Chapter 15

Stochastic erosion of innovators' monopoly power

In this chapter we extend the lab-equipment model of Chapter 14 by adding stochastic erosion of innovators' monopoly power. The motivation is the following.

The model of Chapter 14 assumed that the innovator had perpetual monopoly over the production and sale of the new type of intermediate good. In practice, by legislation patents are of limited duration, 15-20 years. Moreover, it may be difficult to codify exactly the technical aspects of innovations, hence not even within such a limited period do patents give 100% effective protection. While the pharmaceutical industry rely quite much on patents, in many other branches innovative firms use other protection strategies such as concealment of the new technical design. In ICT industries copyright to new software plays a significant role. Still, whatever the protection strategy used, imitators sooner or later find out how to make very close substitutes.

To better accommodate these facts, the present chapter sets up a lab-equipment model where competition in the supply of specialized intermediate goods is more intense than in Chapter 14. For convenience we name the model of Chapter 14 *Model I*. Compared with that model the only difference in the new model is that the duration of monopoly power over the commercial use of an invention is *limited* and *uncertain*. We name the resulting model *Model II*. The notation is the same as in Model I. The analysis is related to the brief discussion of the issues in Acemoglu's Chapter 13.1.6 and in particular to his Exercise 13.13.

First a recapitulation of the technological aspects of the economy.

15.1 The three production sectors

The technology of the economy is the same as in Model I. In the *basic-goods sector* (sector 1) firms combine labor and N_t different intermediate goods to produce a homogeneous output good. The representative firm in the sector has the production function

$$Y_t = A\left(\sum_{i=1}^{N_t} {x_{it}}^{1-\beta}\right) L_t^{\beta}, \qquad A > 0,\ 0 < \beta < 1, \tag{15.1}$$

where Y_t, L_t, and x_{it} denote output of the firm, labor input, and input of intermediate good i, respectively, where $i = 1, 2, ..., N_t$. This sector, as well as the labor market, operate under perfect competition.

The aggregate output of basic goods is used partly for replacing the basic goods, X_t, used in the production of intermediate goods used up in the production of basic goods, partly for consumption, C_t, and partly for investment in R&D, Z_t. Hence, we have

$$Y_t = X_t + C_t + Z_t. \tag{15.2}$$

In the *intermediate-goods sector*, sector 2, at time t there are N_t monopoly firms, each of which supplies a particular already invented intermediate good. Once the technical design for intermediate good i has been invented in sector 3, the inventor enters sector 2 as an innovator. Given the technical design, the innovator can instantly transform a certain number of basic goods into a proportional number of intermediate goods of the invented specialized kind. That is,

$$\text{it takes } \psi x_i \text{ units of the basic good to supply } x_i \text{ units of intermediate good } i, \tag{15.3}$$

where ψ is a positive constant. The transformation requires no labor. Thus, ψ is both the marginal and the average cost of supplying the intermediate good i. This transformation technology applies to all intermediate goods, $i = 1, 2, ..., N_t$, and all t. Hence, the X_t in (15.2) satisfies

$$X_t \equiv \psi \sum_{i=1}^{N_t} x_{it} \equiv \psi Q_t, \tag{15.4}$$

where Q_t is the total supply of intermediate goods, all of which are used up in the production of basic goods.

For a limited period after the invention has been made, through secrecy or imperfect patenting the inventor maintains monopoly power over the commercial use of the invention. The length of this period is uncertain, see below.

In the R&D sector, sector 3, new "technical designs" (blueprints) for making new specialized intermediate goods are invented. The uncertainty associated with R&D is "ideosyncratic". On average it takes an input of $1/\eta$ units of the basic good, and *nothing else*, to obtain one successful R&D outcome (an invention) per time unit. There is free entry to the R&D activity. Ignoring indivisibilities, the aggregate number of new technical designs (inventions) in the economy per time unit is

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \eta Z_t, \qquad \eta > 0, \ \eta \text{ constant}, \tag{15.5}$$

where, as noted above, Z_t is the aggregate R&D investment in terms of basic goods delivered to sector 3 per time unit. As also noted above, after an invention has been made, the inventor enters sector 2 as an innovator and begins supplying the new intermediate good to firms in sector 1.

15.2 Temporary monopoly

To begin with the innovator has a monopoly over the production and sale of the new intermediate good. This may be in the form of a more or less effective patent (free of charge) or copyright to software or simply by secrecy and concealment of the new technical design. But sooner or later imitators find out how to make very close substitutes. There is *uncertainty* as to how long the monopoly position of an innovator lasts.

We assume the cessation of monopoly power follows a Poisson process with an exogenous "arrival" rate $\lambda > 0$, the same for all monopolies.[1] The "event" which "arrives" sooner or later is "exposure to unbounded competition". Independently of how long the monopoly position for firm i has been maintained, the probability that it breaks down in the next time interval of length Δt is approximately $\lambda \cdot \Delta t$ for Δt "small". Equivalently, if T denotes the remaining lifetime of the monopoly status of intermediate good i, then the probability that $T > \tau$ is $e^{-\lambda\tau}$ for all $\tau > 0$. Further, the expirations of the different monopolies are stochastically independent. The expected duration of a monopoly is $\int_0^\infty x\lambda e^{-\lambda x}dx = 1/\lambda$. We shall refer to the parameter λ as the Poisson expiration rate.

