Chapter 14

The lab-equipment model

In the learning-by-doing and learning-by-investing models of chapters 12 and 13, technical progress comes as a by-product of the production activity and is considered an externality. This is just one mechanism behind technical progress. Another branch of growth theory focuses on technical progress as evolving from purposeful decisions by firms in search of monopoly profits on innovations. This branch of growth theory is called *innovation-based growth theory*.

Recall the definition of *technical knowledge* as a list of instructions about how different inputs can be combined to produce a certain output. For example it could be a principle of chemical engineering. Such a list or principle can be copied on the blackboard, in books, in journals, on floppy disks etc. and can, by its nature, be available and used over and over again at arbitrarily many places at the same time. Thus, technical knowledge is a *non-rival* good.[1] At least temporarily, however, new technical knowledge may be *temporarily excludable* by patents, secrecy, or copyright so that the innovator can maintain a monopoly on the commercial use of new technical knowledge for some time.

The lab-equipment model (based on Paul Romer, AER 1987) is the simplest model within the class of models focusing on *horizontal* innovations. This term refers to inventions of *new* types of goods, i.e., new "technical designs" in the language of Romer. The present model considers invention of new technical designs for input goods, but a more general framework would include new types of consumption goods as well.[2] The rising number of vari-

[1] Even though a particular *medium* on which a copy of a list of inctructions is placed is a rival good, it can usually be reproduced at very low cost in comparison with the cost of making additions to the stock of technical knowledge.

[2] For a model where the new goods are new consumption goods, see Acemoglu, Chapter 13, Section 13.4.

eties of goods contributes to productivity via *increased division of labor and specialization* in society. Thus this class of models is known as "increasing-variety models".

In Acemoglu's Chapter 13, Section 13.1, the lab-equipment model is presented in a version containing two knife-edge conditions in the form of arbitrary parameter links. In the present text we present the lab-equipment model without these parameter links. In addition, the presentation below goes more into detail with the national income aspects of the model and with the interaction between the financing needs of R&D labs and the saving by the households.

14.1 Overview of the economy

We consider a closed market economy. The activities in the economy can be subdivided into three sectors:

1. The *basic-goods* sector which operates under conditions of perfect competition and free entry.

2. The specialized *intermediate-goods* sector which operates under conditions of monopolistic competition and barriers to entry.

3. The R&D sector inventing *new technical designs* and operating under conditions of perfect competition and free entry.

All produced goods are non-durable goods. There is no physical capital (durable produced means of production) in the economy. All firms are profit maximizers.

14.1.1 The sectorial production functions

In the *basic-goods sector,* sector 1, firms combine labor and N_t different intermediate goods to produce a homogeneous output good. The representative firm in the sector has the production function

$$Y_t = A\left(\sum_{i=1}^{N_t} {x_{it}}^{1-\beta}\right) L_t^{\beta}, \qquad A > 0,\ 0 < \beta < 1, \tag{14.1}$$

where Y_t is output in the sector, A is a positive constant, x_{it} is input of intermediate good i ($i = 1, 2, ..., N_t$), N_t is the number of different types of intermediate goods available at time t, and L_t is labor input.[3] To avoid arbitrary parameter links, we do not introduce Acemoglu's assumption that the technical coefficient A happens to equal $1/(1-\beta)$. Sector 1 is the only sector that uses labor.

Basic goods have three alternative uses. They can be used a) for consumption, C_t; b) as raw material, X_t, to be converted into specialized intermediate goods (in Danish "halvfabrikata"); and c) as investment, Z_t, in R&D. Hence,

$$Y_t = C_t + X_t + Z_t. \tag{14.2}$$

In the *specialized intermediate-goods sector*, sector 2, at time t there are N_t monopoly firms, each of which supplies a particular already invented intermediate good. Once the technical design for intermediate good i has been invented in sector 3 (see below), the inventor takes out (free of charge) a perpetual patent on the commercial use of this design and enters sector 2 as an innovator. Given the technical design, the innovator can instantly transform a certain number of basic goods into a proportional number of intermediate goods of the invented specialized kind. Specifically, at every time t it takes ψx_i units of the basic good to supply x_i units of intermediate good i :

$$\psi x_i \text{ units of the basic good } \curvearrowright \; x_i \text{ units of intermediate good } i, \tag{14.3}$$

where ψ is a positive constant. We may think of the new technical design as a computer code which, once in place, just requires pressing a key on a computer in order activate the desired number of transformations. The computer cost is negligible and the transformation requires no labor.

