Chapter 11

AK and reduced-form AK models. Consumption taxation.

In his Chapter 11 Acemoglu discusses simple fully-endogenous growth models in the form of Ramsey-style *AK* and *reduced-form AK models*, respectively. The name "AK" refers to a special feature of the aggregate production function, namely the absence of diminishing returns to capital. We present the AK story within a Ramsey (i.e., representative agent) framework. A characteristic result from AK models is that they have *no* transitional dynamics.

With the aim of synthesizing the formal characteristics of such models, this lecture note gives an account of the common features of AK models (Section 11.1) and reduced-form AK models (Section 11.2), respectively. Finally, for later application we discuss in Section 11.3 conditions under which consumption taxation is not distortionary.

11.1 General equilibrium dynamics in the simple AK model

In the simple AK model (Acemoglu, Ch. 11.1) we consider a fully automatized economy where the aggregate production function is

$$Y(t) = AK(t), \qquad A > 0. \tag{11.1}$$

Thus there are constant returns to capital, not diminishing returns, and labor is no longer a production factor. This section provides a detailed proof that when we embed this technology in a Ramsey framework with perfect competition, the model generates balanced growth *from the beginning.* So there will be no transitional dynamics.

We consider a closed economy with perfect competition and no government sector. The dynamic resource constraint for the economy is

$$\dot{K}(t) = Y(t) - c(t)L(t) - \delta K(t) = AK(t) - c(t)L(t) - \delta K(t), \qquad K(0) > 0 \text{ given}, \tag{11.2}$$

where $L(t)$ is the population size. After having found the equilibrium interest rate to be $r = A - \delta$, we find the equilibrium growth rate of per capita consumption to be

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r - \rho) \equiv \frac{1}{\theta}(A - \delta - \rho) \equiv g_c, \tag{11.3}$$

a constant. To ensure positive growth we impose the parameter restriction

$$A - \delta > \rho. \tag{A1}$$

And to ensure boundedness of discounted utility (and thereby a possibility of satisfying the transversality condition of the representative household) we impose the additional parameter restriction:

$$\rho - n > (1 - \theta)g_c. \tag{A2}$$

Reordering gives

$$r = \theta g_c + \rho > g_c + n, \tag{11.4}$$

where the equality is due to (11.3).

Solving the linear differential equation (11.3) gives

$$c(t) = c(0)e^{g_c t}, \tag{11.5}$$

where $c(0)$ is unknown so far (because c is not a predetermined variable). We shall find $c(0)$ by appealing to the household's transversality condition,

$$\lim_{t\to\infty} a(t)e^{-(r-n)t} = 0, \tag{TVC}$$

where $a(t)$ is per capita financial wealth at time t. Recalling the No-Ponzi-Game condition,

$$\lim_{t\to\infty} a(t)e^{-(r-n)t} \geq 0, \tag{NPG}$$

we see that the transversality condition is equivalent to the No-Ponzi-Game condition being not over-satisfied.

Defining $k(t) \equiv K(t)/L(t)$, the dynamic resource constraint, (11.2), is in per-capita terms

$$\dot{k}(t) = (A - \delta - n)k(t) - c(0)e^{g_c t}, \qquad k(0) > 0 \text{ given}, \tag{11.6}$$

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where we have inserted (12.24). The solution to this linear differential equation is (cf. Appendix to Chapter 3)

$$k(t) = \left(k(0) - \frac{c(0)}{r-n-g_c}\right)e^{(r-n)t} + \frac{c(0)}{r-n-g_c}e^{g_c t}, \qquad r \equiv A - \delta. \quad (11.7)$$

In our closed-economy framework with no public debt, $a(t) = k(t)$. So the question is: When will the time path (11.7) satisfy (TVC) with $a(t) = k(t)$? To find out, we multiply by the discount factor $e^{-(r-n)t}$ on both sides of (11.7) to get

$$k(t)e^{-(r-n)t} = k(0) - \frac{c(0)}{r-n-g_c} + \frac{c(0)}{r-n-g_c}e^{-(r-g_c-n)t}.$$

Thus, in view of the assumption (A2), (11.4) holds and thereby the last term on the right-hand side vanishes for $t \to \infty$. Hence

$$\lim_{t\to\infty} k(t)e^{-(r-n)t} = k(0) - \frac{c(0)}{r-n-g_c}.$$

From this we see that the representative household satisfies (TVC) if and only if it chooses

$$c(0) = (r-n-g_c)k(0). \quad (11.8)$$

This is the equilibrium solution for the household's chosen per capita consumption at time $t = 0$. If the household instead had chosen $c(0) < (r-n-g_c)k(0)$, then $\lim_{t\to\infty} k(t)e^{-(r-n)t} > 0$ and so the household would not satisfy (TVC) but instead be over-saving. And if it had chosen $c(0) > (r-n-g_c)k(0)$, then $\lim_{t\to\infty} k(t)e^{-(r-n)t} < 0$ and so the household would be over-consuming and violate (NPG) (hence also (TVC)).

