Lecture Notes in Economic Growth

Christian Groth

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Preface

This is a collection of earlier separate lecture notes in Economic Growth. The notes have been used in recent years in the course Economic Growth within the Master's Program in Economics at the Department of Economics, University of Copenhagen.

Compared with the earlier versions of the lecture notes some chapters have been extended and in some cases divided into several chapters. In addition, discovered typos and similar have been corrected. In some of the chapters a terminal list of references is at present lacking.

The lecture notes are in no way intended as a substitute for the textbook: D. Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press, 2009. The lecture notes are meant to be read along with the textbook. Some parts of the lecture notes are alternative presentations of stuff also covered by the textbook, while many other parts are complementary in the sense of presenting additional material. Sections marked by an asterisk, *, are cursory reading.

For constructive criticism I thank Niklas Brønager, class instructor since 2012, and plenty of earlier students. No doubt, obscurities remain. Hence, I very much welcome comments and suggestions of any kind relating to these lecture notes.

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Christian Groth

Chapter 1

Introduction to economic growth

This introductory lecture is a refresher on basic concepts.

Section 1.1 defines Economic Growth as a field of economics. In Section 1.2 formulas for calculation of compound average growth rates in discrete and continuous time are presented. Section 1.3 briefly presents two sets of stylized facts. Finally, Section 1.4 discusses, in an informal way, the different concepts of cross-country income convergence. In his introductory Chapter 1, §1.5, Acemoglu briefly touches upon these concepts.

1.1 The field

Economic growth analysis is the study of what factors and mechanisms determine the time path of *productivity* (a simple index of productivity is output per unit of labor). The focus is on

- productivity levels and
- productivity growth.

1.1.1 Economic growth theory

Economic growth theory endogenizes productivity growth via considering human capital accumulation (formal education as well as learning-by-doing) and endogenous research and development. Also the conditioning role of geography and juridical, political, and cultural institutions is taken into account.

Although for practical reasons, economic growth theory is often stated in terms of easily measurable variables like per capita GDP, the term "economic growth" may be interpreted as referring to something deeper. We could think of "economic growth" as the widening of the opportunities of human beings to lead freer and more worthwhile lives.

To make our complex economic environment accessible for theoretical analysis we use economic models. What *is* an economic model? It is a way of organizing one's thoughts about the economic functioning of a society. A more specific answer is to define an economic model as a conceptual structure based on a set of mathematically formulated assumptions which have an economic interpretation and from which empirically testable predictions can be derived. In particular, an economic growth model is an economic model concerned with productivity issues. The union of connected and non-contradictory models dealing with economic growth and the theorems derived from these constitute an *economic growth theory*. Occasionally, intense controversies about the validity of different growth theories take place.

The terms "New Growth Theory" and "endogenous growth theory" refer to theory and models which attempt at explaining sustained per capita growth as an outcome of internal mechanisms in the model rather than just a reflection of exogenous technical progress as in "Old Growth Theory".

Among the themes addressed in this course are:

- How is the world income distribution evolving?
- Why do living standards differ so much across countries and regions? Why are some countries 50 times richer than others?
- Why do per capita growth rates differ over long periods?
- What are the roles of human capital and technology innovation in economic growth? Getting the questions right.
- Catching-up and increased speed of communication and technology diffusion.
- Economic growth, natural resources, and the environment (including the climate). What are the limits to growth?
- Policies to ignite and sustain productivity growth.
- The prospects of growth in the future.

The course concentrates on *mechanisms* behind the evolution of productivity in the industrialized world. We study these mechanisms as integral parts of dynamic general equilibrium models. The exam is a test of the extent to which the student has acquired understanding of these models, is able to evaluate them, from both a theoretical and empirical perspective, and is able to use them to analyze specific economic questions. The course is calculus intensive.

1.1.2 Some long-run data

Let Y denote real GDP (per year) and let N be population size. Then Y/N is GDP per capita. Further, let g_Y denote the average (compound) growth rate of Y per year since 1870 and let $g_{Y/N}$ denote the average (compound) growth rate of Y/N per year since 1870. Table 1.1 gives these growth rates for four countries.

	g_Y	$g_{Y/N}$
Denmark	2,67	1,87
UK	1,96	1,46
USA	3,40	1,89
Japan	3,54	2,54

Table 1.1: Average annual growth rate of GDP and GDP per capita in percent, 1870–2006. Discrete compounding. Source: Maddison, A: The World Economy: Historical Statistics, 2006, Table 1b, 1c and 5c.

Figure 1.1 displays the time path of annual GDP and GDP per capita in Denmark 1870-2006 along with regression lines estimated by OLS (logarithmic scale on the vertical axis). Figure 1.2 displays the time path of GDP per capita in UK, USA, and Japan 1870-2006. In both figures the average annual growth rates are reported. In spite of being based on exactly the same data as Table 1.1, the numbers are slightly different. Indeed, the numbers in the figures are slightly lower than those in the table. The reason is that discrete compounding is used in Table 1.1 while continuous compounding is used in the two figures. These two alternative methods of calculation are explained in the next section.

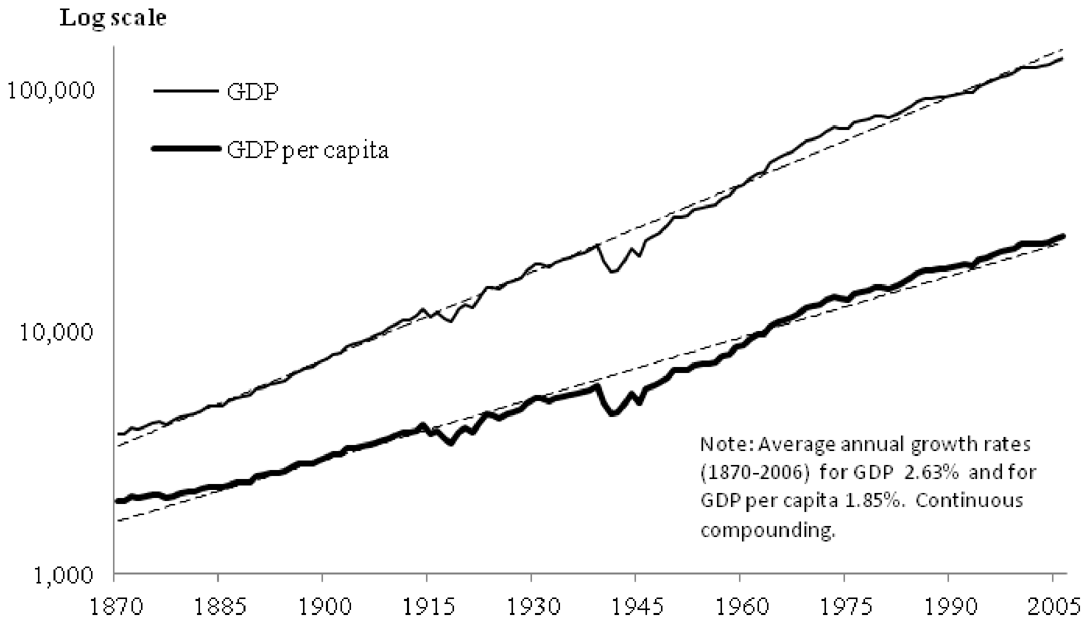


Figure 1.1: GDP and GDP per capita (1990 International Geary-Khamis dollars) in Denmark, 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

1.2 Calculation of the average growth rate

1.2.1 Discrete compounding

Let y denote aggregate labor productivity, i.e., $y \equiv Y/L$, where L is employment. The average growth rate of y from period 0 to period t, with discrete compounding, is that G which satisfies

$$\begin{aligned} y_t &= y_0(1+G)^t, \qquad t = 1, 2, ... , \qquad \text{or} && (1.1) \\ 1+G &= (\frac{y_t}{y_0})^{1/t}, \text{ i.e.,} \\ G &= (\frac{y_t}{y_0})^{1/t} - 1. && (1.2) \end{aligned}$$

"Compounding" means adding the one-period "net return" to the "principal" before adding next period's "net return" (like with interest on interest, also called "compound interest"). Obviously, G will generally be quite different from the arithmetic average of the period-by-period growth rates. To

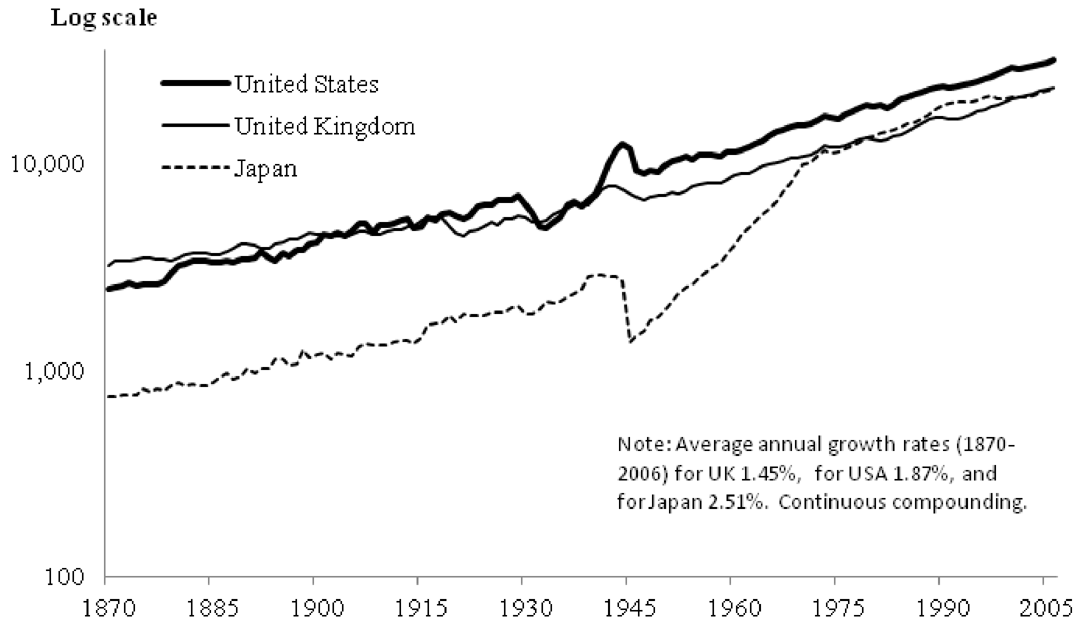


Figure 1.2: GDP per capita (1990 International Geary-Khamis dollars) in UK, USA and Japan, 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

underline this, G is sometimes called the "average compound growth rate" or the "geometric average growth rate".

Using a pocket calculator, the following steps in the calculation of G may be convenient. Take logs on both sides of (1.1) to get

$$\begin{aligned} \ln \frac{y_t}{y_0} &= t \ln(1+G) \Rightarrow \\ \ln(1+G) &= \frac{\ln \frac{y_t}{y_0}}{t} \Rightarrow \qquad (1.3) \\ G &= \text{antilog}\left(\frac{\ln \frac{y_t}{y_0}}{t}\right) - 1. \qquad (1.4) \end{aligned}$$

Note that t in the formulas (1.2) and (1.4) equals the number of periods *minus 1.*

1.2.2 Continuous compounding

The average growth rate of y, with continuous compounding, is that g which satisfies

$$y_t = y_0 e^{gt}, \tag{1.5}$$

where e denotes the Euler number, i.e., the base of the natural logarithm.[1] Solving for g gives

$$g = \frac{\ln \frac{y_t}{y_0}}{t} = \frac{\ln y_t - \ln y_0}{t}. \tag{1.6}$$

The first formula in (1.6) is convenient for calculation with a pocket calculator, whereas the second formula is perhaps closer to intuition. Another name for g is the "exponential average growth rate".

Again, the t in the formula equals the number of periods minus 1.

Comparing with (1.3) we see that $g = \ln(1 + G) < G$ for $G > 0$. Yet, by a first-order Taylor approximation about $G = 0$ we have

$$g = \ln(1 + G) \approx G \quad \text{for } G \text{ "small"}. \tag{1.7}$$

For a given data set the G calculated from (1.2) will be slightly above the g calculated from (1.6), cf. the mentioned difference between the growth rates in Table 1.1 and those in Figure 1.1 and Figure 1.2. The reason is that a given growth force is more powerful when compounding is continuous rather than discrete. Anyway, the difference between G and g is usually unimportant. If for example G refers to the annual GDP growth rate, it will be a small number, and the difference between G and g immaterial. For example, to $G = 0.040$ corresponds $g \approx 0.039$. Even if $G = 0.10$, the corresponding g is 0.0953. But if G stands for the inflation rate and there is high inflation, the difference between G and g will be substantial. During hyperinflation the monthly inflation rate may be, say, $G = 100\%$, but the corresponding g will be only 69%.

Which method, discrete or continuous compounding, is preferable? To some extent it is a matter of taste or convenience. In period analysis discrete compounding is most common and in continuous time analysis continuous compounding is most common.

For calculation with a pocket calculator the continuous compounding formula, (1.6), is slightly easier to use than the discrete compounding formulas, whether (1.2) or (1.4).

To avoid too much sensitiveness to the initial and terminal observations, which may involve measurement error or depend on the state of the business

[1] Unless otherwise specified, whenever we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

cycle, one can use an OLS approach to the trend coefficient, g, in the following regression:

$$\ln Y_t = \alpha + gt + \varepsilon_t.$$

This is in fact what is done in Fig. 1.1.

1.2.3 Doubling time

How long time does it take for y to double if the growth rate with discrete compounding is G? Knowing G, we rewrite the formula (1.3):

$$t = \frac{\ln \frac{y_t}{y_0}}{\ln(1+G)} = \frac{\ln 2}{\ln(1+G)} \approx \frac{0.6931}{\ln(1+G)}.$$

With $G = 0.0187$, cf. Table 1.1, we find

$$t \approx 37.4 \text{ years},$$

meaning that productivity doubles every 37.4 years.

How long time does it take for y to double if the growth rate with continuous compounding is g? The answer is based on rewriting the formula (1.6):

$$t = \frac{\ln \frac{y_t}{y_0}}{g} = \frac{\ln 2}{g} \approx \frac{0.6931}{g}.$$

Maintaining the value 0.0187 also for g, we find

$$t \approx \frac{0.6931}{0.0187} \approx 37.1 \text{ years}.$$

Again, with a pocket calculator the continuous compounding formula is slightly easier to use. With a lower g, say $g = 0.01$, we find doubling time equal to 69.1 years. With $g = 0.07$ (think of China since the 1970's), doubling time is about 10 years! Owing to the compounding exponential growth is extremely powerful.

1.3 Some stylized facts of economic growth

1.3.1 The Kuznets facts

A well-known characteristic of modern economic growth is structural change: unbalanced sectorial growth. There is a massive reallocation of labor from agriculture into industry (manufacturing, construction, and mining) and further into services (including transport and communication). The shares of

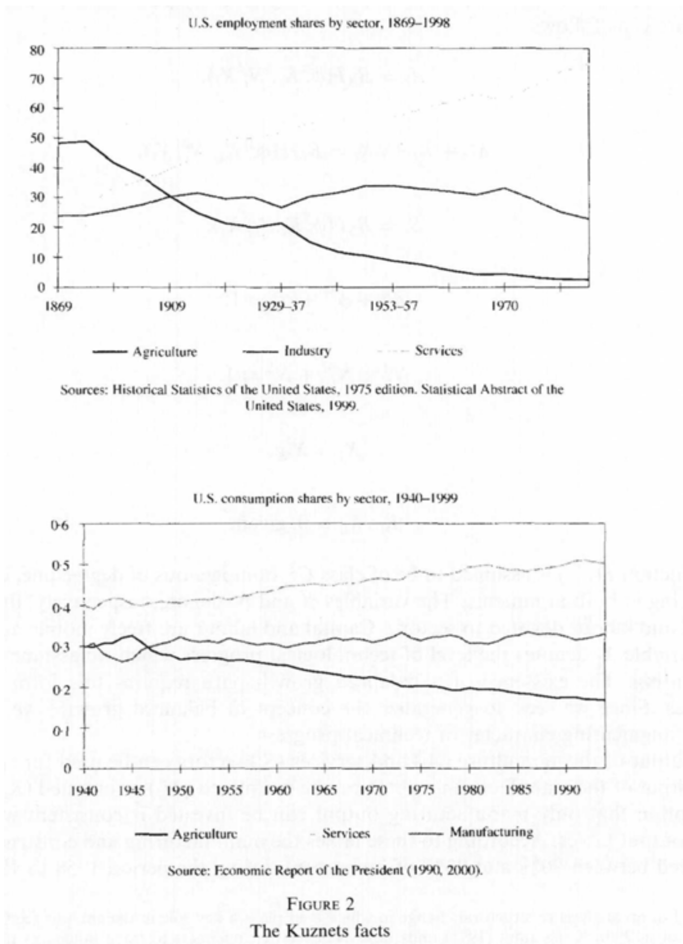


Figure 1.3: The Kuznets facts. Source: Kongsamut et al., Beyond Balanced Growth, Review of Economic Studies, vol. 68, Oct. 2001, 869-82.

total consumption expenditure going to these three sectors have moved similarly. Differences in the demand elasticities with respect to income seem the main explanation. These observations are often referred to as the *Kuznets facts* (after Simon Kuznets, 1901-85, see, e.g., Kuznets 1957).

The two graphs in Figure 1.3 illustrate the Kuznets facts.