Thus, ψ is both the marginal and the average cost of supplying the intermediate good i. This transformation technology applies to all intermediate goods, $i = 1, 2, ..., N_t$, and all t. Hence, the X_t in (14.2) satisfies

$$X_t \equiv \psi \sum_{i=1}^{N_t} x_{it} \equiv \psi Q_t, \tag{14.4}$$

where Q_t is the total supply of intermediate goods, all of which are used up in the production of basic goods. Apart from introducing a specific symbol, Q_t, for this total supply of intermediate goods, our notation is the same as Acemoglu's, Chapter 13. Yet, to help intuition, we think of variety as something discrete rather than a continuum and use summation across varieties as in (14.1) and (14.4) whereas Acemoglu's uses integrals.

[3] By an "intermediate good" is meant a non-durable means of production (like materials and energy) used up in the single production process while "capital" means a durable means of production (like a machine).

The model gives a "truncated" picture of the R&D sector, sector 3, as fictional research labs that transform incoming basic goods (now considered as R&D "equipment") into a random stream of research successes. A research success is an invention of a technical design (blueprint) for making a new specialized intermediate good. There is free entry to R&D activity. The uncertainty associated with R&D is "ideosyncratic" (unsystematic, diversifiable) and the economy is "large". On average it takes an input flow of $1/\eta$ units of the basic good, and *nothing else*, to obtain one successful R&D outcome (an invention) per time unit. By the law of large numbers, the aggregate number of new technical designs (inventions) in the economy per time unit equals the expected number. With time continuous and ignoring indivisibilities,[4] we can therefore write

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \eta Z_t, \qquad \eta > 0, \ \eta \text{ constant}, \tag{14.5}$$

where, as noted above, Z_t is the aggregate research input per time unit and η is "research productivity". Since the payoff to the outlay, Z_t, on R&D comes in the future, this outlay makes up an *investment*. Although the invested basic goods are non-durable goods, the resulting new technical knowledge is durable.

At first sight this whole production setup may seem peculiar. In sector 2 as well as sector 3, parts of the output from sector 1 is used as input to be transformed into specialized intermediate goods and new technical designs, respectively. But there is no labor input in sector 2 and sector 3. Formulating the three kinds of production in the economy in this manner is a convenient way of saving notation and is typical in this type of models.[5] A more realistic full-fledged description of the production structure would start with a production function, with both labor and intermediate goods as inputs, in each sector. Then an assumption could be imposed that the production functions are the same, apart from allowing the total factor productivity to vary across the sectors (only if $1/\psi = \eta = 1$, would the total factor productivities be the same). Setting the model up that way would fit intuition better but would also require a more cumbersome notation. Anyway, the conclusions would not be changed.

Before considering agents' behavior, it may be clarifying to do a little national income accounting.

[4] Conceptually, N_t is a discrete variable taking values in $\{1, 2, \dots\}$. Yet, for N_t "large" it is usually acceptable to smooth N_t out as a continuous and differentiable function of t.

[5] At the same time it is the lack of direct research labor in sector 3 that motivates the term "lab-equipment model". And it is the multi-faceted use of output from sector 1 that motivates the term "basic goods".

14.1.2 National income accounting

The production side Using the basic good as our unit of account, all the specialized intermediate goods will in equilibrium have the same price p_t (see Section 14.3.2). We therefore have:

$$\begin{aligned} \text{value added in sector 1} &= Y_t - p_t Q_t, \\ \text{value added in sector 2} &= p_t Q_t - X_t, \\ \text{value added in sector 3} &= V_t \dot{N}_t - Z_t, \end{aligned} \tag{14.6}$$

where V_t is the market value of an innovation and turns out to be independent of time. The aggregate value added, or net national product, is

$$\begin{aligned} NNP_t &= Y_t - p_t Q_t + p_t Q_t - X_t + V_t \dot{N}_t - Z_t \\ &= Y_t - p_t Q_t + p_t Q_t - \psi Q_t + V_t \dot{N}_t - Z_t = Y_t - \psi Q_t, \end{aligned} \tag{14.7}$$

where the last equality comes from $V_t \dot{N}_t - Z_t = 0$ in equilibrium due to the way sector 3 is described. Since there is no capital that depreciates in the economy, gross national product and net national product are the same.

Notice that the production function for Y is a production function neither for NNP nor even for value added in sector 1, but simply for the quantity of produced goods in that sector. It is typical for a multi-sector model with non-durable intermediate goods that the production functions in the different sectors do not describe value added in the sector but the produced quantity.

The income side There are two kinds of income in the economy, wage income and profits. The time-t real wage per unit of labor is denoted w_t and the profit per time unit earned by each monopoly firm in sector 2 is denoted π_t (in equilibrium it turns out to be the same for all the monopoly firms). Profits are immediately paid out to the share owners. Owing to perfect competition and CRS in both sector 1 and sector 3, there is no profit generated in these sectors. The income side of NNP is thereby

$$NNP_t = w_t L_t + \pi_t N_t,$$

since the number of monopoly firms is N_t. Aggregate income is used for consumption and saving,

$$w_t L + \pi N_t = C_t + S_t.$$

The uses of NNP By (14.7) and (14.4), final output can be written

$$NNP_t = Y_t - \psi Q_t = Y_t - X_t = C_t + Z_t, \tag{14.8}$$

that is, as the sum of aggregate consumption and investment. Aggregate saving is

$$S_t = w_t L + \pi N_t - C_t = NNP_t - C_t = Z_t,$$

by (14.8), reflecting that aggregate saving in a closed economy equals aggregate investment, the R&D expense, Z_t.