Substituting the solution for $c(0)$ into (11.7) gives the evolution of $k(t)$ in equilibrium,

$$k(t) = \frac{c(0)}{r-n-g_c}e^{g_c t} = k(0)e^{g_c t}.$$

So from the beginning k grows at the same constant rate as c. Since per capita output is $y \equiv Y/L = Ak$, the same is true for per capita output. Hence, from start the system is in balanced growth (there is no transitional dynamics).

The AK model features one of the simplest kinds of *endogenous growth* one can think of. Growth is endogenous in the model in the sense that there is positive per capita growth in the long run, generated by an internal mechanism in the model (not by exogenous technology growth). The endogenously determined capital accumulation constitutes the mechanism through which sustained per capita growth is generated. It is because the net marginal productivity of capital is assumed constant and, according to (A1), higher than the rate of impatience, ρ, that capital accumulation itself is so powerful.

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11.2 Reduced-form AK models

The models known as reduced-form AK models are a generalization of the simple AK model considered above. In contrast to the simple AK model, where only physical capital is an input, a reduced-form AK model assumes a technology involving at least two different inputs. Yet it is possible that in general equilibrium the aggregate production function ends up implying proportionality between output and some measure of "broad capital", i.e.,

$$Y(t) = B\tilde{K}(t),$$

where B is some endogenously determined positive constant and $\tilde{K}(t)$ is "broad capital". If in addition the real interest rate in general equilibrium ends up being a constant, the model is called a *reduced-form AK model.* In the simple AK model constancy of average productivity of capital is postulated from the beginning. In the reduced-form AK models the average productivity of capital becomes and remains *endogenously* constant over time.

In for instance the "AK model with physical and human capital" in Acemoglu, Ch. 11.2, along the balanced growth path (obtained after an initial phase with specialization in either physical or human capital accumulation) we have[1]

$$Y(t) = F(K(t), h(t)L(t)) = f(\hat{k}^*)h(t)L(t) = f(\hat{k}^*)H(t). \tag{11.9}$$

Define

$$\tilde{K}(t) \equiv K(t) + H(t) = \text{ "broad capital"}.$$

Then

$$\tilde{K}(t) \equiv (\frac{K(t)}{H(t)} + 1)H(t) = (\hat{k}^* + 1)H(t).$$

Isolating $H(t)$ and inserting into (11.9) gives

$$Y(t) = \frac{1}{\hat{k}^* + 1}\tilde{K}(t) \equiv B\tilde{K}(t).$$

So at the abstract level it is conceivable that "broad capital", defined as the sum of physical and human capital, can be meaningful. Empirically, however, there is no basis for believing *this* concept of "broad capital to be useful, cf. Exercises V.4 and V.5.

Anyway, a reduced-form AK model ends up with quite similar aggregate relations as those in the simple AK model. Hence the solution procedure

[1] The mentioned initial phase is left unnoticed in Acemoglu. In our notation $k \equiv K/L$ and $\hat{k} \equiv K/H$, while Acemoglu's text has $k \equiv K/H$.

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to find the equilibrium path (see Chapter 12) is quite similar to that in the simple AK model above. Again there will be no transitional dynamics.

The nice feature of AK models is that they provide very simple theoretical examples of endogenous growth. The problematic feature is that they may simplify the technology description *too* much and at best constitute knife-edge cases. More about this in Chapter 13.

11.3 On consumption taxation

As a preparation for the discussion shortly in this course of fiscal policy in relation to economic growth, we shall here try to clarify an aspect of consumption taxation. This is the question: is a consumption tax distortionary - always? never? sometimes?

The answer is the following.

1. Suppose labor supply is *elastic* (due to leisure entering the utility function). Then a consumption tax (whether constant or time-dependent) is generally distortionary (not neutral). This is because it reduces the effective opportunity cost of leisure by reducing the amount of consumption forgone by working one hour less. Indeed, the tax makes consumption goods more expensive and so the amount of consumption that the agent can buy for the hourly wage becomes smaller. The substitution effect on leisure of a consumption tax is thus positive, while the income and wealth effects will be negative. Generally, the net effect will not be zero, but it can be of any sign; it may be small in absolute terms.

2. Suppose labor supply is *inelastic* (no trade-off between consumption and leisure). Then, at least in the type of growth models we consider in this course, a constant (time-independent) consumption tax acts as a lump-sum tax and is thus non-distortionary. If the consumption tax is *time-dependent*, however, a distortion of the *intertemporal* aspect of household decisions tends to arise.

