Chapter 12

Learning by investing: two versions

The *learning-by-investing model,* sometimes called the *learning-by-doing model,* is one of the basic endogenous growth models. By basic is meant that the model specifies not only the technological aspects of the economy but also the market structure and the household sector, including household preferences. As in much other endogenous growth theory, the modeling of the household sector follows Ramsey and assumes the existence of a representative infinitely-lived household. Since this results in a simple determination of the long-run interest rate (the modified golden rule), the analyst can in a first approach concentrate on the main issue, technological change, without being detracted by aspects secondary to this issue.

In the present model learning from investment experience and diffusion across firms of the resulting new technical knowledge (positive externalities) play a key role.

There are two popular alternative versions of the model. The distinguishing feature is whether the learning parameter (see below) is less than one or equal to one. The first case corresponds to (a simplified version of) a model by Nobel laureate Kenneth Arrow (1962). The second case has been drawn attention to by Paul Romer (1986) who assumes that the learning parameter equals one. These two contributions start out from a common framework which we now present.[1]

[1] This lecture note also contains, in the appendix, a refresher on the concepts of a saddle point and saddle point stability.

12.1 The common framework

We consider a closed economy with firms and households interacting under conditions of perfect competition. Later, a government attempting to internalize the positive investment externality is introduced.

Let there be N firms in the economy (N "large"). Suppose they all have the same neoclassical production function, F, with CRS. Firm no. i faces the technology

$$Y_{it} = F(K_{it}, A_t L_{it}), \qquad i = 1, 2, ..., N, \tag{12.1}$$

where the economy-wide technology level A_t is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$A_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{12.2}$$

where I_s^n is aggregate net investment and $K_t = \sum_i K_{it}$.[2]

The idea is that investment − the production of capital goods − as an unintended *by-product* results in *experience* or what we may call on-the-job *learning*. Experience allows producers to recognize opportunities for process and quality improvements. In this way knowledge is achieved about how to use the new capital goods efficiently and how to produce them in a cost-efficient way. This includes learning how to improve their design so that in combination with labor they are more productive and better satisfy the needs of the users. As formulated by Arrow:

> "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).[3]

The learning is assumed to benefit producers in many lines of production in the economy: learning by doing, learning by watching, learning by using. There are knowledge spillovers across firms and these spillovers are

[2] With arbitrary units of measurement for labor and output, the hypothesis is $A_t = BK_t^{\lambda}$, $B > 0$. In (12.2) measurement units are chosen such that $B = 1$.

[3] Concerning empirical evidence of learning-by-doing and learning-by-investing, see Chapter 13. The citation of Arrow indicates that it was rather experience from cumulative *gross* investment he had in mind as the basis for learning. Yet the hypothesis in (12.2) is the more popular one - seemingly for no better reason than that it leads to simpler dynamics. Another way in which (12.2) deviates from Arrow's original ideas is by assuming that technical progress is disembodied rather than embodied, an important distinction we defined in Chapter 2.2.

reasonably fast relative to the time horizon relevant for growth theory. In our macroeconomic approach both F and A are in fact assumed to be exactly the same for all firms in the economy. That is, in this specification the firms producing consumption-goods benefit from the learning just as much as the firms producing capital-goods.

The parameter λ indicates the elasticity of the general technology level, A, with respect to cumulative aggregate net investment and is named the "learning parameter". Whereas Arrow assumes $\lambda < 1$, Romer focuses on the case $\lambda = 1$. The case of $\lambda > 1$ is ruled out since it would lead to explosive growth (infinite output in finite time) and is therefore not plausible.

12.1.1 The individual firm

In the simple Ramsey model we assumed that households directly own the capital goods in the economy and rent them out to the firms. When discussing learning-by-investment, it somehow fits the intuition better if we (realistically) assume that the firms generally own the capital goods they use. They then finance their capital investment by issuing shares and bonds. Households' financial wealth then consists of these shares and bonds.

Consider firm i. There is perfect competition in all markets. So the firm is a price taker. Its problem is to choose a production and investment plan which maximizes the present value, V_i, of expected future cash-flows. Thus the firm chooses $(L_{it}, I_{it})_{t=0}^{\infty}$ to maximize

$$V_{i0} = \int_0^\infty \left[F(K_{it}, A_t L_{it}) - w_t L_{it} - I_{it}\right] e^{-\int_0^t r_s ds} dt$$

subject to $\dot{K}_{it} = I_{it} - \delta K_{it}$. Here w_t and I_t are the real wage and gross investment, respectively, at time t, r_s is the real interest rate at time s, and $\delta \geq 0$ is the capital depreciation rate. Rising marginal capital installation costs and other kinds of adjustment costs are assumed minor and can be ignored. It can be shown that in this case the firm's problem is equivalent to maximization of current pure profits in every short time interval. So, as hitherto, we can describe the firm as just solving a series of static profit maximization problems.

We suppress the time index when not needed for clarity. At any date firm i maximizes current pure profits, $\Pi_i = F(K_i, AL_i) - (r + \delta)K_i - wL_i$. This leads to the first-order conditions for an interior solution:

$$\begin{aligned} \partial \Pi_i / \partial K_i &= F_1(K_i, AL_i) - (r + \delta) = 0, \qquad (12.3) \\ \partial \Pi_i / \partial L_i &= F_2(K_i, AL_i)A - w = 0. \end{aligned}$$

Behind (12.3) is the presumption that each firm is small relative to the economy as a whole, so that each firm's investment has a negligible effect on the economy-wide technology level A_t. Since F is homogeneous of degree one, by Euler's theorem,[4] the first-order partial derivatives, F_1 and F_2, are homogeneous of degree 0. Thus, we can write (12.3) as

$$F_1(k_i, A) = r + \delta, \tag{12.4}$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (12.4) determines k_i uniquely. From (12.4) follows that the chosen capital-labor ratio, k_i, will be the same for all firms, say $\bar{k}$.