An investor (household) who contemplates to finance the R&D activity of a prospective innovator now faces a double risk, first the risk that the R&D is unsuccessful for a long time, second the risk that, when finally it is successful, the monopoly profits on the resulting innovation will only last

[1] This approach builds on Barro and Sala-i-Martin (1995).

for a short time. The model assumes, however, that all uncertainty is *idiosyncratic,* that is, the stochastic events that an R&D lab is successful in a certain time interval and that an innovator looses her monopoly position in a certain time interval are uncorrelated across R&D labs, innovators, and time and are in fact not correlated with anything in the economy. Assuming a "large" number of both R&D labs and intermediate-goods firms still being monopolies, investors can eliminate any risk by diversifying their investment as described in Chapter 14. Of course, this whole setup is an abstraction and can at best be considered a benchmark case.

As labor supply is a constant, L, clearing in the labor market implies $L_t = L$. We insert this into the production function (15.1) of the representative firm in sector 1. Maximizing profit, at time t this firm then demands $x_i(p_{it}) = (A(1-\beta))^{1/\beta} Lp_{it}^{-1/\beta}$ units of intermediate good i per time unit, $i = 1, 2, ..., N_t$. As long as innovator i is still a monopolist, she faces this downward-sloping demand curve with price elasticity $-1/\beta$ and sets the price, p_{it}, such that $MR = MC$ (marginal revenue = marginal cost). With the basic good as our numeraire, this amounts to

$$(1 - \frac{1}{1/\beta})p_{it} = \psi.$$

Solving for p_{it}, we get

$$p_{it} = (1 + \text{markup}) \cdot MC = \frac{1}{1-\beta}\psi \equiv p.$$

Thereby, as long as innovator i is still a monopolist, the sales of intermediate good i is

$$x_i(p_{it}) = x_i(p) = (A(1-\beta))^{1/\beta} Lp^{-1/\beta} = \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv x^{(m)}, \quad (15.6)$$

for $i = 1, 2, \ldots, N_t$. We shall refer to $x^{(m)}$ as the *monopoly supply* of a specific intermediate good.

The corresponding total revenue per time unit is $(\psi/(1-\beta)) \cdot x^{(m)}$ and the total cost is $\psi \cdot x^{(m)}$. The earned profit per time unit is thus

$$\pi_{it} = (p - \psi)x_i(p) = (\frac{\psi}{1-\beta} - \psi)x^{(m)} = \frac{\beta}{1-\beta}\psi x^{(m)} \equiv \pi^{(m)}, \quad (15.7)$$

for $i = 1, 2, \ldots, N_t$. The formulas for $x^{(m)}$ and $\pi^{(m)}$ are the same as those for x and π, respectively, in Model I, cf. Chapter 14.

As described above, however, sooner or later innovator i loses the monopoly. Thereafter intermediate good i is supplied under conditions of perfect competition and its price, p_{it}, is driven down to the competitive market price level = marginal cost = ψ. Since marginal cost, ψ, is also *average* cost, the profit vanishes. The aggregate sales of intermediate good i, now supplied by many competitors, are

$$x_i(p_{it}) = x_i(\psi) = \left(\frac{A(1-\beta)}{\psi}\right)^{1/\beta} L \equiv (1-\beta)^{-1/\beta} x^{(m)} \equiv x^{(c)} > x^{(m)}, \quad (15.8)$$

where $x^{(c)}$ will be referred to as the *competitive supply* of a specific intermediate good. The inequality in (15.8) follows from $0 < \beta < 1$. Economically, the inequality in (15.8) reflects that the demand depends negatively on the price, which is lower under perfect competition.

To summarize: In view of production and cost symmetry, each intermediate good supplied under monopolistic conditions is supplied in the amount, $x^{(m)}$, and each intermediate good supplied under competitive conditions is supplied in the larger amount, $x^{(c)}$. That is,

$$x_i = \begin{cases} x^{(m)} \text{ if } i \text{ is still a monopoly,} \\ x^{(c)} \text{ if } i \text{ is no longer a monopoly,} \end{cases} \quad (15.9)$$

where $x^{(m)}$ and $x^{(c)}$ are given in (15.6) and (15.8), respectively.

15.3 The aggregate production function in equilibrium

Substituting (15.9) into (15.1), we can write output in sector 1 as

$$Y_t = A\left[N_t^{(m)}(x^{(m)})^{1-\beta} + N_t^{(c)}(x^{(c)})^{1-\beta}\right] L^\beta, \quad (15.10)$$

where $N_t^{(m)}$ is the number of intermediate good types that at time t are still supplied under monopolistic conditions and $N_t^{(c)}$ is the number of intermediate good types that have become competitive. For each t we have

$$N_t = N_t^{(m)} + N_t^{(c)}. \quad (15.11)$$

There are now *two* state variables in the model. There is therefore scope for transitional dynamics, as we shall see soon.

