we be sure that the current technology leader can avoid being undercut by the previous incumbent when charging the monopoly price? No, only if the innovation is *drastic.* By this is meant that the step size, γ, is large enough so that even if the previous incumbent is ready to just charge the marginal cost, $r+\delta$, then she is competed out by the new firm i charging the monopoly price $\frac{1}{\alpha}(r+\delta)$ for supplying the more efficient version of the capital good.

[3] Jones and Vollrath implicitly assume $\delta = 0$ in Section 5.2-4, which is not in harmony with their Section 5.1 and the rest of the book.

To fix ideas, we simplifying assume that all innovations are drastic.[4] The price set by the monopolist is then the monopoly price and the accounting profit is

$$\pi_i = (\frac{1}{\alpha} - 1)(r + \delta)x_i. \tag{6}$$

From now, for simplicity we will refer to this as just the "profit" of firm i.

2.3 Preliminary observations regarding equilibrium

Before going into detail with the R&D sector, it is convenient to combine some elements from Sector 1 and 2 under the assumption of market clearing with perfectly flexible prices.

To supply x_i version-i units of capital, the monopolist in Sector 2 needs x_i units of raw capital. So the demand for raw capital goods is $K^d = x_i$. The supply of raw capital goods is simply the currently available stock of raw capital, i.e., $K^s = K$. For an arbitrary t, we thus have in equilibrium,

$$x_i = K. \tag{7}$$

Substituting this into (1) yields

$$Y = K^{\alpha}(A_i L_Y)^{1-\alpha}. \tag{8}$$

This is the aggregate production function in Sector 1 in equilibrium at time t where version i represents the leading-edge technology.

Starting with (4), we then have

$$\frac{\partial Y}{\partial x_i} = \alpha x_i^{\alpha-1}(A_i L_Y)^{1-\alpha} = \alpha K^{\alpha-1}(A_i L_Y)^{1-\alpha} = \alpha \frac{Y}{K} = \frac{\partial Y}{\partial K} = p = \frac{1}{\alpha}(r + \delta),$$

where the second equality comes from (7), the third and fourth from (8), and the last from (5) combined with (4). It follows that

$$r + \delta = \alpha^2 \frac{Y}{K} = \alpha \frac{\partial Y}{\partial K} < \frac{\partial Y}{\partial K} = \frac{\partial Y}{\partial x_i}. \tag{9}$$

Reading this from the right to the left, we see that, in equilibrium, the marginal productivity of capital of the currently superior quality, $\partial Y / \partial x_i$, is above the cost, $r + \delta$, per unit

[4] If an innovation is *nondrastic,* then to discourage the incumbent from staying in the market, the newcommer has to charge a sufficiently low price, the *limit price.* Although this will be below the monopoly price, it will still be above the marginal cost (which is the same as for the incumbent). The reason is the higher efficiency associated with the new technology. So, also in case of non-drastic innovations will "creative" destruction take place.

of raw capital, by a factor $1/\alpha > 1$. This is due to the monopoly pricing of the capital input. Under perfect competition capital would be demanded up to the point where its marginal productivity equals the competitive cost, $r + \delta$, per unit of capital. In contrast, here capital is demanded only up to the point where its marginal productivity equals the capital cost dictated by a capital goods supplier with market power.

Substituting (9) and (7) into (6) gives

$$\pi_i = (\frac{1}{\alpha} - 1)(r + \delta)K = \frac{1-\alpha}{\alpha}\alpha^2 Y = (1-\alpha)\alpha Y \equiv \pi. \tag{10}$$

3 Sector 3: R&D

The model assumes, naturally, that there is uncertainty in R&D. Let t_i be the point in time at which the current leading-edge technology was invented and let t_{i+1} be the unknown future point in time where the next upward jump on the quality ladder takes place. Then the length of the time interval (t_i, t_{i+1}) – the "waiting time" – is a stochastic variable.

3.1 The "research technology"

The R&D process is modelled as an *inhomogeneous Poisson process.*

3.1.1 The single R&D lab

Consider a single R&D lab which is active in the time interval (t_i, t_{i+1}). By definition, *within* this time interval the lab does not face the event of another lab "coming first". Let $\ell_A(t)$ denote the input of R&D labor per time unit at time $t \in (t_i, t_{i+1})$ and let arrival of a "success" mean arrival of the event that the considered lab makes a "viable" invention (by "viable" we mean "not duplicated"). The model then introduces four assumptions:

(i) The *success arrival rate* (per time unit) at time t is $\bar{\eta}(t)\ell_A(t)$, where $\bar{\eta}(t)$ is an economy-wide "research productivity", which by the lab is perceived as exogenous.[5]

This means that the probability of success within a short time interval "from now", conditional on no other labs "coming first", is approximately proportional to the length

[5] To get conformity with notation in the exercises, we have replaced the textbook's $\bar{\mu}$ by $\bar{\eta}$.