12.1.2 The household

The representative household (or family dynasty) has $L_t = L_0 e^{nt}$ members each of which supplies one unit of labor inelastically per time unit, $n \geq 0$. The household has CRRA instantaneous utility with parameter $\theta > 0$. The pure rate of time preference is a constant, ρ. The flow budget identity in per head terms is

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given},$$

where a is per head financial wealth. The NPG condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0.$$

The resulting consumption-saving plan implies that per head consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho),$$

and the transversality condition that the NPG condition is satisfied with strict equality. In general equilibrium of our closed economy without natural resources and government debt, a_t will equal K_t/L_t.

[4] Recall that a function $f(x, y)$ defined in a domain D is homogeneous of degree h if for all (x, y) in D, $f(\lambda x, \lambda y) = \lambda^h f(x, y)$ for all $\lambda > 0$. If a differentiable function $f(x, y)$ is homogeneous of degree h, then (i) $x f_1'(x, y) + y f_2'(x, y) = h f(x, y)$, and (ii) the first-order partial derivatives, $f_1'(x, y)$ and $f_2'(x, y)$, are homogeneous of degree $h - 1$.

12.1.3 Equilibrium in factor markets

In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $\sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital-labor ratio, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{12.5}$$

As a consequence we can use (12.4) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, A_t) - \delta. \tag{12.6}$$

That is, whereas in the firm's first-order condition (12.4) causality goes from r_t to k_{it}, in (12.6) causality goes from k_t to r_t. Note also that in our closed economy with no natural resources and no government debt, a_t will equal k_t.

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, A) L_i = \sum_i F(k, A) L_i \quad \text{(by (12.1) and (12.5))} \\ &= F(k, A) \sum_i L_i = F(k, A) L = F(K, AL) = F(K, K^\lambda L) \quad \text{(by (12.2))}, \quad (12.7) \end{aligned}$$

where we have several times used that F is homogeneous of degree one.

12.2 The arrow case: $\lambda < 1$

The Arrow case is the robust case where the learning parameter satisfies $0 < \lambda < 1$. The method for analyzing the Arrow case is analogue to that used in the study of the Ramsey model with exogenous technical progress. In particular, aggregate capital per unit of effective labor, $\tilde{k} \equiv K/(AL)$, is a key variable. Let $\tilde{y} \equiv Y/(AL)$. Then

$$\tilde{y} = \frac{F(K, AL)}{AL} = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{12.8}$$

We can now write (12.6) as

$$r_t = f'(\tilde{k}_t) - \delta, \tag{12.9}$$

where $\tilde{k}_t$ is pre-determined.

12.2.1 Dynamics

From the definition $\tilde{k} \equiv K/(AL)$ follows

$$\begin{aligned} \frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - \lambda\frac{\dot{K}}{K} - n \quad \text{(by (12.2))} \\ &= (1-\lambda)\frac{Y - C - \delta K}{K} - n = (1-\lambda)\frac{\tilde{y} - \tilde{c} - \delta\tilde{k}}{\tilde{k}} - n, \quad \text{where } \tilde{c} \equiv \frac{C}{AL} \equiv \frac{c}{A}. \end{aligned}$$

Multiplying through by $\tilde{k}$ we have

$$\dot{\tilde{k}} = (1-\lambda)(f(\tilde{k}) - \tilde{c}) - [(1-\lambda)\delta + n]\,\tilde{k}. \tag{12.10}$$

In view of (12.9), the Keynes-Ramsey rule implies

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left(f'(\tilde{k}) - \delta - \rho\right). \tag{12.11}$$

Defining $\tilde{c} \equiv c/A$, now follows

$$\begin{aligned} \frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{A}}{A} = \frac{\dot{c}}{c} - \lambda\frac{\dot{K}}{K} = \frac{\dot{c}}{c} - \lambda\frac{Y - cL - \delta K}{K} = \frac{\dot{c}}{c} - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}) \\ &= \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}). \end{aligned}$$

Multiplying through by $\tilde{c}$ we have

$$\dot{\tilde{c}} = \left[\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(f(\tilde{k}) - \tilde{c} - \delta\tilde{k})\right]\tilde{c}. \tag{12.12}$$

The two coupled differential equations, (12.10) and (12.12), determine the evolution over time of the economy.

Phase diagram

Figure 12.1 depicts the phase diagram. The $\dot{\tilde{k}} = 0$ locus comes from (12.10), which gives

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}, \tag{12.13}$$

where we realistically may assume that $\delta + n/(1-\lambda) > 0$. As to the $\dot{\tilde{c}} = 0$ locus, we have

$$\begin{aligned} \dot{\tilde{c}} &= 0 \text{ for } \tilde{c} = f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda\theta}(f'(\tilde{k}) - \delta - \rho) \\ &= f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda}g_c \equiv c(\tilde{k}) \quad \text{(from (12.11))}. \end{aligned} \tag{12.14}$$