1.3.2 Kaldor's stylized facts

Surprisingly, in spite of the Kuznets facts, the evolution at the *aggregate* level in developed countries is by many economists seen as roughly described by what is called Kaldor's "stylized facts" (after the Hungarian-British economist Nicholas Kaldor, 1908-1986, see, e.g., Kaldor 1957, 1961)[2]:

1. Real output per man-hour grows at a more or less constant rate over fairly long periods of time. (Of course, there are short-run fluctuations superposed around this trend.)

2. The stock of physical capital per man-hour grows at a more or less constant rate over fairly long periods of time.

3. The ratio of output to capital shows no systematic trend.

4. The rate of return to capital shows no systematic trend.

5. The income shares of labor and capital (in the national accounting sense, i.e., including land and other natural resources), respectively, are nearly constant.

6. The growth rate of output per man-hour differs substantially across countries.

These claimed regularities do certainly not fit all developed countries equally well. Although Solow's growth model (Solow, 1956) can be seen as the first successful attempt at building a model consistent with Kaldor's "stylized facts", Solow once remarked about them: "There is no doubt that they are stylized, though it is possible to question whether they are facts" (Solow, 1970). But the Kaldor "facts" do at least seem to fit the US and UK quite well, see, e.g., Attfield and Temple (2010). The sixth Kaldor fact is, of course, well documented empirically (a nice summary is contained in Pritchett, 1997).

Kaldor also proposed hypotheses about the links between growth in the different sectors (see, e.g., Kaldor 1967):

a. Productivity growth in the manufacturing and construction sectors is enhanced by output growth in these sectors (this is also known as Verdoorn's Law). Increasing returns to scale and learning by doing are the main factors behind this.

b. Productivity growth in agriculture and services is enhanced by output growth in the manufacturing and construction sectors.

1.4 Concepts of income convergence

The two most popular across-country income convergence concepts are "β convergence" and "σ convergence".

[2] Kaldor presented his six regularities as "a stylised view of the facts".

1.4.1 β convergence vs. σ convergence

Definition 1 *We say that β convergence occurs for a given selection of countries if there is a tendency for the poor (those with low income per capita or low output per worker) to subsequently grow faster than the rich.*

By "grow faster" is meant that the growth rate of per capita income (or per worker output) is systematically higher.

In many contexts, a more appropriate convergence concept is the following:

Definition 2 *We say that σ convergence, with respect to a given measure of dispersion, occurs for a given collection of countries if this measure of dispersion, applied to income per capita or output per worker across the countries, declines systematically over time. On the other hand, σ divergence occurs, if the dispersion increases systematically over time.*

The reason that σ convergence must be considered the more appropriate concept is the following. In the end, it is the question of increasing or decreasing dispersion across countries that we are interested in. From a superficial point of view one might think that β convergence implies decreasing dispersion and vice versa, so that β convergence and σ convergence are more or less equivalent concepts. But since the world is not deterministic, but stochastic, this is not true. Indeed, β convergence is only a necessary, not a sufficient condition for σ convergence. This is because over time some reshuffling among the countries is always taking place, and this implies that there will always be some extreme countries (those initially far away from the mean) that move closer to the mean, thus creating a negative correlation between initial level and subsequent growth, in spite of equally many countries moving from a middle position toward one of the extremes.[3] In this way β convergence may be observed at the same time as there is no σ convergence; the mere presence of random measurement errors implies a bias in this direction because a growth rate depends negatively on the initial measurement and positively on the later measurement. In fact, β convergence may be consistent with σ *divergence* (for a formal proof of this claim, see Barro and Sala-i-Martin, 2004, pp. 50-51 and 462 ff.; see also Valdés, 1999, p. 49-50, and Romer, 2001, p. 32-34).

[3] As an intuitive analogy, think of the ordinal rankings of the sports teams in a league. The dispersion of rankings is constant by definition. Yet, no doubt there will allways be some tendency for weak teams to rebound toward the mean and of champions to revert to mediocrity. (This example is taken from the first edition of Barro and Sala-i-Martin, *Economic Growth*, 1995; I do not know why, but the example has been deleted in the second edition from 2004.)

Hence, it is wrong to conclude from β convergence (poor countries tend to grow faster than rich ones) to σ convergence (reduced dispersion of per capita income) without any further investigation. The mistake is called "regression towards the mean" or "Galton's fallacy". Francis Galton was an anthropologist (and a cousin of Darwin), who in the late nineteenth century observed that tall fathers tended to have not as tall sons and small fathers tended to have taller sons. From this he falsely concluded that there was a tendency to averaging out of the differences in height in the population. Indeed, being a true aristocrat, Galton found this tendency pitiable. But since his conclusion was mistaken, he did not really have to worry.

Since σ convergence comes closer to what we are ultimately looking for, from now, when we speak of just "income convergence", σ convergence is understood.

In the above definitions of σ convergence and β convergence, respectively, we were vague as to what kind of selection of countries is considered. In principle we would like it to be a representative sample of the "population" of countries that we are interested in. The population could be all countries in the world. Or it could be the countries that a century ago had obtained a certain level of development.

One should be aware that historical GDP data are constructed retrospectively. Long time series data have only been constructed for those countries that became relatively rich during the after-WWII period. Thus, if we as our sample select the countries for which long data series exist, a so-called *selection bias* is involved which generates a spurious convergence. A country which was poor a century ago will only appear in the sample if it grew rapidly over the next 100 years. A country which was relatively rich a century ago will appear in the sample unconditionally. This selection bias problem was pointed out by DeLong (1988) in a criticism of widespread false interpretations of Maddison's long data series (Maddison 1982).

1.4.2 Measures of dispersion

Our next problem is: *what* measure of dispersion is to be used as a useful descriptive statistics for σ convergence? Here there are different possibilities. To be precise about this we need some notation. Let

$$\begin{aligned} y &\equiv \frac{Y}{L}, \qquad \text{and} \\ q &\equiv \frac{Y}{N}, \end{aligned}$$

where Y = real GDP, L = employment, and N = population. If the focus is on living standards, Y/N, is the relevant variable.[4] But if the focus is on (labor) productivity, it is Y/L, that is relevant. Since most growth models focus on Y/L rather than Y/N, let os take y as our example.

One might think that the standard deviation of y could be a relevant measure of dispersion when discussing whether σ convergence is present or not. The *standard deviation* of y across n countries in a given year is

$$\sigma_y \equiv \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}, \tag{1.8}$$

where

$$\bar{y} \equiv \frac{\sum_i y_i}{n}, \tag{1.9}$$

i.e., $\bar{y}$ is the average output per worker. However, if this measure were used, it would be hard to find *any* group of countries for which there is income convergence. This is because y tends to grow over time for most countries, and then there is an inherent tendency for the variance also to grow; hence also the square root of the variance, σ_y, tends to grow. Indeed, suppose that for all countries, y is doubled from time t_1 to time t_2. Then, automatically, σ_y is also doubled. But hardly anyone would interpret this as an increase in the income inequality across the countries.

Hence, it is more adequate to look at the standard deviation of *relative* income levels:

$$\sigma_{y/\bar{y}} \equiv \sqrt{\frac{1}{n}\sum_i(\frac{y_i}{\bar{y}} - 1)^2}. \tag{1.10}$$

This measure is the same as what is called the *coefficient of variation*, CV_y, usually defined as

$$CV_y \equiv \frac{\sigma_y}{\bar{y}}, \tag{1.11}$$

that is, the standard deviation of y standardized by the mean. That the two measures are identical can be seen in this way:

$$\frac{\sigma_y}{\bar{y}} \equiv \frac{\sqrt{\frac{1}{n}\sum_i(y_i - \bar{y})^2}}{\bar{y}} = \sqrt{\frac{1}{n}\sum_i(\frac{y_i - \bar{y}}{\bar{y}})^2} = \sqrt{\frac{1}{n}\sum_i(\frac{y_i}{\bar{y}} - 1)^2} \equiv \sigma_{y/\bar{y}}.$$

[4] Or perhaps better, Q/N, where $Q \equiv GNP \equiv GDP - rD - wF$. Here, rD, denotes net interest payments on foreign debt and wF denotes net labor income of foreign workers in the country.

The point is that the coefficient of variation is "scale free", which the standard deviation itself is not.

Instead of the coefficient of variation, another scale free measure is often used, namely the standard deviation of $\ln y$, i.e.,

$$\sigma_{\ln y} \equiv \sqrt{\frac{1}{n}\sum_i (\ln y_i - \ln y^*)^2}, \tag{1.12}$$

where

$$\ln y^* \equiv \frac{\sum_i \ln y_i}{n}. \tag{1.13}$$

Note that y^* is the geometric average, i.e., $y^* \equiv \sqrt[n]{y_1 y_2 \cdots y_n}$. Now, by a first-order Taylor approximation of $\ln y$ around $y = \bar{y}$, we have

$$\ln y \approx \ln \bar{y} + \frac{1}{\bar{y}}(y - \bar{y})$$

Hence, as a very rough approximation we have $\sigma_{\ln y} \approx \sigma_{y/\bar{y}} = CV_y$, though this approximation can be quite poor (cf. Dalgaard and Vastrup, 2001). It may be possible, however, to defend the use of $\sigma_{\ln y}$ in its own right to the extent that y tends to be approximately lognormally distributed across countries.

Yet another possible measure of income dispersion across countries is the *Gini index* (see for example Cowell, 1995).

1.4.3 Weighting by size of population

Another important issue is whether the applied dispersion measure is based on a *weighting of the countries by size of population*. For the world as a whole, when no weighting by size of population is used, then there is a slight tendency to income divergence according to the $\sigma_{\ln q}$ criterion (Acemoglu, 2009, p. 4), where q is per capita income ($\equiv Y/N$). As seen by Fig. 4 below, this tendency is not so clear according to the CV_q criterion. Anyway, when there *is* weighting by size of population, then in the last twenty years there has been a tendency to income convergence at the global level (Sala-i-Martin 2006; Acemoglu, 2009, p. 6). With weighting by size of population (1.12) is modified to

$$\sigma^w_{\ln q} \equiv \sqrt{\sum_i w_i (\ln q_i - \ln q^*)^2},$$

where

$$w_i = \frac{N_i}{N} \quad \text{and} \quad \ln q^* \equiv \sum_i w_i \ln q_i.$$

1.4.4 Unconditional vs. conditional convergence

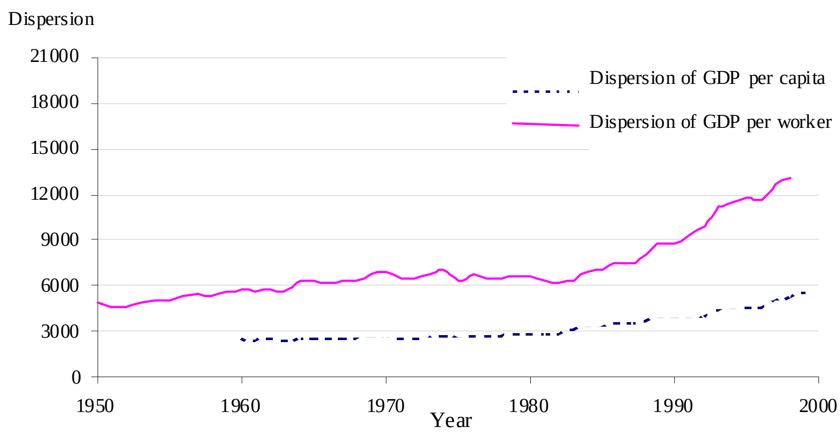
Yet another distinction in the study of income convergence is that between unconditional (or absolute) and conditional convergence. We say that a large heterogeneous group of countries (say the countries in the world) show *unconditional* income convergence if income convergence occurs for the whole group without conditioning on specific characteristics of the countries. If income convergence occurs only for a subgroup of the countries, namely those countries that in advance share the same "structural characteristics", then we say there is *conditional* income convergence. As noted earlier, when we speak of just income "convergence", income "σ convergence" is understood. If in a given context there might be doubt, one should of course be explicit and speak of unconditional or conditional σ convergence. Similarly, if the focus for some reason is on β convergence, we should distinguish between unconditional and conditional β convergence.

What the precise meaning of "structural characteristics" is, will depend on what model of the countries the researcher has in mind. According to the Solow model, a set of relevant "structural characteristics" are: the aggregate production function, the initial level of technology, the rate of technical progress, the capital depreciation rate, the saving rate, and the population growth rate. But the Solow model, as well as its extension with human capital (Mankiw et al., 1992), is a model of a closed economy with exogenous technical progress. The model deals with "within-country" convergence in the sense that the model predicts that a closed economy being initially below or above its steady state path, will over time converge towards its steady state path. It is far from obvious that this kind of model is a good model of cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and some countries are pushing the technological frontier further out, while others try to imitate and catch up.

1.4.5 A bird's-eye view of the data

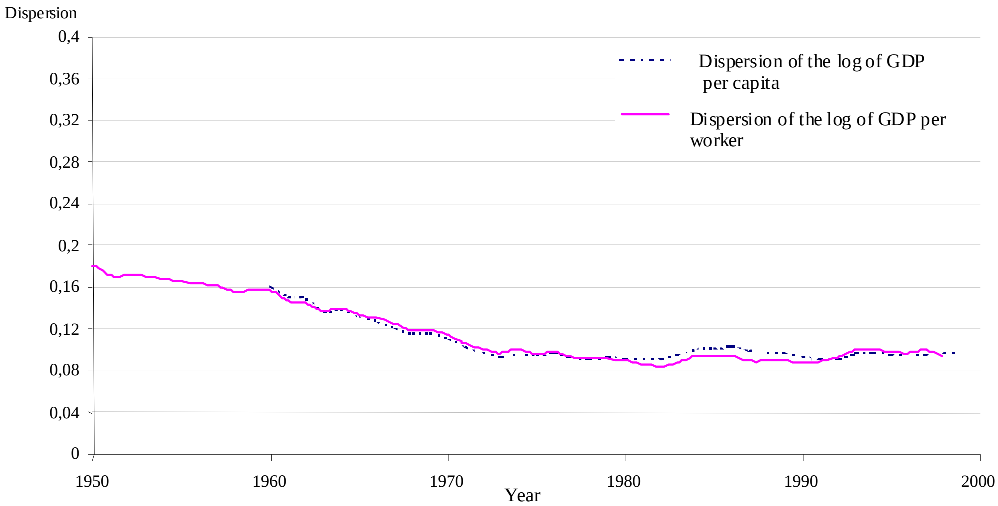
In the following no serious econometrics is attempted. We use the term "trend" in an admittedly loose sense.

Figure 1.4 shows the time profile for the standard deviation of y itself for 12 EU countries, whereas Figure 1.5 and Figure 1.6 show the time profile of the standard deviation of $\log y$ and the time profile of the coefficient of variation, respectively. Comparing the upward trend in Figure 1.4 with the downward trend in the two other figures, we have an illustration of the fact that the movement of the standard deviation of y itself does not capture



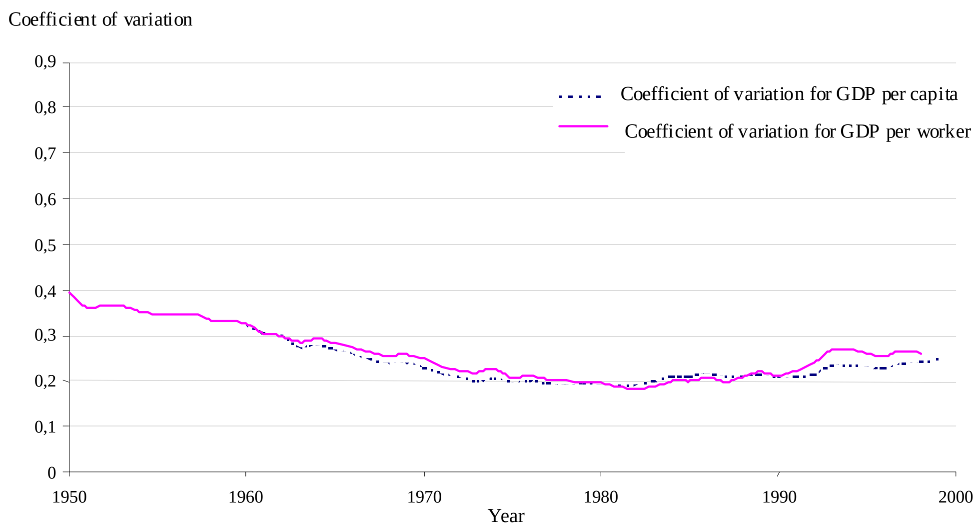
Remarks: Germany is not included in GDP per worker. GDP per worker is missing for Sweden and Greece in 1950, and for Portugal in 1998. The EU comprises Belgium, Denmark, Finland, France, Greece, Holland, Ireland, Italy, Luxembourg, Portugal, Spain, Sweden, Germany, the UK and Austria.
Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

Figure 1.4: Standard deviation of GDP per capita and per worker across 12 EU countries, 1950-1998.



Remarks: Germany is not included in GDP per worker. GDP per worker is missing for Sweden and Greece in 1950, and for Portugal in 1998. The EU comprises Belgium, Denmark, Finland, France, Greece, Holland, Ireland, Italy, Luxembourg, Portugal, Spain, Sweden, Germany, the UK and Austria.
Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

Figure 1.5: Standard deviation of the log of GDP per capita and per worker across 12 EU countries, 1950-1998.