With the help of (15.11) and (15.8), we may rewrite (15.10):

$$\begin{aligned} Y_t &= A\left[(N_t - N_t^{(c)})(x^{(m)})^{1-\beta} + N_t^{(c)}(1-\beta)^{-(1-\beta)/\beta}(x^{(m)})^{1-\beta}\right]L^\beta \\ &= A\left[N_t - N_t^{(c)} + N_t^{(c)}(1-\beta)^{-(1-\beta)/\beta}\right](x^{(m)})^{1-\beta}L^\beta \\ &= A\left[N_t + ((1-\beta)^{-(1-\beta)/\beta} - 1)N_t^{(c)}\right](x^{(m)})^{1-\beta}L^\beta \\ &= A\left[1 + ((1-\beta)^{-(1-\beta)/\beta} - 1)\frac{N_t^{(c)}}{N_t}\right](x^{(m)})^{1-\beta}L^\beta N_t. \qquad (15.12) \end{aligned}$$

Aggregate output is seen to depend on $N_t^{(c)}/N_t$. If the dynamics are such that $N_t^{(c)}/N_t$ tends to a positive constant, then Y_t will tend to be proportional to the produced "input", N_t, since $x^{(m)}$ is a constant, cf. (15.6). Therefore, the model is likely capable of generating fully endogenous growth, driven by R&D. We come back to this below.

In the case of universal and perpetual monopoly power, $N_t^{(c)} = 0$ and so (15.12) reduces to $Y_t = A(x^{(m)})^{1-\beta}L^\beta N_t \equiv Y_t^{(m)}$, which is the equilibrium output of basic goods in Model I. Substituting this into the expression (15.12), we see that

$$Y_t = \left[1 + ((1-\beta)^{-(1-\beta)/\beta} - 1)\frac{N_t^{(c)}}{N_t}\right]Y_t^{(m)} > Y_t^{(m)}. \qquad (15.13)$$

While in Model I, the Poisson expiration rate, λ, is nil, hence $N_t^{(c)} = 0$ for all t, here wqe have $\lambda > 0$ so that $N_t^{(c)} > 0$. This means that a fraction of the intermediate goods are supplied at a price equal to marginal cost thus inducing efficient use of these. Thereby productivity is enhanced and we get $Y_t > Y_t^{(m)}$ as shown by (15.12) (where $(1-\beta)^{-(1-\beta)/\beta} > 1$ in view of $0 < \beta < 1$).

15.4 The no-arbitrage condition under uncertainty*

All uncertainty is assumed to be ideosyncratic. By diversified investment in R&D lotteries and the stock market, the risk-averse households can therefore eliminate any risk and obtain the risk-free rate of return, r_t, with certainty. The appropriate discount rate for calculating the present value of expected future profits in any monopoly i is this risk-free rate, r_t. Consequently, ruling out speculative bubbles, the market value of monopoly i at time t is

$$V_{it} = \int_t^\infty E_t(\pi_{i\tau}) e^{-\int_t^\tau r_s ds} d\tau, \tag{15.14}$$

where $\pi_{i\tau}$ is the time-τ profit flow, now a stochastic variable as seen from time $t < \tau$:

$$\pi_{i\tau} = \begin{cases} \pi^{(m)} \text{ if firm } j \text{ is still a monopolist at time } \tau, \\ 0 \text{ otherwise.} \end{cases}$$

Expected profit flow at time τ, as seen from time t, is

$$E_t(\pi_{i\tau}) = \pi^{(m)} e^{-\lambda(\tau-t)} + 0 \cdot (1 - e^{-\lambda(\tau-t)}) = \pi^{(m)} e^{-\lambda(\tau-t)}. \tag{15.15}$$

Substituting into (15.14), we get

$$V_{it} = \pi^{(m)} \int_t^\infty e^{-\int_t^\tau (r_s+\lambda) ds} d\tau \equiv V_t. \tag{15.16}$$

This market value is the same for all intermediate goods i which at time t still retain monopoly. The expression (15.16) gives the market value in a *certainty-equivalent* form. On the one hand the integral in (15.16) "treats" the monopoly profit stream as if it were perpetual, on the other hand this future potential profit is discounted at an effective discount rate, $r_\tau + \lambda$, taking into account the probability, $e^{-\lambda(\tau-t)}$, that at time τ the ability to earn this profit has disappeared. The V_t in (15.16) is an observable variable given that the firm is still a monopoly (otherwise it has market value equal to nil). The uncertainty is about profits in the future and the discount rate for these equals the risk-free interest rate plus a risk premium, here equal to λ, which is the approximate conditional probability that the monopoly status breaks down in the time interval $(\tau, \tau + 1]$, given it is retained up to time τ.[2]

At this point we face the question: how is the risk-free interest rate, r_t, determined? To approach an answer, it is useful to derive the no-arbitrage condition which is implicit in (15.14). It may help intuition to think of r_t as the interest rate on a market for safe loans.

By differentiating (15.16) w.r.t. t, using Leibniz's' formula,[3] we get

$$\frac{\pi^{(m)} + \dot{V}_t^{(+)}}{V_t} = r_t + \lambda, \tag{15.17}$$

[2] This is known from the theory of a Poisson process.
[3] See Appendix A.

where $\dot{V}_t^{(+)}$ is the conditional capital gain, that is, the increase per time unit in the market value of the monopoly firm at time t, conditional on its monopoly position remaining in place also in the next moment. This formula equalizes the instantaneous *conditional* rate of return per time unit on shares in monopoly firms to the risk-free interest rate plus a premium reflecting the risk that the monopoly position expires within the next instant.

Alternatively we may derive the no-arbitrage condition (15.17) without appealing to Leibniz's' formula (which may not be part of the reader's standard math toolbox). This alternative approach has the advantage of being more intuitive. Let

$u_t \equiv$ the firm's earnings in the time interval $(t, t + \Delta t)$, given that the firm is still a monopolist at time t.