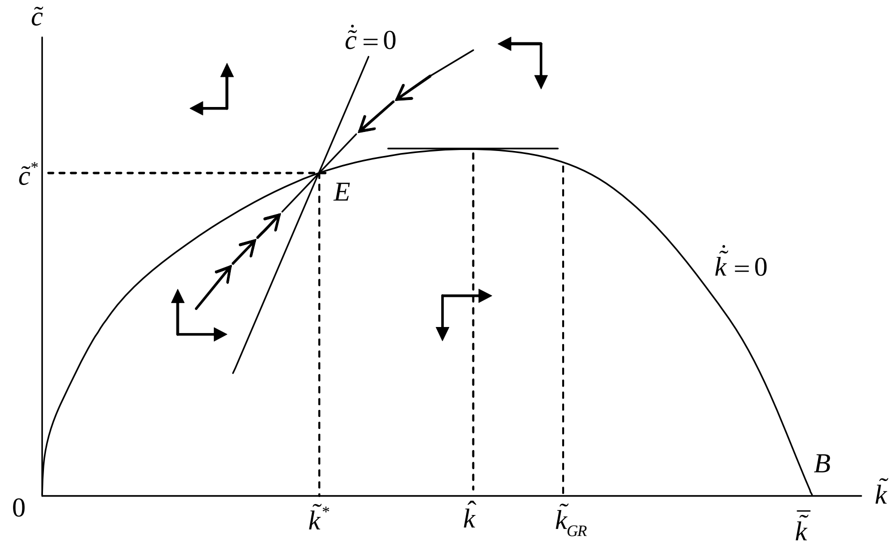

Figure 12.1: Phase diagram for the Arrow model.

Before determining the slope of the $\dot{\tilde{c}} = 0$ locus, it is convenient to consider the steady state, $(\tilde{k}^*, \tilde{c}^*)$.

Steady state

In a steady state $\tilde{c}$ and $\tilde{k}$ are constant so that the growth rate of C as well as K equals $\dot{A}/A + n$, i.e.,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{A}}{A} + n = \lambda \frac{\dot{K}}{K} + n.$$

Solving gives

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda}.$$

Thence, in a steady state

$$g_c = \frac{\dot{C}}{C} - n = \frac{n}{1-\lambda} - n = \frac{\lambda n}{1-\lambda} \equiv g_c^*, \quad \text{and} \tag{12.15}$$

$$\frac{\dot{A}}{A} = \lambda \frac{\dot{K}}{K} = \frac{\lambda n}{1-\lambda} = g_c^*. \tag{12.16}$$

The steady-state values of r and $\tilde{k}$, respectively, will therefore satisfy, by (12.11),

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g_c^* = \rho + \theta \frac{\lambda n}{1-\lambda}. \tag{12.17}$$

To ensure existence of a steady state we assume that the private marginal product of capital is sufficiently sensitive to capital per unit of effective labor, from now called the "capital intensity":

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \delta + \rho + \theta\frac{\lambda n}{1-\lambda} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

The transversality condition of the representative household is that $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$, where a_t is per capita financial wealth. In general equilibrium $a_t = k_t \equiv \tilde{k}_t A_t$, where A_t in steady state grows according to (12.16). Thus, in steady state the transversality condition can be written

$$\lim_{t\to\infty} \tilde{k}^* e^{(g_c^* - r^* + n)t} = 0. \tag{TVC}$$

For this to hold, we need

$$r^* > g_c^* + n = \frac{n}{1-\lambda}, \tag{12.18}$$

by (12.15). In view of (12.17), this is equivalent to

$$\rho - n > (1-\theta)\frac{\lambda n}{1-\lambda}, \tag{A2}$$

which we assume satisfied.

As to the slope of the $\dot{\tilde{c}} = 0$ locus we have, from (12.14),

$$c'(\tilde{k}) = f'(\tilde{k}) - \delta - \frac{1}{\lambda}(\tilde{k}\frac{f''(\tilde{k})}{\theta} + g_c) > f'(\tilde{k}) - \delta - \frac{1}{\lambda}g_c, \tag{12.19}$$

since $f'' < 0$. At least in a small neighborhood of the steady state we can sign the right-hand side of this expression. Indeed,

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* = \rho + \theta g_c^* - \frac{1}{\lambda}g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} = \rho - n - (1-\theta)\frac{\lambda n}{1-\lambda} > 0, \tag{12.20}$$

by (12.15) and (A2). So, combining with (12.19), we conclude that $c'(\tilde{k}^*) > 0$. By continuity, in a small neighborhood of the steady state, $c'(\tilde{k}) \approx c'(\tilde{k}^*) > 0$. Therefore, close to the steady state, the $\dot{\tilde{c}} = 0$ locus is positively sloped, as indicated in Figure 12.1.

Still, we have to check the following question: In a neighborhood of the steady state, which is steeper, the $\dot{\tilde{c}} = 0$ locus or the $\dot{\tilde{k}} = 0$ locus? The slope of the latter is $f'(\tilde{k}) - \delta - n/(1-\lambda)$, from (12.13). At the steady state this slope is

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* \in (0, c'(\tilde{k}^*)),$$

in view of (12.20) and (12.19). The $\dot{\tilde{c}} = 0$ locus is thus steeper. So, the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus from below and can only cross once.

The assumption (A1) ensures existence of a $\tilde{k}^* > 0$ satisfying (12.17). As Figure 12.1 is drawn, a little more is implicitly assumed namely that there exists a $\hat{k} > 0$ such that the *private* net marginal product of capital equals the steady-state growth rate of output, i.e.,

$$f'(\hat{k}) - \delta = (\frac{\dot{Y}}{Y})^* = (\frac{\dot{A}}{A})^* + \frac{\dot{L}}{L} = \frac{\lambda n}{1-\lambda} + n = \frac{n}{1-\lambda}, \tag{12.21}$$

where we have used (12.16). Thus, the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \hat{k}$ is horizontal and $\hat{k} > \tilde{k}^*$ as indicated in the figure.