Remarks: Germany is not included in GDP per worker. GDP per worker is missing for Sweden and Greece in 1950, and for Portugal in 1998. The EU comprises Belgium, Denmark, Finland, France, Greece, Holland, Ireland, Italy, Luxembourg, Portugal, Spain, Sweden, Germany, the UK and Austria.
Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

Figure 1.6: Coefficient of variation of GDP per capita and GDP per worker across 12 EU countries, 1950-1998.

income convergence. To put it another way: although there seems to be conditional income convergence with respect to the two scale-free measures, Figure 1.4 shows that this tendency to convergence is *not* so strong as to produce a narrowing of the absolute distance between the EU countries.[5]

Figure 1.7 shows the time path of the coefficient of variation across 121 countries in the world, 22 OECD countries and 12 EU countries, respectively. We see the lack of unconditional income convergence, but the presence of conditional income convergence. One should not over-interpret the observation of convergence for the 22 OECD countries over the period 1950-1990. It is likely that this observation suffer from the selection bias problem mentioned in Section 1.4.1. A country that was poor in 1950 will typically have become a member of OECD only if it grew relatively fast afterwards.

[5] Unfortunately, sometimes misleading graphs or texts to graphs about across-country income convergence are published. In the collection of exercises, Chapter 1, you are asked to discuss some examples of this.

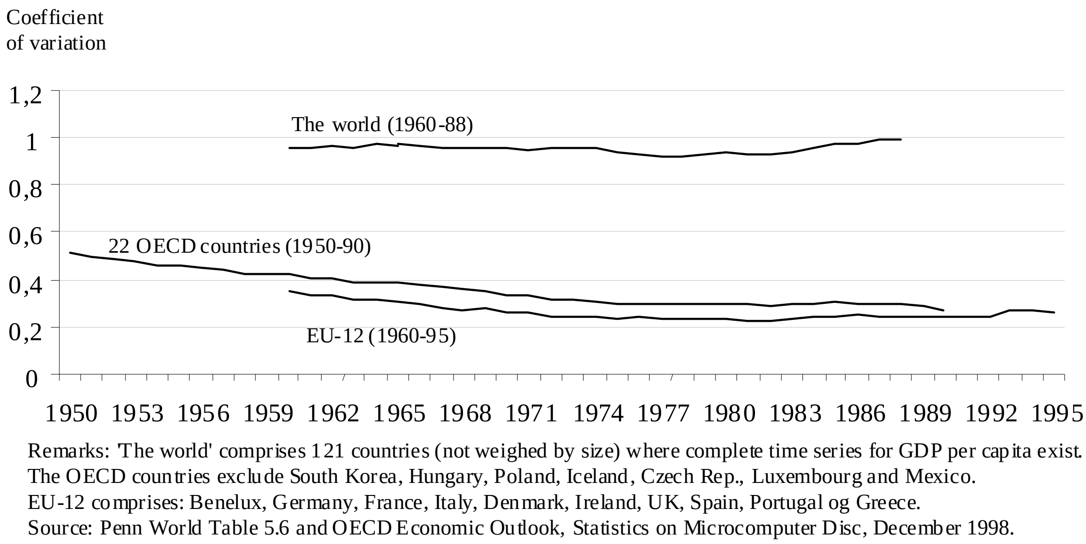


Figure 1.7: Coefficient of variation of income per capita across different sets of countries.

1.4.6 Other convergence concepts

Of course, just considering the time profile of the first and second moments of a distribution may sometimes be a poor characterization of the evolution of the distribution. For example, there are signs that the distribution has polarized into *twin peaks* of rich and poor countries (Quah, 1996a; Jones, 1997). Related to this observation is the notion of club convergence. If income convergence occurs *only* among a subgroup of the countries that to some extent share the same initial conditions, then we say there is *club-convergence.* This concept is relevant in a setting where there are *multiple* steady states toward which countries can converge. At least at the theoretical level multiple steady states can easily arise in overlapping generations models. Then the initial condition for a given country matters for which of these steady states this country is heading to. Similarly, we may say that *conditional club-convergence* is present, if income convergence occurs *only* for a subgroup of the countries, namely countries sharing similar structural characteristics (this may to some extent be true for the OECD countries) *and*, within an interval, similar initial conditions.

Instead of focusing on income convergence, one could study *TFP conver-*

gence at aggregate or industry level.[6] Sometimes the less demanding concept of *growth rate convergence* is the focus.

The above considerations are only of a very elementary nature and are only about descriptive statistics. The reader is referred to the large existing literature on concepts and econometric methods of relevance for characterizing the evolution of world income distribution (see Quah, 1996b, 1996c, 1997, and for a survey, see Islam 2003).

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Chapter 2

Review of technology

The aim of this chapter is, first, to introduce the terminology concerning firms' technology and technological change used in the lectures and exercises of this course. At a few points I deviate somewhat from definitions in Acemoglu's book. Section 1.3 can be used as a formula manual for the case of CRS.

Second, the chapter contains a brief discussion of the somewhat controversial notions of a representative firm and an aggregate production function.

Regarding the distinction between discrete and continuous time analysis, most of the definitions contained in this chapter are applicable to both.

2.1 The production technology

Consider a two-factor production function given by

$$Y = F(K, L), \tag{2.1}$$

where Y is output (value added) per time unit, K is capital input per time unit, and L is labor input per time unit ($K \geq 0$, $L \geq 0$). We may think of (2.1) as describing the output of a firm, a sector, or the economy as a whole. It is in any case a very simplified description, ignoring the heterogeneity of output, capital, and labor. Yet, for many macroeconomic questions it may be a useful first approach. Note that in (2.1) not only Y but also K and L represent *flows,* that is, quantities per unit of time. If the time unit is one year, we think of K as measured in machine hours per year. Similarly, we think of L as measured in labor hours per year. Unless otherwise specified, it is understood that the rate of utilization of the production factors is constant over time and normalized to one for each production factor. As explained in Chapter 1, we can then use the same symbol, K, for the *flow* of capital services as for the *stock* of capital. Similarly with L.

2.1.1 A neoclassical production function

By definition, K and L are non-negative. It is generally understood that a production function, $Y = F(K, L)$, is *continuous* and that $F(0, 0) = 0$ (no input, no output). Sometimes, when specific functional forms are used to represent a production function, that function may not be defined at points where $K = 0$ or $L = 0$ or both. In such a case we adopt the convention that the domain of the function is understood extended to include such boundary points whenever it is possible to assign function values to them such that continuity is maintained. For instance the function $F(K, L) = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$, is not defined at $(K, L) = (0, 0)$. But by assigning the function value 0 to the point $(0, 0)$, we maintain both continuity and the "no input, no output" property, cf. Exercise 2.4.

We call the production function *neoclassical* if for all (K, L), with $K > 0$ and $L > 0$, the following additional conditions are satisfied:

(a) $F(K, L)$ has continuous first- and second-order partial derivatives satisfying:

$$F_K > 0, \quad F_L > 0, \tag{2.2}$$

$$F_{KK} < 0, \quad F_{LL} < 0. \tag{2.3}$$

(b) $F(K, L)$ is strictly quasiconcave (i.e., the level curves, also called isoquants, are strictly convex to the origin).

In words: (a) says that a neoclassical production function has continuous substitution possibilities between K and L and the *marginal productivities* are positive, but diminishing in own factor. Thus, for a given number of machines, adding one more unit of labor, adds to output, but less so, the higher is already the labor input. And (b) says that every isoquant, $F(K, L) = \bar{Y}$, has a strictly convex form qualitatively similar to that shown in Figure 2.1.[1] When we speak of for example F_L as the marginal *productivity* of labor, it is because the "pure" partial derivative, $\partial Y/\partial L = F_L$, has the denomination of a productivity (output units/yr)/(man-yrs/yr). It is quite common, however, to refer to F_L as the marginal *product* of labor. Then a unit marginal increase in the labor input is understood: $\Delta Y \approx (\partial Y/\partial L)\Delta L = \partial Y/\partial L$ when $\Delta L = 1$. Similarly, F_K can be interpreted as the marginal *productivity* of capital or as the marginal *product* of capital. In the latter case it is understood that $\Delta K = 1$, so that $\Delta Y \approx (\partial Y/\partial K)\Delta K = \partial Y/\partial K$.

[1] For any fixed $\bar{Y} \geq 0$, the associated *isoquant* is the level set $\{(K, L) \in \mathbb{R}_+ | \ F(K, L) = \bar{Y}\}$.

The definition of a neoclassical production function can be extended to the case of n inputs. Let the input quantities be $X_1, X_2, \ldots, X_n$ and consider a production function $Y = F(X_1, X_2, \ldots, X_n)$. Then F is called neoclassical if all the marginal productivities are positive, but diminishing, and F is strictly quasiconcave (i.e., the upper contour sets are strictly convex, cf. Appendix A).

Returning to the two-factor case, since $F(K, L)$ presumably depends on the level of technical knowledge and this level depends on time, t, we might want to replace (2.1) by

$$Y_t = F^t(K_t, L_t), \tag{2.4}$$

where the superscript on F indicates that the production function may shift over time, due to changes in technology. We then say that $F^t(\cdot)$ is a neoclassical production function if it satisfies the conditions (a) and (b) for all pairs (K_t, L_t). *Technological progress* can then be said to occur when, for K_t and L_t held constant, output increases with t.

For convenience, to begin with we skip the explicit reference to time and level of technology.

The marginal rate of substitution Given a neoclassical production function F, we consider the isoquant defined by $F(K, L) = \bar{Y}$, where $\bar{Y}$ is a positive constant. The *marginal rate of substitution*, MRS_{KL}, of K for L at the point (K, L) is defined as the absolute slope of the isoquant at that point, cf. Figure 2.1. The equation $F(K, L) = \bar{Y}$ defines K as an implicit function of L. By implicit differentiation we find $F_K(K, L)dK/dL + F_L(K, L) = 0$, from which follows

$$MRS_{KL} \equiv -\frac{dK}{dL}_{|Y=\bar{Y}} = \frac{F_L(K, L)}{F_K(K, L)} > 0. \tag{2.5}$$

That is, MRS_{KL} measures the amount of K that can be saved (approximately) by applying an extra unit of labor. In turn, this equals the ratio of the marginal productivities of labor and capital, respectively.[2] Since F is neoclassical, by definition F is strictly quasi-concave and so the marginal rate of substitution is diminishing as substitution proceeds, i.e., as the labor input is further increased along a given isoquant. Notice that this feature characterizes the marginal rate of substitution for any neoclassical production function, whatever the returns to scale (see below).

[2] The subscript $|Y = \bar{Y}$ in (2.5) indicates that we are moving along a given isoquant, $F(K, L) = \bar{Y}$. Expressions like, e.g., $F_L(K, L)$ or $F_2(K, L)$ mean the partial derivative of F w.r.t. the second argument, evaluated at the point (K, L).

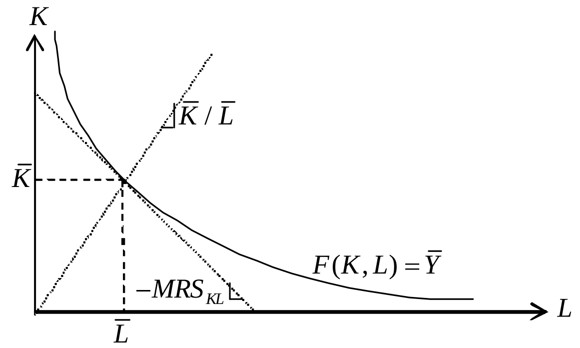

Figure 2.1: MRS_{KL} as the absolute slope of the isoquant.

When we want to draw attention to the dependency of the marginal rate of substitution on the factor combination considered, we write $MRS_{KL}(K, L)$. Sometimes in the literature, the marginal rate of substitution between two production factors, K and L, is called the *technical* rate of substitution (or the technical rate of transformation) in order to distinguish from a consumer's marginal rate of substitution between two consumption goods.

As is well-known from microeconomics, a firm that minimizes production costs for a given output level and given factor prices, will choose a factor combination such that MRS_{KL} equals the ratio of the factor prices. If $F(K, L)$ is homogeneous of degree q, then the marginal rate of substitution depends only on the factor proportion and is thus the same at any point on the ray $K = (\bar{K}/\bar{L})L$. That is, in this case the expansion path is a straight line.

The Inada conditions A continuously differentiable production function is said to satisfy the *Inada conditions*[3] if

$$\lim_{K\to 0} F_K(K, L) = \infty, \lim_{K\to\infty} F_K(K, L) = 0, \tag{2.6}$$

$$\lim_{L\to 0} F_L(K, L) = \infty, \lim_{L\to\infty} F_L(K, L) = 0. \tag{2.7}$$

In this case, the marginal productivity of either production factor has no upper bound when the input of the factor becomes infinitely small. And the marginal productivity is gradually vanishing when the input of the factor increases without bound. Actually, (2.6) and (2.7) express *four* conditions, which it is preferable to consider separately and label one by one. In (2.6) we have two *Inada conditions for MPK* (the marginal productivity of capital), the first being a *lower*, the second an *upper* Inada condition for MPK. And

[3] After the Japanese economist Ken-Ichi Inada, 1925-2002.

in (2.7) we have two *Inada conditions for MPL* (the marginal productivity of labor), the first being a *lower*, the second an *upper* Inada condition for MPL. In the literature, when a sentence like "the Inada conditions are assumed" appears, it is sometimes not made clear which, and how many, of the four are meant. Unless it is evident from the context, it is better to be explicit about what is meant.

The definition of a neoclassical production function we gave above is quite common in macroeconomic journal articles and convenient because of its flexibility. There are textbooks that define a neoclassical production function more narrowly by including the Inada conditions as a requirement for calling the production function neoclassical. In contrast, in this course, when in a given context we need one or another Inada condition, we state it explicitly as an additional assumption.

2.1.2 Returns to scale

If all the inputs are multiplied by some factor, is output then multiplied by the same factor? There may be different answers to this question, depending on circumstances. We consider a production function $F(K, L)$ where $K > 0$ and $L > 0$. Then F is said to have *constant returns to scale* (CRS for short) if it is homogeneous of degree one, i.e., if for all (K, L) and all $\lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

As all inputs are scaled up or down by some factor > 1, output is scaled up or down by the same factor.[4] The assumption of CRS is often defended by the *replication argument.* Before discussing this argument, lets us define the two alternative "pure" cases.

The production function $F(K, L)$ is said to have *increasing returns to scale* (IRS for short) if, for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) > \lambda F(K, L).$$

That is, IRS is present if, when all inputs are scaled up by some factor $>$ 1, output is scaled up by *more* than this factor. The existence of gains by specialization and division of labor, synergy effects, etc. sometimes speak in support of this assumption, at least up to a certain level of production. The assumption is also called the *economies of scale* assumption.

[4] In their definition of a neoclassical production function some textbooks add constant returns to scale as a requirement besides (a) and (b). This course follows the alternative terminology where, if in a given context an assumption of constant returns to scale is needed, this is stated as an additional assumption.

Another possibility is *decreasing returns to scale* (DRS). This is said to occur when for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) < \lambda F(K, L).$$

That is, DRS is present if, when all inputs are scaled up by some factor, output is scaled up by *less* than this factor. This assumption is also called the *diseconomies of scale* assumption. The underlying hypothesis may be that control and coordination problems confine the expansion of size. Or, considering the "replication argument" below, DRS may simply reflect that behind the scene there is an additional production factor, for example land or a irreplaceable quality of management, which is tacitly held fixed, when the factors of production are varied.

EXAMPLE 1 The production function

$$Y = AK^{\alpha}L^{\beta}, \qquad A > 0, 0 < \alpha < 1, 0 < \beta < 1, \tag{2.8}$$

where A, α, and β are given parameters, is called a *Cobb-Douglas production function*. The parameter A depends on the choice of measurement units; for a given such choice it reflects "efficiency", also called the "total factor productivity". Exercise 2.2 asks the reader to verify that (2.8) satisfies (a) and (b) above and is therefore a neoclassical production function. The function is homogeneous of degree $\alpha + \beta$. If $\alpha + \beta = 1$, there are CRS. If $\alpha + \beta < 1$, there are DRS, and if $\alpha + \beta > 1$, there are IRS. Note that α and β must be less than 1 in order not to violate the diminishing marginal productivity condition. □

EXAMPLE 2 The production function

$$Y = \min(AK, BL), \qquad A > 0, B > 0, \tag{2.9}$$

where A and B are given parameters, is called a *Leontief production function* or a *fixed-coefficients production function;* A and B are called the *technical coefficients*. The function is not neoclassical, since the conditions (a) and (b) are not satisfied. Indeed, with this production function the production factors are not substitutable at all. This case is also known as the case of *perfect complementarity* between the production factors. The interpretation is that already installed production equipment requires a fixed number of workers to operate it. The inverse of the parameters A and B indicate the required capital input per unit of output and the required labor input per unit of output, respectively. Extended to many inputs, this type of production function is often used in multi-sector input-output models (also called Leontief models).