14.1.3 The potential for sustained productivity growth

Already the production function (14.1) conveys the basic idea of an "increasing-variety model". In equilibrium we get $x_{it} = x_t$ for all i since the intermediate goods enter symmetrically in this production function and end up having the same price (see below). Thereby, (14.1) becomes

$$Y = AN_t x_t^{\beta} L_t^{1-\beta} = A(N_t x_t)^{\beta} N_t^{1-\beta} L_t^{1-\beta} \equiv f(N_t x_t, N_t, L_t),$$

where $N_t x_t$ is the total input of intermediate goods and $f_2 > 0$. We see that

$$\frac{\partial Y_t}{\partial N_t}|_{N_t x_t = \text{const.}} = f_2(N_t x_t, N_t, L_t) > 0.$$

This says that for a given total input, $N_t x_t$, of intermediate goods, and a given L_t, the higher the number of varieties (with which follows a lower x_t of each intermediate since $N_t x_t$ is given), the more productive is this total input of intermediate goods. "Variety is productive". There are "gains to division of labor and specialization in society". The number of input varieties, N_t, can thus be interpreted as a measure of the level of productivity-enhancing knowledge.[6] Note also that the function f displays a form of increasing returns to scale with respect to *three* "inputs": intermediate goods, $N_t x_t$, variety, N_t, and labor, L_t.

14.2 Households and the labor market

There are L households, all alike, with infinite horizon and preference parameters $\theta > 0$ and ρ. Each household supplies inelastically one unit of labor

[6] There exists a related class of models where growth (measured in terms of produced economic value) is driven by increasing variety of *consumption* goods rather than increasing variety of input goods. These models are sometimes called "love of variety" models. See Acemoglu, Section 13.4.

per time unit. Let c_t denote per capita consumption C_t/L. A household chooses a plan $(c_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{c_t{}^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= r_t a_t + w_t - c_t, \qquad a_0 \text{ given}, \\ \lim_{t\to\infty} a_t e^{-\int_0^t r_s ds} &\geq 0, \end{aligned} \tag{14.9}$$

where a_t equals per capita financial wealth. In equilibrium

$$a_t = \frac{V_t N_t}{L},$$

because the only asset with market value in the economy is equity shares in the monopoly firms the value of which equals the market value per technical design multiplied by the number of technical designs available. As accounted for in Section 14.3.3, the risk-averse households ($u'' < 0$) can fully diversify any risk so as to obtain the rate of return, r_t, with certainty on all their saving.

The first-order conditions for the consumption-saving problem lead to the Keynes-Ramsey rule

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{14.10}$$

The necessary transversality condition is that the No-Ponzi-Game condition (14.9) is satisfied with equality.

The labor market

There is perfect competition and complete real wage flexibility in the labor market. For every t, the supply of labor is L, a constant. The demand for labor, L_t, comes from the basic-goods sector (as the two other sectors do not use labor). In equilibrium,

$$L_t = L. \tag{14.11}$$

14.3 Firms' behavior

To save notation, in the description below, we take (14.11) for granted.

14.3.1 The competitive producers of basic goods

At every t the representative firm in the basic-goods sector maximizes profit under perfect competition:

$$\max_{L,x_1,x_2,\ldots,x_N} \Pi_t = A\left(\sum_{i=1}^{N_t} x_{it}^{1-\beta}\right)L^{\beta} - \sum_{i=1}^{N} p_{it}x_{it} - w_t L. \tag{14.12}$$

The first-order conditions are, for every t,

$$\partial\Pi_t/\partial L = \partial Y_t/\partial L - w_t = \beta Y_t/L - w_t = 0 \tag{14.13}$$

and

$$\partial\Pi_t/\partial x_{it} = \partial Y_t/\partial x_{it} - p_{it} = A(1-\beta)x_{it}^{-\beta}L^{\beta} - p_{it} = 0, \qquad i = 1,2,\ldots,N_t.$$

This gives the demand for intermediate good i:

$$x_{it} = \left(\frac{p_{it}}{A(1-\beta)L^{\beta}}\right)^{-1/\beta} = \left[A(1-\beta)\right]^{1/\beta} L p_{it}^{-1/\beta}, \qquad i = 1,2,\ldots,N_t. \tag{14.14}$$

The price elasticity of demand, $\mathrm{E}\ell_{p_i}x_i$, for intermediate good i is thus $-1/\beta$. This reflects that the elasticity of substitution between the specialized intermediate goods in (14.12) is $1/(1-(1-\beta)) = 1/\beta$. This elasticity is above 1. Hence, while the specialized intermediate goods are not perfect substitutes, they are sufficiently substitutable for a monopolistic competition equilibrium in sector 2 to exist, as we shall now see.