Note, however, that $\hat{k}$ is not the golden-rule capital intensity. The latter is the capital intensity, $\tilde{k}_{GR}$, at which the *social* net marginal product of capital equals the steady-state growth rate of output (see Appendix). If $\tilde{k}_{GR}$ exists, it will be larger than $\hat{k}$ as indicated in Figure 12.1. To see this, we now derive a convenient expression for the social marginal product of capital. From (12.7) we have

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= F_1(\cdot) + F_2(\cdot)\lambda K^{\lambda-1}L = f'(\tilde{k}) + F_2(\cdot)K^{\lambda}L(\lambda K^{-1}) \quad \text{(by (12.8))} \\
&= f'(\tilde{k}) + (F(\cdot) - F_1(\cdot)K)\,\lambda K^{-1} \quad \text{(by Euler's theorem)} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda}L - f'(\tilde{k})K)\lambda K^{-1} \quad \text{(by (12.8) and (12.2))} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda-1}L - f'(\tilde{k}))\lambda = f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} > f'(\tilde{k}).
\end{aligned}$$

in view of $\tilde{k} = K/(K^{\lambda}L) = K^{1-\lambda}L^{-1}$ and $f(\tilde{k})/\tilde{k} - f'(\tilde{k}) > 0$. As expected, the positive externality makes the social marginal product of capital larger than the private one. Since we can also write $\partial Y/\partial K = (1-\lambda)f'(\tilde{k}) + \lambda f(\tilde{k})/\tilde{k}$, we see that $\partial Y/\partial K$ is (still) a decreasing function of $\tilde{k}$ since both $f'(\tilde{k})$ and $f(\tilde{k})/\tilde{k}$ are decreasing in $\tilde{k}$. So the golden rule capital intensity, $\tilde{k}_{GR}$, will be that capital intensity which satisfies

$$f'(\tilde{k}_{GR}) + \lambda\frac{f(\tilde{k}_{GR}) - \tilde{k}_{GR}f'(\tilde{k}_{GR})}{\tilde{k}_{GR}} - \delta = \left(\frac{\dot{Y}}{Y}\right)^* = \frac{n}{1-\lambda}.$$

To ensure there exists such a $\tilde{k}_{GR}$, we strengthen the right-hand side inequality in (A1) by the assumption

$$\lim_{\tilde{k}\to\infty}\left(f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}}\right) < \delta + \frac{n}{1-\lambda}. \tag{A3}$$

This, together with (A1) and $f'' < 0$, implies existence of a unique $\tilde{k}_{GR}$, and in view of our additional assumption (A2), we have $0 < \tilde{k}^* < \hat{k} < \tilde{k}_{GR}$, as displayed in Figure 12.1.

Stability

The arrows in Figure 12.1 indicate the direction of movement, as determined by (12.10) and (12.12)). We see that the steady state is a *saddle point*. Moreover, the dynamic system is *saddle-point stable*.[5] The dynamic system has one pre-determined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$. The saddle path is not parallel to the jump variable axis. We claim that for a given $\tilde{k}_0 > 0$, (i) the initial value of $\tilde{c}_0$ will be the ordinate to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path; (ii) over time the economy will move along the saddle path towards the steady state. Indeed, this time path is consistent with all conditions of general equilibrium, including the transversality condition (TVC). And the path is the *only* technically feasible path with this property. Indeed, all the divergent paths in Figure 12.1 can be ruled out as equilibrium paths because they can be shown to violate the transversality condition of the household.

In the long run c and $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k}^*)A$ grow at the rate $\lambda n/(1-\lambda)$, which is positive if and only if $n > 0$. This is an example of *endogenous growth* in the sense that the positive long-run per capita growth rate is generated through an internal mechanism (learning) in the model (in contrast to exogenous technology growth as in the Ramsey model with exogenous technical progress).

12.2.2 Two types of endogenous growth

As also touched upon elsewhere in these lecture notes, it is useful to distinguish between two types of endogenous exponential growth. *Fully endogenous* exponential growth occurs when the long-run growth rate of c is positive without support from growth in any exogenous factor; the Romer case, to be considered in the next section, provides an example. *Semi-endogenous* exponential growth occurs if growth is endogenous but a positive per capita growth rate can not be maintained in the long run without the support from growth in some exogenous factor (for example exogenous growth in the labor force). Clearly, in the Arrow version of learning by investing, exponential growth is "only" semi-endogenous. The technical reason for this is the assumption that the learning parameter, λ, is below 1, which implies diminishing marginal returns to capital at the aggregate level. As a consequence,

[5] A formal definition is given in Appendix B.

if and only if $n > 0$, do we have $\dot{c}/c > 0$ in the long run. In line with this, $\partial g_y^*/\partial n > 0$.[6]

The key role of population growth derives from the fact that although there are diminishing marginal returns to capital at the aggregate level, there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. To put it differently: when there are increasing returns to K and L together, growth in the labor force not only counterbalances the falling marginal productivity of aggregate capital (this counter-balancing role reflects the direct complementarity between K and L), but also upholds sustained productivity growth via the learning mechanism.

Note that in the semi-endogenous growth case, $\partial g_y^*/\partial \lambda = n/(1-\lambda)^2 > 0$ for $n > 0$. That is, a higher value of the learning parameter implies higher per capita growth in the long run, when $n > 0$. Note also that $\partial g_y^*/\partial \rho = 0$ $= \partial g_y^*/\partial \theta$, that is, in the semi-endogenous growth case, preference parameters do not matter for the long-run per capita growth rate. As indicated by (12.15), the long-run growth rate is tied down by the learning parameter, λ, and the rate of population growth, n. Like in the simple Ramsey model, however, it can be shown that preference parameters matter for the *level* of the growth path. For instance (12.17) shows that $\partial \tilde{k}^*/\partial \rho < 0$ so that more patience (lower ρ) imply a higher $\tilde{k}^*$ and thereby a higher $y_t = f(\tilde{k}^*)A_t$.