In aggregate analysis neoclassical production functions, allowing substitution between capital and labor, are more popular than Leontief functions. But sometimes the latter are preferred, in particular in short-run analysis with focus on the use of already installed equipment where the substitution possibilities are limited.[5] As (2.9) reads, the function has CRS. A generalized form of the Leontief function is $Y = \min(AK^{\gamma}, BL^{\gamma})$, where $\gamma > 0$. When $\gamma < 1$, there are DRS, and when $\gamma > 1$, there are IRS. □

The replication argument The assumption of CRS is widely used in macroeconomics. The model builder may appeal to the *replication argument.* To explain the content of this argument we have to first clarify the distinction between rival and nonrival inputs or more generally the distinction between rival and nonrival goods. A good is *rival* if its character is such that one agent's use of it inhibits other agents' use of it at the same time. A pencil is thus rival. Many production inputs like raw materials, machines, labor etc. have this property. In contrast, however, technical knowledge like a farmaceutical formula or an engineering principle is *nonrival.* An unbounded number of factories can simultaneously use the same farmaceutical formula.

The replication argument now says that by, conceptually, doubling all the rival inputs, we should always be able to double the output, since we just "replicate" what we are already doing. One should be aware that the CRS assumption is about *technology* in the sense of functions linking inputs to outputs − limits to the *availability* of input resources is an entirely different matter. The fact that for example managerial talent may be in limited supply does not preclude the thought experiment that *if* a firm could double all its inputs, including the number of talented managers, then the output level could also be doubled.

The replication argument presupposes, first, that *all* the relevant inputs are explicit as arguments in the production function; second, that these are changed equiproportionately. This, however, exhibits the weakness of the replication argument as a defence for assuming CRS of our present production function, $F(\cdot)$. One could easily make the case that besides capital and labor, also land is a necessary input and should appear as a separate argument.[6] If an industrial firm decides to duplicate what it has been doing, it needs a piece of land to build another plant like the first. Then, on the basis of the replication argument we should in fact expect DRS w.r.t. capital and labor alone. In manufacturing and services, empirically, this and other possible

[5] Cf. Section 2.4.

[6] We think of "capital" as producible means of production, whereas "land" refers to non-producible natural resources, including for example building sites.

sources for departure from CRS may be minor and so many macroeconomists feel comfortable enough with assuming CRS w.r.t. K and L alone, at least as a first approximation. This approximation is, however, less applicable to poor countries, where natural resources may be a quantitatively important production factor.

There is a further problem with the replication argument. Strictly speaking, the CRS claim is that by changing all the inputs equiproportionately by *any* positive factor, λ, which does not have to be an integer, the firm should be able to get output changed by the same factor. Hence, the replication argument requires that indivisibilities are negligible, which is certainly not always the case. In fact, the replication argument is more an argument *against* DRS than *for* CRS in particular. The argument does not rule out IRS due to synergy effects as size is increased.

Sometimes the replication line of reasoning is given a more subtle form. This builds on a useful *local* measure of returns to scale, named the *elasticity of scale*.

The elasticity of scale* To allow for indivisibilities and mixed cases (for example IRS at low levels of production and CRS or DRS at higher levels), we need a local measure of returns to scale. One defines the *elasticity of scale*, $\eta(K, L)$, of F at the point (K, L), where $F(K, L) > 0$, as

$$\eta(K,L) = \frac{\lambda}{F(K,L)}\frac{dF(\lambda K, \lambda L)}{d\lambda} \approx \frac{\Delta F(\lambda K, \lambda L)/F(K,L)}{\Delta\lambda/\lambda}, \text{ evaluated at } \lambda = 1. \tag{2.10}$$

So the elasticity of scale at a point (K, L) indicates the (approximate) percentage increase in output when both inputs are increased by 1 percent. We say that

$$\text{if } \eta(K,L) \begin{cases} > 1, \text{ then there are locally } \textit{IRS,} \\ = 1, \text{ then there are locally } \textit{CRS,} \\ < 1, \text{ then there are locally } \textit{DRS.} \end{cases} \tag{2.11}$$

The production function *may* have the same elasticity of scale everywhere. This is the case if and only if the production function is homogeneous. If F is homogeneous of degree h, then $\eta(K, L) = h$ and h is called the *elasticity of scale parameter*.

Note that the elasticity of scale at a point (K, L) will always equal the sum of the partial output elasticities at that point:

$$\eta(K,L) = \frac{F_K(K,L)K}{F(K,L)} + \frac{F_L(K,L)L}{F(K,L)}. \tag{2.12}$$

This follows from the definition in (2.10) by taking into account that

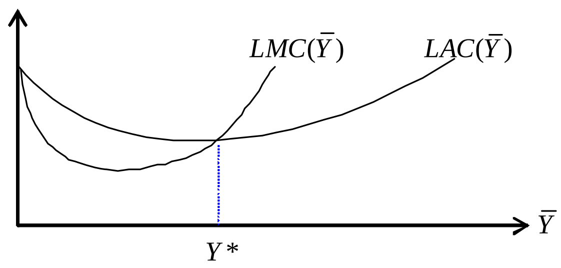

Figure 2.2: Locally CRS at optimal plant size.

$$\begin{aligned}\frac{dF(\lambda K, \lambda L)}{d\lambda} &= F_K(\lambda K, \lambda L)K + F_L(\lambda K, \lambda L)L \\ &= F_K(K, L)K + F_L(K, L)L, \text{ when evaluated at } \lambda = 1.\end{aligned}$$

Figure 2.2 illustrates a popular case from introductory economics, an average cost curve which from the perspective of the individual firm (or plant) is U-shaped: at low levels of output there are falling average costs (thus IRS), at higher levels rising average costs (thus DRS).[7] Given the input prices, w_K and w_L, and a specified output level, $\bar{Y}$, we know that the cost minimizing factor combination $(\bar{K}, \bar{L})$ is such that $F_L(\bar{K}, \bar{L})/F_K(\bar{K}, \bar{L}) = w_L/w_K$. It is shown in Appendix A that the elasticity of scale at $(\bar{K}, \bar{L})$ will satisfy:

$$\eta(\bar{K}, \bar{L}) = \frac{LAC(\bar{Y})}{LMC(\bar{Y})}, \tag{2.13}$$

where $LAC(\bar{Y})$ is average costs (the minimum unit cost associated with producing $\bar{Y}$) and $LMC(\bar{Y})$ is marginal costs at the output level $\bar{Y}$. The L in LAC and LMC stands for "long-run", indicating that both capital and labor are considered variable production factors within the period considered. At the optimal plant size, Y^*, there is equality between LAC and LMC, implying a unit elasticity of scale, that is, locally we have CRS. That the long-run average costs are here portrayed as rising for $\bar{Y} > Y^*$, is not essential for the argument but may reflect either that coordination difficulties are inevitable or that some additional production factor, say the building site of the plant, is tacitly held fixed.

Anyway, we have here a more subtle replication argument for CRS w.r.t. K and L at the aggregate level. Even though technologies may differ across plants, the surviving plants in a competitive market will have the same average costs at the optimal plant size. In the medium and long run, changes in

[7] By a "firm" is generally meant the company as a whole. A company may have several "manufacturing plants" placed at different locations.

aggregate output will take place primarily by entry and exit of optimal-size plants. Then, with a large number of relatively small plants, each producing at approximately constant unit costs for small output variations, we can without substantial error assume constant returns to scale at the aggregate level. So the argument goes. Notice, however, that even in this form the replication argument is not entirely convincing since the question of indivisibility remains. The optimal plant size may be large relative to the market − and is in fact so in many industries. Besides, in this case also the perfect competition premise breaks down.

2.1.3 Properties of the production function under CRS

The empirical evidence concerning returns to scale is mixed. Notwithstanding the theoretical and empirical ambiguities, the assumption of CRS w.r.t. capital and labor has a prominent role in macroeconomics. In many contexts it is regarded as an acceptable approximation and a convenient simple background for studying the question at hand.

Expedient inferences of the CRS assumption include:

(i) marginal costs are constant and equal to average costs (so the right-hand side of (2.13) equals unity);

(ii) if production factors are paid according to their marginal productivities, factor payments exactly exhaust total output so that pure profits are neither positive nor negative (so the right-hand side of (2.12) equals unity);

(iii) a production function known to exhibit CRS and satisfy property (a) from the definition of a neoclassical production function above, will automatically satisfy also property (b) and consequently *be* neoclassical;

(iv) a neoclassical two-factor production function with CRS has always $F_{KL} > 0$, i.e., it exhibits "direct complementarity" between K and L;

(v) a two-factor production function known to have CRS and to be twice continuously differentiable with positive marginal productivity of each factor everywhere in such a way that all isoquants are strictly convex to the origin, *must* have *diminishing* marginal productivities everywhere.[8]

[8]Proofs of these claims can be found in intermediate microeconomics textbooks and in the Appendix to Chapter 2 of my Lecture Notes in Macroeconomics.

A principal implication of the CRS assumption is that it allows a reduction of dimensionality. Considering a neoclassical production function, $Y = F(K, L)$ with $L > 0$, we can under CRS write $F(K, L) = LF(K/L, 1) \equiv Lf(k)$, where $k \equiv K/L$ is called the *capital-labor ratio* (sometimes the *capital intensity*) and $f(k)$ is the *production function in intensive form* (sometimes named the per capita production function). Thus output per unit of labor depends only on the capital intensity:

$$y \equiv \frac{Y}{L} = f(k).$$

When the original production function F is neoclassical, under CRS the expression for the marginal productivity of capital simplifies:

$$F_K(K, L) = \frac{\partial Y}{\partial K} = \frac{\partial \left[Lf(k)\right]}{\partial K} = Lf'(k)\frac{\partial k}{\partial K} = f'(k). \qquad (2.14)$$

And the marginal productivity of labor can be written

$$\begin{aligned} F_L(K, L) &= \frac{\partial Y}{\partial L} = \frac{\partial \left[Lf(k)\right]}{\partial L} = f(k) + Lf'(k)\frac{\partial k}{\partial L} \\ &= f(k) + Lf'(k)K(-L^{-2}) = f(k) - f'(k)k. \qquad (2.15) \end{aligned}$$

A neoclassical CRS production function in intensive form always has a positive first derivative and a negative second derivative, i.e., $f' > 0$ and $f'' < 0$. The property $f' > 0$ follows from (2.14) and (2.2). And the property $f'' < 0$ follows from (2.3) combined with

$$F_{KK}(K, L) = \frac{\partial f'(k)}{\partial K} = f''(k)\frac{\partial k}{\partial K} = f''(k)\frac{1}{L}.$$

For a neoclassical production function with CRS, we also have

$$f(k) - f'(k)k > 0 \text{ for all } k > 0, \qquad (2.16)$$

in view of $f(0) \geq 0$ and $f'' < 0$. Moreover,

$$\lim_{k \to 0} \left[f(k) - f'(k)k\right] = f(0). \qquad (2.17)$$

Indeed, from the mean value theorem[9] we know there exists a number $a \in (0, 1)$ such that for any given $k > 0$ we have $f(k) - f(0) = f'(ak)k$. From this follows $f(k) - f'(ak)k = f(0) < f(k) - f'(k)k$, since $f'(ak) > f'(k)$ by $f'' < 0$.

[9]This theorem says that if f is continuous in $[\alpha, \beta]$ and differentiable in (α, β), then there exists at least one point γ in (α, β) such that $f'(\gamma) = (f(\beta) - f(\alpha))/(\beta - \alpha)$.

In view of $f(0) \geq 0$, this establishes (2.16). And from $f(k) > f(k) - f'(k)k > f(0)$ and continuity of f follows (2.17).

Under CRS the Inada conditions for MPK can be written

$$\lim_{k \to 0} f'(k) = \infty, \qquad \lim_{k \to \infty} f'(k) = 0. \tag{2.18}$$

In this case standard parlance is just to say that "f satisfies the Inada conditions".

An input which must be positive for positive output to arise is called an *essential input*; an input which is not essential is called an *inessential input.* The second part of (2.18), representing the upper Inada condition for MPK under CRS, has the implication that *labor* is an essential input; but capital need not be, as the production function $f(k) = a + bk/(1+k)$, $a > 0, b > 0$, illustrates. Similarly, under CRS the upper Inada condition for MPL implies that *capital* is an essential input. These claims are proved in Appendix C. Combining these results, when *both* the upper Inada conditions hold and CRS obtain, then both capital and labor are essential inputs.[10]

Figure 2.3 is drawn to provide an intuitive understanding of a neoclassical CRS production function and at the same time illustrate that the lower Inada conditions are more questionable than the upper Inada conditions. The left panel of Figure 2.3 shows output per unit of labor for a *CRS neoclassical production function* satisfying the Inada conditions for MPK. The $f(k)$ in the diagram could for instance represent the Cobb-Douglas function in Example 1 with $\beta = 1-\alpha$, i.e., $f(k) = Ak^{\alpha}$. The right panel of Figure 2.3 shows a non-neoclassical case where only two alternative *Leontief techniques* are available, technique 1: $y = \min(A_1k, B_1)$, and technique 2: $y = \min(A_2k, B_2)$. In the exposed case it is assumed that $B_2 > B_1$ and $A_2 < A_1$ (if $A_2 \geq A_1$ at the same time as $B_2 > B_1$, technique 1 would not be efficient, because the same output could be obtained with less input of at least one of the factors by shifting to technique 2). If the available K and L are such that $k < B_1/A_1$ or $k > B_2/A_2$, some of either L or K, respectively, is idle. If, however, the available K and L are such that $B_1/A_1 < k < B_2/A_2$, it is efficient to *combine* the two techniques and use the fraction μ of K and L in technique 1 and the remainder in technique 2, where $\mu = (B_2/A_2 - k)/(B_2/A_2 - B_1/A_1)$. In this way we get the "labor productivity curve" OPQR (the envelope of the two techniques) in Figure 2.3. Note that for $k \to 0$, MPK stays equal to $A_1 < \infty$, whereas for all $k > B_2/A_2$, $MPK = 0$. A similar feature remains true, when we consider *many,* say n, alternative efficient Leontief techniques available. Assuming these techniques cover a considerable range w.r.t. the B/A ratios,

[10] Given a Cobb-Douglas production function, both production factors are essential whether we have DRS, CRS, or IRS.

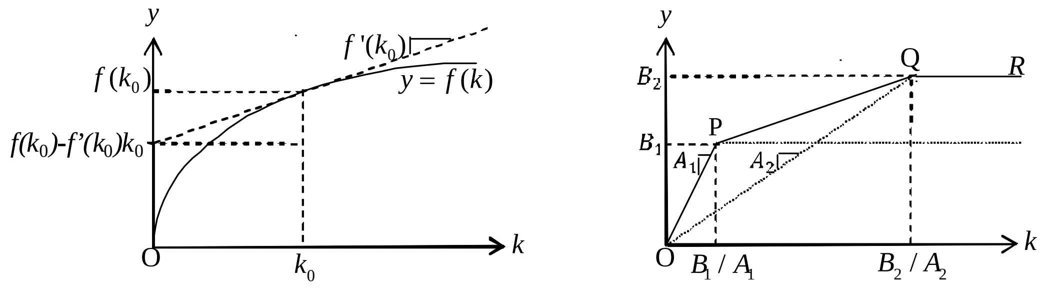

Figure 2.3: Two labor productivity curves based on CRS technologies. Left: neoclassical technology with Inada conditions for MPK satisfied; the graphical representation of MPK and MPL at $k = k_0$.as $f'(k_0)$ and $f(k_0)- f'(k_0)k_0$ are indicated. Right: a combination of two efficient Leontief techniques.

we get a labor productivity curve looking more like that of a neoclassical CRS production function. On the one hand, this gives some intuition of what lies behind the assumption of a neoclassical CRS production function. On the other hand, it remains true that for all $k > B_n/A_n$, $MPK = 0$,[11] whereas for $k \to 0$, MPK stays equal to $A_1 < \infty$, thus questioning the lower Inada condition.

The implausibility of the lower Inada conditions is also underlined if we look at their implication in combination with the more reasonable upper Inada conditions. Indeed, the four Inada conditions taken *together* imply, under CRS, that output has no upper bound when either input goes to infinity for fixed amount of the other input (see Appendix C).

2.2 Technological change

When considering the movement over time of the economy, we shall often take into account the existence of *technological change*. When technological change occurs, the production function becomes time-dependent. Over time the production factors tend to become more productive: more output for given inputs. To put it differently: the isoquants move inward. When this is the case, we say that the technological change displays *technological progress*.

[11] Here we assume the techniques are numbered according to ranking with respect to the size of B.

Concepts of neutral technological change

A first step in taking technological change into account is to replace (2.1) by (2.4). Empirical studies typically specialize (2.4) by assuming that technological change take a form known as *factor-augmenting* technological change:

$$Y_t = F(a_t K_t, b_t L_t), \tag{2.19}$$

where F is a (time-independent) neoclassical production function, Y_t, K_t, and L_t are output, capital, and labor input, respectively, at time t, while a_t and b_t are time-dependent efficiencies of capital and labor, respectively, reflecting technological change. In macroeconomics an even more specific form is often assumed, namely the form of *Harrod-neutral technological change.*[12] This amounts to assuming that a_t in (2.19) is a constant (which we can then normalize to one). So only b_t, which we will then denote T_t, is changing over time, and we have

$$Y_t = F(K_t, T_t L_t). \tag{2.20}$$

The efficiency of labor, T_t, is then said to indicate the *technology level.* Although one can imagine natural disasters implying a fall in T_t, generally T_t tends to rise over time and then we say that (2.20) represents *Harrod-neutral technological progress.* An alternative name for this is *labor-augmenting* technological progress (technological change acts *as if* the labor input were augmented).