There will be no opportunities for arbitrage if the expected instantaneous unconditional rate of return per time unit on shares in the monopoly firm equals the required rate of return which is the risk-free interest rate, r_t. This amounts to the condition

$$\frac{\lim_{\Delta t \to 0} \frac{E_t(u_t \Delta t)}{\Delta t}}{V_t} = r_t. \tag{15.18}$$

The firm's earnings in the time interval $(t, t + \Delta t]$ is approximately $u_t \Delta t$. This is a stochastic variable and its expected value as seen from time t is

$$E_t(u_t \Delta t) \approx \lambda \Delta t(-V_t) + (1 - \lambda \Delta t)(\pi^{(m)} + \dot{V}_t^{(+)})\Delta t. \tag{15.19}$$

Indeed, V_t is the capital loss in case the monopoly position ceases and $\lambda \Delta t$ is the approximate probability that this event occurs within the time interval $(t, t + \Delta t]$, given that at time t it has not yet occurred. Similarly, $1 - \lambda \Delta t$ is the approximate probability that a monopoly position retained up to time t remains in force at least up to time $t + \Delta t$. And $(\pi^{(m)} + \dot{V}_t^{(+)})\Delta t$ is the total return in that case. Now, (15.19) can be written:

$$\begin{aligned} E_t(u_t \Delta t) &\approx -\lambda \Delta t V_t + (\pi^{(m)} + \dot{V}_t^{(+)})\Delta t - \lambda(\pi^{(m)} + \dot{V}_t^{(+)})(\Delta t)^2 \qquad (15.20) \\ &= (\pi^{(m)} + \dot{V}_t^{(+)} - \lambda V_t)\Delta t - \lambda(\pi^{(m)} + \dot{V}_t^{(+)})(\Delta t)^2 \Rightarrow \\ \frac{E_t(u_t \Delta t)}{\Delta t} &\approx \pi^{(m)} + \dot{V}_t^{(+)} - \lambda V_t - \lambda(\pi^{(m)} + \dot{V}_t^{(+)})\Delta t \\ &\to \pi^{(m)} + \dot{V}_t^{(+)} - \lambda V_t \text{ for } \Delta t \to 0. \end{aligned}$$

Hence, the condition (15.18) implies the no-arbitrage condition

$$\frac{\lim_{\Delta t \to 0} \frac{E_t(u_t \Delta t)}{\Delta t}}{V_t} = \frac{\pi^{(m)} + \dot{V}_t^{(+)} - \lambda V_t}{V_t} = r_t. \tag{15.21}$$

Reordering, we see that this is the same condition as (15.17).

15.5 The equilibrium rate of return when R&D is active

At the aggregate level, by the law of large numbers, the cost of making $\dot{N}_t$ inventions per time unit at time t is $Z_t = \dot{N}_t/\eta$ basic goods per time unit. The expected cost per invention is thus $1/\eta$. An equilibrium with active R&D therefore requires[4]

$$V_t = 1/\eta \equiv V. \tag{15.22}$$

So the market value of a monopoly firm is constant as long as the monopoly position is upheld. The conditional capital gain, $\dot{V}_t^{(+)}$, is therefore zero, whereby substituting (15.22) into (15.21) and applying (15.7) yields

$$r_t = \eta\pi^{(m)} - \lambda = \eta\frac{\beta}{1-\beta}\psi x^{(m)} - \lambda \equiv r^* \equiv r^{(m)} - \lambda < r^{(m)} \equiv \eta\pi^{(m)}, \tag{15.23}$$

where $r^{(m)}$ is the equilibrium interest rate that would apply in case of perpetual monopoly as in Model I.

Like in Model I, the equilibrium interest rate in Model II is thus from the beginning a constant, r^*. In view of $\lambda > 0$, (15.23) shows that $r^* < r^{(m)}$. Because of the limited duration of monopoly power in our present model, the expected rate of return on investing in R&D is smaller than in the case of no erosion of monopoly power as in Model I.

The description of the household sector is as in Model I, except that now per capita financial wealth is

$$a_t = \frac{N_t^{(m)} V_t}{L}.$$

Indeed, now only $N_t^{(m)} = N_t - N_t^{(c)}$ firms have positive market value, namely the firms that supply intermediate goods under monopolistic conditions. The households' first-order condition lead to the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r^* - \rho) = \frac{1}{\theta}(\eta\pi^{(m)} - \lambda - \rho) \equiv g_c^* < g_c^{(m)}, \tag{15.24}$$

where $g_c^{(m)}$ is the per capita consumption growth rate from Model I, the case of perpetual monopoly.

In order to have a model with *growth*, we assume parameters are such that $g_c^* > 0$. In addition, to avoid unbounded utility and help fulfillment of

[4] A more detailed argument is given in Chapter 14.

the households' transversality condition, we assume $\rho > (1-\theta)g_c^*$. These two conditions amount to the parameter restrictions

$$\eta\pi^{(m)} > \lambda + \rho, \qquad \text{and} \tag{A1}$$

$$\rho > (1-\theta)g_c^* = (1-\theta)\frac{1}{\theta}(\eta\pi^{(m)} - \lambda - \rho), \tag{A2}$$

respectively, where $\pi^{(m)} \equiv \beta(1-\beta)^{-1}\psi x^{(m)} > 0$ (from (15.7)), with $x^{(m)} \equiv ((A(1-\beta)^2)/\psi)^{1/\beta} L > 0$ (from (15.6)). The set of parameter combinations satisfying these two conditions is not empty. Indeed, for arbitrary values of η and the parameters entering $\pi^{(m)}$, choose for instance $\rho = 0$ and $\lambda > 0$ so that (A1) is satisfied. Then $g_c^* > 0$, and (A2) is satisfied for any $\theta > 1$.