This suggests that although taxes and subsidies do not have long-run growth effects, they can have *level* effects.

In this model there is clearly a motivation for government intervention due to the positive externality of private investment. But details about the design of government policy vis-a-vis this externality will in this lecture note only be discussed in relation to the Romer case of $\lambda = 1$, which is simpler and to which we now return.

12.3 Romer's limiting case: $\lambda = 1$, $n = 0$

We now consider the limiting case $\lambda = 1$. We should think of it as a thought experiment because, by most observers, the value 1 is considered an unrealistically high value for the learning parameter. Moreover, in combination with $n > 0$, the value 1 will lead to a forever rising per capita growth rate which does not accord the economic history of the industrialized world over more

[6]Note, however, that the model, and therefore (12.15), presupposes $n \geq 0$. If $n < 0$, the steady-state formulas in Section 12.2 are no longer valid. The formula in (12.16) would for instance imply a *decreasing* level of technical knowledge, which, at least in a modern economy, is implausible.

than a century. To avoid a forever rising growth rate, we therefore introduce the parameter restriction $n = 0$.

The resulting model turns out to be extremely simple and at the same time it gives striking results (both circumstances have probably contributed to its popularity).

First, with $\lambda = 1$ we get $A = K$ and so the equilibrium interest rate is, by (12.6),

$$r = F_1(k, K) - \delta = F_1(1, L) - \delta \equiv \bar{r},$$

where we have divided the two arguments of $F_1(k, K)$ by $k \equiv K/L$ and again used Euler's theorem. Note that the interest rate is constant "from the beginning" and independent of the historically given initial value of K, K_0. The aggregate production function is now

$$Y = F(K, KL) = F(1, L)K, \qquad L \text{ constant}, \tag{12.22}$$

and is thus *linear* in the aggregate capital stock.[7] In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. Thereby the Romer model belongs to the class of *reduced-form AK models*, that is, models where in general equilibrium the interest rate and the aggregate output-capital ratio are necessarily constant over time whatever the initial conditions.

The method for analyzing an AK model is different from the one used for a diminishing returns model as above.

12.3.1 Dynamics

The Keynes-Ramsey rule now takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv \gamma, \tag{12.23}$$

which is also constant "from the beginning". To ensure positive growth, we assume

$$F_1(1, L) - \delta > \rho. \tag{A1'}$$

And to ensure bounded intertemporal utility (and thereby a possibility of satisfying the transversality condition of the representative household), it is assumed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A2'}$$

[7] Acemoglu, p. 400, writes this as $Y = \tilde{f}(L)K$.

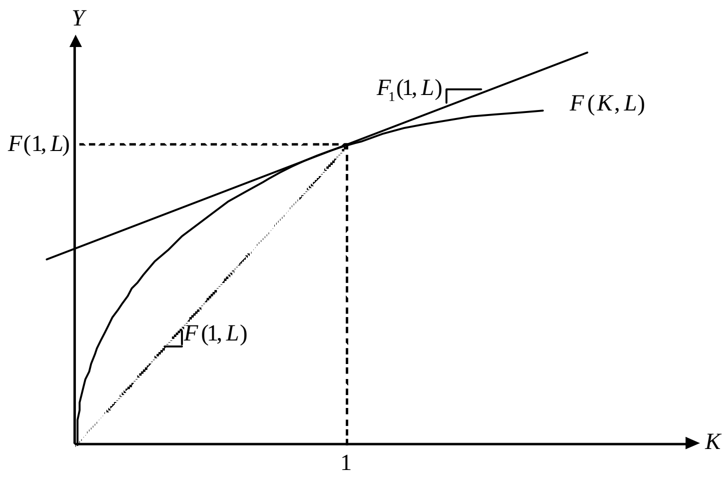

Figure 12.2: Illustration of the fact that for L given, $F(1, L) > F_1(1, L)$.

Solving the linear differential equation (12.23) gives

$$c_t = c_0 e^{\gamma t}, \tag{12.24}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by applying the households' transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

First, note that the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1, L)K - cL - \delta K,$$

or, in per-capita terms,

$$\dot{k} = [F(1, L) - \delta]\, k - c_0 e^{\gamma t}. \tag{12.25}$$

In this equation it is important that $F(1, L) - \delta - \gamma > 0$. To understand this inequality, note that, by (A2'), $F(1, L) - \delta - \gamma > F(1, L) - \delta - \bar{r} = F(1, L) - F_1(1, L) = F_2(1, L)L > 0$, where the first equality is due to $\bar{r} = F_1(1, L) - \delta$ and the second is due to the fact that since F is homogeneous of degree 1, we have, by Euler's theorem, $F(1, L) = F_1(1, L) \cdot 1 + F_2(1, L)L > F_1(1, L) > \delta$, in view of (A1'). The key property $F(1, L) - F_1(1, L) > 0$ is illustrated in Figure 12.2.