14.3.2 The monopolist suppliers of intermediate goods

In principle the decision problem of monopolist i is the following. Subject to the demand function (14.14), a price and quantity path $(p_{i\tau}, x_{i\tau})_{\tau=t}^{\infty}$ should be chosen so as to maximize the value of the firm (the present value of future cash flows):

$$V_{it} = \int_t^{\infty} \pi_{i\tau} e^{-\int_t^{\tau} r_s ds} d\tau, \tag{14.15}$$

where $\pi_{i\tau}$ is the profit at time τ,

$$\pi_{i\tau} = (p_{i\tau} - \psi)x_{i\tau}, \tag{14.16}$$

and where the discount discount rate is r_s, the risk-free interest rate.

Since there is in this intertemporal problem no interdependence across time, the problem reduces to a series of static problems, one for each τ:

$$\max_{p_i} \pi_i = (p_i - \psi)x_i \quad \text{s.t. (14.14).}$$

To solve for p_i, we could substitute the constraint into the expression for π_i, take the derivative w.r.t. p_i, and then equalize the result to zero.

Alternatively, we may use the rule that the profit maximizing price of a monopolist is the price at which marginal revenue equals marginal cost, $MR = MC$. This is the more intuitive route we will take. We have

$$TR \text{ (= total revenue)} = p_i x_i = p_i(x_i)x_i,$$

where $p_i(x_i)$ denotes the maximum price at which the amount x_i can be sold. Thus, by the product rule,

$$\begin{aligned} MR &= \frac{dTR}{dx_i} = p_i(x_i) + x_i p_i'(x_i) = p_i\left(1 + \mathrm{E}\ell_{x_i} p_i\right) \\ &\equiv p_i\left(1 + \frac{1}{\mathrm{E}\ell_{p_i} x_i}\right) = p_i\left(1 + \frac{1}{-1/\beta}\right) = p_i\left(1 - \beta\right), \end{aligned}$$

from (14.14). Marginal cost is $MC = \psi$. So the profit maximizing price is

$$p_i = \frac{\psi}{1-\beta} \equiv p > \psi. \tag{14.17}$$

Owing to monopoly power, the price is above MC; the mark-up factor (or "degree of monopoly") is $1/(1-\beta)$. As expected, a lower absolute price elasticity of demand, $1/\beta$, results in a higher mark-up.

Since the elasticity of demand w.r.t. the price is independent of the quantity demanded and since MC is constant, the chosen price is time independent. Moreover the price is the same for all $i = 1, 2, \ldots, N$. Substitution into (14.14), (14.16), and (14.15), gives

$$x_{it} = \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv x, \text{ for all } i, \tag{14.18}$$

$$\pi_{it} = (p_{it} - \psi)x_{it} = \left(\frac{\psi}{1-\beta} - \psi\right)x = \frac{\beta}{1-\beta}\psi x \equiv \pi \text{ for all } i, \text{ and} \tag{14.19}$$

$$V_{it} = \int_t^\infty \pi_{is} e^{-\int_t^s r_\tau d\tau} ds = \pi \int_t^\infty e^{-\int_t^s r_\tau d\tau} ds \equiv V_t \text{ for all } i, \tag{14.20}$$

respectively. We see that all the monopoly firms sell the same quantity x, earn the same profit, π, and have the same market value, V_t. In addition, (14.18) and (14.19) show that x and π are constant over time. We will soon see that so is V_t.

The reduced-form aggregate production function in the economy

Note that although we have skipped the two arbitrary parameter links, $A = 1/(1-\beta)$ and $\psi = 1-\beta$, applied by Acemoglu, the resulting expressions for p, x, π, and V_t are tractable anyway.[7] So is the implied result for gross output in the basic-goods sector:

$$Y_t = AN_t x^{1-\beta} L^{\beta} = AN_t \left(\frac{A(1-\beta)^2}{\psi} \right)^{\frac{1-\beta}{\beta}} L \equiv \hat{A} N_t L, \tag{14.21}$$

where we have inserted (14.18) into (14.1) and defined

$$\hat{A} \equiv A \left(\frac{A(1-\beta)^2}{\psi} \right)^{\frac{1-\beta}{\beta}} .$$

The value added in the sector is

$$Y_t - pQ_t = \hat{A} N_t L - pN_t x = (\hat{A} L - px) N_t,$$

where p and x are constants given in (14.17) and (14.18), respectively.