of this time interval:

$$P(\text{success}\,|(t, t+\Delta t)) = \bar{\eta}(t)\ell_A(t)\Delta t + o(\Delta t) \approx \bar{\eta}(t)\ell_A(t)\Delta t, \tag{11}$$

where $o(\Delta t)$ is standard symbol for a function, the value of which declines faster than its argument, here Δt, when the latter approaches zero, that is, $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0$. Thereby $\lim_{\Delta t \to 0} P(\text{success}|(t, t+\Delta t))\,/\Delta t = \bar{\eta}(t)\ell_A(t)$. We may say: the difference between $P(\text{success}|(t, t+\Delta t))$ and $\bar{\eta}(t)\ell_A(t)\Delta t$ has "order of magnitude less than Δt".

(ii) There is stochastic independence across time within the time interval (t_i, t_{i+1}).

Digression on Poisson processes If $\bar{\eta}(t)\ell_A(t)$ were a *constant*, equal to $\lambda > 0$, say, then the R&D process would be a *homogeneous Poisson process* with arrival rate λ. With T denoting the waiting time from time t and onward until a success arrives, then, again conditional on no other labs "coming first", the probability that T exceeds $\tau > 0$ would be $P(T > \tau) = e^{-\lambda\tau}$. Moreover, assuming the lab, in case of success, continues researching for yet another quality improvement, the number, m, of success arrivals within a time interval of length Δt would follow a *Poisson distribution,* that is,

$$P(m = a\,|(t, t+\Delta t)) = e^{-\lambda\Delta t}(\lambda\Delta t)^a/a!, \tag{*}$$

where $a = 0, 1, 2, \ldots$, and $a! \equiv a \cdot (a-1) \cdot (a-2) \cdot \cdots \cdot 1$, $0! = 1$. The expectation of m is $\lambda\Delta t$, and the variance is the same.

In the present model, however, both $\bar{\eta}(t)$ and $\ell_A(t)$ will generally be time dependent. The R&D process is assumed to be an *inhomogeneous Poisson process* with arrival rate $\lambda(t) = \bar{\eta}(t)\ell_A(t)$. This means that the probability of the event $m = a$ is as in (*) except that $\lambda\Delta t$ should be replaced by $\int_t^{t+\Delta t} \lambda(s)ds$. If Δt is "small", the expectation of m thus equals

$$\int_t^{t+\Delta t} \lambda(s)ds \approx \lambda(t)\Delta t, \tag{**}$$

and the same holds true for the variance. □

In accordance with (*) and (**), under the assumption that the lab continues its research throughout the time interval $(t, t+\Delta t)$, the expected number of success arrivals is

$$E(m\,|\bar{\eta}(t)\ell_A(t), (t, t+\Delta t)\,) \approx \bar{\eta}(t)\ell_A(t)\Delta t.$$

In case $\bar{\eta}(t)\ell_A(t)$ were a *constant* during the considered time interval, we could replace the "≈" by "=".

Before proceeding, a reservation seems appropriate. The assumption (ii) is a kind of "no memory" assumption since it ignores learning over time within the lab. This seems problematic. Indeed, R&D should be considered a cumulative process. The only excuse for assumption (ii) is the need for simplicity in a first approach.

3.1.2 Aggregate R&D and the evolution of technology

The third assumption concerning R&D deals with the economy-wide R&D where many labs are involved:

(iii) Research outcomes are stochastically independent across R&D labs.

Let $L_A(t)$ denote the aggregate input of research labor at time t, i.e., $L_A(t) \equiv \sum \ell_A(t)$. We then have

$$P(\text{success}\,|(t, t+\Delta t)) \approx \bar{\eta}(t)L_A(t)\Delta t. \tag{12}$$

Thus, with $M(t)$ denoting the aggregate number of success arrivals in the time interval $(t, t+1)$, and letting our time unit be "small", the following approximation holds for the expected aggregate number of success arrivals over the time interval $(t, t+1)$ is:

$$E(M(t)\,|\bar{\eta}(t)L_A(t), (t, t+1)\,) \approx \bar{\eta}(t)L_A(t). \tag{13}$$

Finally, the fourth assumption is about how economy-wide "research productivity" is determined:

(iv) $\bar{\eta}(t) = \eta A_i^{\varphi-1}L_A(t)^{-\xi}$, $\eta > 0, \varphi \leq 1, 0 \leq \xi < 1$. Here $\varphi - 1$ is the elasticity of research productivity w.r.t. the accumulated "stock of knowledge" at time t, measured by A_i, and ξ is the degree of R&D overlap in the economy.[6] There are "many" labs in the economy, and the individual labs rightly perceive their influence on $\bar{\eta}(t)$ to be negligible.