The solution of a general linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$, is

$$x(t) = (x(0) - \frac{c}{a+h})e^{-at} + \frac{c}{a+h}e^{ht}. \qquad (12.26)$$

Thus the solution to (12.25) is

$$k_t = (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1,L) - \delta - \gamma}e^{\gamma t}. \qquad (12.27)$$

To check whether (TVC) is satisfied we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L) - \delta - \gamma}e^{(\gamma-\bar{r})t} \\ &\rightarrow (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma})e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \rightarrow \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A2'). But $\bar{r} = F_1(1, L) - \delta < F(1, L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1, L) - \delta - \gamma)k_0. \qquad (12.28)$$

If c_0 is less than this, there will be over-saving and (TVC) is violated ($a_t e^{-\bar{r}t} \rightarrow \infty$ for $t \rightarrow \infty$, since $a_t = k_t$). If c_0 is higher than this, both the NPG and (TVC) are violated ($a_t e^{-\bar{r}t} \rightarrow -\infty$ for $t \rightarrow \infty$).

Inserting the solution for c_0 into (12.27), we get

$$k_t = \frac{c_0}{F(1,L) - \delta - \gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1, L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional dynamics).

This is a case of *fully endogenous growth* in the sense that the long-run growth rate of c is positive without the support by growth in any exogenous factor. This outcome is due to the absence of diminishing returns to aggregate capital, which is implied by the assumed high value of the learning parameter. But the empirical foundation for this high value is weak, to say the least, cf. Chapter 13. A further drawback of this special version of the learning model is that the results are *non-robust*. With λ slightly less than 1, we are back in the Arrow case and growth peters out, since $n = 0$. With λ slightly above 1, it can be shown that growth becomes explosive: infinite output in finite time![8]

[8] See Appendix B in Chapter 13.

The Romer case, $\lambda = 1$, is thus a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which *apriori* can take any value within an interval. Second, the imposed value leads to non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system.

Note that the *causal structure* in the long run in the diminishing returns case is different than in the AK-case of Romer. In the diminishing returns case the steady-state growth rate is determined first, as g_c^* in (12.15), then r^* is determined through the Keynes-Ramsey rule and, finally, Y/K is determined by the technology, given r^*. In contrast, the Romer case has Y/K and r directly given as $F(1, L)$ and $\bar{r}$, respectively. In turn, $\bar{r}$ determines the (constant) equilibrium growth rate through the Keynes-Ramsey rule.

12.3.2 Economic policy in the Romer case

In the AK case, that is, the fully endogenous growth case, we have $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$. Thus, preference parameters *matter* for the long-run growth rate and not "only" for the *level* of the upward-sloping time path of per capita output. This suggests that taxes and subsidies can have *long-run* growth effects. In any case, in this model there is a motivation for government intervention due to the positive externality of private investment. This motivation is present whether $\lambda < 1$ or $\lambda = 1$. Here we concentrate on the latter case, for no better reason than that it is simpler. We first find the social planner's solution.

The social planner

Recall that by a *social planner* we mean a fictional "all-knowing and all-powerful" decision maker who maximizes an objective function under no other constraints than what follows from technology and initial resources. The social planner faces the aggregate production function (12.22) or, in per capita terms, $y_t = F(1, L)k_t$. The social planner's problem is to choose $(c_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= F(1, L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, && (12.29) \\ k_t &\geq 0 \text{ for all } t > 0. && (12.30) \end{aligned}$$

The current-value Hamiltonian is

$$H(k, c, \eta, t) = \frac{c^{1-\theta}}{1-\theta} + \eta\left(F(1, L)k - c - \delta k\right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \tag{12.31}$$

$$\frac{\partial H}{\partial k} = \eta(F(1, L) - \delta) = -\dot{\eta} + \rho\eta. \tag{12.32}$$

We guess that also the transversality condition,

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \tag{12.33}$$

must be satisfied by an optimal solution.[9] This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really an optimal solution.

Log-differentiating w.r.t. t in (12.31) and combining with (12.32) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1, L) - \delta - \rho) \equiv \gamma_{SP}. \tag{12.34}$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal product of capital is $\partial y_t/\partial k_t = F(1, L) > F_1(1, L)$. To ensure bounded intertemporal utility we sharpen (A2') to

$$\rho > (1-\theta)\gamma_{SP}. \tag{A2''}$$

To find the time path of k_t, note that the dynamic resource constraint (12.29) can be written

$$\dot{k}_t = (F(1, L) - \delta)k_t - c_0 e^{\gamma_{SP} t},$$

in view of (12.34). By the general solution formula (12.26) this has the solution

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t}. \tag{12.35}$$

[9] The proviso implied by saying "guess" is due to the fact that optimal control theory does not guarantee that this "standard" transversality condition is necessary for optimality in *all* infinite horizon optimization problems.

In view of (12.32), in an interior optimal solution the time path of the adjoint variable η is

$$\eta_t = \eta_0 e^{-[(F(1,L)-\delta-\rho]t},$$

where $\eta_0 = c_0^{-\theta} > 0$, by (12.31). Thus, the conjectured transversality condition (12.33) implies

$$\lim_{t\to\infty} k_t e^{-(F(1,L)-\delta)t} = 0, \tag{12.36}$$

where we have eliminated η_0. To ensure that this is satisfied, we multiply k_t from (12.35) by $e^{-(F(1,L)-\delta)t}$ to get

$$\begin{aligned} k_t e^{-(F(1,L)-\delta)t} &= k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} + \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{[\gamma_{SP}-(F(1,L)-\delta)]t} \\ &\to k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} \text{ for } t \to \infty, \end{aligned}$$

since, by (A2"), $\gamma_{SP} < \rho + \theta\gamma_{SP} = F(1,L) - \delta$ in view of (12.34). Thus, (12.36) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma_{SP})k_0. \tag{12.37}$$

Inserting this solution for c_0 into (12.35), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{\gamma_{SP}t} = k_0 e^{\gamma_{SP}t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, our candidate for the social planner's solution is from start in balanced growth (there is no transitional dynamics).

