Short Note 1

At least in these countries, therefore, the potential coordination failure laid bare by OLG models does not seem to have been operative in practice.

4.4 The functional distribution of income

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The neoclassical theory

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How the labor income share depends on the capital-labor ratio

To begin with we ignore technological progress and write aggregate output as $Y = F(K, L)$, where F is neoclassical with CRS. From Euler's theorem follows that $F(K, L) = F_1K + F_2L = f'(k)K + (f(k) - kf'(k))L$, where $k \equiv K/L$. In equilibrium under perfect competition we have

$$Y = \hat{r}K + wL,$$

where $\hat{r} = r + \delta$ is the cost per unit of capital input and w is the real wage, i.e., the cost per unit of labor input. The labor income share is

$$\frac{wL}{Y} = \frac{f(k) - kf'(k)}{f(k)} \equiv \frac{w(k)}{f(k)} \equiv SL(k) = \frac{wL}{\hat{r}K + wL} = \frac{\frac{w/\hat{r}}{k}}{1 + \frac{w/\hat{r}}{k}},$$

where the function $SL(\cdot)$ is the share of labor function and $w/\hat{r}$ is the factor price ratio.

Suppose that capital tends to grow faster than labor so that k rises over time. Unless the production function is Cobb-Douglas, this will under perfect competition affect the labor income share. But apriori it is not obvious in what direction. If the proportionate rise in the factor price ratio $w/\hat{r}$ is greater (smaller) than that in k, then SL goes up (down). Indeed, if we let $\mathrm{E}\ell_x g(x)$ denote the elasticity of a function $g(x)$ w.r.t. x, then

$$SL'(k) \gtreqless 0 \text{ for } \mathrm{E}\ell_k \frac{w}{\hat{r}} \gtreqless 1,$$

respectively.

Usually, however, the inverse elasticity is considered, namely $\mathrm{E}\ell_{w/\hat{r}}k$. This elasticity, which indicates how sensitive the cost minimizing capital-labor ratio, k, is to a given factor price ratio $w/\hat{r}$, coincides with the *elasticity of* *factor substitution* (for a general definition, see below). The latter is often denoted σ. Since in the CRS case, σ will be a function of only k, we write $\mathrm{E}\ell_{w/\hat{r}}k = \sigma(k)$. We therefore have

$$SL'(k) \gtreqless 0 \text{ for } \sigma(k) \lesseqgtr 1,$$

respectively. If F is Cobb-Douglas, i.e., $Y = K^{\alpha}L^{1-\alpha}, 0 < \alpha < 1$, we have $\sigma(k) \equiv 1$, cf. the next section. In this case variation in k does not change the labor income share under perfect competition. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for industrialized economies, but the bulk of studies seems to conclude with $\sigma(k) < 1$ (see Section 4.5).

Now, let us add Harrod-neutral technical progress to the discussion. So we write aggregate output as $Y = F(K, TL)$, where F is neoclassical with CRS, and $T = T_t = T_0(1+g)^t$. Then the labor income share is

$$\frac{wL}{Y} = \frac{w/T}{Y/(TL)} \equiv \frac{\tilde{w}}{\tilde{y}}.$$

The above formulas still hold if we replace k by $\tilde{k} \equiv K/(TL)$ and w by $\tilde{w} \equiv w/T$. While k empirically is clearly growing, $\tilde{k} \equiv k/T$ is not necessarily so because also T is increasing.

As we have seen, Kaldor's stylized facts essentially means that, apart from short-run fluctuations, $\tilde{k}$ and therefore also $\hat{r}$ and the labor income share tend to be constant over time, independently of the sign of $\sigma(\tilde{k}) - 1$. Given the production function f, the elasticity of substitution between capital and labor does not depend on the presence or absence of Harrod-neutral technical progress, but only on the function itself. This is because under Harrod-neutrality, the technology level T only appears as a multiplicative factor to L, whereby T cancels out in the calculation of the elasticity of factor substitution.