If the function F in (2.20) is homogeneous of degree one (so that the technology exhibits CRS w.r.t. capital and labor), we may write

$$\tilde{y}_t \equiv \frac{Y_t}{T_t L_t} = F(\frac{K_t}{T_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0.$$

where $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$ (habitually called the "effective" capital intensity or, if there is no risk of confusion, just the capital intensity). In rough accordance with a general trend in aggregate productivity data for industrialized countries we often assume that T grows at a constant rate, g, so that in discrete time $T_t = T_0(1 + g)^t$ and in continuous time $T_t = T_0 e^{gt}$, where $g > 0$. The popularity in macroeconomics of the hypothesis of labor-augmenting technological progress derives from its consistency with Kaldor's "stylized facts", cf. Chapter 4.

There exists two alternative concepts of neutral technological progress. *Hicks-neutral* technological progress is said to occur if technological development is such that the production function can be written in the form

$$Y_t = T_t F(K_t, L_t), \tag{2.21}$$

[12]The name refers to the English economist Roy F. Harrod, 1900−1978.

where, again, F is a (time-independent) neoclassical production function, while T_t is the growing technology level.[13] The assumption of Hicks-neutrality has been used more in microeconomics and partial equilibrium analysis than in macroeconomics. If F has CRS, we can write (2.21) as $Y_t = F(T_tK_t, T_tL_t)$. Comparing with (2.19), we see that in this case Hicks-neutrality is equivalent with $a_t = b_t$ in (2.19), whereby technological change is said to be *equally factor-augmenting*.

Finally, in a kind of symmetric analogy with (2.20), *Solow-neutral* technological progress[14] is often in textbooks presented by a formula like:

$$Y_t = F(T_tK_t, L_t). \tag{2.22}$$

Another name for the same is *capital-augmenting* technological progress (because here technological change acts as if the capital input were augmented). Solow's original concept[15] of neutral technological change is not well portrayed this way, however, since it is related to the notion of *embodied* technological change and capital of *different vintages,* see below.

It is easily shown (Exercise 2.5) that the Cobb-Douglas production function (2.8) satisfies all three neutrality criteria at the same time, if it satisfies one of them (which it does if technological change does not affect α and β). It can also be shown that within the class of neoclassical CRS production functions the Cobb-Douglas function is the only one with this property (see Exercise 4.? in Chapter 4).

Note that the neutrality concepts do not say anything about the *source* of technological progress, only about the quantitative form in which it materializes. For instance, the occurrence of Harrod-neutrality should not be interpreted as indicating that the technological change emanates specifically from the labor input in some sense. Harrod-neutrality only means that technological innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* in the form (2.20). Similarly, if indeed an improvement in the quality of the labor input occurs, this "labor-specific" improvement may be manifested in a higher A_t, B_t, or both.

Before proceeding, we briefly comment on how the capital stock, K_t, is typically measured. While data on gross investment, I_t, is available in national income and product accounts, data on K_t usually is not. One ap-

[13] The name refers to the English economist and Nobel Prize laureate John R. Hicks, 1904−1989.

[14] The name refers to the American economist and Nobel Prize laureate Robert Solow (1924−).

[15] Solow (1960).

proach to the measurement of K_t is the *perpetual inventory method* which builds upon the accounting relationship

$$K_t = I_{t-1} + (1-\delta)K_{t-1}. \tag{2.23}$$

Assuming a constant capital depreciation rate δ, backward substitution gives

$$K_t = I_{t-1}+(1-\delta)\left[I_{t-2} + (1-\delta)K_{t-2}\right] = \ldots = \sum_{i=1}^{T}(1-\delta)^{i-1}I_{t-i}+(1-\delta)^T K_{t-T}. \tag{2.24}$$

Based on a long time series for I and an estimate of δ, one can insert these observed values in the formula and calculate K_t, starting from a rough conjecture about the initial value K_{t-T}. The result will not be very sensitive to this conjecture since for large T the last term in (2.24) becomes very small.

Embodied vs. disembodied technological progress

There exists an additional taxonomy of technological change. We say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The new technology is incorporated in the design of newly produced equipment, but this equipment will not participate in subsequent technological progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Thus investment becomes an important driving force in productivity increases.

We way formalize embodied technological progress by writing capital accumulation in the following way:

$$K_{t+1} - K_t = q_t I_t - \delta K_t, \tag{2.25}$$

where I_t is gross investment in period t, i.e., $I_t = Y_t - C_t$, and q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising q so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in *efficiency units*. In aggregate models C and I are produced with the same technology, the aggregate production function. From this together with (2.25) follows that q capital goods can be produced at the same minimum cost as one consumption good. Hence, the equilibrium price, p, of capital goods in terms of the consumption good must equal the inverse of q, i.e., $p = 1/q$. The output-capital ratio in value terms is $Y/(pK) = QY/K$.

Note that even if technological change does not directly appear in the production function, that is, even if for instance (2.20) is replaced by $Y_t = F(K_t, L_t)$, the economy may experience a rising standard of living when q is growing over time.

In contrast, *disembodied technological change* occurs when new technical and organizational knowledge increases the combined productivity of the production factors independently of when they were constructed or educated. If the K_t appearing in (2.20), (2.21), and (2.22) above refers to the total, historically accumulated capital stock as calculated by (2.24), then the evolution of T in these expressions can be seen as representing disembodied technological change. All vintages of the capital equipment benefit from a rise in the technology level T_t. No new investment is needed to benefit.

Based on data for the U.S. 1950-1990, and taking quality improvements into account, Greenwood et al. (1997) estimate that embodied technological progress explains about 60% of the growth in output per man hour. So, empirically, *embodied* technological progress seems to play a dominant role. As this tends not to be fully incorporated in national income accounting at fixed prices, there is a need to adjust the investment levels in (2.24) to better take estimated quality improvements into account. Otherwise the resulting K will not indicate the capital stock measured in efficiency units.

2.3 The concepts of a representative firm and an aggregate production function

Many macroeconomic models make use of the simplifying notion of a *representative firm.* By this is meant a fictional firm whose production "represents" aggregate production (value added) in a sector or in society as a whole.

Suppose there are n firms in the sector considered or in society as a whole. Let F^i be the production function for firm i so that $Y_i = F^i(K_i, L_i)$, where Y_i, K_i, and L_i are output, capital input, and labor input, respectively, $i = 1, 2, \ldots, n$. Further, let $Y = \Sigma_{i=1}^n Y_i$, $K = \Sigma_{i=1}^n K_i$, and $L = \Sigma_{i=1}^n L_i$. Ignoring technological change, suppose these aggregate variables in a given society turn out to be related through some production function, $F^*(\cdot)$, in the following way:

$$Y = F^*(K, L).$$

Then $F^*(K, L)$ is called the *aggregate production function* or the production function of the *representative* firm. It is *as if* aggregate production is the result of the behavior of such a single firm.

A simple example where the aggregate production function is well-defined is the following. Suppose that all firms have the *same* production function, i.e., $F^i(\cdot) = F(\cdot)$, so that $Y_i = F(K_i, L_i)$, $i = 1, 2, \ldots, n$. If in addition F has CRS, we then have

$$Y_i = F(K_i, L_i) = L_i F(k_i, 1) \equiv L_i f(k_i),$$

where $k_i \equiv K_i/L_i$. Hence, facing given factor prices, cost minimizing firms will choose the same capital intensity $k_i = k$, for all i. From $K_i = kL_i$ then follows $\sum_i K_i = k \sum_i L_i$ so that $k = K/L$. Thence,

$$Y \equiv \sum Y_i = \sum L_i f(k_i) = f(k) \sum L_i = f(k)L = F(k,1)L = F(K,L).$$

In this (trivial) case the aggregate production function is well-defined and turns out to be exactly the same as the identical CRS production functions of the individual firms.

Allowing for the existence of *different* production functions at firm level, we may define the aggregate production function as

$$\begin{aligned} F(K,L) &= \max_{(K_1,L_1,\ldots,K_n,L_n)\geq 0} F^1(K_1,L_1) + \cdots + F^n(K_n,L_n) \\ \text{s.t. } \sum_i K_i &\leq K, \quad \sum_i L_i \leq L. \end{aligned}$$

Allowing also the existence of different output goods, different capital goods, and different types of labor makes the issue more intricate, of course. Yet, if firms are price taking profit maximizers and there are nonincreasing returns to scale, we at least know that the aggregate outcome is *as if,* for given prices, the firms jointly maximize aggregate profit on the basis of their combined production technology. The problem is, however, that the conditions needed for this to imply existence of an aggregate production function which is *well-behaved* (in the sense of inheriting simple qualitative properties from its constituent parts) are restrictive.

Nevertheless macroeconomics often treats aggregate output as a single homogeneous good and capital and labor as being two single and homogeneous inputs. There was in the 1960s a heated debate about the problems involved in this, with particular emphasis on the aggregation of different kinds of equipment into one variable, the capital stock "K". The debate is known as the "Cambridge controversy" because the dispute was between a group of economists from Cambridge University, UK, and a group from Massachusetts Institute of Technology (MIT), which is located in Cambridge, USA. The former group questioned the theoretical robustness of several of the neoclassical tenets, including the proposition that rising aggregate capital intensity tends to be associated with a falling rate of interest. Starting at the disaggregate level, an association of this sort is not a logical necessity because, with different production functions across the industries, the relative prices of produced inputs tend to change, when the interest rate changes. While acknowledging the possibility of "paradoxical" relationships, the latter group maintained that in a macroeconomic context they are likely to cause devastating problems only under exceptional circumstances. In the end this is a matter of empirical assessment.[16]

To avoid complexity and because, for many important issues in growth theory, there is today no well-tried alternative, we shall in this course most of the time use aggregate constructs like "Y", "K", and "L" as simplifying devices, hopefully acceptable in a first approximation. There are cases, however, where some disaggregation is pertinent. When for example the role of imperfect competition is in focus, we shall be ready to disaggregate the production side of the economy into several product lines, each producing its own differentiated product. We shall also touch upon a type of growth models where a key ingredient is the phenomenon of "creative destruction" meaning that an incumbent technological leader is competed out by an entrant with a qualitatively new technology.

Like the representative firm, the *representative household* and the *aggregate consumption function* are simplifying notions that should be applied only when they do not get in the way of the issue to be studied. The importance of budget constraints may make it even more difficult to aggregate over households than over firms. Yet, *if* (and that is a big if) all households have the *same constant* propensity to consume out of income, aggregation is straightforward and the representative household is a meaningful concept. On the other hand, if we aim at understanding, say, the *interaction* between lending and borrowing households, perhaps via financial intermediaries, the representative household is not a useful starting point. Similarly, if the theme is conflicts of interests between firm owners and employees, the existence of *different* types of households should be taken into account.

[16] In his review of the Cambridge controversy Mas-Colell (1989) concluded that: "What the 'paradoxical' comparative statics [of disaggregate capital theory] has taught us is simply that modelling the world as having a single capital good is not *a priori* justified. So be it."

2.4 Long-run vs. short-run production functions*

Is the substitutability between capital and labor the same "ex ante" and "ex post"? By ex ante is meant "when plant and machinery are to be decided upon" and by ex post is meant "after the equipment is designed and constructed". In the standard neoclassical competitive setup there is a presumption that also after the construction and installation of the equipment in the firm, the ratio of the factor inputs can be fully adjusted to a change in the relative factor price. In practice, however, when some machinery has been constructed and installed, its functioning will often require a more or less fixed number of machine operators. What can be varied is just the *degree of utilization* of the machinery. That is, after construction and installation of the machinery, the choice opportunities are no longer described by the neoclassical production function but by a Leontief production function,

$$Y = \min(Au\bar{K}, BL), \qquad A > 0, B > 0, \tag{2.26}$$

where $\bar{K}$ is the size of the installed machinery (a fixed factor in the short run) measured in efficiency units, u is its utilization rate ($0 \leq u \leq 1$), and A and B are given technical coefficients measuring efficiency.

So in the short run the choice variables are u and L. In fact, essentially only u is a choice variable since efficient production trivially requires $L = Au\bar{K}/B$. Under "full capacity utilization" we have $u = 1$ (each machine is used 24 hours per day seven days per week). "Capacity" is given as $A\bar{K}$ per week. Producing efficiently at capacity requires $L = A\bar{K}/B$ and the marginal product by increasing labor input is here nil. But if demand, Y^d, is *less* than capacity, satisfying this demand efficiently requires $u = Y^d/(A\bar{K}) < 1$ and $L = Y^d/B$. As long as $u < 1$, the marginal productivity of labor is a *constant*, B.

The various efficient input proportions that are possible *ex ante* may be approximately described by a neoclassical CRS production function. Let this function on intensive form be denoted $y = f(k)$. When investment is decided upon and undertaken, there is thus a choice between alternative efficient pairs of the technical coefficients A and B in (2.26). These pairs satisfy

$$f(k) = Ak = B. \tag{2.27}$$

So, for an increasing sequence of k's, $k_1, k_2, \ldots, k_i, \ldots$, the corresponding pairs are $(A_i, B_i) = (f(k_i)/k_i, f(k_i))$, $i = 1, 2, \ldots$[17] We say that ex ante,

[17] The points P and Q in the right-hand panel of Fig. 2.3 can be interpreted as con-

depending on the relative factor prices as they are "now" and are expected to evolve in the future, a suitable technique, (A_i, B_i), is chosen from an opportunity set described by the given neoclassical production function. But ex post, i.e., when the equipment corresponding to this technique is installed, the production opportunities are described by a Leontief production function with $(A, B) = (A_i, B_i)$.

In the picturesque language of Phelps (1963), technology is in this case *putty-clay*. Ex ante the technology involves capital which is "putty" in the sense of being in a malleable state which can be transformed into a range of various machinery requiring capital-labor ratios of different magnitude. But once the machinery is constructed, it enters a "hardened" state and becomes "clay". Then factor substitution is no longer possible; the capital-labor ratio at full capacity utilization is fixed at the level $k = B_i/A_i$, as in (2.26). Following the terminology of Johansen (1972), we say that a putty-clay technology involves a "long-run production function" which is neoclassical and a "short-run production function" which is Leontief.

In contrast, the standard neoclassical setup assumes the same range of substitutability between capital and labor ex ante and ex post. Then the technology is called *putty-putty*. This term may also be used if ex post there is at least *some* substitutability although less than ex ante. At the opposite pole of putty-putty we may consider a technology which is *clay-clay*. Here neither ex ante nor ex post is factor substitution possible. Table 2.1 gives an overview of the alternative cases.

Table 2.1. Technologies classified according to factor substitutability ex ante and ex post

	Ex post substitution	
Ex ante substitution	possible	impossible
possible	putty-putty	putty-clay
impossible		clay-clay

The putty-clay case is generally considered the realistic case. As time proceeds, technological progress occurs. To take this into account, we may replace (2.27) and (2.26) by $f(k_t, t) = A_t k_t = B_t$ and $Y_t = \min(A_t u_t \bar{K}_t, B_t L_t)$, respectively. If a new pair of Leontief coefficients, (A_{t_2}, B_{t_2}), efficiency-dominates its predecessor (by satisfying $A_{t_2} \geq A_{t_1}$ and $B_{t_2} \geq B_{t_1}$ with at

structed this way from the neoclassical production function in the left-hand panel of the figure.

least one strict equality), it may pay the firm to invest in the new technology at the same time as some old machinery is scrapped. Real wages tend to rise along with technological progress and the scrapping occurs because the revenue from using the old machinery in production no longer covers the associated labor costs.

The clay property ex-post of many technologies is important for short-run analysis. It implies that there may be non-decreasing marginal productivity of labor up to a certain point. It also implies that in its investment decision the firm will have to take expected future technologies and future factor prices into account. For many issues in long-run analysis the clay property ex-post may be less important, since over time adjustment takes place through new investment.

2.5 Literature notes

As to the question of the empirical validity of the constant returns to scale assumption, Malinvaud (1998) offers an account of the econometric difficulties associated with estimating production functions. Studies by Basu (1996) and Basu and Fernald (1997) suggest returns to scale are about constant or decreasing. Studies by Hall (1990), Caballero and Lyons (1992), Harris and Lau (1992), Antweiler and Treffler (2002), and Harrison (2003) suggest there are quantitatively significant increasing returns, either internal or external. On this background it is not surprising that the case of IRS (at least at industry level), together with market forms different from perfect competition, has in recent years received more attention in macroeconomics and in the theory of economic growth.

Macroeconomists' use of the value-laden term "technological progress" in connection with technological change may seem suspect. But the term should be interpreted as merely a label for certain types of shifts of isoquants in an abstract universe. At a more concrete and disaggregate level analysts of course make use of more refined notions about technological change, recognizing for example not only benefits of new technologies, but also the risks, including risk of fundamental mistakes (think of the introduction and later abandonment of asbestos in the construction industry).

An informative history of technology is ...

Embodied technological progress, sometimes called investment-specific technological progress, is explored in, for instance, Solow (1960), Greenwood et al. (1997), and Groth and Wendner (2014). Hulten (2001) surveys the literature and issues related to measurement of the direct contribution of capital accumulation and technological change, respectively, to productivity growth.

Conditions ensuring that a representative household is admitted and the concept of Gorman preferences are discussed in Acemoglu (2009). Another useful source, also concerning the conditions for the representative firm to be a meaningful notion, is Mas-Colell et al. (1995). For general discussions of the limitations of representative agent approaches, see Kirman (1992) and Gallegati and Kirman (1999). Reviews of the "Cambridge Controversy" are contained in Mas-Colell (1989) and Felipe and Fisher (2003). The last-mentioned authors find the conditions required for the well-behavedness of these constructs so stringent that it is difficult to believe that actual economies are in any sense close to satisfy them. For a less distrustful view, see for instance Ferguson (1969), Johansen (1972), Malinvaud (1998), Jorgenson et al. (2005), and Jones (2005).