The next step is to check whether our candidate solution satisfies a set of *sufficient* conditions for an optimal solution. Here we can use *Mangasarian's theorem* which, applied to a problem like this, with one control variable and one state variable, says that the following conditions are sufficient:

(a) Concavity: The Hamiltonian is jointly concave in the control and state variables, here c and k.

(b) Non-negativity: There is for all $t \geq 0$ a non-negativity constraint on the state variable; and the co-state variable, η, is non-negative for all $t \geq 0$.

(c) TVC: The candidate solution satisfies the transversality condition $\lim_{t\to\infty} k_t\eta_t e^{-\rho t} = 0$, where $\eta_t e^{-\rho t}$ is the discounted co-state variable.

In the present case we see that the Hamiltonian is a sum of concave functions and therefore is itself concave in (k, c). Further, from (12.30) we see that condition (b) is satisfied. Finally, our candidate solution is constructed so as to satisfy condition (c). The conclusion is that our candidate solution *is* an optimal solution. We call it the SP allocation.

Implementing the SP allocation in the market economy

Returning to the market economy, we assume there is a policy maker, say the government, with only two activities. These are (i) paying an investment subsidy, s, to the firms so that their capital costs are reduced to

$$(1-s)(r+\delta)$$

per unit of capital per time unit; (ii) financing this subsidy by a constant consumption tax rate τ.

Let us first find the size of s needed to establish the SP allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-s)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-s)(r+\delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-s)(r+\delta),$$

where $k \equiv K/L$, which is pre-determined from the supply side. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k, K)}{1-s} - \delta = \frac{F_1(1, L)}{1-s} - \delta, \tag{12.38}$$

again using Euler's theorem.

It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1, L) - \delta$. If we can obtain $r = F(1, L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required s thus satisfies

$$r = \frac{F_1(1, L)}{1-s} - \delta = F(1, L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1,L)}{F(1,L)} = \frac{F(1,L) - F_1(1,L)}{F(1,L)} = \frac{F_2(1,L)L}{F(1,L)}.$$

In case $Y_i = K_i^{\alpha}(AL_i)^{1-\alpha}$, $0 < \alpha < 1$, $i = 1,\ldots, N$, this gives $s = 1 - \alpha$.

It remains to find the required consumption tax rate τ. The tax revenue will be τcL, and if the government budget should be balanced at every instant,[10] the *required* tax revenue is

$$\mathcal{T} = s(r + \delta)K = (F(1,L) - F_1(1,L))\, K = \tau cL.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{\mathcal{T}}{cL} = \frac{F(1,L) - F_1(1,L)}{c/k} = \frac{F(1,L) - F_1(1,L)}{F(1,L) - \delta - \gamma_{SP}} > 0, \qquad (12.39)$$

where we have used that the proportionality in (12.37) between c and k holds for all $t \geq 0$. Substituting (12.34) into (12.39), the solution for τ can be written

$$\tau = \frac{\theta\,[F(1,L) - F_1(1,L)]}{(\theta - 1)(F(1,L) - \delta) + \rho} = \frac{\theta F_2(1,L)L}{(\theta - 1)(F(1,L) - \delta) + \rho}.$$

The required tax rate on consumption is thus a constant. It therefore does not distort the consumption/saving decision on the margin, cf. Chapter 11.

It follows that the allocation obtained by this subsidy-tax policy *is* the SP allocation. A policy, here the policy (s, τ), which in a decentralized system induces the SP allocation, is called a *first-best policy.*

12.4 Appendix

A. The golden-rule capital intensity in the Arrow case

In our discussion of the Arrow model in Section 12.2 (where $0 < \lambda < 1$), we claimed that the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that effective capital-labor ratio at which the social net marginal productivity of capital equals the steady-state growth rate of output. In this respect the Arrow model with endogenous technical progress is similar to the standard neoclassical growth model with exogenous technical progress.

The claim corresponds to a very general theorem, valid also for models with many capital goods and non-existence of an aggregate production function. This theorem says that the highest sustainable path for consumption

[10] We say "if" because in a growing economy there is scope for some persistent deficit financing without threatening fiscal sustainability.

per unit of labor in an economy will be that path which results from those techniques which profit maximizing firms choose under perfect competition when the real interest rate equals the steady-state growth rate of GNP (see Gale and Rockwell, 1975).

To prove our claim, note that in steady state, (12.14) holds whereby consumption per unit of labor (here the same as per capita consumption in view of L = labor force = population) can be written

$$\begin{aligned} c_t &\equiv \tilde{c}_t A_t = \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] K_t^{\lambda} \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \left(K_0 e^{\frac{n}{1-\lambda}t}\right)^{\lambda} \quad (\text{by } g_K^* = \frac{n}{1-\lambda}) \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \left((\tilde{k}L_0)^{\frac{1}{1-\lambda}} e^{\frac{n}{1-\lambda}t}\right)^{\lambda} \quad (\text{from } \tilde{k} = \frac{K_t}{K_t^{\lambda} L_t} = \frac{K_t^{1-\lambda}}{L_t} \text{ also for } t = 0) \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \tilde{k}^{\frac{\lambda}{1-\lambda}} L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t} \equiv \varphi(\tilde{k}) L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}, \end{aligned}$$

defining $\varphi(\tilde{k})$ in the obvious way.