The only uncertainty assumed present in the economy is the uncertainty related to research outcomes in the individual labs. According to Assumption (iii) these research

[6] The motivation for choosing the exponent on A to be $\varphi - 1$ rather than just φ, as in the horizontal innovations model, is that each innovation in the present model generates a rise in A that is proportionate to A and thus becomes larger and larger.

We have replaced the textbook's θ by η in order to reserve θ to denote a preference parameter when specifying the household sector in the model. We have further replaced the textbook's $1 - \lambda$ by ξ.

outcomes are stochastically independent across labs.[7] In economists' jargon, uncertainty is thus "idiosyncratic", allowing investors to reduce their risk by diversification, as we shall see below.

Before detailing that aspect, some observations about the aggregate research outcome *per time unit* are pertinent. Let $A(t)$ indicate the labour efficiency associated with the leading-edge technology at time t. Thus, in the present situation $A(t) = A_i$. With $M(t)$ success arrivals in the time interval $(t, t+1)$, we then have

$$\begin{aligned}
A(t+1) &= A(t)(1+\gamma)^{M(t)} \Rightarrow \\
\ln A(t+1) &= \ln A(t) + M(t)\ln(1+\gamma) \Rightarrow \\
\frac{A(t+1)-A(t)}{A(t)} &\approx \ln A(t+1) - \ln A(t) = M(t)\ln(1+\gamma) \Rightarrow \\
E_t\frac{A(t+1)-A(t)}{A(t)} &= E_t(M(t))\ln(1+\gamma) \approx \bar{\eta}(t)L_A(t)\ln(1+\gamma), \qquad (14)
\end{aligned}$$

where E_t is the expectation operator conditional on the current Poisson arrival rate $\bar{\eta}(t)L_A(t)$, and $E_tM(t)$ is a shorthand for the left-hand side of (13). Besides, "$\approx$" in the last line follows from the approximation in (13). (One should not here introduce γ as an approximation to $\log(1+\gamma)$ because that would require γ to be "small" which need not be true here. Imagine for instance that the focus is on a series of "big" communication innovations: electrical telegraphs, telephone, cell phone, internet, Skype. The time elapsed between the innovations may be many years, but each new innovation is "large".)

3.2 The economics of R&D

3.2.1 Demand for R&D labor

As noted in Section 2.1, P_A is the market value of the right to use the technical design i corresponding to innovation i. In other words, P_A is the market value of a successful research outcome. Let us consider the situation from the point of view of an R&D lab which is active in the time interval between innovation $i-1$ and innovation i. The lab's demand for R&D labor is

$$\ell_A^d = \begin{cases} \infty \text{ if } w < P_A\bar{\eta}, \\ \text{undetermined if } w = P_A\bar{\eta}, \\ 0 \text{ if } w > P_A\bar{\eta}. \end{cases} \qquad (15)$$

[7] There are no economy-wide risk factors in the model (say earthquakes, economic recession, shocks to terms of trade).

Here $P_A\bar{\eta}$ can be viewed as the value of the expected payoff per worker per time unit = value of "marginal product" of R&D labor = $P_A\partial(\bar{\eta}L_A)/\partial L_A$ from (13). For the lab to be willing to hire R&D workers, we must have $w \leq P_A\partial(\bar{\eta}L_A)/\partial L_A = P_A\bar{\eta}$, *if* R&D firms behave in a risk-neutral manner. As the next sub-section will argue, that is what they will do.

3.2.2 The financing of R&D

There is a time lag of *random* length between a research lab's outlay on R&D and the arrival of a successful research outcome, an invention. During this period, which in principle has no upper bound, the individual R&D lab is incurring sunk costs and has no revenue at all. R&D is thus risky, and continuous refinancing is needed until the research is successful.

However, since the uncertainty is "ideosyncratic", and the economy is "large" and has "many" R&D labs, the risk can be diversified. R&D labs as well as the monopolist in Sector 2 can behave in a risk-neutral manner. In equilibrium all investors will receive a rate of return equal to the risk-free interest rate.

The easiest approach to the financing issue is to assume that R&D labs finance their current expense, wL_A, by issuing equity shares that pay no dividend until success arrives. A part of households' saving is via mutual funds (that are assumed to have no administration costs) channeled to the many different R&D labs. When success arrives, the mutual funds collect a return which can take two alternative forms. Either the return is in the form of a share of the sales price, P_A, of the patent (which the successful lab receives free of charge). Or the return is in the form of shares in the profit, π, if the R&D lab decides itself to enter Sector 2 and supply the new version of the capital good services as a monopolist. For simplicity we assume that the mutual funds manage the *total* household saving and thus allocate only a part of it to R&D. The remaining part is used to buy equity shares issued by the incumbent monopolist to finance the purchases of raw capital goods in the market for these. Finally, the mutual funds pay out to their risk-averse investors, the households, a rate of return equal to the risk-free rate of interest.