To understand answer 2, consider a Ramsey household with inelastic labor supply. Suppose the household faces a time-varying consumption tax rate $\tau_t > 0$. To obtain a consumption level per time unit equal to c_t per capita, the household has to spend

$$\bar{c}_t = (1 + \tau_t)c_t$$

units of account (in real terms) per capita. Thus, spending $\bar{c}_t$ per capita per time unit results in the per capita consumption level

$$c_t = (1 + \tau_t)^{-1}\bar{c}_t. \tag{11.10}$$

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In order to concentrate on the consumption tax as such, we assume the tax revenue is simply given back as lump-sum transfers and that there are no other government activities. Then, with a balanced government budget, we have

$$x_t L_t = \tau_t c_t L_t,$$

where x_t is the per capita lump-sum transfer, exogenous to the household, and L_t is the size of the representative household.

Assuming CRRA utility with parameter $\theta > 0$, the instantaneous per capita utility can be written

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} = \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta}.$$

In our standard notation the household's intertemporal optimization problem, in continuous time, is then to choose $(\bar{c}_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned}
U_0 &= \int_0^\infty \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\
\bar{c}_t &\geq 0, \\
\dot{a}_t &= (r_t - n)a_t + w_t + x_t - \bar{c}_t, \quad a_0 \text{ given}, \\
\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s - n)ds} &\geq 0.
\end{aligned}$$

From now, we let the timing of the variables be implicit unless needed for clarity. The current-value Hamiltonian is

$$H = \frac{(1+\tau)^{\theta-1}\bar{c}^{1-\theta} - 1}{1-\theta} + \lambda\left[(r-n)a + w + x - \bar{c}\right],$$

where λ is the co-state variable associated with financial per capita wealth, a. An interior optimal solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial \bar{c}} = (1+\tau)^{\theta-1}\bar{c}^{-\theta} - \lambda = 0, \text{ so that } (1+\tau)^{\theta-1}\bar{c}^{-\theta} = \lambda, \quad (11.11)$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} + (\rho - n)\lambda, \quad (11.12)$$

and a transversality condition which amounts to

$$\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s - n)ds} = 0. \quad (11.13)$$

We take logs in (11.11) to get

$$(\theta - 1)\log(1+\tau) - \theta \log \bar{c} = \log \lambda.$$

Differentiating w.r.t. time, taking into account that $\tau = \tau_t$, gives

$$(\theta - 1)\frac{\dot{\tau}}{1+\tau} - \theta\frac{\dot{\bar{c}}}{\bar{c}} = \frac{\dot{\lambda}}{\lambda} = \rho - r.$$

By ordering, we find the growth rate of consumption spending,

$$\frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right].$$

Using (11.10), this gives the growth rate of consumption,

$$\frac{\dot{c}}{c} = \frac{\dot{\bar{c}}}{\bar{c}} - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right] - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}(r - \frac{\dot{\tau}}{1+\tau} - \rho).$$

Assuming firms maximize profit under perfect competition, in equilibrium the real interest rate will satisfy

$$r = \frac{\partial Y}{\partial K} - \delta. \tag{11.14}$$

But the *effective* real interest rate, $\hat{r}$, faced by the consuming household, is

$$\hat{r} = r - \frac{\dot{\tau}}{1+\tau} \lesseqqgtr r \text{ for } \dot{\tau} \gtreqqless 0,$$

respectively. If for example the consumption tax is increasing, then the effective real interest rate faced by the consumer is smaller than the market real interest rate, given in (11.14), because saving implies postponing consumption and future consumption is more expensive due to the higher consumption tax rate.

The conclusion is that a time-varying consumption tax rate is distortionary. It implies a wedge between the intertemporal rate of transformation faced by the consumer, reflected by $\hat{r}$, and the intertemporal rate of transformation available in the technology of society, indicated by r in (11.14). On the other hand, *if* the consumption tax rate is constant, the consumption tax is non-distortionary when there is no utility from leisure.

A remark on tax smoothing

In models with transitional dynamics it is often so that maintaining constant tax rates is inconsistent with maintaining a balanced government budget. Is the implication of this that we should recommend the government to let tax rates be continually adjusted so as to maintain a forever balanced budget?

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No! As the above example as well as business cycle theory suggest, maintaining tax rates constant ("tax smoothing"), and thereby allowing government deficits and surpluses to arise, will generally make more sense. In itself, a budget deficit is not worrisome. It only becomes worrisome if it is not accompanied later by sufficient budget surpluses to avoid an exploding government debt/GDP ratio to arise. This requires that the tax rates taken together have a *level* which in the long run matches the level of government expenses.

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