It is often assumed that capital depreciation can be described as geometric (in continuous time exponential) evaporation of the capital stock. This formula is popular in macroeconomics, more so because of its simplicity than its realism. An introduction to more general approaches to depreciation is contained in, e.g., Nickell (1978).

2.6 References

(incomplete)

Chapter 3

Continuous time analysis

Because dynamic analysis is generally easier in continuous time, growth models are often stated in continuous time. This chapter gives an account of the conceptual aspects of continuous time analysis. Appendix A considers simple growth arithmetic in continuous time. And Appendix B provides solution formulas for linear first-order differential equations.

3.1 The transition from discrete time to continuous time

We start from a discrete time framework. The run of time is divided into successive periods of equal length, taken as the time-unit. Let us here index the periods by $i = 0, 1, 2, ...$ Thus financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given,}$$

where s_i is (net) saving in period i.

3.1.1 Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t + \Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t + \Delta t)$

approach zero, i.e., let $\Delta t \to 0$. When $a(\cdot)$ is a differentiable function, we have

$$\lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t+\Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *derivative of* $a(\cdot)$ at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{3.1}$$

where $s(t)$ is the saving flow at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit choose one year. Going back to discrete time we have that if wealth grows at a constant rate $g > 0$ per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, \dots . \tag{3.2}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period,

$$a_i = a_0(1 + \frac{g}{n})^i. \tag{3.3}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (3.3) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0 \left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$ and thus $m \to \infty$. Then, in the limit there is continuous compounding and it can be shown that

$$a(t) = a_0 e^{gt}, \tag{3.4}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1 + 1/m)^m \simeq 2.7182818285...$.

The formula (3.4) is the continuous-time analogue to the discrete time formula (3.2) with annual compounding. A geometric growth factor is replaced by an exponential growth factor.

We can also view the formulas (3.2) and (3.4) as the solutions to a difference equation and a differential equation, respectively. Thus, (3.2) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (3.4) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{3.5}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (3.5) is called the *basic accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.

3.1.2 Compound interest and discounting

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro on a drawing account in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{3.6}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *interest intensity*, respectively, or the nominal and real *force of interest*.

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (3.5) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time analysis, we use compound discounting. We simply reverse the accumulation formula and go from the compounded or terminal value to the present value $a(t_0)$. Similarly, given a consumption plan, $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt}dt, \tag{3.7}$$

presupposing a constant interest rate. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (3.7) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau} dt.$$

In (3.7) $c(t)$ is discounted by $e^{-rt} \approx (1+r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (**??**) where it is c_{t-1} that is discounted by $(1+r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t-1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

3.2 The allowed range for parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{3.8}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (3.8) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

3.3 Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations*. What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units*. Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t)+I(t)$ or $K(t)+\dot{K}(t)$ are thus illegitimate. But one can write $K(t+\Delta t) \approx K(t)+(I(t)-\delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during*

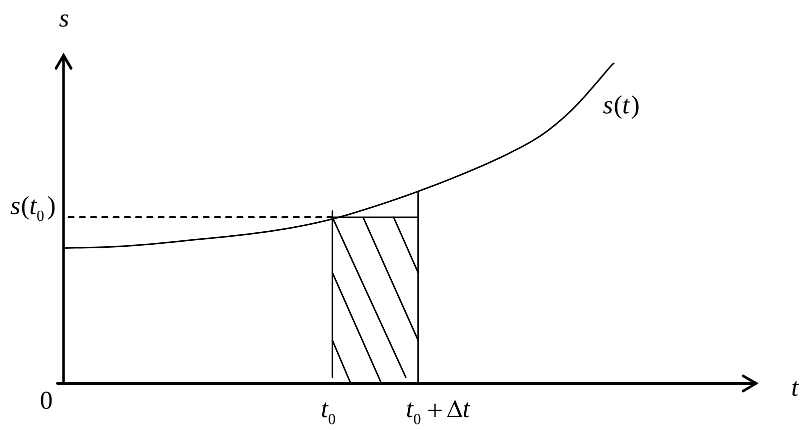

Figure 3.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0+\Delta t$ is $\approx$ the hatched area.

this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0+\Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0)\cdot\Delta t$, cf. the hatched area in Figure 3.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i)\cdot 4$ per year.

3.4 The choice between discrete and continuous time analysis

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. And it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. And for example centralized asset markets equilibrate almost instantaneously and respond immediately to new information. For such markets a formulation in continuous time seems preferable.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{3.9}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1 + \alpha)x_t$, where $1 + \alpha < 0$. The solution is $x_t = (1 + \alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1 + \alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

It should be added, however, that this potential problem *can* always be avoided within discrete time models by choosing a sufficiently *short* period length. Indeed, the solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (3.9) the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (3.9). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, seems preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.

3.5 Appendix

A. Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small.

B. Solution formulas for linear differential equations of first order

For a general differential equation of first order, $\dot{x}(t) = \varphi(x(t), t)$, with $x(t_0) = x_{t_0}$ and where φ is a continuous function, we have, at least for t in an interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$,

$$x(t) = x_{t_0} + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau. \qquad (*)$$

To get a confirmation, calculate $\dot{x}(t)$ from (*).

For the special case of a linear differential equation of first order, $\dot{x}(t) + a(t)x(t) = b(t)$, we can specify the solution. Three sub-cases of rising complexity are:

1. $\dot{x}(t)+ax(t)=b$, with $a\neq 0$ and initial condition $x(t_0)=x_{t_0}$. Solution:

$$x(t)=(x_{t_0}-x^*)e^{-a(t-t_0)}+x^*,\ \text{where } x^*=\frac{b}{a}.$$

If $a=0$, we get, directly from (*), the solution $x(t)=x_{t_0}+bt$.[1]

2. $\dot{x}(t)+ax(t)=b(t)$, with initial condition $x(t_0)=x_{t_0}$. Solution:

$$x(t)=x_{t_0}e^{-a(t-t_0)}+e^{-a(t-t_0)}\int_{t_0}^{t}b(s)e^{a(s-t_0)}ds.$$

Special case: $b(t)=ce^{ht}$, with $h\neq -a$ and initial condition $x(t_0)=x_{t_0}$. Solution:

$$x(t)=x_{t_0}e^{-a(t-t_0)}+e^{-a(t-t_0)}c\int_{t_0}^{t}e^{(a+h)(s-t_0)}ds=(x_{t_0}-\frac{c}{a+h})e^{-a(t-t_0)}+\frac{c}{a+h}e^{h(t-t_0)}.$$

3. $\dot{x}(t)+a(t)x(t)=b(t)$, with initial condition $x(t_0)=x_{t_0}$. Solution:

$$x(t)=x_{t_0}e^{-\int_{t_0}^{t}a(\tau)d\tau}+e^{-\int_{t_0}^{t}a(\tau)d\tau}\int_{t_0}^{t}b(s)e^{\int_{t_0}^{s}a(\tau)d\tau}ds.$$

Special case: $b(t)=0$. Solution:

$$x(t)=x_{t_0}e^{-\int_{t_0}^{t}a(\tau)d\tau}.$$

Even more special case: $b(t)=0$ and $a(t)=a$, a constant. Solution:

$$x(t)=x_{t_0}e^{-a(t-t_0)}.$$

Remark 1 For $t_0=0$, most of the formulas will look simpler.

Remark 2 To check whether a suggested solution *is* a solution, calculate the time derivative of the suggested solution and add an arbitrary constant. By appropriate adjustment of the constant, the final result should be a replication of the original differential equation together with its initial condition.

[1] Some non-linear differential equations can be transformed into this simple case. For simplicity let $t_0=0$. Consider the equation $\dot{y}(t)=\alpha y(t)^{\beta}$, $y_0>0$ given, $\alpha\neq 0,\beta\neq 1$ (a Bernoulli equation). To find the solution for $y(t)$, let $x(t)\equiv y(t)^{1-\beta}$. Then, $\dot{x}(t)$ $=(1-\beta)y(t)^{-\beta}\dot{y}(t)=(1-\beta)y(t)^{-\beta}\alpha y(t)^{\beta}=(1-\beta)\alpha$. The solution for this is $x(t)$ $=x_0+(1-\beta)\alpha t$, where $x_0=y_0^{1-\beta}$. Thereby the solution for $y(t)$ is $y(t)=x(t)^{1/(1-\beta)}$ $=\left(y_0^{1-\beta}+(1-\beta)\alpha t\right)^{1/(1-\beta)}$, which is defined for $t>-y_0^{1-\beta}/((1-\beta)\alpha$.

Chapter 4

Balanced growth theorems

In this chapter we shall discuss three fundamental propositions about balanced growth. In view of the generality of the propositions, they have a broad field of application.

The chapter covers the stuff in Acemoglu's §2.7.3. Our propositions 1 and 2 are slight extensions of part 1 and 2, respectively, of what Acemoglu calls Uzawa's Theorem I (Acemoglu, 2009, p. 60). Proposition 3 essentially corresponds to what Acemoglu calls Uzawa's Theorem II (Acemoglu, 2009, p. 63).

4.1 Balanced growth and constancy of key ratios

First we shall define the terms "steady state" and "balanced growth" as they are usually defined in growth theory. With respect to "balanced growth" this implies a minor deviation from the way Acemoglu briefly defines it informally on his page 57. The main purpose of the present chapter is to lay bare the connections between these two concepts as well as their relation to the hypothesis of Harrod-neutral technical progress and Kaldor's stylized facts.

4.1.1 The concepts of steady state and balanced growth

A basic equation in many one-sector growth models for a closed economy in continuous time is

$$\dot{K} = I - \delta K = Y - C - \delta K \equiv S - \delta K, \tag{4.1}$$

where K is aggregate capital, I aggregate gross investment, Y aggregate output, C aggregate consumption, S aggregate gross saving ($\equiv Y - C$), and

$\delta \geq 0$ is a constant physical capital depreciation rate.

Usually, in the theoretical literature on dynamic models, a *steady state* is defined in the following way:

Definition 3 *A steady state of a dynamic model is a stationary solution to the fundamental differential equation(s) of the model.*

Or briefly: a steady state is a stationary point of a dynamic process.

Let us take the Solow growth model as an example. Here gross saving equals sY, where s is a constant, $0 < s < 1$. Aggregate output is given by a neoclassical production function, F, with CRS and Harrod-neutral technical progress: $Y = F(K, AL) = ALF(\tilde{k}, 1) \equiv TLf(\tilde{k})$, where L is the labor force, A is the level of technology, and $\tilde{k} \equiv K/(AL)$ is the (effective) capital intensity. Moreover, $f' > 0$ and $f'' < 0$. Solow assumes $L(t) = L(0)e^{nt}$ and $A(t) = A(0)e^{gt}$, where $n \geq 0$ and $g \geq 0$ are the constant growth rates of the labor force and technology, respectively. By log-differentiating $\tilde{k}$ w.r.t. t,[1] we end up with the *fundamental differential equation* ("law of motion") of the Solow model:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + g + n)\tilde{k}. \tag{4.2}$$

Thus, in the Solow model, a (non-trivial) steady state is a $\tilde{k}^* > 0$ such that, if $\tilde{k} = \tilde{k}^*$, then $\dot{\tilde{k}} = 0$.

The most common definition in the literature of balanced growth for an aggregate economy is the following:

Definition 4 *A balanced growth path is a path $(Y, K, C)_{t=0}^{\infty}$ along which the quantities Y, K, and C are positive and grow at constant rates (not necessarily positive and not necessarily the same).*

Acemoglu, however, defines (p. 57) balanced growth in the following way: "balanced growth refers to an allocation where output grows at a constant rate and capital-output ratio, the interest rate, and factor shares remain constant". My problem with this definition is that it mixes growth of quantities with distribution aspects (interest rate and factor income shares). And it is not made clear what is meant by the output-capital ratio if the relative price of capital goods is changing over time. So I stick to the standard definition above which is known to function well in many different contexts.

[1] Or by directly using the fraction rule, see Appendix A to Chapter 3.

4.1.2 A general result about balanced growth

We now leave the specific Solow model. The interesting fact is that, given the dynamic resource constraint (4.1), we have *always* that if there is balanced growth with positive gross saving, then the ratios Y/K and C/Y are constant (by "*always*" is meant: independently of how saving is determined and of how the labor force and technology change). And also the other way round: as long as gross saving is positive, constancy of the Y/K and C/Y ratios is enough to ensure balanced growth. So balanced growth and constancy of key ratios are essentially equivalent.

This is a very practical general observation. And since Acemoglu does not state any balanced growth theorem at this general level, we shall do it, in a precise way, here, together with a proof. Letting g_x denote the growth rate of the (positively valued) variable x, i.e., $g_x \equiv \dot{x}/x$, we claim:

Proposition 1 *(the balanced growth equivalence theorem). Let* $(Y, K, C)_{t=0}^{\infty}$ *be a path along which* Y, K*,* C*, and* $S \equiv Y - C$ *are positive for all* $t \geq 0$*. Then, given the accumulation equation (4.1), the following holds:*

(i) if there is balanced growth, then $g_Y = g_K = g_C$*, and the ratios* Y/K *and* C/Y *are constant;*

(ii) if Y/K *and* C/Y *are constant, then* Y, K*, and* C *grow at the same constant rate, i.e., not only is there balanced growth, but the growth rates of* Y, K*, and* C *are the same.*

Proof Consider a path $(Y, K, C)_{t=0}^{\infty}$ along which Y, K, C, and $S \equiv Y - C$ are positive for all $t \geq 0$. (i) Assume there is balanced growth. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (4.1), we have that $S/K = g_K + \delta$ is constant, implying

$$g_S = g_K. \tag{4.3}$$

Further, since $Y = C + S$,

$$\begin{aligned} g_Y &= \frac{\dot{Y}}{Y} = \frac{\dot{C}}{Y} + \frac{\dot{S}}{Y} = g_C\frac{C}{Y} + g_S\frac{S}{Y} = g_C\frac{C}{Y} + g_K\frac{S}{Y} \qquad \text{(by (4.3))} \\ &= g_C\frac{C}{Y} + g_K\frac{Y-C}{Y} = \frac{C}{Y}(g_C - g_K) + g_K. \end{aligned} \tag{4.4}$$

Now, let us provisionally assume that $g_K \neq g_C$. Then (4.4) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{4.5}$$

a constant, so that $g_Y = g_C$. But this result implies, by (4.5), that $C/Y = 1$, i.e., $C = Y$. In view of (4.1), however, this outcome contradicts the given condition that $S > 0$. Hence, our provisional assumption is wrong, and we have $g_K = g_C$. By (4.4), this implies $g_Y = g_K = g_C$, but now without the condition $C/Y = 1$ being implied. It follows that Y/K and C/K are constant. Then, also $C/Y = (Y/K)/(C/K)$ is constant.

(ii) Suppose Y/K and C/Y are constant. Then $g_Y = g_K = g_C$, so that C/K is a constant. We now show that this implies that g_K is constant. Indeed, from (4.1), $S/Y = 1 - C/Y$, so that also S/Y is constant. It follows that $g_S = g_Y = g_K$, so that S/K is constant. By (4.1),

$$\frac{S}{K} = \frac{\dot{K} + \delta K}{K} = g_K + \delta,$$

so that g_K is constant. This, together with constancy of Y/K and C/Y, implies that also g_Y and g_C are constant. □

Remark. It is part (i) of the proposition which requires the assumption $S > 0$ for all $t \geq 0$. If $S = 0$, we would have $g_K = -\delta$ and $C \equiv Y - S = Y$, hence $g_C = g_Y$ for all $t \geq 0$. Then there would be balanced growth if the common value of g_C and g_Y had a constant growth rate. This growth rate, however, could easily differ from that of K. Suppose $Y = AK^\alpha L^{1-\alpha}$, $g_A = \gamma$ and $g_L = n$ (γ and n constants). Then we would have $g_C = g_Y = \gamma - \alpha\delta + (1-\alpha)n$, which could easily be strictly positive and thereby different from $g_K = -\delta \leq 0$ so that (i) no longer holds. □

The nice feature is that this proposition holds for *any* model for which the simple dynamic resource constraint (4.1) is valid. No assumptions about for example CRS and other technology aspects or about market form are involved. Further, the proposition suggests that if one accepts Kaldor's stylized facts as a description of the past century's growth experience, and if one wants a model consistent with them, one should construct the model such that it can generate balanced growth. For a model to be capable of generating balanced growth, however, technological progress must be of the Harrod-neutral type (i.e., be labor-augmenting), at least in a neighborhood of the balanced growth path. For a fairly general context (but of course not as general as that of Proposition 1), this was shown already by Uzawa (1961). The next section presents a modernized version of Uzawa's contribution.

4.2 The crucial role of Harrod-neutrality

Let the aggregate production function be

$$Y(t) = \tilde{F}(K(t), L(t); t). \tag{4.6}$$

The only technology assumption needed is that $\tilde{F}$ has CRS w.r.t. the first two arguments ($\tilde{F}$ need not be neoclassical for example). As a representation of technical progress, we assume $\partial\tilde{F}/\partial t > 0$ for all $t \geq 0$ (i.e., as time proceeds, unchanged inputs result in more and more output). We also assume that the labor force evolves according to

$$L(t) = L(0)e^{nt}, \tag{4.7}$$

where n is a constant. Further, non-consumed output is invested and so (4.1) is the dynamic resource constraint of the economy.