3.2.3 No-arbitrage condition regarding P_A

How is under these (idealized) conditions the market value, $P_A(t)$, of a patent at time t determined in equilibrium? In view of the risk-neutral behavior by the participants in the financial markets, equilibrium requires that $P_A(t)$ satisfies the no-arbitrage condition

$$P_A(t)r(t) = \pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t). \tag{16}$$

Here $\dot{P}_A(t)$ ($\equiv dP_A(t)/dt$) is the incumbent monopolist's expected capital gain per time unit *conditional* on the monopoly position remaining in place also in the next moment. The alternative possible situation is that the monopoly position is lost due to the arrival of an innovating firm with a more productive version of the capital good. In that case the total value $P_A(t)$ is lost.

The whole right-hand side of (16) indicates the expected return per time unit on holding the patent instead of selling it and investing in the loan market. To understand this, consider a small time interval $(t, t + \Delta t)$. As seen from time t, two outcomes are possible. Either the monopoly position, and hence $P_A(t)$, is lost. According to (12), the probability of that event is approximately $\bar{\eta}(t)L_A(t)\Delta t$. Alternatively, the incumbent's monopoly remains in place over the time interval, in which case the total revenue is $(\pi(t) + \dot{P}_A(t))\Delta t$. The probability of that event is approximately $1 - \bar{\eta}(t)L_A(t)\Delta t$.

Consequently, if $z(t)$ denotes the total return per time unit on holding the patent instead of selling it, the expected return over the time interval $(t, t+\Delta t)$ is approximately

$$\begin{aligned} E_t(z(t)\Delta t) &\approx \bar{\eta}(t)L_A(t)\Delta t(-P_A(t)) + (1 - \bar{\eta}(t)L_A(t)\Delta t)\left[\pi(t) + \dot{P}_A(t)\right]\Delta t \\ &= \left[\pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t)\right]\Delta t - \bar{\eta}(t)L_A(t)\left[\pi(t) + \dot{P}_A(t)\right](\Delta t)^2. \end{aligned}$$

Dividing through by Δt, we get

$$\begin{aligned} \frac{E_t(z(t)\Delta t)}{\Delta t} &= E_t(z(t)) = \pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t) - \bar{\eta}(t)L_A(t)\left[\pi(t) + \dot{P}_A(t)\right]\Delta t \\ &\to \pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t) \quad \text{for} \quad \Delta t \to 0. \end{aligned}$$

Thus, the right-hand side of (16) does indeed represent the expected return per time unit on holding the patent instead of selling it. And the left-hand side of (16) is the return obtained by selling the patent and investing in a safe loan market. Under risk neutrality, for given expectations, the market price $P_A(t)$ adjusts so as to equalize the two sides of (16).

The no-arbitrage condition (16) plays a key role in the determination of the risk-free interest rate in general equilibrium, cf. point (v) of Lemma 1 below. Before proceeding, for purposes of intuition, it may be useful to consider the no-arbitrage condition from additional angles.

We may rewrite the no-arbitrage condition (16) in "required rate of return" form:

$$\frac{\pi(t) + \dot{P}_A(t)}{P_A(t)} = r(t) + \bar{\eta}(t)L_A(t). \tag{17}$$

Here, the instantaneous *conditional* rate of return per time unit on shares in the monopoly firm is equalized to the "required rate of return" in the sense of the minimum expected rate of return justifying staying in the Sector-2 business. This minimum rate of return is the sum of the risk-free interest rate and a premium reflecting the risk that the monopoly position expires within the next instant.

Yet another useful way of thinking about the no-arbitrage condition is in the form of the present value of expected future accounting profits:

$$P_A(t) = \int_t^\infty \pi(s)e^{-\int_t^s(r(\tau)+\bar{\eta}(\tau)L_A(\tau))d\tau}ds. \tag{18}$$

The right-hand side here makes up the *fundamental value* of the patent at time t, given the expected future risk-adjusted interest rates, $r(\tau) + \bar{\eta}(\tau)L_A(\tau)$. Indeed, (16) can be considered a differential equation for the function $P_A(t)$. The solution to this differential equation, presupposing that there are no bubbles, *is* (18) (the proof is similar to that in the appendix of Short Note 2). The convenience of (18) is that, given the expected future accounting profits and risk-adjusted interest rates, the formula directly tells us the market value of the incumbent monopolist's patent. If, for instance, π grows at a constant rate g_π, and r and $\bar{\eta}L_A$ are constant, then (18) can be written

$$\begin{aligned} P_A(t) &= \int_t^\infty \pi(t)e^{g_\pi(s-t)}e^{-(r+\bar{\eta}L_A)(s-t)}ds = \pi(t)\int_t^\infty e^{-(r+\bar{\eta}L_A-g_\pi)(s-t)}ds \\ &= \pi(t)\frac{1}{r+\bar{\eta}L_A-g_\pi}. \end{aligned} \tag{19}$$

