Economic Growth.

April 11, 2016. Christian Groth

Additional exercises for Section VII in the collection of exercises as of Febr. 15, 2016

VII.10 *Basic elements of the Romer-Jones model.* Consider a closed economy with population $L = L_0 e^{nt}$, $n \geq 0$. Labor is homogeneous. Each member of the population supplies one unit of labor per time unit. There are three production sectors:

Firms in *Sector 1* produce *final goods* (consumption goods and "raw" capital goods) in the amount Y_t per time unit, under perfect competition.

Firms in *Sector 2* supply *specialized capital goods* (here also called intermediate goods) that are *rented out* to firms in Sector 1 in the aggregate amount X_t per time unit, under conditions of monopolistic competition and barriers to entry.

Firms in *Sector 3* perform R&D to develop *technical designs* ("blueprints") for new specialized capital goods under conditions of perfect competition and free entry.

Note that from the point of view of Sector 1, the delivery of inputs from Sector 2 appears as an input of "intermediate goods" (in the sense of goods that cannot be stored). Note also that from a national-income accounting point of view, the specialized capital goods supplied by Sector 2 are no less "final goods" than the output of Sector 1. Yet, for convenience we reserve the term *final goods* for the output of Sector 1 while we often will refer to the *specialized capital goods* as the intermediate goods.[1] As to the raw capital goods produced in sector 1, it is easiest to imagine that they are *sold* to either sector-2 firms or to households that then rent them out to sector-2 firms.

There is a labor market and a market for risk-free loans. Both markets have perfect competition. Let the market wage be denoted w_t and the interest rate r_t. The economy is "large" and there are many R&D labs in the economy. There is no uncertainty and agents have perfect foresight. All firms are profit maximizers. Time is continuous.

[1] This is as in the textbook.

Sector 1. The representative firm in Sector 1 has the production function

$$Y_t = L_{Yt}^{1-\alpha} \sum_{j=1}^{A_t} x_{jt}^{\alpha}, \qquad 0 < \alpha < 1, \tag{1}$$

where Y_t is the produced quantity per time unit, x_{jt} is input of the specialized capital goods good j ($j = 1, 2, ..., A_t$), A_t is the number of different types of specialized capital goods available in the economy at time t, and L_{Yt} is labor input. At any point in time, the representative firm maximizes profit, taking the range of "varieties" as given.

The output of final goods is used partly for consumption, $C_t \equiv c_t L_t$, partly for investment in raw capital, I_{Kt} :

$$Y_t = C_t + I_{Kt} = c_t L_t + \dot{K}_t + \delta K_t, \qquad \delta \geq 0, \tag{2}$$

where K_t is the stock of raw capital goods in the economy and δ is the capital depreciation rate.

From now on, the explicit dating of the variables is omitted unless needed for clarity. Let the final good be the numeraire and let p_j denote the rental rate per time unit for using the specialized capital good j.

a) Derive the FOCs of the representative firm in Sector 1 at some arbitrary t. What is the price elasticity of the firm's demand for the intermediate good j, $j = 1, 2, ..., A$?

b) Suppose $p_j = p$, $\forall j$ (below this is shown to be true in equilibrium). Show that the assumed production function, (1), in this case is in conformity with the classical idea from Adam Smith that "there are gains by specialization and division of labor" or, with another formulation, "variety is productive". *Hint:* check how a rise in A affects Y for given L_Y and given total input of intermediates, $X = \sum_{j=1}^{A} x_j$.

Sector 2. After having invented the technical design j, the successful R&D lab in Sector 3 has taken out (free of charge) a perpetual patent on the commercial use of this design. Suppose the lab subsequently sells the patent to somebody else at the price P_A. The buyer then establishes a new firm, *firm* j, in Sector 2 and begins supplying the new intermediate good corresponding to this design, that is, the intermediate good j. The buyer can at any point in time resell the patent to somebody else at the going market value of the invention (the present discounted value of expected future accounting profits).

Given the technical design j, firm j can effortless transform final goods into intermediate goods of type j simply by pressing a button on a computer, thereby activating a computer code. The following linear transformation rule applies to all $j = 1,\ldots, A$:

it takes $x_j > 0$ units of raw capital to supply x_j units of intermediate good j.

c) Being a monopolist, firm j is a price setter. Find the profit maximizing p_j and show that it is the same for all firms in the sector, thus allowing us to write $p_j = p$, $\forall j$.

Sector 3. All R&D labs in Sector 3 face the same linear "research technology":

$$\#\text{ inventions per time unit } = \bar{\eta}\ell_A,$$

where $\bar{\eta}$ is productivity in R&D, which the individual R&D lab takes as given. Consider a given R&D lab. Let P_A denote the market value of an invention at current time t. The lab's demand for labor is

$$\ell_A^d = \begin{cases} \infty \text{ if } w < P_A\bar{\eta}, \\ \text{undetermined if } w = P_A\bar{\eta}, \\ 0 \text{ if } w > P_A\bar{\eta}. \end{cases} \tag{*}$$

At the economy-wide level, the R&D productivity is determined the following way:

$$\bar{\eta} = \eta A^{\varphi} L_A^{-\xi}, \qquad \eta > 0, \varphi \leq 1, 0 \leq \xi < 1, \tag{**}$$

where L_A is aggregate employment in Sector 3, $L_Y + L_A = L$, and φ and ξ are parameters.

d) Explain (*) and interpret (**), including the parameters.