As alluded to earlier, there are empiricists who reject Kaldor's "facts" as a general tendency. For instance Piketty (2014) claims that the effective capital-labor ratio $\tilde{k}$ has an upward trend, temporarily braked by two world wars and the Great Depression in the 1930s. If so, the sign of $\sigma(\tilde{k}) - 1$ becomes decisive for in what direction wL/Y will move. Piketty interprets the econometric literature as favoring $\sigma(\tilde{k}) > 1$, which means there should be downward pressure on wL/Y. This source behind a falling wL/Y can be questioned, however. Indeed, $\sigma(\tilde{k}) > 1$ contradicts the more general empirical view referred to above. According to Summers (2014), Piketty's interpretation relies on conflating gross and net returns to capital.

Immigration

Here is another example that illustrates the importance of the size of $\sigma(\tilde{k})$. Consider an economy with perfect competition and a given aggregate capital stock K and technology level T (entering the production function in the labor-augmenting way as above). Suppose that for some reason, immigration, say, aggregate labor supply, L, shifts up and full employment is maintained by the needed real wage adjustment. In what direction will aggregate labor income $wL = \tilde{w}(\tilde{k})TL$ then change? The effect of the larger L is to some extent offset by a lower w brought about by the lower effective capital-labor ratio. Indeed, in view of $d\tilde{w}/d\tilde{k} = -\tilde{k}f''(\tilde{k}) > 0$, we have $\tilde{k} \downarrow$ implies $w \downarrow$ for fixed T. So we cannot apriori sign the change in wL. The following relationship can be shown (Exercise 4.??), however:

$$\frac{\partial(wL)}{\partial L} = (1 - \frac{\alpha(\tilde{k})}{\sigma(\tilde{k})})w \gtreqless 0 \text{ for } \alpha(\tilde{k}) \lesseqgtr \sigma(\tilde{k}), \tag{4.26}$$

respectively, where $a(\tilde{k}) \equiv \tilde{k}f'(\tilde{k})/f(\tilde{k})$ is the output elasticity w.r.t. capital which under perfect competition equals the gross capital income share. It follows that the larger L will not be fully offset by the lower w as long as the elasticity of factor substitution, $\sigma(\tilde{k})$, exceeds the gross capital income share, $\alpha(\tilde{k})$. This condition seems confirmed by most of the empirical evidence (see, e.g., Antras 2004 and Chirinko 2008).

The elasticity of factor substitution*

We shall here discuss the concept of elasticity of factor substitution at a more general level. Fig. 4.6 depicts an isoquant, $F(K, L) = \bar{Y}$, for a given neoclassical production function, $F(K, L)$, which need not have CRS. Let MRS denote the marginal rate of substitution of K for L, i.e., $MRS = F_L(K, L)/F_K(K, L)$.[9] At a given point (K, L) on the isoquant curve, MRS is given by the absolute value of the slope of the tangent to the isoquant at that point. This tangent coincides with that isocost line which, given the factor prices, has minimal intercept with the vertical axis while at the same time touching the isoquant. In view of $F(\cdot)$ being neoclassical, the isoquants are by definition strictly convex to the origin. Consequently, MRS is rising along the curve when L decreases and thereby K increases. Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio K/L, as a function of

[9] When there is no risk of confusion as to what is up and what is down, we use MRS as a shorthand for the more correct MRS_{KL}.

MRS. If we let MRS rise along the given isoquant, the corresponding value of the ratio K/L will also rise.