This present-value formula is useful for intuitive interpretation of effects of everything-else-equal shifts in the interest rate, r, in the expected number of innovations per time unit, $\bar{\eta}L_A$, and in the growth rate of the profit:

$$\begin{aligned} r \ &\uparrow \ \Rightarrow P_A(t)\downarrow \text{ due to stronger discounting,} \\ \bar{\eta}L_A \ &\uparrow \ \Rightarrow P_A(t)\downarrow \text{ due to lower expected duration of monopoly,} \\ g_\pi \ &\uparrow \ \Rightarrow P_A(t)\uparrow \text{ because investors like fast-growing dividends.} \end{aligned}$$

A final comment: We have throughout presumed that a new technological breakthrough means that the monopoly position of the incumbent is lost. Could the incumbent not bid for the patent offered to the market by the successful R&D lab? Yes, it could. But new potential entrepreneurs will always (in this model) be willing to bid *more.* The incumbent faces the problem that the gain by investing in the new technology is partly destroyed since she looses the existing profits earned. This point is known as *Arrow's replacement effect* (Arrow, 1962).

4 Equilibrium in the labor market

The labor market is competitive. There is an inelastic labor supply of size $L = L_0 e^{nt}$. Equilibrium in the labor market thus requires that

$$L_Y + L_A = L = L_0 e^{nt}.$$

In equilibrium with *active* R&D ($L_A > 0$), we must have

$$w = P_A \bar{\eta},$$

in view of (15). Since labor is homogeneous, the equilibrium wage, w, must also equal marginal productivity of labor in Sector 1 at full employment:

$$w = \frac{\partial Y}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y}.$$

Combining the two last equations gives

$$P_A \bar{\eta} \equiv P_A \eta A^{\varphi - 1} L_A^{-\xi} = (1-\alpha)\frac{Y}{L_Y}. \tag{20}$$

5 Balanced growth

In the non-stochastic Romer-Jones model of horizontal innovations with Ramsey households, cf. Short Note 2, we have, under certain parameter restrictions that in the long run, the system converges to a BGP with the property that $g_y = g_c = g_k = g_A = constant > 0$. In analogy with this, we may think of the present model as portraying a system which, in a stochastic sense, in the long run approaches a path with the property that the *average* growth rates of y, c, k, and A over long time horizons are both constant and equal:

$$Eg_y = Eg_c = Eg_k = Eg_A = constant > 0. \tag{21}$$

The constancy of Eg_A means that on average there is exponential growth in input quality. This corresponds to "Moore's law": the observation that, over the history of computing hardware, the efficiency of microprocessors has approximately doubled every two years.[8] Indeed, a constant doubling time is equivalent to exponential growth. But a *two* years' doubling time is, of course, much *faster* exponential growth than what we see anywhere regarding productivity at a more aggregate level.[9]

5.1 An approximating deterministic BGP

We now take a bird's eye view and look at the long-run evolution *as if* the level of labor efficiency, A, evolves in a "smooth" deterministic way as a function of time and has actual growth rate, $g_A \equiv dA(t)/dt$, equal to the expected constant long-run growth rate, Eg_A.[10] By (14) we see that this amounts to

$$g_A = Eg_A = \bar{\eta}(t)L_A(t)\ln(1+\gamma) = \eta A(t)^{\varphi-1}L_A(t)^{1-\xi}\ln(1+\gamma) = \textit{constant} > 0, \quad (22)$$

where the last equality comes from Assumption (iv) (with $A_i = A(t)$) in Section 3.1.2 about how $\bar{\eta}(t)$ is determined.

When the evolution of A is "smooth", so is that of y, c, and k. In the present context we define an "approximating deterministic BGP" as a deterministic path along which

$$g_y = g_c = g_k = g_A, \quad (23)$$

where g_A is constant and satisfies (22). It is well-known that if $\varphi = 1$ and $n > 0$ or $\varphi < 1$ and $n = 0$, no deterministic BGP can exist (in the first case because the growth rates will continue to be rising over time, in the latter case because the needed sustained growth in L_A to compensate for the declining $A^{\varphi-1}$ will be absent). In the following lemma we therefore only need consider the combinations $\varphi = 1$ together with $n = 0$ and $\varphi < 1$ together with $n > 0$.

LEMMA 1. Let $\varphi \leq 1, n \geq 0, \eta > 0, 0 \leq \xi < 1$. Consider an approximating deterministic BGP. Let the associated g_A have the value $g_A^* > 0$. It holds that:

[8] Gordon E. Moore was co-founder of the micro-electronics industry firm Intel in the late 1960s.

[9] Two years' doubling time is equivalent to a constant growth rate of 35 percent per year ($g = (\ln 2)/2 = 0.35$).

[10] Given the time unit, say one year, and given the proportionate size, γ, of the step increases, this "even out" of the growth path of A seems more acceptable, the "larger" is A (the denominator in the calculation of the growth rate), and the more frequent are the step increases, cf. the law of large numbers.