Since the economy is "large", we can at the economy-wide level perceive $A = A(t)$ as a continuous and "smooth" function of time.

e) Write down an expression for $\dot{A} \equiv dA(t)/dt$ as seen from the point of view of the economy as a whole?

Now consider *general equilibrium* (GE) at an arbitrary point in time.

f) Find expressions for x_j $(= x)$ and π_j $(= \pi)$ in terms of L_Y and r. Comment.

g) Show that $x = K/A$, $Y = L_Y^{1-\alpha} A x^{\alpha} = K^{\alpha}(AL)^{1-\alpha}$, and $\partial Y/\partial x = \alpha Y/K = \partial Y/\partial K$.

h) Find r in terms of Y/K. Compare with $\partial Y/\partial K - \delta$. Comment.

i) Show that $\pi = (1-\alpha)\alpha Y/A$.

j) P_A is an asset price. Write down the no-arbitrage condition that P_A must satisfy. Make sure whether you think a risk premium should enter the no-arbitrage condition or not. State the reason for your choice.

VII.11 *The Romer-Jones model – further elements.* We consider the same framework and use the same notation as in Problem VII.10 (it is an advantage if you have already solved that problem).

a) Show that if in GE, $L_A > 0$, then

$$w = P_A \bar{\eta} = (1-\alpha)Y/L_Y. \qquad (***)$$

From now on assume $\varphi < 1$ and $n > 0$ and suppose that the equilibrium path followed by the economy is a BGP.

b) Determine g_A along the BGP and denote the solution g_A^*. *Hint:* use that the growth rate of L_A along a BGP must equal a certain parameter.

c) r must be constant along a BGP. Why? *Hint:* use your result from h) of Problem VII.10 and your general knowledge about BGPs in an economy satisfying (2) of Problem VII.10.

d) Show that $g_{P_A} = n$ and $P_A = \pi/(r-n)$. *Hint:* first, take logs and time derivatives in (***); next apply the no-arbitrage condition from j) of Problem VII.10; finally, use your result from c).

The interest rate r can not be found without information about the saving behavior of the households; that issue is postponed to Problem VII.12. Nevertheless, we will here derive a general formula for $s_R \equiv L_A/L \in (0,1)$ in terms of r along a BGP.

e) Do that! *Hint:* start from (***) and then apply the result concerning P_A at d).

Suppose that R&D is supported by a subsidy $\sigma \in (0,1)$ such that the cost per unit of R&D labor is $(1-\sigma)w$.

f) For a given r, how, if at all, may the subsidy affect s_R along a BGP? *Hint:* consider how (***) is modified.

g) Might subsidizing R&D be a good idea? Why or why not?

VII.12 *The Romer-Jones model with Ramsey households.* We consider the same framework and use the same notation as in the problems VII.10 and VII.11 (it is an advantage if you have already solved these problems). We concentrate on the case $\varphi < 1$ and $n > 0$.

To "close" the model, we shall be specific about the household sector. Suppose there is a fixed number of infinitely-lived households, all alike. Each household has $L_t = L_0 e^{nt}$ members, $n > 0$, and each member supplies inelastically one unit of labor per time unit. We normalize the number of households to be one. Given $\theta > 0$ and $\rho > 0$, the household's problem is to choose a plan $(c_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{c_t{}^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given}, \\ \lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} &\geq 0, \end{aligned}$$

where a_t is per capita financial wealth.

a) Express a_t in terms of P_{At}, $A(t)$, and K_t as defined in the Romer-Jones model, cf. Problem VII.10.

Assume that $\rho - n > (1-\theta)g_A^*$, where g_A^* is the solution for g_A along a BGP in the Romer-Jones model, cf. b) of Problem VII.11.

b) On the basis of the first-order conditions for the household's problem, write down a formula for the growth rate of c_t in terms of the interest rate at time t.

c) Assume the economy is in balanced growth. Then $g_c = g_A^* = (1-\xi)n/(1-\varphi)$. Show this.

d) Solve for r and s_R, respectively, along the BGP. *Hint:* use what you know from d) of Problem VII.11 combined with e) of that problem.

e) How, if at all, does a subsidy $\sigma \in (0,1)$ to R&D, financed by a lump-sum tax, such that the cost per unit of R&D labor is $(1-\sigma)w$, affect s_R and g_c, respectively, along the BGP? *Hint:* a look at f) of Problem VII.11 might be useful. Comment.

f) Does the subsidy affect the BGP? Why or why not?

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