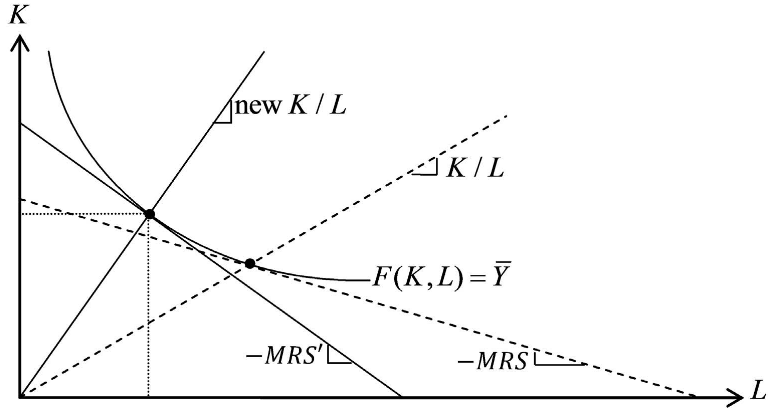

Figure 4.6: Substitution of capital for labor as the marginal rate of substitution increases from MRS to MRS'.

The *elasticity of substitution* between capital and labor is defined as the elasticity of the ratio K/L with respect to MRS when we move along a given isoquant, evaluated at the point (K, L). Let this elasticity be denoted $\tilde{\sigma}(K, L)$. Thus,

$$\tilde{\sigma}(K,L) = \frac{MRS}{K/L}\frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} = \frac{\frac{d(K/L)}{K/L}}{\frac{dMRS}{MRS}}_{|Y=\bar{Y}}. \tag{4.27}$$

Although the elasticity of factor substitution is a characteristic of the technology as such and is here defined without reference to markets and factor prices, it helps the intuition to refer to factor prices. At a cost-minimizing point, MRS equals the factor price ratio $w/\hat{r}$. Thus, the *elasticity of factor substitution* will under cost minimization coincide with *the percentage increase in the ratio of the cost-minimizing factor ratio induced by a one percentage increase in the inverse factor price ratio, holding the output level unchanged.*[10] The elasticity of factor substitution is thus a positive number and reflects how sensitive the capital-labor ratio K/L is under cost minimization to an increase in the factor price ratio $w/\hat{r}$ for a given output level. The less curvature the isoquant has, the greater is the elasticity of factor substitution. In an analogue way, in consumer theory one considers the elasticity of

[10]This characterization is equivalent to interpreting the elasticity of substitution as the percentage *decrease* in the factor ratio (when moving along a given isoquant) induced by a one-percentage *increase* in the *corresponding* factor price ratio.

substitution between two consumption goods or between consumption today and consumption tomorrow, cf. Chapter 3. In that context the role of the given isoquant is taken over by an indifference curve. That is also the case when we consider the intertemporal elasticity of substitution in labor supply, cf. the next chapter.

Calculating the elasticity of substitution between K and L at the point (K, L), we get

$$\tilde{\sigma}(K,L) = -\frac{F_K F_L (F_K K + F_L L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]}, \tag{4.28}$$

where all the derivatives are evaluated at the point (K, L). When $F(K, L)$ has CRS, the formula (4.28) simplifies to

$$\tilde{\sigma}(K,L) = \frac{F_K(K,L)F_L(K,L)}{F_{KL}(K,L)F(K,L)} = -\frac{f'(k)\left(f(k) - f'(k)k\right)}{f''(k)kf(k)} \equiv \sigma(k), \tag{4.29}$$

where $k \equiv K/L$.[11] We see that under CRS, the elasticity of substitution depends only on the capital-labor ratio k, not on the output level. We will now consider the case where the elasticity of substitution is independent also of the capital-labor ratio.

4.5 The CES production function*

It can be shown[12] that if a neoclassical production function with CRS has a constant elasticity of factor substitution different from one, it must be of the form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}}, \tag{4.30}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function* (CES for Constant Elasticity of Substitution). For a given choice of measurement units, the parameter A reflects efficiency (or what is known as *total factor productivity*) and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (4.30) is not defined when either K or L (or both) equal 0. We can circumvent this problem by extending the domain of

[11] The formulas (4.28) and (4.29) are derived in Appendix D.

[12] See, e.g., Arrow et al. (1961).