So both gross and net output in the basic-goods sector are proportional to the number of intermediate-goods varieties (in some sense an index of the endogenous level of technical knowledge in society). Moreover, a similar proportionality will hold for the net national product, NNP. Indeed, according to (14.8),

$$NNP_t = Y_t - \psi Q_t = \hat{A} N_t L - \psi N_t x = (\hat{A} L - \psi x) N_t. \tag{14.22}$$

This is a first signal that the model is likely to end up as a reduced-form AK model with N ("knowledge capital") acting as the capital variable.

Now to the R&D firms of sector 3.

14.3.3 R&D firms

In Section 1.1 we expressed the aggregate number of new technical designs (inventions) per time unit this way:

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \eta Z_t, \qquad \eta > 0, \; \eta \text{ constant}, \tag{*}$$

[7] With his two parameter links Acemoglu obtains $A/\psi = (1-\beta)^{-2}$ from which follows the simple formulas $x_{it} = L$ and $\pi_{it} = \beta L$ for all i and all t. Although these formulas are, of course, simpler, they are "dangerous" when one wants to calculate, for instance, $\partial \pi / \partial \beta$, in order to assess the effect of a rise in β (the output elasticity w.r.t. labor) on the monopoly profit π.

where Z_t is the R&D investment (in terms of basic goods) and η is "research productivity". What is the microeconomic story behind this?

There is a "large" number of R&D labs and free entry and exit. All R&D labs operate under the same conditions with regard to "research technology". The following simplifying assumptions are made. The random R&D outcomes are:

(i) uncorrelated across time (*no memory*),

(ii) uncorrelated across the R&D labs,

(iii) uncorrelated with any variable in the economy, and

(iv) there is *no overlap* in research.

The "no memory" assumption, (i), ignores learning over time within the lab which seems a quite drastic assumption; indeed, innovation should be considered a cumulative process. Assumption (ii) seems drastic as well, since some learning across R&D labs is likely. In combination, the assumptions (i), (ii), and (iii) sum up to what is called "ideosyncratic" uncertainty. The "no overlap" assumption, (iv), amounts to assuming that inventions can go in so many directions that the likelihood of different research labs chasing and making the same invention is negligible. So we can find the aggregate increase in "knowledge" simply by summing the contributions by the individual research labs.

The "research technology"

The "research technology" faced by the individual R&D labs can be described as a *Poisson process*. The expected number of successful research outcomes (inventions) per time unit is proportional to the flow input of basic goods into the lab.

Consider an arbitrary R&D lab, j, at time t, $j = 1, 2, \ldots, J_t$, where J_t is "large". Let z_{jt} be the amount of basic goods the lab devotes to research per time unit. There is an instantaneous *success arrival rate*, η, per unit invested such that, given the research flow z_{jt}, the success arrival rate (= expected number of inventions per time unit) at time t, is

$$\eta_{jt} = \eta z_{jt}, \qquad \eta > 0. \tag{14.23}$$

The Poisson parameter, η, measures "research productivity". The interpretation of η_{it} is that if n_{jt} denotes the number of success arrivals in the time interval $(t, t+\Delta t]$, then

$$\eta_{jt} = \lim_{\Delta t \to 0} \frac{E_t(n_{jt} \,| z_{jt}, \Delta t)}{\Delta t}, \tag{14.24}$$

where E_t is the conditional expectation operator at time t.

At the aggregate level, since, by assumption, there is no overlap in research,

$$\frac{\Delta N_t}{\Delta t} = \frac{\sum_j (n_{jt})}{\Delta t} \approx \frac{E_t\left(\sum_j n_{jt} \left| (z_{jt})_{j=1}^{J_t}, \Delta t\right)\right.}{\Delta t} = \sum_j \frac{E_t(n_{jt} \,| z_{jt}, \Delta t)}{\Delta t}.$$

Appealing to the law of large numbers, we replace "≈" by "=", ignore indivisibilities, and take limits:

$$\dot{N}_t = \lim_{\Delta t \to 0} \frac{\Delta N_t}{\Delta t} = \sum_j \lim_{\Delta t \to 0} \frac{E_t(n_{jt} \,| z_{jt}, \Delta t)}{\Delta t} = \sum_j \eta_{jt} = \eta \sum_j z_{jt} = \eta Z_t, \tag{14.25}$$

which is (*). The third equality in (14.25) comes from (14.24), the fourth from (14.23), and the last from the definition of aggregate R&D input, Z_t.

The financing of R&D

There is a time lag of *random* length between a research lab's outlay on R&D and the arrival of a successful research outcome, an invention. During this period, which in principle has no upper bound, the R&D lab is incurring sunk costs and has no revenue at all. R&D is thus risky and continuous refinancing is needed until the research is successful.

Under certain conditions, the required financing of R&D will nevertheless be available. To clarify this, consider first the situation *ex post* a successful research outcome. When a successful research outcome arrives, the inventor takes out (free of charge) a perpetual patent on the commercial use of the invention. This gives the invention the market value, V_t, the same for all research labs, cf. (14.20). The inventor can realize this market value either by licensing the right to use the invention commercially or by directly herself entering sector 2 as a monopolist supplier of the new good made possible by the invention. To fix ideas, we assume the latter always takes place.