(A1) requires that the "growth engine" of the economic system, as determined in particular by η, A, and L, is "powerful enough" for growth to arise. Below we return to what exactly constitutes the growth engine in this model. Suffice it to say here that increases in η, A, and L augment the strength of the growth engine (thereby making (A1) more likely to hold) while a rise in ψ reduces the strength of the growth engine (thereby making (A1) less likely to hold).[5]

15.6 Transitional dynamics*

Given that cessations of individual monopolies follow the assumed independent Poisson processes with expiration rate λ, the aggregate number of transitions per time unit from monopoly to competitive status follow a Poisson process with arrival rate $\lambda N_t^{(m)}$. The expected number of transitions per time unit from monopoly to competitive status is then

$$E_t\dot{N}_t^{(c)} = \lambda N_t^{(m)}.$$

Assuming $N_t^{(m)}$ is "large", the difference between actual and expected transitions per time unit will be negligible (by the law of large numbers), and we simply write

$$\dot{N}_t^{(c)} = \lambda N_t^{(m)} = \lambda(N_t - N_t^{(c)}). \tag{15.25}$$

Let the fraction of intermediate goods supplied under competitive conditions be denoted $s_t \equiv N_t^{(c)}/N_t$ (s for "share") and let $g_x \equiv \dot{x}_t/x_t$ for any positively-valued variable x_t. Then, taking logs and differentiating w.r.t. t,

[5] In view of $r^{(m)} \equiv \eta\beta(1-\beta)^{-1}\psi(A(1-\beta)^2)^{1/\beta}\psi^{-1/\beta}L$, this role of ψ is due to $1-1/\beta < 0$.

we get

$$
\begin{aligned}
g_s &= g_{N^{(c)}} - g_N = \lambda \frac{N_t - N_t^{(c)}}{N_t^{(c)}} - g_N = \lambda(s_t^{-1} - 1) - g_N \\
&= \lambda s_t^{-1} - (\lambda + g_N) \gtreqless 0 \quad \text{for} \quad s_t \lesseqgtr \frac{\lambda}{\lambda + g_N}, \qquad (15.26)
\end{aligned}
$$

where the second equality is implied by (15.25).

The general law of movement of N_t is given by (15.5), which, together with (15.2) and (15.13) and the definition $\tilde{c}_t \equiv \frac{c_t}{N_t}$, implies that

$$
\begin{aligned}
\dot{N}_t &= \eta Z_t = \eta(Y_t - X_t - C_t) = \eta \left\{ Y_t - \psi(N_t^{(m)} x^{(m)} + N_t^{(c)} x^{(c)}) - C_t \right\} \\
&= \eta \left\{ \left(1 + ((1-\beta)^{-(1-\beta)/\beta} - 1) \frac{N_t^{(c)}}{N_t} \right) Y_t^{(m)} - \psi((N_t - N_t^{(c)}) x^{(m)} + N_t^{(c)} x^{(c)}) - c_t L \right\} \\
&= \eta \left\{ (1 + ((1-\beta)^{-(1-\beta)/\beta} - 1) s_t) \frac{Y_t^{(m)}}{N_t} - \psi x^{(m)} + \psi(x^{(m)} - x^{(c)}) s_t) - \tilde{c}_t L \right\} N_t \\
&= \eta \left\{ \left(1 + ((1-\beta)^{-(1-\beta)/\beta} - 1) s_t \right) A (x^{(m)})^{1-\beta} L^\beta - \psi x^{(m)} \right. \\
&\quad \left. + \psi \left[x^{(m)} - (1-\beta)^{-1/\beta} x^{(m)} \right] s_t) - \tilde{c}_t L \right\} N_t \qquad \text{(by (15.8))} \\
&= \eta \left\{ A (x^{(m)})^{1-\beta} L^\beta - \psi x^{(m)} \right. \\
&\quad \left. + \left[((1-\beta)^{-(1-\beta)/\beta} - 1) A (x^{(m)})^{1-\beta} L^\beta - \psi((1-\beta)^{-1/\beta} - 1) x^{(m)} \right] s_t - \tilde{c}_t L \right\} N_t \\
&\equiv \eta \left(B_1 + B_2 s_t - \tilde{c}_t L \right) N_t,
\end{aligned}
$$

where the constants B_1 and B_2 are implicitly defined. The growth rate of N_t can thus be written

$$
g_N = \eta \left(B_1 + B_2 s_t - \tilde{c}_t L \right). \qquad (15.27)
$$

We now construct the implied dynamic system in the endogenous variables s_t and $\tilde{c}_t$. From (15.26) follows $\dot{s}_t = \lambda - (\lambda + g_N) s_t$, which combined with (15.27) yields

$$
\dot{s}_t = \lambda - (\lambda + \eta (B_1 + B_2 s_t - \tilde{c}_t L)) s_t. \qquad (15.28)
$$

Similarly, from $\tilde{c}_t \equiv c_t / N_t$ follows $\dot{\tilde{c}}_t / \tilde{c}_t = g_c - g_N = g_c^* - g_N$, by (15.24). So,

$$
\dot{\tilde{c}}_t = (g_c^* - \eta (B_1 + B_2 s_t - \tilde{c}_t L)) \tilde{c}_t. \qquad (15.29)
$$

The differential equations (15.28) and (15.29) constitute a dynamic system with two endogenous variables, s_t and $\tilde{c}_t$, the first of which is predetermined while the second is a jump variable (forward-looking variable).