Proposition 2 *(Uzawa's balanced growth theorem) Let* $(Y(t), K(t), C(t))_{t=0}^{\infty}$*, where* $0 < C(t) < Y(t)$ *for all* $t \geq 0$*, be a path satisfying the capital accumulation equation (4.1), given the CRS-production function (5.2) and the labor force path in (4.7). Then:*

(i) a necessary condition for this path to be a balanced growth path is that along the path it holds that

$$Y(t) = \tilde{F}(K(t), L(t); t) = \tilde{F}(K(t), A(t)L(t); 0), \tag{4.8}$$

where $A(t) = e^{gt}$ *with* $g \equiv g_Y - n$*;*

(ii) for any $g > 0$ *such that there is a* $q > g + n + \delta$ *with the property that* $\tilde{F}(1, k^{-1}; 0) = q$ *for some* $k > 0$ *(i.e., at any* t*, hence also* $t = 0$*, the production function* $\tilde{F}$ *in (5.2) allows an output-capital ratio equal to* q*), a sufficient condition for the existence of a balanced growth path with output-capital ratio* q*, is that the technology can be written as in (4.8) with* $A(t) = e^{gt}$*.*

Proof (i)[2] Suppose the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path. By definition, g_K and g_Y are then constant, so that $K(t) = K(0)e^{g_K t}$ and $Y(t) = Y(0)e^{g_Y t}$. We then have

$$Y(t)e^{-g_Y t} = Y(0) = \tilde{F}(K(0), L(0); 0) = \tilde{F}(K(t)e^{-g_K t}, L(t)e^{-nt}; 0), \tag{4.9}$$

[2]This part draws upon Schlicht (2006), who generalized a proof in Wan (1971, p. 59) for the special case of a constant saving rate.

where we have used (5.2) with $t = 0$. In view of the precondition that $S(t) \equiv Y(t) - C(t) > 0$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (4.9) then implies

$$Y(t) = \tilde{F}(K(t)e^{g_Y t}e^{-g_K t}, L(t)e^{g_Y t}e^{-nt}; 0) = \tilde{F}(K(t), e^{(g_Y - n)t}L(t); 0).$$

We see that (4.8) holds for $A(t) = e^{gt}$ with $g \equiv g_Y - n$.

(ii) Suppose (4.8) holds with $A(t) = e^{gt}$. Let $g > 0$ be given such that there is a $q > g + n + \delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$. Then our first claim is that with $K(0) = kL(0)$, $s \equiv (g + n + \delta)/q$, and $S(t) = sY(t)$, (4.1), (4.7), and (4.8) imply $Y(t)/K(t) = q$ for all $t \geq 0$. Indeed, by construction

$$\frac{Y(0)}{K(0)} = \frac{\tilde{F}(K(0), L(0); 0)}{K(0)} = \tilde{F}(1, k^{-1}; 0) = q = \frac{\delta + g + n}{s}. \tag{4.10}$$

It follows that $sY(0)/K(0) - \delta = g + n$. So, by (4.1), we have $\dot{K}(0)/K(0) = sY(0)/K(0) - \delta = g + n$, implying that K initially grows at the same rate as effective labor input, $A(t)L(t)$. Then, in view of $\tilde{F}$ being homogeneous of degree one w.r.t. its first two arguments, also Y grows initially at this rate. As an implication, the ratio Y/K does not change, but remains equal to the right-hand side of (4.10) for all $t \geq 0$. Consequently, K and Y continue to grow at the same constant rate, $g + n$. As $C = (1 - s)Y$, C grows forever also at this constant rate. Hence, the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path, as was to be proved. □

The form (4.8) indicates that along a balanced growth path, technical progress must be purely "labor augmenting", that is, Harrod-neutral. It is in this case convenient to define a new CRS function, F, by $F(K(t), A(t)L(t)) \equiv \tilde{F}(K(t), A(t)L(t); 0)$. Then (i) of the proposition implies that at least along the balanced growth path, we can rewrite the production function this way:

$$Y(t) = \tilde{F}(K(t), L(t); t) = F(K(t), A(t)L(t)), \tag{4.11}$$

where $A(t) = e^{gt}$ with $g \equiv g_Y - n$.

It is important to recognize that the occurrence of Harrod-neutrality says nothing about what the *source* of technological progress is. Harrod-neutrality should not be interpreted as indicating that the technological progress emanates specifically from the labor input. Harrod-neutrality only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest* *itself* at the aggregate level in the form (4.11).[3]

What is the intuition behind the Uzawa result that for balanced growth to be possible, technical progress must have the purely labor-augmenting form? First, notice that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output. In contrast, in simple macro models labor is a non-produced production factor which (at least in this context) grows in an exogenous way. Second, because of CRS, the original formulation, (5.2), of the production function implies that

$$1 = \tilde{F}(\frac{K(t)}{Y(t)}, \frac{L(t)}{Y(t)}; t). \tag{4.12}$$

Now, since capital is accumulated non-consumed output, it inherits the trend in output such that $K(t)/Y(t)$ must be constant along a balanced growth path (this is what Proposition 1 is about). Labor does not inherit the trend in output; indeed, the ratio $L(t)/Y(t)$ is free to adjust as time proceeds. When there is technical progress ($\partial\tilde{F}/\partial t > 0$) along a balanced growth path, this progress must manifest itself in the form of a changing $L(t)/Y(t)$ in (4.12) as t proceeds, precisely because $K(t)/Y(t)$ *must* be constant along the path. In the "normal" case where $\partial\tilde{F}/\partial L > 0$, the needed change in $L(t)/Y(t)$ is a *fall* (i.e., a rise in $Y(t)/L(t)$). This is what (4.12) shows. Indeed, the fall in $L(t)/Y(t)$ must exactly offset the effect on $\tilde{F}$ of the rising t, when there is a fixed capital-output ratio.[4] It follows that along the balanced growth path, $Y(t)/L(t)$ is an increasing implicit function of t. If we denote this function $A(t)$, we end up with (4.11).

The generality of Uzawa's theorem is noteworthy. The theorem assumes CRS, but does not presuppose that the technology is neoclassical, not to speak of satisfying the Inada conditions.[5] And the theorem holds for exogenous as well as endogenous technological progress. It is also worth mentioning that the proof of the sufficiency part of the theorem is *constructive*. It provides a method to construct a hypothetical balanced growth path (BGP from now).[6]

A simple implication of the Uzawa theorem is the following. Interpreting the $A(t)$ in (4.8) as the "level of technology", we have:

[3]For a CRS Cobb-Douglas production function with technological progress, Harrod-neutrality is present whenever the output elasticity w.r.t capital (often denoted α) is constant over time.

[4]This way of presenting the intuition behind the Uzawa result draws upon Jones and Scrimgeour (2008).

[5]Many accounts of the Uzawa theorem, including Jones and Scrimgeour (2008), presume a neoclassical production function, but the theorem is much more general.

[6]Part (ii) of Proposition 2 is ignored in Acemoglu's book.

COROLLARY Along a BGP with positive gross saving and the technology level, $A(t)$, growing at the rate g, output grows at the rate $g+n$ while labor productivity, $y \equiv Y/L$, and consumption per unit of labor, $c \equiv C/L$, grow at the rate g.

Proof That $g_Y = g+n$ follows from (i) of Proposition 2. As to the growth rate of labor productivity we have

$$y_t = \frac{Y(0)e^{g_Y t}}{L(0)e^{nt}} = y(0)e^{(g_Y - n)t} = y(0)e^{gt}.$$

Finally, by Proposition 1, along a BGP with $S > 0$, c must grow at the same rate as y. □

We shall now consider the implication of Harrod-neutrality for the income shares of capital and labor when the technology is neoclassical and markets are perfectly competitive.

4.3 Harrod-neutrality and the functional income distribution

There is one facet of Kaldor's stylized facts we have so far not related to Harrod-neutral technical progress, namely the long-run "approximate" constancy of both the income share of labor, wL/Y, and the rate of return to capital. At least with neoclassical technology, profit maximizing firms, and perfect competition in the output and factor markets, these properties are inherent in the combination of constant returns to scale, balanced growth, and the assumption that the relative price of capital goods (relative to consumption goods) equals one. The latter condition holds in models where the capital good is nothing but non-consumed output, cf. (4.1).[7]

To see this, we start out from a neoclassical CRS production function with Harrod-neutral technological progress,

$$Y(t) = F(K(t), A(t)L(t)). \tag{4.13}$$

With $w(t)$ denoting the real wage at time t, in equilibrium under perfect competition the labor income share will be

$$\frac{w(t)L(t)}{Y(t)} = \frac{\frac{\partial Y(t)}{\partial L(t)}L(t)}{Y(t)} = \frac{F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)}. \tag{4.14}$$

[7] The reader may think of the "corn economy" example in Acemoglu, p. 28.

In this simple model, without natural resources, capital (gross) income equals non-labor income, $Y(t) - w(t)L(t)$. Hence, if $r(t)$ denotes the (net) rate of return to capital at time t, then

$$r(t) = \frac{Y(t) - w(t)L(t) - \delta K(t)}{K(t)}. \tag{4.15}$$

Denoting the capital (gross) income share by $\alpha(t)$, we can write this $\alpha(t)$ (in equilibrium) in three ways:

$$\begin{aligned}
\alpha(t) &\equiv \frac{Y(t) - w(t)L(t)}{Y(t)} = \frac{(r(t)+\delta)K(t)}{Y(t)}, \\
\alpha(t) &= \frac{F(K(t), A(t)L(t)) - F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)} = \frac{F_1(K(t), A(t)L(t))K(t)}{Y(t)}, \\
\alpha(t) &= \frac{\frac{\partial Y(t)}{\partial K(t)} K(t)}{Y(t)},
\end{aligned} \tag{4.16}$$

where the first row comes from (4.15), the second from (4.13) and (4.14), the third from the second together with Euler's theorem.[8] Comparing the first and the last row, we see that in equilibrium

$$\frac{\partial Y(t)}{\partial K(t)} = r(t) + \delta.$$

In this condition we recognize one of the first-order conditions in the representative firm's profit maximization problem under perfect competition, since $r(t) + \delta$ can be seen as the firm's required gross rate of return.[9]

In the absence of uncertainty, the equilibrium real interest rate in the bond market must equal the rate of return on capital, $r(t)$. And $r(t) + \delta$ can then be seen as the firm's cost of disposal over capital per unit of capital per time unit, consisting of interest cost plus capital depreciation.

Proposition 3 *(factor income shares and rate of return under balanced growth) Let the path* $(K(t), Y(t), C(t))_{t=0}^{\infty}$ *be a BGP in a competitive economy with the production function (4.13) and with positive saving. Then, along the BGP, the* $\alpha(t)$ *in (4.16) is a constant,* $\alpha \in (0,1)$*. The labor income share will be* $1-\alpha$ *and the (net) rate of return on capital will be* $r = \alpha q - \delta$*, where* q *is the constant output-capital ratio along the BGP.*

[8] From Euler's theorem, $F_1 K + F_2 AL = F(K, AL)$, when F is homogeneous of degree one.

[9] With natural resources, say land, entering the set of production factors, the formula, (4.15), for the rate of return to capital should be modified by subtracting rents from the numerator.

Proof By CRS we have $Y(t) = F(K(t), A(t)L(t)) = A(t)L(t)F(\tilde{k}(t), 1) \equiv A(t)L(t)f(\tilde{k}(t))$. In view of part (i) of Proposition 2, by balanced growth, $Y(t)/K(t)$ is some constant, q. Since $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t)$ and $f'' < 0$, this implies $\tilde{k}(t)$ constant, say equal to $\tilde{k}^*$. But $\partial Y(t)/\partial K(t) = f'(\tilde{k}(t))$, which then equals the constant $f'(\tilde{k}^*)$ along the BGP. It then follows from (4.16) that $\alpha(t) = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, where $0 < \alpha$ follows from $f' > 0$ and $\alpha < 1$ from the fact that $q = Y/K = f(\tilde{k}^*)/\tilde{k}^* > f'(\tilde{k}^*)$, in view of $f'' < 0$ and $f(0) \geq 0$. Then, by the first equality in (4.16), $w(t)L(t)/Y(t) = 1 - \alpha(t) = 1 - \alpha$. Finally, by (4.15), the (net) rate of return on capital is $r = (1 - w(t)L(t)/Y(t))Y(t)/K(t) - \delta = \alpha q - \delta$. □

This proposition is of interest by displaying a link from balanced growth to constancy of factor income shares and the rate of return, that is, some of the "stylized facts" claimed by Kaldor. Note, however, that although the proposition implies constancy of the income shares and the rate of return, it does not *determine* them, except in terms of α and q. But both q and, generally, α are endogenous and depend on $\tilde{k}^*$,[10] which will generally be unknown as long as we have not specified a theory of saving. This takes us to theories of aggregate saving, for example the simple Ramsey model, cf. Chapter 8 in Acemoglu's book.

4.4 What if technological change is embodied?

In our presentation of technological progress above we have implicitly assumed that all technological change is *disembodied*. And the way the propositions 1, 2, and 3, are formulated assume this.

As noted in Chapter 2, *disembodied technological change* occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again the aggregate dynamic resource constraint (4.1) and the production function (5.2):

$$\begin{aligned} \dot{K}(t) &= Y(t) - C(t) - \delta K(t), && (*) \\ Y(t) &= \tilde{F}(K(t), L(t); t), \qquad \partial \tilde{F}/\partial t > 0. && (**) \end{aligned}$$

Here $Y(t) - C(t)$ is aggregate gross investment, $I(t)$. For a given level of $I(t)$, the resulting amount of new capital goods per time unit ($\dot{K}(t) + \delta K(t)$), measured in efficiency units, is independent of *when* this investment occurs. It is

[10] As to α, there is of course a trivial exception, namely the case where the production function is Cobb-Douglas and α therefore is a given parameter.

thereby not affected by technological progress. Similarly, the interpretation of $\partial\tilde{F}/\partial t > 0$ in (**) is that the higher technology level obtained as time proceeds results in higher productivity of *all* capital and labor. Thus also firms that have only old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.[11]

In contrast, we say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.

As also noted in Chapter 2, we may represent embodied technological progress (also called investment-specific technological change) by writing capital accumulation in the following way,

$$\dot{K}(t) = q(t)I(t) - \delta K(t), \tag{4.17}$$

where $I(t)$ is gross investment at time t and $q(t)$ measures the "quality" (productivity) of newly produced investment goods. The increasing level of technology implies increasing $q(t)$ so that a given level of investment gives rise to a greater and greater additions to the capital stock, K, measured in efficiency units. As in our aggregate framework, q capital goods can be produced at the same minimum cost as one consumption good, we have $p \cdot q = 1$, where p is the equilibrium price of capital goods in terms of consumption goods. So embodied technological progress is likely to result in a steady decline in the relative price of capital equipment, a prediction confirmed by the data (see, e.g., Greenwood et al., 1997).

This raises the question how the propositions 1, 2, and 3 fare in the case of embodied technological progress. The answer is that a generalized version of Proposition 1 goes through. Essentially, we only need to replace (4.1) by (4.17) and interpret K in Proposition 1 as the *value* of the capital stock, i.e., we have to replace K by $\tilde{K} = pK$.

But the concept of Harrod-neutrality no longer fits the situation without further elaboration. Hence to obtain analogies to Proposition 2 and Proposition 3 is a more complicated matter. Suffice it to say that with em-

[11] In the standard versions of the Solow model and the Ramsey model it is assumed that all technological progress has this form - for no other reason than that this is by far the simplest case to analyze.

bodied technological progress, the class of production functions that are consistent with balanced growth is smaller than with disembodied technological progress.

4.5 Concluding remarks

In the Solow model as well as in many other models with disembodied technological progress, a steady state and a balanced growth path imply each other. Indeed, they are two sides of the same process. There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technical change). Therefore, it is recommendable always to maintain a terminological distinction between the two concepts, steady state and balanced growth.[12]

Note that the definition of balanced growth refers to *aggregate* variables. At the same time as there is balanced growth at the aggregate level, *structural change* may occur. That is, a changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009, Chapter 20).

In view of the key importance of Harrod-neutrality, a natural question is: has growth theory uncovered any *endogenous* tendency for technical progress to converge to Harrod-neutrality? Fortunately, in his Chapter 15 Acemoglu outlines a theory about a mechanism entailing such a tendency, the theory of "directed technical change". Jones (2005) suggests an alternative mechanism.

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[12]Here we depart from Acemoglu, p. 65, where he says that he will use the two terms "interchangingly". We also depart from Barro and Sala-i-Martin (2004, pp. 33-34) who *define* a steady state as synonymous with a balanced growth path as the latter was defined above.

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Chapter 5

The concepts of TFP and growth accounting: Some warnings

5.1 Introduction

This chapter discusses the concepts of Total Factor Productivity, TFP, and TFP growth, and ends up with three warnings regarding uncritical use of them.

First, however, we should provide a precise definition of the TFP *level* which is in fact a tricky concept. Unfortunately, Acemoglu (p. 78) does not make a clear distinction between TFP *level* and TFP *growth*. Moreover, Acemoglu's point of departure (p. 77) assumes *a priori* that the way the production function is time-dependent can be represented by a one-dimensional index, $A(t)$. The TFP concept and the applicability of growth accounting are, however, not limited to this case.