Now consider the situation *ex ante* an R&D investment is decided.

CLAIM 1 Given the market value, V_t, of an invention, the expected payoff per time unit per unit of basic goods invested in R&D is $V_t\eta$.

Proof Consider an arbitrary R&D lab j. The probability of a successful research outcome in a "small" time interval $(t, t+\Delta t]$ is approximately $\eta_{jt}\Delta t$. And the probability that more than one successful research outcome arrives in the time interval is negligible. We thus have

$$E_t\,(\text{R\&D payoff}\,|z_{jt}, \Delta t\,) \approx V_t\eta_{jt}\Delta t + 0\cdot(1-\eta_{jt}\Delta t) = V_t\eta_{jt}\Delta t. \qquad (14.26)$$

Substituting (14.23) into this and dividing through by $z_{jt}\Delta t$ gives

$$\frac{E_t\,(\text{R\&D payoff}\,|z_{jt}, \Delta t\,)}{z_{jt}\Delta t} \approx \frac{V_t\eta z_{jt}\Delta t}{z_{jt}\Delta t} = V_t\eta.$$

Letting $\Delta t \to 0$, "$\approx$" can in the limit be replaced by "$=$", thus confirming the claim. □

Considering equilibrium in the loan market, we have:

CLAIM 2 Let $\sum_j z_{jt} = Z_t$. (i) In any equilibrium in the loan market, whether with $Z_t = 0$ or $Z_t > 0$, we have

$$V_t\eta \leq 1. \qquad (14.27)$$

(ii) In any equilibrium in the loan market where $Z_t > 0$, we have

$$V_t\eta = 1. \qquad (14.28)$$

Proof. (i) Suppose that, contrary to (14.27), we have $V_t\eta > 1$. By Claim 1, the expected R&D payoff per time unit per unit cost of R&D is then higher than the R&D cost and so expected pure profit by doing R&D is positive. The flow demand for finance to R&D firms will therefore be unbounded. The flow supply of finance, ultimately coming from household saving, is, however, bounded and thus there is excess demand for funds and thereby not equilibrium.[8] Thus $V_t\eta > 1$ can be ruled out as an equilibrium and this leaves (14.27) as the only possible state in an equilibrium.

(ii) Consider an equilibrium with $Z_t > 0$. Since it is an equilibrium, (14.27) must hold. By way of contradiction, let us imagine there is strict inequality in (14.27). Then all R&D firms will choose $z_{jt} = 0$ and we reach the conclusion that $Z_t = 0$, thus contradicting that $Z_t > 0$. So there can not be strict inequality in (14.27) and we are left with (14.28) as the only possible state in an equilibrium with $Z_t > 0$. □

[8]For the sake of intuition, allow disequilibrium to exist in the very short run. Then the excess demand for funds drives the interest rate, r_t, up, thus lowering V_t (cf. (14.20)) until $V_t\eta = 1$.

It follows from Claim 2 that when the market value of inventions satisfy (14.28), the cost of doing R&D is on average exactly covered by the expected payoff. In return for putting one unit of account at the disposal of a research lab, the household gets a payoff of V_t if the research turns out to be successful and zero otherwise. In expected value the payoff per time unit is one unit of account. It is as if the household buys a lottery ticket offered by the R&D lab to finance its current R&D costs. The lottery prize consists of shares of stock giving the right to the future monopoly profits if the current research is successful within one time unit. The lottery is "fair" because the cost of participating equals the expected payoff. In spite of being risk averse ($u''(c) < 0$), the households are willing to participate because the uncertainty is "ideosyncratic" and the economy is "large". This allows the households to avoid the risk by spreading their investment over a variety of R&D labs, i.e., by diversifying their investment.

14.3.4 Equilibrium in the loan market

What must the size of the equilibrium real interest rate, r_t, in the loan market be? This rate must satisfy the following no-arbitrage relation vis-a-vis the instantaneous rate of return on shares in sector-2 firms supplying specialized intermediate goods:

$$r_t = \frac{\pi + dV_t/dt}{V_t}, \tag{14.29}$$

where π is the constant dividend (assuming all profit is paid out to the share owners) and dV_t/dt is the capital gain (positive or negative) on holding shares. As an implication of Claim 2, in an equilibrium with $Z_t > 0$, the market value of any invention is

$$V_t = 1/\eta \equiv V,$$

a constant. So $dV_t/dt = 0$, and (14.29) simplifies to

$$r_t = \frac{\pi}{1/\eta} = \eta\pi \equiv r, \tag{14.30}$$

where π is determined by (14.19). That is, along an equilibrium path with $Z_t > 0$, the interest rate is *constant* and *determined* by (14.30).