(i) If $\varphi = 1$ and $n = 0$, then $L_A(t) = L_A$, a positive constant, $\bar{\eta} = \eta$, and $g_A^* = \eta L_A^{1-\xi} \ln(1+\gamma)$.

(ii) If $\varphi < 1$ and $n > 0$, then $g_{L_Y} = g_{L_A} = n$ and $g_A^* = \frac{1-\xi}{1-\varphi} n$.

(iii) $\bar{\eta} L_A = g_A^* / \ln(1+\gamma)$.

(iv) $g_{P_A} = g_Y = g_A^* + n = g_\pi$.

(v) $r = \alpha \bar{\eta} L_Y - (1 - \ln(1+\gamma))\bar{\eta} L_A + n$.

Proof. (i) Apply (22). (ii) That $g_{L_Y} = g_{L_A} = n$ follows by the same reasoning as in Short Note 2, Section 5.2. As to g_A^*, "take logs and time derivatives" in (22) and then solve for g_A. (iii) In (22), let $g_A = g_A^*$, and solve for $\bar{\eta} L_A$. (iv) Multiplying through by L_A in (20) gives

$$P_A \bar{\eta} L_A = (1-\alpha) Y \frac{L_A}{L_Y}, \tag{24}$$

where, along the BGP, by (iii), $\bar{\eta} L_A$ is constant, and, by (i) and (ii), so is L_A/L_Y. Hence, $g_{P_A} = g_Y = g_\pi$, where the last equality comes from (10). Moreover, $Y = yL$ so that $g_Y = g_y + n = g_A^* + n$, where the last equality follows from (23) in combination with $g_A = g_A^*$. (v) From the no-arbitrage condition (16), we have along the BGP that $r = (\pi + \dot{P}_A)/P_A - \bar{\eta} L_A = \alpha \bar{\eta} L_Y + g_A^* + n - \bar{\eta} L_A$, by (10), (20), (iii), and (iv). □

5.2 The representative household

To determine L_A and g_A along the BGP, we need more knowledge of the real interest rate, which in turn requires taking household behavior into account.

As in connection with the horizontal innovations model in Short Note 2, we assume a representative household with infinite horizon, rate of time preference equal to ρ, and CRRA instantaneous utility with parameter $\theta > 0$. The household's per head consumption will thus satisfy the Keynes-Ramsey rule

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{25}$$

and the per head financial wealth, $a(t)$, of the household will satisfy the transversality condition

$$\lim_{t\to\infty} a(t) e^{-\int_0^t (r(s)-n)ds} = 0. \tag{26}$$

Per head financial wealth is

$$a(t) \equiv \frac{K(t) + P_A(t)}{L(t)} = \frac{(K(0) + P_A(0))e^{(g_A^*+n)t}}{L(0)e^{nt}} = \frac{(K(0) + P_A(0))e^{g_A^* t}}{L(0)} \equiv a(0)e^{g_A^* t} \tag{27}$$

along the BGP, in view of (23) and (iv) of Lemma 1.

Now to the final solution of the model, along the approximating deterministic BGP. Here we have to distinguish between two alternative cases, the fully-endogenous growth case versus the semi-endogenous growth case.

5.3 The fully-endogenous growth case: $\varphi = 1$ and $n = 0$

This is the case studied in the pioneering article by Aghion and Howitt (1992). For simplicity we ignore the duplication externality and set $\xi = 0$ (as also Aghion and Howitt do).

As in Short Note 2 on the horizontal innovations model, a first step in the analysis is to pin down the relationship between the interest rate and the constant employment in R&D along the approximating deterministic BGP, as this relationship is manifested at the production side. From (v) of Lemma 1, with $n = 0$, we have along the BGP,

$$r = \alpha\eta L - \alpha\eta L_A + (\ln(1+\gamma) - 1)\eta L_A, \qquad \text{(by (i) and (iii) of Lemma 1)} \tag{28}$$

where, in addition to (v) and (i) from the lemma, we have used that $L_Y + L_A = L$.

The second step in the analysis is to pin down a second relationship between the interest rate and the constant employment in R&D, this time involving households' behavior. Isolating r in (25) along a BGP immediately gives

$$r = \rho + \theta g_c^* = \rho + \theta g_A^* = \rho + \theta\eta L_A \ln(1+\gamma), \tag{29}$$

where the second equality comes from (23) and the third from (iii) of Lemma 1; an asterisk signifies that a value along the BGP is considered. Equalizing the right-hand sides of (28) and (29) and rearranging gives

$$L_A = \frac{\alpha\eta L - \rho}{[(\theta - 1)\ln(1+\gamma) + 1 + \alpha]\,\eta} \equiv L_A^*. \tag{30}$$

By (i) of Lemma 1, with $\xi = 0$, we finally get

$$g_A^* = \eta L_A^* \ln(1+\gamma) = \frac{(\alpha\eta L - \rho)\ln(1+\gamma)}{(\theta - 1)\ln(1+\gamma) + 1 + \alpha}. \tag{31}$$

These results have been derived under the pre-condition that the transversality condition of the representative household is satisfied along the BGP and that L_A is positive.