15.7 Long-run growth

In a steady state ($\dot{s}_t = 0 = \dot{\tilde{c}}_t$), by definition of $\tilde{c}_t$, we must have $g_N = g_c$, where $g_c = g_c^*$, cf. (15.24). In steady state, therefore, $g_N = g_c^*$. Consequently, in view of (15.26), the steady-state value of s_t is

$$s^* = \frac{\lambda}{\lambda + g_c^*}. \tag{15.30}$$

Finally, the steady-state value of $\tilde{c}_t$ is $\tilde{c}^* = (B_1 + B_2 s^* - g_c^*/\eta)/L$.

In the steady state there is balanced growth in the sense that Y_t, c_t, N_t, $N_t^{(c)}$, and Y_t grow at the same constant rate as c_t, namely the rate g_c^* given in (15.24). This follows from the constancy of $\tilde{c}$ and s ($\equiv N^{(c)}/N$) in steady state together with the expression (15.13) for the aggregate production function in sector 1.

Moreover, the total supply of intermediate goods per time unit in the steady state is

$$\begin{aligned} Q_t &= N_t^{(m)} x^{(m)} + N_t^{(c)} x^{(c)} = (N_t - N_t^{(c)}) x^{(m)} + N_t^{(c)} x^{(c)} \\ &= \left[(1 - s^*) x^{(m)} + s^* x^{(m)}\right] N_t. \end{aligned}$$

So, in the steady state, Q_t is proportional to N_t. And by (15.2), the delivery of basic goods to sector 2 is $X_t = \psi Q_t$ per time unit, which is thus also proportional to N_t in the steady state. Hence, in steady state both Q_t and X_t grow at the same rate as N_t, the rate g_c^*.

The same is true for the R&D investment. Indeed, in steady state, $Z_t = \dot{N}_t/\eta = g_N^* N_t/\eta$. As shown in Appendix B, where also a phase diagram is sketched, the steady state is a saddle point. An only half-finished dynamic analysis in that appendix suggests that for any given initial $N_0^{(c)}/N_0 \in (0, 1)$, there exists a unique solution to the model and it converges to the steady state for $t \to \infty$, that is, saddle-point stability prevails.

So also Model II generates fully endogenous growth. The long-run per capita growth rate equals g_c^*, defined in (15.24). What makes fully endogenous growth possible is again that the "growth engine" of the economy features constant returns to scale w.r.t. producible inputs. Recall that the *growth engine* of a model is defined as the set of input-producing sectors using their own output as input. The present model can be reduced to two sectors that make up the growth engine. Indeed, the factor $Y_t^{(m)}$ in the aggregate production function of sector 1 given in (15.13) can be written

$$Y_t^{(m)} = A(N_t x^{(m)})^{1-\beta} L^\beta N_t^\beta = A \left(X_t^{(m)}/\psi\right)^{1-\beta} L^\beta N_t^\beta, \tag{15.31}$$

where $N_t x^{(m)}$ is the total input of intermediates in the production of basic goods in case of universal monopoly power as in Model I, and $X_t^{(m)}$ is the corresponding required input of basic goods as raw material in its own production, cf. (15.4). In this way sector 2 can be considered integrated in sector 1. On this basis, the so delineated sector 1, together with sector 3, constitutes the growth engine of the present model. Basic goods, $Y = X + C + Z$, and technical knowledge, represented by the number, N, of varieties of intermediate goods, are the two kinds of output that enter their own production as inputs. Sector 1 delivers the input flow X to itself and the input flow Z to sector 3. And sector 3 delivers the input flow N to sector 1. The production functions (15.13) (with $N_t^{(c)}/N_t = s^*$ and $Y_t^{(m)}$ written as in (15.31)) and (15.5) show that in steady state there are constant returns to scale w.r.t. these two producible inputs. It is this property that generates fully endogenous growth in the model.

The long-run per capita growth rate depends on those parameters that also appear in Model I in qualitatively the same way as in that model, see Chapter 14. In Model II, however, the long-run per capita growth rate is smaller than in Model I with perpetual monopolies, cf. (15.24). This is due to the new parameter, the Poisson expiration rate λ. Indeed, (15.24) indicates that a larger λ, i.e., a smaller expected duration, $1/\lambda$, of the status as a monopolist, implies a lower per capita growth rate, g_c^*. The reason is that the erosion of monopoly power implies less protection of private ownership of the inventions. This reduces the private profitability of R&D and thereby the incentive to do R&D.

15.8 Economic policy

At the theoretical level the analysis exposes the presence of static and dynamic distortions. Compared with perpetual monopoly, erosion of monopoly power *mitigates* the *static inefficiency problem* arising from prices above marginal cost, as described in Section 15.3. But erosion of monopoly power *aggravates* the underinvestment in R&D and thereby the *dynamic distortion* in the system. In this way long-run growth (within these multiple-sector AK-style models) is reduced even *more*, relative to the social optimum, than in the case of perpetual monopolies.

