VII.14 *The original Romer model of horizontal innovations.* The "original Romer model" is from Romer (JPE, 1990) and is a limiting case of the model of the exercises VII.10 - VII.13, namely the case $\varphi = 1$. Since in this case, $n > 0$ would lead to a forever rising per capita growth rate, which is implausible, Romer concentrates on the case $n = 0$. A further difference from the general framework in VII.10 - VII.13 is that for simplicity, Romer ignores the "stepping on toes" effect, thus concentrating on the case $\xi = 0$. The household sector consists of Ramsey households in the same way as in Exercise VII.12, but with $n = 0$.[2] To include economic policy in the model, we assume there is a subsidy $\sigma \in (0, 1)$ to R&D, financed by a lump-sum tax.

Notation is as in VII.10 - VII.13. The final good is the numeraire. The time index t on time-dependent variables is omitted unless needed for clarity; $\forall j$ means $j = 1, 2, \ldots, A$. In compact form the basic relations of the model are:

$$Y = L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha}, \qquad 0 < \alpha < 1, \tag{1}$$

$$\frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L_Y} = w, \qquad \frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} = p_j, \qquad \forall j, \tag{Y-FOCs}$$

$$Y = C + I_K = cL + \dot{K} + \delta K, \ \delta \geq 0, \ K_0 > 0 \text{ given}, \ L > 0, \ \text{constant}, \tag{2}$$

$$p_j = \frac{1}{\alpha}(r+\delta) \equiv p, \qquad \forall j, \tag{3}$$

$$\pi_j = (\frac{1}{\alpha} - 1)(r+\delta)x_j = (\frac{1}{\alpha} - 1)(r+\delta)x \equiv \pi, \ \forall j, \tag{3'}$$

$$\dot{A} = \bar{\eta} L_A \equiv \eta A L_A, \qquad \eta > 0, \quad A_0 > 0 \text{ given}, \tag{4}$$

$$(1-\sigma)w \geqq P_A \partial \dot{A}/\partial L_A = P_A \bar{\eta}, \text{ with " = " if } L_A > 0, \tag{5}$$

$$P_A r = \pi + \dot{P}_A, \tag{6}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho), \tag{7}$$

$$\lim_{t\to\infty} a_t e^{-\int_0^t r_s ds} = 0, \qquad a_t \equiv \frac{K_t + P_{At}A_t}{L}. \tag{8}$$

[2] To be true, not exactly "in the same way", since in the original Romer model the representative household's labor supply, L, contains both skilled and unskilled labor, but in fixed exogenous amounts. Unskilled labor is only used in the final goods sector, while skilled labor is used in both that sector and the R&D sector. As education is not considered (skills come as "manna from heaven"), and the R&D sector demands no unskilled labor, the conclusions from the model are essentially the same as if labor were homogeneous. Hence, we here simplify and assume homogenous labor.

In general equilibrium with active R&D ($L_A > 0$) we thus have (cf. VII.10 - VII.11)

$$
\begin{aligned}
(K^d &=)\ Ax \ = K\ (= K^s), && (9)\\
L_Y + L_A &= L, && (10)\\
Y &= K^{\alpha}(AL_Y)^{1-\alpha}, && (11)\\
\frac{1}{\alpha}(r+\delta) &= \frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha}(\frac{K}{A})^{\alpha-1} = \alpha\frac{Y}{K} = \frac{\partial Y}{\partial K}, && (12)\\
\pi &= (1-\alpha)\alpha\frac{Y}{A}, && (13)\\
w &= (1-\alpha)\frac{Y}{L_Y} = \frac{P_A \eta A}{1-\sigma}. && (14)
\end{aligned}
$$

a) Briefly, interpret (3), (5), (6), (7), (8), and (14).

b) Show that sustained exponential growth in technical knowledge, measured by A, is technically feasible.

The method of solving the model for balanced growth is somewhat different from the method in the Jones case $\varphi < 1$ and $n > 0$.

c) Show that P_A must be constant along a BGP. *Hint:* apply (14).

d) Determine L_A in balanced growth. *Hint:* on the basis of the no-arbitrage condition for P_A, find an expression for r in terms of L_Y by applying the result from c) together with (13) and (14); next, find another expression for r in terms of L_A from the Keynes-Ramsey rule and your general knowledge about balanced growth; finally, combine.

e) Determine the per capita growth rate g_c^* in balanced growth.

f) For an equilibrium path to be a BGP with $L_A > 0$, household impatience must be sufficiently *low*. Find a necessary and sufficient condition for household impatience to be consistent with $L_A > 0$. *Hint:* apply your result for L_A in balanced growth.

From now, assume this condition is satisfied.

g) For balanced growth with the growth rate g_c^* found at e) to really be an equilibrium path, the household's TVC must be satisfied along the path. This requires that $r > g_c^*$ along the path. Show this. A necessary and sufficient condition for this to hold is that household impatience is *high* enough and satisfies $\rho > (1-\theta)g_c^*$. Show this. *Hint:* apply the Keynes-Ramsey rule.

From now, assume the condition $\rho > (1-\theta)g_c^*$ is satisfied.

h) Show that it is always theoretically possible that this condition holds at the same time as the condition from f) holds.

i) Indicate the sign of the effect on g_c^* of a rise in ρ, θ, η, L, and σ, respectively. In each case, give some intuition.

j) The conclusions from this model are somewhat different from what we get in the Jones case $\varphi < 1$. Comment on the differences.

k) It can be shown that an active government needs two policy instruments to implement the optimal resource allocation in the economy, given that discounted utility of the representative household is the optimality criterion. In combination with the R&D subsidy, a certain additional subsidy is needed as ingredient of an optimal policy in the Romer model. What additional subsidy (also financed by a lump-sum tax) could this be? Why?

VII.15 **(= V.3)** *Human capital and catching up* Consider a country which is fully integrated in the world market for goods and financial capital. Suppose that the real interest rate in the world market is a constant, $r > 0$. Let the aggregate production function be $Y_t = F(K_t, A_t h L_t)$ (standard notation). The technology level A_t evolves according to the catching-up hypothesis

$$\frac{\dot{A}_t}{A_t} = \xi \frac{\tilde{A}_t}{A_t},$$

where $\xi > 0$, and $\tilde{A}_t = \tilde{A}_0 e^{gt}$ is the world frontier technology level, $g > 0$.[3] We assume $A_0 < \tilde{A}_0$ and $0 < \xi < g$.

a) Will the country's technology *level* in the long run be able to catch up? *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$, has the solution $x(t) = (x_0 - x^*)e^{-at} + x^*$, where $x^* = b/a$; let $x(t) \equiv A_t/\tilde{A}_t$ and express the growth rate of x in terms of x, ξ, and g.

[3] Cf. Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

b) Show that the country's technology *growth rate* in the long run *will* be able to catch up?

Suppose the country *is* already "in the long run". Suppose further that the inhabitants spend S years of early life in school and then enters the labor market with a human capital $h(S) = S^{\eta}$, where $\eta > 0$. We assume that life expectancy approximately equals the inverse of the country's mortality rate, m, which we assume has for a long time been essentially constant.

c) Suppose S is chosen so as to maximize individual human wealth. Find h. *Hint:* $h'(S)/h(S) = r + m - g$.

d) Let the catching-up ability be an increasing function of aggregate human capital, $H = Nh$, where N is the size of the adult population (whether still in school or not), i.e., $\xi = \xi(H)$, $\xi' > 0$. Can a general health improvement in the country in the long run help in catching up with respect to the *level* of technology? Why or why not?