To ensure that the transversality condition (26) with $n = 0$, in combination with (27), holds along the BGP, we need the assumption that $\rho > (1-\theta)g_A^*$. Inserting (31), and rearranging, gives the requirement

$$\rho > \frac{(1-\theta)\alpha\eta L}{1+\alpha}\ln(1+\gamma). \tag{A1-f}$$

To ensure $L_A^* > 0$, we assume

$$\rho < \alpha\eta L \quad \text{and, if } \theta < 1, \text{ then } 0 < \gamma \leq e - 1. \tag{A2}$$

Empirics generally find $\theta \geq 1$.

Imposing both (A1-f) and (A2) in present case where $\varphi = 1$ and $n = 0 = \xi$, there is a meaningful BGP solution to the model. The solution features "fully endogenous" exponential growth. This per capita growth is generated by an internal mechanism, through which labor is allocated to R&D. And the exponential per capita growth is maintained without support of growth in any exogenous factor.

Among other things, one can make comparative static analysis on the result in (31). For instance, not surprisingly, $\partial g_A^*/\partial\rho < 0$, $\partial g_A^*/\partial\theta < 0$, and $\partial g_A^*/\partial\eta > 0$, $\partial g_A^*/\partial\gamma > 0$.

We also see that $\partial g_A^*/\partial L > 0$. The "fully endogenous" growth case thus implies a *scale effect on growth,* which is an empirically problematic feature.

5.4 The semi-endogenous growth case: $\varphi < 1$, $n > 0$, and $\xi \in [0, 1)$

The order in which we find g_A^* and L_A^*/L is now reversed. The growth rate of A along the approximating deterministic BGP was found already in (ii) of Lemma 1, which displays the standard semi-endogenous growth result emphasized by Jones. We repeat the result here:

$$g_A^* = \frac{(1-\xi)n}{1-\varphi} > 0, \tag{32}$$

as $n > 0$.

Contrary to the fully-endogenous growth case, here the relative step increase, γ, does *not* affect the expected growth rate of A. This is due to Assumption (iv) in Section 3.1.2 about how the economy-wide research productivity, $\bar{\eta}$, is determined. In view of the exponent $\varphi - 1$ on A_i being negative when $\varphi < 1$, in Assumption (iv), a larger γ implies that the upward jumps in A reduce the economy-wide research productivity, $\bar{\eta}$, by more than otherwise. In the long run this means a larger expected waiting time before the next technological breakthrough.

It remains to solve for the fraction of labor in research, $s_R \equiv L_A/L$, along the approximating deterministic BGP. The solution for s_R is important for the analysis of how the *level* of y and c along the BGP depends on parameters and economic policy. From (v) of Lemma 1,

$$\begin{aligned} r - n + (1 - \ln(1+\gamma))\bar{\eta} L_A &= \alpha \bar{\eta} L_Y \Rightarrow \frac{r-n}{\alpha \bar{\eta} L_A} + \frac{1 - \ln(1+\gamma)}{\alpha} = \frac{L_Y}{L_A} \Rightarrow \\ \frac{L_A/L}{L_Y/L} &\equiv \frac{s_R}{1 - s_R} = \frac{1}{\frac{r-n}{\alpha \bar{\eta} L_A} + \frac{1-\ln(1+\gamma)}{\alpha}} \Rightarrow \\ s_R &= \frac{1}{1 + \frac{r-n}{\alpha \bar{\eta} L_A} + \frac{1-\ln(1+\gamma)}{\alpha}}. \end{aligned} \tag{33}$$

This result is essentially the same as (5.33) in Jones and Vollrath, since they have $\mu \equiv \bar{\eta} L_A$ and implicitly use the "approximation" $\gamma \approx \ln(1+\gamma)$ (which we have avoided because γ may be "large" as argued at the end of Section 3.1). Anyway, the result is only a step towards a solution because both $\mu \equiv \bar{\eta} L_A$ and r are endogenous variables in the general equilibrium of the model. Fortunately, however, we have (iii) of Lemma 1, so that (33) can be written

$$s_R = \frac{1}{1 + \ln(1+\gamma)\frac{r-n}{\alpha g_A^*} + \frac{1-\ln(1+\gamma)}{\alpha}}. \tag{34}$$

Given our household description, along the approximating deterministic BGP, r must equal $\rho + \theta g_A^*$, which, inserted into (34), gives the final solution for s_R :

$$s_R = \frac{1}{1 + \frac{1}{\alpha}\left(\ln(1+\gamma)(\frac{\rho-n}{g_A^*} + \theta - 1) + 1\right)} \equiv s_R^*, \tag{35}$$

where (32) can be inserted.