At the empirical level, for instance Jones and Williams (1998) estimate that R&D investment in the U.S. economy is only about a fourth of the social optimum. So government intervention seems definitely motivated.

A social planner

Letting $g^{(m)}$ denote the growth rate under perpetual monopoly as in Model I, we have

$$g_c^* < g^{(m)} < g_{SP} = \frac{1}{\theta}\left(\eta\frac{\beta}{1-\beta}\psi x^{(c)} - \rho\right), \tag{15.32}$$

where $x^{(c)}$ is the competitive supply of each intermediate-good type, defined in (15.8), and g_{SP} is the optimal growth rate from the point of view of an "all-knowing and all-powerful" social planner with the same criterion function as that of the representative household. The first inequality in (15.32) was shown above and the second is shown in Exercise VII.4. While the formal derivation of the social planner's solution is dealt with in that exercise, here we shall consider the issues in more intuitive terms.

The first policy problem is that in the market economy, the invented specialized intermediate goods are, at least to begin with, priced above the private marginal cost, ψ, which is also the social marginal cost. Consequently, under laissez faire, these goods are not supplied and used up to the point where their marginal productivity equals their social marginal cost. A "free" potential productivity gain is left unexploited in the economy.

A second problem is that this "static distortion" leads to a "dynamic distortion". Indeed, the fact that "too little" of the specialized intermediate goods is demanded means that the market for each variety is "too small". This results in too little profits to the suppliers of these goods, hence too little market value of inventions, that is, too little remuneration of the R&D activity. Consequently, there is too little incentive to do R&D, and even the growth rate $g^{(m)}$ in model I ends up smaller than the social optimum. On top of this comes in Model II that the imperfect protection of innovations reduces the incentive to do R&D further, and the growth rate ends up even lower than in Model I.

Returning to the static distortion, from the social planner's point of view the aggregate production function in the basic-goods sector can in the reduced form be written

$$Y_t = A(N_t x)^{1-\beta}L^{\beta}N_t^{\beta} = A\psi^{-(1-\beta)}X_t{}^{1-\beta}L^{\beta}N_t^{\beta},$$

which is analogue to (15.31). Given $Y_t = X_t + C_t + Z_t$, for fixed t, the social planner wants to choose the "raw material" input X_t so as to maximize what is left for final use, $C_t + Z_t$, i.e., consumption plus investment. The first-order condition is

$$\frac{\partial(Y_t - X_t)}{\partial X_t} = (1-\beta)A\psi^{-(1-\beta)}X_t{}^{-\beta}L^{\beta}N_t^{\beta} - 1 = 0,$$

and obviously $\partial^2(Y_t - X_t)/(\partial X_t^2) < 0$. Solving for X_t gives

$$X_t = \left(\frac{A(1-\beta)}{\psi}\right)^{1/\beta} \psi L N_t = \psi N_t x^{(c)},$$

where the last equality comes from (15.8). This shows that society should, as expected, supply each of the N_t intermediate good types in the competitive amount $x^{(c)}$ rather than supply $N_t^{(m)}$ of them in the amount $x^{(m)} < x^{(c)}$ as in Model II or, even worse, supply all N_t intermediate good types in the amount $x^{(m)}$ as in Model I.

Policy instruments

To counteract the monopolist price distortion and encourage demand for monopolized intermediate goods, a subsidy at constant rate σ to purchases of monopolized intermediate goods will work. By setting $\sigma = \beta$, the monopoly pricing is exactly neutralized from the point of view of the buyer who will have to pay $(1-\sigma)p = (1-\sigma)\psi/(1-\beta) = \psi$, which is the marginal cost of supplying the good. This solves the static efficiency problem.

In Model I, solving this problem can be shown to automatically solve, indirectly, also the dynamic efficiency problem. In Model II, solving the static efficiency problem will also encourage R&D but, because of the imperfect protection of innovations, not to the extent needed to get the optimal (first-best) solution. A second policy instrument is needed. A direct stimulus in the form of a subsidy to R&D investment is called for.

By comparing with the social planner's allocation, it is possible to find exact formulas for this R&D subsidy rate as well as non-distortionary financing such that the social planner's allocation is implemented in a decentralized way. Taxation on consumption and labor income are workable in these models.

Dilemmas in the design of patent systems

There are many dilemmas regarding how to design patent systems. Model II above illustrates one of them, namely the question what the period length of patents should be. The inverse of λ can be interpreted as a measure of the average duration of patents. A larger λ (shorter duration) reduces static inefficiency in an economy described by Model II but it also aggravates the underinvestment in R&D and thereby increases the dynamic inefficiency in the economy. We could more generally interpret λ as reflecting strictness of antitrust policy and the conclusion would be similar.

Going outside the present specific model, there are many further aspects to take into account, e.g., spill-over effects of R&D and intensional knowledge sharing, which we shall not consider here. A survey is contained in Hall and Harhoff (2012). We end this chapter by a citation from Wikipedia (07-05-2015):

> Legal scholars, economists, scientists, engineers, activists, policymakers, industries, and trade organizations have held differing views on patents and engaged in contentious debates on the subject. Recent criticisms primarily from the scientific community focus on the core tenet of the intended utility of patents, as now some argue they are retarding innovation. Critical perspectives emerged in the nineteenth century, and recent debates have discussed the merits and faults of software patents, nanotechnology patents and biological patents. These debates are part of a larger discourse on intellectual property protection which also reflects differing perspectives on copyright.