We look for that value of $\tilde{k}$ at which this steady-state path for c_t is at the highest technically feasible level. The positive coefficient, $L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}$, is the only time-dependent factor and can be ignored since it is exogenous. The problem is thereby reduced to the static problem of maximizing $\varphi(\tilde{k})$ with respect to $\tilde{k} > 0$. We find

$$\begin{aligned} \varphi'(\tilde{k}) &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\right] \tilde{k}^{\frac{\lambda}{1-\lambda}} + \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \frac{\lambda}{1-\lambda} \tilde{k}^{\frac{\lambda}{1-\lambda}-1} \\ &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda}) + \left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \frac{n}{1-\lambda})\right) \frac{\lambda}{1-\lambda}\right] \tilde{k}^{\frac{\lambda}{1-\lambda}} \\ &= \left[(1-\lambda)f'(\tilde{k}) - (1-\lambda)\delta - n + \lambda \frac{f(\tilde{k})}{\tilde{k}} - \lambda(\delta + \frac{n}{1-\lambda})\right] \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \\ &= \left[(1-\lambda)f'(\tilde{k}) - \delta + \lambda \frac{f(\tilde{k})}{\tilde{k}} - \frac{n}{1-\lambda}\right] \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \equiv \psi(\tilde{k}) \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda}, \quad (12.40) \end{aligned}$$

defining $\psi(\tilde{k})$ in the obvious way. The first-order condition for the problem, $\varphi'(\tilde{k}) = 0$, is equivalent to $\psi(\tilde{k}) = 0$. After ordering this gives

$$f'(\tilde{k}) + \lambda \frac{f(\tilde{k}) - \tilde{k} f'(\tilde{k})}{\tilde{k}} - \delta = \frac{n}{1-\lambda}. \quad (12.41)$$

We see that

$$\varphi'(\tilde{k}) \gtreqless 0 \quad \text{for} \quad \psi(\tilde{k}) \gtreqless 0,$$

respectively. Moreover,

$$\psi'(\tilde{k}) = (1-\lambda)f''(\tilde{k}) - \lambda \frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}^2} < 0,$$

in view of $f'' < 0$ and $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$. So a $\tilde{k} > 0$ satisfying $\psi(\tilde{k}) = 0$ is the unique maximizer of $\varphi(\tilde{k})$. By (A1) and (A3) in Section 12.2 such a $\tilde{k}$ exists and is thereby the same as the $\tilde{k}_{GR}$ we were looking for.

The left-hand side of (12.41) equals the social marginal productivity of capital and the right-hand side equals the steady-state growth rate of output. At $\tilde{k} = \tilde{k}_{GR}$ it therefore holds that

$$\frac{\partial Y}{\partial K} - \delta = \left(\frac{\dot{Y}}{Y}\right)^*.$$

This confirms our claim in Section 12.2 about $\tilde{k}_{GR}$.

Remark about the absence of a golden rule in the Romer model. In the Romer model, which has constant L, the golden rule is not a well-defined concept for the following reason. Along any balanced growth path we have from (12.29),

$$g_k \equiv \frac{\dot{k}_t}{k_t} = F(1, L) - \delta - \frac{c_t}{k_t} = F(1, L) - \delta - \frac{c_0}{k_0},$$

because g_k $(= g_K)$ is by definition constant along a balanced growth path, whereby also c_t/k_t must be constant. We see that g_k is decreasing linearly from $F(1, L) - \delta$ to $-\delta$ when c_0/k_0 rises from nil to $F(1, L)$. So choosing among alternative technically feasible balanced growth paths is inevitably a choice between starting with low consumption to get high growth forever or starting with high consumption to get low growth forever. Given any $k_0 > 0$, the alternative possible balanced growth paths will therefore sooner or later cross each other in the $(t, \ln c)$ plane. Hence, there exists no balanced growth path which for all $t \geq 0$ has c_t higher than along any other technically feasible balanced growth path. So no golden rule path exists. This is a general property of AK and reduced-form AK models.

B. Saddle-point stability

This appendix is a refresher on the concept of saddle-point stability, a concept which perplexes many people.

Consider a *two-dimensional* dynamic system (two coupled first-order differential equations). Suppose the system has a steady state which is a *saddle point* (which is the case if and only if the two eigenvalues of the associated Jacobian matrix, evaluated at the steady state, are of opposite sign). Then, so far, either presence or absence of saddle-point stability is possible. And which of the two cases occur can not be diagnosed from the two differential equations in isolation. One has to consider the boundary conditions. Here is a complete definition of (local) saddle-point stability.

DEFINITION. A steady state of a two-dimensional dynamic system is (locally) *saddle-point stable* if:

1. the steady state is a saddle point;
2. one of the two endogenous variables is predetermined while the other is a jump variable;
3. the saddle path is not parallel to the jump variable axis; and
4. there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

Thus, to establish saddle-point stability, all four properties must be verified. If for instance point 1 and 2 hold, but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \to \infty$.

To say that the steady state is saddle-point stable is synonymous with saying that the *dynamic system* is saddle-point stable.

For an n-*dimensional* dynamic system (n coupled first-order differential equations, $n \geq 2$) the concepts of a saddle point and saddle-point stability are defined via a generalization of point 1 to 4. As to point 1: A steady state of an n-dimensional dynamic system is called a *saddle point* if all eigenvalues of the associated Jacobian matrix have non-zero real parts and at least two of the eigenvalues, have real parts of opposite sign. As to point 2: The number of predetermined variables in the system equals the number of eigenvalues with negative real part (and the number of jump variables in the system consequently equals the number of eigenvalues with positive real part). The generalization of points 3 and 4 is more nerdy, so we refer to, for instance, Hirsch and Smale, *Differential equations, dynamic systems, and linear algebra*, Academic Press, 1974.