To ensure that $Z_t > 0$ and thereby positive growth is present in the economy, we need that the parameters are such that households *do* save. In view of the Keynes-Ramsey rule, this requires $r > \rho$ which in turn, by (14.30), requires a sufficiently high research productivity

$$\eta > \rho/\pi. \tag{A1}$$

What ensures that household saving and R&D investment match each other? Let aggregate financial wealth at time t be denoted $\mathcal{A}_t$. Then, in an equilibrium with $Z_t > 0$,

$$a_t L \equiv \mathcal{A}_t = V N_t = \frac{1}{\eta} N_t.$$

In view of $\dot{N}_t = \eta Z_t$ from (14.5), we therefore have

$$\dot{\mathcal{A}}_t = V \dot{N}_t = \frac{1}{\eta} \dot{N}_t = \frac{1}{\eta} \eta Z_t = Z_t. \tag{14.31}$$

By definition, households' aggregate saving, S_t, equals the increase in financial wealth per time unit, i.e., $S_t = \dot{\mathcal{A}}_t$.[9] Substituting this into (14.31), we see that the investment, Z_t, and saving, S_t, are two sides of the same coin when the interest rate takes the equilibrium value $\eta\pi$ in (14.30), and full employment, as in (14.11), is ensured through real wage flexibility.

To understand that there are neither losers nor winners in this saving-investment process, it may help intuition to imagine that all the saving, $S_t \Delta t$, in a short time interval $(t, t + \Delta t]$ first goes to large mutual funds (that have no administrative costs). These mutual funds instantly use the receipts to buy lottery tickets offered by R&D labs to cover current R&D costs. For the mutual funds taken together this involves an exchange of the outlay $S_t \Delta t$ for shares giving the right to the future monopoly profits associated with those research labs that turn out to be successful in the time interval considered. By the law of large numbers the inventions by these labs have exactly the same value as the outlay. Indeed, by (14.31), we have

$$V \dot{N}_t \Delta t = S_t \Delta t.$$

From then on, holding diversified shares in the monopolies supplying the newly invented intermediate goods gives the normal rate of return in the economy, r. A fraction of the R&D labs have not been successful in the time interval considered (and the financing to them has thereby been lost). But others have been successful and made an invention. The unequal occurrence of failures and successes across the many different R&D labs is neutralized when it comes to the payout to the customers, i.e., the households who have deposits in the mutual funds.

As an alternative financing setup, suppose that the R&D labs offer project contracts of the following form. A contract stipulates that the investor pays

[9] In this model households' gross saving equals their net saving since there are no assets that depreciate.

the lab $1/\eta$ units of account per time unit until a successful research outcome arrives. The corresponding liability of the lab is, when achieving success and becoming an entrepreneur in sector 2, to let the subsequent permanent profit stream earned on the invention go to the investor. By Claim 1, such R&D contracts have no market value. But after a successful R&D outcome there is a capital gain in the sense that the contracts become shares in the hands of the investors giving permanent dividends equal to π per time unit and thus having a market value equal to $V = 1/\eta$ forever.

Note that as the model is formulated, there is *no value added* in the R&D sector, as was also mentioned in connection with (14.7) in Section 14.1.2. Instead, the value that at the aggregate level comes out as $V\dot{N}_t$ is just a cost free one-to-one instantaneous transformation of Z_t which is a part of the value added created *in the basic-goods sector*. It is ultimately *this* value added that households' saving pays for.

14.4 General equilibrium of an economy satisfying (A1)

The assumption (A1) ensures a research productivity high enough to provide a rate of return exceeding the rate of time preference and thereby induce the household saving needed for R&D investment, Z_t, to be positive. And from (14.30) we know that along an equilibrium path with $Z_t > 0$, and therefore $\dot{N} > 0$, the interest rate (= the rate of return in the economy) is a constant, r. Then the Keynes-Ramsey rule, (14.10), yields

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}(\eta\pi - \rho) \equiv g_c, \tag{14.32}$$

where π is given (14.19). To ensure that the path considered with $\dot{N} > 0$ is really capable of being an equilibrium path, we need the parameter restriction

$$\rho > (1-\theta)g_c = (1-\theta)\frac{1}{\theta}(\eta\pi - \rho), \tag{A2}$$

since otherwise the transversality condition of the household could not be satisfied.[10]

From (14.21) and (14.22) we know that along an equilibrium path, gross as well as net output in the basic-goods sector are proportional to the stock of "knowledge capital", N_t. Moreover, the analysis of the previous section

[10] Another aspect of this is that (A2) ensures that the utility integral U_0 is bounded and thereby allows maximization in the first place.

shows that the preliminary national income accounting sketched in Section 14.1.2 is correct. Hence, by (14.22), also the aggregate value added in the economy as a whole, NNP, is proportional to N_t. Indeed,

$$NNP_t = Y_t - \psi Q_t = \hat{A} N_t L - \psi N_t x = (\hat{A} L - \psi x) N_t \equiv \bar{A} N_t.$$

So the model does indeed belong to the class of reduced-form AK models.