15.9 Appendix

A. Deriving (15.17) **on the basis of Leibniz's formula**
We shall apply *Leibniz's formula*[6] which says:

$$\begin{aligned} F(t) &= \int_{a(t)}^{b(t)} f(\tau,t)d\tau \Rightarrow \\ F'(t) &= f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(\tau,t)}{\partial t}d\tau. \end{aligned}$$

In the present case we have from (15.16), $V_t = \pi^{(m)}F(t)$, where

$$F(t) = \int_t^{\infty} e^{-\int_t^{\tau}(r_s+\lambda)ds}d\tau,$$

whereby $b(t) = \infty$ and $a(t) = t$, so that $b'(t) = 0$ and $a'(t) = 1$. We get $\dot{V}_t^{(+)} = \pi^{(m)}F'(t)$, that is,

$$\begin{aligned} \frac{\dot{V}_t^{(+)}}{\pi^{(m)}} &= F'(t) = 0 - e^{-\int_t^{t}(r_s+\lambda)ds} + \int_t^{\infty} e^{-\int_t^{\tau}(r_s+\lambda)ds}(r_t+\lambda)d\tau \\ &= -1 + (r_t+\lambda)F(t) = -1 + (r_t+\lambda)\frac{V_t}{\pi^{(m)}}. \end{aligned}$$

[6] For details, see for instance Sydsæter et al. (2008).

Reordering gives

$$\frac{\pi^{(m)} + \dot{V}_t^{(+)}}{V_t} = r_t + \lambda,$$

which is the no-arbitrage condition (15.17).

B. Stability analysis

The Jacobian matrix, evaluated in the steady state, is

$$\begin{aligned} J^* &= \begin{bmatrix} \partial \dot{s}/\partial s & \partial \dot{s}/\partial \tilde{c} \\ \partial \dot{\tilde{c}}/\partial s & \partial \dot{\tilde{c}}/\partial \tilde{c} \end{bmatrix}_{|(s,\tilde{c})=(s^*,\tilde{c}^*)} \\ &= \begin{bmatrix} -(\lambda + g_N + \eta B_2 s^*) & \eta L u^* \\ -\eta B_2 \tilde{c}^* & \eta L \tilde{c}^* \end{bmatrix}. \end{aligned}$$

The determinant of this matrix is

$$\det J^* = -(\lambda + g_N + \eta B_2 s^*)\eta L \tilde{c}^* + \eta L u^* \eta B_2 \tilde{c}^* = -(\lambda + g_N)\eta L \tilde{c}^* < 0.$$

Hence, the eigenvalues are of opposite sign and the steady state is a saddle point. A possible configuration of the phase diagram is sketched in Fig. 15.1. In the steady state the TVC of the households is satisfied in that

$$\begin{aligned} a_t e^{-r^* t} &= \frac{N_t^{(m)} V}{L} e^{-r^* t} = \frac{N_t^{(m)}}{L\eta} e^{-r^* t} = \frac{N_t - N_t^{(c)}}{L\eta} e^{-r^* t} \\ &= \frac{(1 - s_t) N_t}{L\eta} e^{-r^* t} = \frac{(1 - s^*) N_0 e^{g_c^* t}}{L\eta} e^{-r^* t} \to 0 \text{ for } t \to \infty, \end{aligned}$$

since $r^* \equiv r^{(m)} - \lambda$ so that (A2) combined with (15.24) implies $r^* > g_c^*$. The TVC is therefore also satisfied along the unique converging path.

15.10 References

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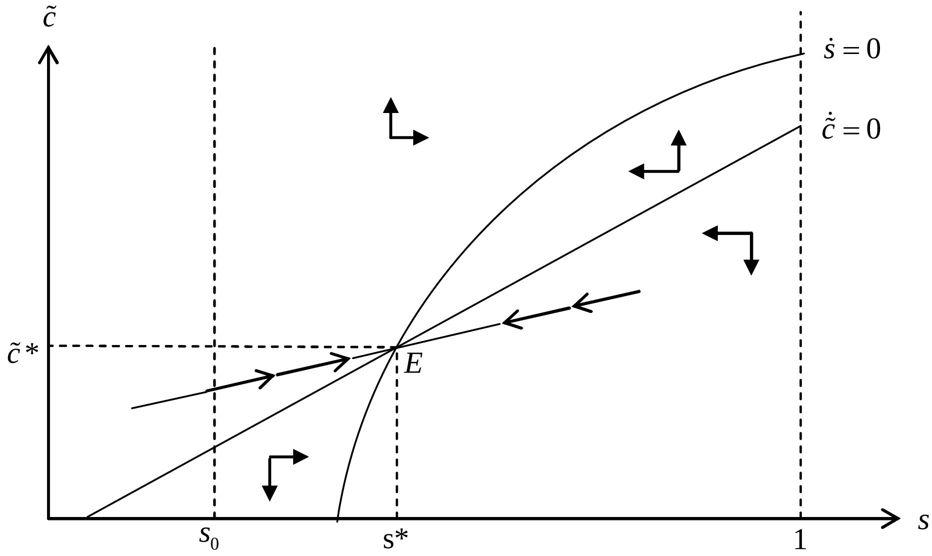

Figure 15.1: Phase diagram.