To ensure that the transversality condition (26), in combination with (27), holds along the BGP, we need the same parameter restriction as in the "fully-endogenous growth" case above and in the horizontal innovations model of Short Note 2, namely that

$$\rho - n > (1-\theta) g_A^*, \tag{A1-s}$$

with g_A^* given by (32). Moreover, with this parameter restriction we automatically have $\rho + \theta g_A^*$ $(= r^*) > \rho$ which, according to the Keynes-Ramsey rule, is needed for $g_c^* > 0$ to be an outcome in balanced growth. In addition, given $g_A^* > 0$, (A1-s) is equivalent to the factor $((\rho - n)/g_A^* + \theta - 1)$ in (35) being positive.

On the basis of the formula (35), long-run level effects on s_R^* of different parameter shifts can be studied. The roles of the parameters ρ, θ, n, φ, and ξ are qualitatively similar to their roles in the horizontal innovations model. A new feature compared with the horizontal innovations model is the appearance of the relative step increase, γ, in the formula − and with a *negative* effect on the equilibrium allocation of labor to R&D. The explanation is related to that of the absence of an effect on g_A^* from γ given above.

Like in the horizontal innovations model (cf. Exercise VII.7), level effects on $y^*(t)$ and $c^*(t)$ of parameter shifts are a bit more complicated than the level effects on s_R^*. Indeed, a shift in s_R has ambiguous effects on both $y^*(t)$ and $c^*(t)$ along a BGP.

6 Concluding remarks

In extended versions of the Schumpeterian model, there are many different types of capital goods. Each of these types are produced in its own product line represented by a point on a horizontal axis. For each of these points there is then a vertical "quality ladder" along which the quality improvements of each capital good type take place, based on new technical designs developed in corresponding specific subsets of R&D labs. Overall labor efficiency, A, then becomes an average of the leading-edge qualities in the different product lines. As an implication of this "averaging" across *many* product lines, it is common in the literature to completely "smooth out" the evolution of A and, appealing to the law of large numbers, assume away any uncertainty at the aggregate level. Thereby, a deterministic streamlined description of the economy, with $g_A = Eg_A$ at the aggregate level, is upheld.

Obviously, the present model is in many respects very abstract. For instance, it does not consider the mutual relationship between private R&D and the evolution of basic science and higher education at universities.

Another limitation is the simplifying assumption that the innovator has *perpetual* monopoly over the production and sale of the new version of the capital good. In practice, by legislation, patents are of limited duration, 15-20 years. Moreover, it may be difficult to codify exactly the technical aspects of innovations, hence not even within such a limited period do patents give 100% effective protection. While the pharmaceutical industry rely quite much on patents, in many other branches innovative firms use other protection strategies such as *concealment* of the new technical design. In ICT industries copyright to new software plays a significant role. Still, whatever the protection strategy used, imitators sooner or later find out how to make very close substitutes.

To better accommodate such facts, models have been developed where the duration of monopoly power over the commercial use of an invention is *limited* and *uncertain.* For instance, Barro and Sala-i-Martin (20014, Ch. 6.2) present a model with stochastic erosion of the innovator's monopoly power. The model exposes the policy dilemma regarding the design of patents. Both *static* and *dynamic* distortions are involved. Compared with perpetual monopoly, shorter duration of patents *mitigates* the *static inefficiency problem* arising from prices above marginal cost. Shorter duration of patents also make it easier and less expensive to build on previous discoveries. On the other hand, there is the problem that shorter duration of patents may *aggravate* the *dynamic distortion* deriving from the "surplus appropriability problem" illustrated in Jones and Vollrath, p. 134: there may be too little private incentive to invest in R&D.

At the empirical level, Jones and Williams (1998) estimate that R&D investment in the U.S. economy is only about a fourth of the social optimum. So government intervention seems motivated. But how should it be done? According to Paul Romer (2000) it may be a better growth policy strategy to support education in science and engineering than to support specific R&D activities.

There are many further aspects to take into account, e.g., spill-over effects of R&D and intensional knowledge sharing, which we shall not consider here. A survey is contained in Hall and Harhoff (2012). We end this Short Note by a citation from Wikipedia (07-05-2015):

> Legal scholars, economists, scientists, engineers, activists, policymakers, industries, and trade organizations have held differing views on patents and engaged in contentious debates on the subject. Recent criticisms primarily from the scientific community focus on the core tenet of the intended utility of patents, as now some argue they are retarding innovation. Critical perspectives emerged in the nineteenth century, and recent debates have discussed the merits and faults of software patents, nanotechnology patents, and biological patents. These debates are part of a larger discourse on intellectual property protection which also reflects differing perspectives on copyright.

7 Literature

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