14.4.1 The balanced growth path

From the general theory of reduced-form AK models with Ramsey households, we know that the "capital" variable of the model, here "knowledge capital", N_t, will grow at the same constant rate as per capita consumption already from the beginning. In the present case the latter growth rate is given by (14.32). And

$$\dot{N}_t = \eta Z_t = \eta (NNP_t - C_t) = \eta (\bar{A} N_t - c_t L), \tag{14.33}$$

so that

$$g_N \equiv \frac{\dot{N}_t}{N_t} = \eta \left(\bar{A} - \frac{c_t L}{N_t} \right).$$

As $g_N = g_c$, this implies

$$c_t L = (\bar{A} - \frac{g_c}{\eta}) N_t,$$

for all $t \geq 0$. Hence, the until now unknown initial per capita consumption is

$$c_0 = (\bar{A} - \frac{g_c}{\eta}) \frac{N_0}{L}.$$

Labour productivity can be defined as

$$y_t \equiv NNP_t / L = \bar{A} N_t / L. \tag{14.34}$$

hence $g_y = g_N = g_c$.

Thus the model generates fully endogenous balanced growth and there are no transitional dynamics. What makes fully endogenous growth possible is, as usual, that the "growth engine" of the economy features constant returns to scale w.r.t. producible inputs. Generally, as defined in Chapter 13.5, the *growth engine* of a model is the set of input-producing sectors using their own output as an input. After having derived the aggregate production function in sector 1 as expressed in (14.21), sector 2 can be considered integrated in sector 1. On this basis, sector 1 and sector 3 constitute the growth engine in the model. Basic goods, $Y = X + C + Z$, and technical knowledge, represented by the number, N, of varieties of intermediate goods, are the two kinds of producible inputs. Sector 1 delivers the input flow X to itself and the input flow Z to sector 3. And sector 3 delivers the input N, "knowledge capital", to sector 1. The production functions (14.21) and (14.5) show that there are constant returns to scale w.r.t. these two producible inputs. This is the reason that fully endogenous growth is generated. In view of the absence of transitional dynamics, the model can be classified as a reduced-form two-sector AK model.

14.4.2 Comparative analysis

Given the per capita growth rate in (14.32), we have:

$\partial g_c/\partial\rho = -1/\theta < 0$. Higher impatience $\Rightarrow$ lower propensity to save $\Rightarrow$ less investment in R&D.

$\partial g_c/\partial\theta < 0$. Higher desire for consumption smoothing $\Rightarrow$ attempt to transform some of the higher future consumption possibility into higher consumption today $\Rightarrow$ lower saving $\Rightarrow$ less investment in R&D.

$\partial g_c/\partial A > 0$. Higher factor productivity $\Rightarrow$ higher return on saving $\Rightarrow$ more saving at the aggregate level (the negative substitution effect and wealth effect on consumption dominates the positive income effect) $\Rightarrow$ more investment in R&D. As usual the constant A need not have a narrow technical interpretation. It can reflect the quality of the institutions in society (rule of law etc.) and the level of "social capital".[11]

$\partial g_c/\partial\psi < 0$. Higher production costs of the specialized intermediate goods $\Rightarrow$ higher production costs for basic goods $\Rightarrow$ higher R&D costs $\Rightarrow$ less investment in R&D.[12]

$\partial g_c/\partial\eta > 0$. Higher R&D productivity results in more R&D investment and higher growth.

$\partial g_c/\partial L = (\eta/\theta)\partial\pi/\partial L > 0$. A larger population L implies lower per capita cost, η/L, associated with producing a given amount of new technical knowledge which in turn improves productivity for *all* members of society. This is an implication of knowledge being a nonrival good. In a larger society, with larger markets, the incentive to do R&D is therefore higher. In the present version of the R&D model the result is a higher growth rate perma-

[11] By social capital is meant society's stock of social networks and shared norms that support and maintain confidence, credibility, trust, and trustworthiness.

[12] Acemoglu's Equation (13.20), p. 439, entails that $\partial g_c/\partial\psi = 0$. This is due to the arbitrary parameter link $\psi = 1 - \beta$. This link implies that the effect of increasing the production costs of specialized intermediate goods is not separated from the effect of increasing the elasticity of output of basic goods w.r.t. input of intermediate goods, cf. (14.1).

nently. This is a manifestation of the controversial *strong* scale effect (scale effect on growth), typical for the "first-generation" innovation-based growth models with fully endogenous growth. This strong scale effect, as well as the fully endogenous growth property, is due to a "hidden" knife-edge condition in the specification of the "growth engine", cf. the general discussion in Chapter 13.5 and Exercise VII.5.