Economic Growth.

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Additional exercise for Section VIII in the collection of exercises (as of Febr. 15, 2016)

VIII.4 *An R&D-based growth model with an essential non-renewable resource.* Consider a closed economy with population = labor force = $L = L_0 e^{nt}$. From the perspective of a social planner, the production side of the economy is given by the following equations:

$$\begin{aligned} Y &= K^{\alpha}(AL_Y)^{\beta}R^{\gamma}, && \alpha,\beta,\gamma>0, \quad \alpha+\beta+\gamma=1, && (1)\\ \dot{K} &= Y - cL - \delta K, && \delta>0, \qquad K_0>0 \text{ given}, && (2)\\ \dot{A} &= \eta A^{\varphi}L_A^{1-\xi}, && \eta>0, \quad \varphi<1, \quad 0\leq\xi<1, \quad A_0>0 \text{ given}, && (3)\\ \dot{S} &= -R \equiv -uS, && S_0>0 \text{ given}, && (4)\\ L_Y + L_A &= L = L_0e^{nt}, && n>0. && (5)\end{aligned}$$

Notation is standard. The dating of the time-dependent variables is suppressed, when not needed for clarity. Greek letters stand for constant parameters.

a) Briefly interpret the equations.

Let a balanced growth path (BGP) for this economy be defined as a path along which Y, C, K, A, R, and S are positive and change at constant proportionate rates (some or all of which may be negative). It can be shown that a BGP with $L_A > 0$ and positive gross saving has the following *BGP properties*:

$$\begin{aligned} g_{L_Y} &= g_{L_A} = n, && \text{(i)}\\ g_K &= g_Y = g_C = \text{ a constant}, && \text{(ii)}\\ g_R &= g_S \equiv \dot{S}_t/S_t \equiv -R_t/S_t \equiv -\bar{u} = \text{ a constant} < 0, && \text{(iii)}\\ S_t &> 0 \text{ for all } t>0 \text{ while } S_t \to 0 \text{ for } t\to\infty. && \text{(iv)}\end{aligned}$$

Point (iv) says that the resource stock is never completely depleted, but nothing of the resource is left unutilized forever.

Our first problem is: How far can the model, as it stands, together with the "BGP properties", bring us towards a determination of the *size* of g_c along the BGP? To answer that:

b) Let $y \equiv Y/L$. Show that "take logs and then time derivatives" applied to (1), followed by the BGP assumption, gives a linear equation in $g_y, g_A,$ and u.

c) We know that in the horizontal innovations model without natural resources (i.e., $\gamma = 0$), $g_y = g_A$ along a BGP. Does this hold also when $\gamma > 0$? (*Hint:* use your result from b).) Comment.

d) Whatever your reply to c), show, after having divided through by A in (3), that "take logs and then time derivatives" applied to (3), followed by the BGP assumption, gives a solution for g_A along the BGP.

e) Does the result at d) solve the problem of pinning down the size of g_c along the BGP? Yes or no? Why?

Suppose the household sector consists of a fixed number of infinitely-lived households, all alike, with time preference rate $\rho > 0$ and instantaneous utility function $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$ (if $\theta = 1$, $u(c) = \ln c$). Normalizing the number of households to be one, allows us to speak of a *representative household*. It can be shown that if the criterion function of the social planer is the same as that of the representative household, then the first-order conditions to the social planner's problem imply that the optimal path follows a Keynes-Ramsey rule and a Hotelling rule. When combined with the aggregate production function given in (1) above, these two rules lead to the linear equation

$$(1-\theta)g_c + u = \rho - n.$$

f) Write also the result at b) above as a linear equation in the unknowns g_c and u. Solve the two equations for g_c and u. Denote the solutions g_c^* and u^*, respectively.

For a BGP to be consistent with general equilibrium in the model, parameters must be such that the transversality condition of the representative household is satisfied along the BGP. As usual, the condition $g - n > (1-\theta)g_c^*$ is both necessary and sufficient for this. We assume this condition is satisfied. It can be shown that this condition also ensures that our u^* is *positive*, which is required for the equilibrium BGP to exist, cf. BGP property (iii) above.

g) State a necessary and sufficient condition for $g_c^* > 0$.

Assume the condition is satisfied.

h) Comment on the kind of endogenous growth generated.

i) Sign $\partial g_c^*/\partial n, \partial g_c^*/\partial \rho, \partial g_c^*/\partial \varphi$, and $\partial u^*/\partial \rho$. Briefly give intuition in each case.

j) Briefly evaluate the model.