For convenience, in this chapter we treat time as continuous (although the timing of the variables is indicated merely by a subscript).[1]

5.2 TFP level and TFP growth

Let Y_t denote aggregate output (value added in fixed prices) at time t in a sector or the economy as a whole. Suppose Y_t is determined by the function

$$Y_t = \tilde{F}(K_t, H_t; t), \tag{5.1}$$

[1] I thank Niklas Brønager for useful discussions related to this chapter.

where K_t is an aggregate input of physical capital and H_t an index of quality-adjusted labor input.[2] The "quality-adjustment" of the input of labor (man-hours per year) aims at taking educational level and work experience into account. In fact, both output and the two inputs are aggregates of heterogeneous elements. The involved conceptual and measurement difficulties are huge and there are different opinions in the growth accounting literature about how to best deal with them. Here we ignore these problems. The third argument in (5.1) is time, t, indicating that the production function $\tilde{F}(\cdot\ ,\cdot\ ; t)$ is time-dependent. Thus "shifts in the production function", due to changes in efficiency and technology ("technical change" for short), can be taken into account. We treat time as continuous and assume that $\tilde{F}$ is a neoclassical production function. When the partial derivative of $\tilde{F}$ w.r.t. the third argument is positive, i.e., $\partial\tilde{F}/\partial t > 0$, technical change amounts to technical *progress*. We consider the economy from a purely supply-side perspective.[3]

We shall here concentrate on the fundamentals of TFP and TFP growth. These can in principle be described without taking the heterogeneity and changing quality of the labor input into account. Hence we shall from now on ignore this aspect and simplifying *assume* that labor is homogeneous and labor quality is constant. So (5.1) is reduced to the simpler case,

$$Y_t = \tilde{F}(K_t, L_t; t), \tag{5.2}$$

where L_t is the number of man-hours per year. As to measurement of K_t, some adaptation of the *perpetual inventory method*[4] is typically used, with some correction for under-estimated quality improvements of investment goods in national income accounting. The output measure is (or at least should be) corrected correspondingly, also for under-estimated quality improvements of consumption goods.

[2] Natural resources (land, oil wells, coal in the ground, etc.) constitute a third primary production factor. The role of this factor is in growth accounting often subsumed under K.

[3] Sometimes in growth accounting the left-hand side variable, Y, in (5.2) is the gross product rather than value added. Then non-durable intermediate inputs should be taken into account as a third production factor and enter as an additional argument of $\tilde{F}$ in (5.2). Since non-market production is difficult to measure, the government sector is usually excluded from Y in (5.2). Total Factor Productivity is by some authors called *Multifactor Productivity* and abbreviated MFP.

[4] Cf. Chapter 2.

5.2.1 TFP growth

The notion of Total Factor Productivity at time t, TFP_t, is intended to indicate a *level* of productivity. Nevertheless there is a tendency in the literature to evade a direct definition of this level and instead go straight away to a decomposition of output *growth*. Let us start the same way here but not forget to come back to the issue about what can be meant by the level of TFP.

The growth rate of a variable Z at time t will be denoted $g_{Z,t}$. Taking logs and differentiating w.r.t. t in (5.2) we get

$$\begin{aligned} g_{Y,t} &\equiv \frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left[\tilde{F}_K(K_t, L_t; t)\dot{K}_t + \tilde{F}_L(K_t, L_t; t)\dot{L}_t + \tilde{F}_t(K_t, L_t; t) \cdot 1\right] \\ &= \frac{K_t\tilde{F}_K(K_t, L_t; t)}{Y_t} g_{K,t} + \frac{L_t\tilde{F}_L(K_t, L_t; t)}{Y_t} g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} \\ &\equiv \varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t}, \end{aligned} \tag{5.3}$$

where $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$ are shorthands for $\varepsilon_K(K_t, L_t; t) \equiv \frac{K_t\tilde{F}_K(K_t,L_t;t)}{\tilde{F}(K_t,L_t;t)}$ and $\varepsilon_L(K_t, L_t; t) \equiv \frac{L_t\tilde{F}_L(K_t,L_t;t)}{\tilde{F}(K_t,L_t;t)}$, respectively, that is, the partial output elasticities w.r.t. the two production factors, evaluated at the factor combination (K_t, L_t) at time t. Finally, $\tilde{F}_t(K_t, L_t; t) \equiv \partial\tilde{F}/\partial t$, that is, the partial derivative w.r.t. the third argument of the function $\tilde{F}$, evaluated at the point (K_t, L_t, t).

The equation (5.3) is the *basic growth-accounting relation*, showing how the output growth rate can be decomposed into the "contribution" from growth in each of the inputs and a residual. The TFP *growth rate* is defined as the residual

$$g_{\text{TFP},t} \equiv g_{Y,t} - (\varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} g_{L,t}) = \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t}, \tag{5.4}$$

So the TFP growth rate is what is left when from the output growth rate is subtracted the "contribution" from growth in the factor inputs weighted by the output elasticities w.r.t. these inputs. This is sometimes interpreted as reflecting that part of the output growth rate which is *explained* by technical progress. One should be careful, however, not to identify a descriptive accounting relationship with deeper causality. Without a complete model, at most one can say that the TFP growth rate measures that fraction of output growth that is *not directly attributable* to growth in the capital and labor inputs. So:

> The TFP growth rate can be interpreted as reflecting the "*direct* contribution" to current output growth from current technical change (in a broad sense including learning by doing and organizational improvement).

Let us consider how the actual estimation of $g_{\text{TFP},t}$ can be carried out. The output elasticities w.r.t. capital and labor, $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$, will, under perfect competition and absence of externalities, equal the income shares of capital and labor, respectively. Time series for these income shares and for Y, K, and L, hence also for $g_{Y,t}$, $g_{K,t}$, and $g_{L,t}$, can be obtained (directly or with some adaptation) from national income accounts. This allows straightforward measurement of the residual, $g_{\text{TFP},t}$.[5]

The decomposition in (5.4) was introduced already by Solow (1957). Since the TFP growth rate appears as a residual, it is sometimes called the *Solow residual*. As a residual it may reflect the contribution of many things, some wanted (current technical innovation in a broad sense including organizational improvement), others unwanted (such as varying capacity utilization, omitted inputs, measurement errors, and aggregation bias).

5.2.2 The TFP level

Now let us consider the *level* of TFP, that "something" for which we have calculated its growth rate without yet having defined what it really is. But knowing the growth rate of TFP for all t in a certain time interval, we in fact have a differential equation in the TFP level of the form $dx(t)/dt = g(t)x(t)$, namely:

$$d(\text{TFP}_t)/dt = g_{\text{TFP},t} \cdot \text{TFP}_t.$$

The solution of this simple linear differential equation is[6]

$$\text{TFP}_t = \text{TFP}_0 e^{\int_0^t g_{\text{TFP},\tau} d\tau}. \tag{5.5}$$

For a given initial value $\text{TFP}_0 > 0$ (which may be normalized to 1 if desired), the time path of TFP is determined by the right-hand side of (5.5). Consequently:

> The TFP level at time t can interpreted as reflecting the cumulative "*direct* contribution" to output since time 0 from cumulative technical change since time 0.

[5] Of course, data are in discrete time. So to make actual calculations we have to translate (5.4) into discrete time. The weights $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$ can then be estimated by two-years moving averages of the factor income shares as shown in Acemoglu (2009, p. 79).

[6] See Appendix B of Chapter 3 in these lecture notes or Appendix B to Acemoglu.

Why do we say "*direct* contribution"? The reason is that the cumulative technical change since time 0 may also have an *indirect* effect on output, namely via affecting the output elasticities w.r.t. capital and labor, $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$. Through this channel cumulative technical change affects the role of input growth for output growth. This possible indirect effect over time of technical change is not included in the TFP concept.

To clarify the matter we will compare the TFP calculation under Hicks-neutral technical change with that under other forms of technical change.

5.3 The case of Hicks-neutrality*

In the case of Hicks neutrality, by definition, technical change can be represented by the evolution of a one-dimensional variable, B_t, and the production function in (5.2) can be specified as

$$Y_t = \tilde{F}(K_t, L_t; t) = B_t F(K_t, L_t). \tag{5.6}$$

Here the TFP level is at any time, t, identical to the level of B_t if we normalize the initial values of both B and TFP to be the same, i.e., $\text{TFP}_0 = B_0 > 0$. Indeed, calculating the TFP growth rate, (5.4), on the basis of (5.6) gives

$$g_{\text{TFP},t} = \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} = \frac{\dot{B}_t F(K_t, L_t)}{B_t F(K_t, L_t)} = \frac{\dot{B}_t}{B_t} \equiv g_{B,t}, \tag{5.7}$$

where the second equality comes from the fact that K_t and L_t are kept fixed when the *partial* derivative of $\tilde{F}$ w.r.t. t is calculated. The formula (5.5) now gives

$$\text{TFP}_t = B_0 \cdot e^{\int_0^t g_{B,\tau} d\tau} = B_t.$$

The nice feature of Hicks neutrality is thus that we can write

$$\text{TFP}_t = \frac{\tilde{F}(K_t, L_t; t)}{\tilde{F}(K_t, L_t; 0)} = \frac{B_t F(K_t, L_t)}{B_0 F(K_t, L_t)} = B_t, \tag{5.8}$$

using the normalization $B_0 = 1$. That is:

> *Under Hicks neutrality, current TFP appears as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.*

So in the case of Hicks neutrality the economic meaning of the TFP level is straightforward. The reason is that under Hicks neutrality the output elasticities w.r.t. capital and labor, $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$, are *independent* of technical change.

5.4 Absence of Hicks-neutrality*

The above very intuitive interpretation of TFP is only valid under Hicks-neutral technical change. Neither under general technical change nor even under Harrod- or Solow-neutral technical change (unless the production function is Cobb-Douglas so that both Harrod and Solow neutrality imply Hicks-neutrality), will current TFP appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.

To see this, let us return to the general time-dependent production function in (5.2). Let X_t denote the ratio between the current output level at time t and the hypothetical output level, $\tilde{F}(K_t, L_t; 0)$, that would have obtained with the current inputs of capital and labor in case of no change in the technology since time 0, i.e.,

$$X_t \equiv \frac{\tilde{F}(K_t, L_t; t)}{\tilde{F}(K_t, L_t; 0)}. \tag{5.9}$$

So X_t can be seen as a factor of joint-productivity growth from time 0 to time t evaluated at the time-t input combination.

If this X_t should always indicate the level of TFP at time t, the growth rate of X_t should equal the growth rate of TFP. Generally, it does not, however. Indeed, defining $G(K_t, L_t) \equiv \tilde{F}(K_t, L_t; 0)$, by the rule for the time derivative of fractions[7], we have

$$\begin{aligned}
g_{X,t} &\equiv \frac{d\tilde{F}(K_t, L_t; t)/dt}{\tilde{F}(K_t, L_t; t)} - \frac{dG(K_t, L_t)/dt}{G(K_t, L_t)} \\
&= \frac{1}{Y_t}\left[\tilde{F}_K(K_t, L_t; t)\dot{K}_t + \tilde{F}_L(K_t, L_t; t)\dot{L}_t + \tilde{F}_t(K_t, L_t; t)\cdot 1\right] \\
&\quad - \frac{1}{G(K_t, L_t)}\left[G_K(K_t, L_t)\dot{K}_t + G_L(K_t, L_t)\dot{L}_t\right] \\
&= \varepsilon_K(K_t, L_t; t)g_{K,t} + \varepsilon_L(K_t, L_t; t)g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} \\
&\quad - (\varepsilon_K(K_t, L_t; 0)g_{K,t} + \varepsilon_L(K_t, L_t; 0)g_{L,t}) \qquad (5.10)\\
&= (\varepsilon_K(K_t, L_t; t) - \varepsilon_K(K_t, L_t; 0))\, g_{K,t} + (\varepsilon_L(K_t, L_t; t) - \varepsilon_L(K_t, L_t; 0))g_{L,t} + g_{\text{TFP},t} \\
&\neq g_{\text{TFP},t} \qquad \text{generally,}
\end{aligned}$$

where $g_{\text{TFP},t}$ is given in (5.4). Unless the partial output elasticities w.r.t. capital and labor, respectively, are unaffected by technical change, the conclusion is that TFP_t will differ from our X_t defined in (5.9). So:

[7] See Appendix A to Chapter 3 of these lecture notes.

In the absence of Hicks neutrality, current TFP does not generally appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.

A closer look at X_t vs. $\mathbf{TFP}_t$

As X_t in (5.9) is the time-t output arising from the time-t inputs relative to the fictional time-0 output from the same inputs, we consider X_t along with TFP as two alternative joint-productivity indices. From (5.10) we see that

$$g_{\mathrm{TFP},t} = g_{X,t} - (\varepsilon_K(K_t, L_t; t) - \varepsilon_K(K_t, L_t; 0))\, g_{K,t} - (\varepsilon_L(K_t, L_t; t) - \varepsilon_L(K_t, L_t; 0)) g_{L,t}.$$

So the growth rate of TFP equals the growth rate of the joint-productivity index X corrected for the cumulative impact of technical change since time 0 on the direct contribution to time-t output growth from time-t input growth. This impact comes about when the output elasticities w.r.t. capital and labor, respectively, are affected by technical change, that is, when $\varepsilon_K(K_t, L_t; t) \neq \varepsilon_K(K_t, L_t; 0)$ and/or $\varepsilon_L(K_t, L_t; t) \neq \varepsilon_L(K_t, L_t; 0)$.

Under Hicks-neutral technical change there will be no correction because the output elasticities are *independent* of technical change. In this case TFP coincides with the index X. In the absence of Hicks-neutrality the two indices differ. This is why we in Section 2.2 characterized the TFP level as the cumulative "*direct* contribution" to output since time 0 from cumulative technical change, thus excluding the possible indirect contribution coming about via the potential effect of technical change on the output elasticities w.r.t. capital and labor and thereby on the contribution to output from input growth.

Given that the joint-productivity index X is the more intuitive joint-productivity measure, why is TFP the more popular measure? There are at least two reasons for this. First, it can be shown that the TFP measure has more convenient balanced growth properties. Second, X is more difficult to measure. To see this we substitute (5.3) into (5.10) to get

$$g_{X,t} = g_{Y,t} - (\varepsilon_K(K_t, L_t; 0) g_{K,t} + \varepsilon_L(K_t, L_t; 0) g_{L,t}). \tag{5.11}$$

The relevant output elasticities, $\varepsilon_K(K_t, L_t; 0) \equiv \frac{K_t \tilde{F}_K(K_t, L_t; 0)}{\tilde{F}(K_t, L_t; 0)}$ and $\varepsilon_L(K_t, L_t; 0) \equiv \frac{L_t \tilde{F}_L(K_t, L_t; 0)}{\tilde{F}(K_t, L_t; 0)}$, are hypothetical constructs, referring to the technology as it was at time 0, but with the factor combination observed at time t, not at time 0. The nice thing about the Solow residual is that under the assumptions of perfect competition and absence of externalities, it allows measurement by using data on prices and quantities alone, that is, without knowledge of the production function. To evaluate g_X, however, we need estimates of the hypothetical output elasticities, $\varepsilon_K(K_t, L_t; 0)$ and $\varepsilon_L(K_t, L_t; 0)$. This requires knowledge about how the output elasticities depend on the factor combination and time, respectively, that is, knowledge about the production function.

Now to the warnings concerning application of the TFP measure.

5.5 Three warnings

Balanced growth at the aggregate level, hence Harrod neutrality, seems to characterize the growth experience of the UK and US over at least a century (Kongsamut et al., 2001; Attfield and Temple, 2010). At the same time the aggregate elasticity of factor substitution is generally estimated to be significantly less than one (see, e.g., Antras, 2004). This amounts to rejection of the Cobb-Douglas specification of the aggregate production function and so, at the aggregate level, Harrod neutrality rules out Hicks neutrality.

Warning 1 Since Hicks-neutrality is empirically doubtful at the aggregate level, TFP_t can often *not* be identified with the simple intuitive joint-productivity measure X_t, defined in (5.9) above.

Warning 2 When Harrod neutrality obtains, relative TFP growth rates across sectors or countries can be quite deceptive.

Suppose there are n countries and that country i has the aggregate production function

$$Y_{it} = F^{(i)}(K_{it}, A_t L_{it}) \qquad i = 1, 2, ..., n,$$

where $F^{(i)}$ is a neoclassical production function with CRS and A_t is the level of labor-augmenting technology which, for simplicity, we assume shared by all the countries (these are open and "close" to each other). So technical progress is Harrod-neutral. Let the growth rate of A be a constant $g > 0$. Many models imply that $\tilde{k}_i \equiv K_{it}/(A_t L_{it})$ tends to a constant, $\tilde{k}_i^*$, in the long run, which we assume is also the case here. Then, for $t \to \infty$, $k_{it} \equiv K_{it}/L_{it} \equiv \tilde{k}_{it} A_t$ where $\tilde{k}_{it} \to \tilde{k}_i^*$ and $y_{it} \equiv Y_{it}/L_{it} \equiv \tilde{y}_{it} A_t$ where $\tilde{y}_{it} \to \tilde{y}_i^* = f^{(i)}(\tilde{k}_i^*)$; here $f^{(i)}$ is the production function on intensive form. So in the long run g_{k_i} and g_{y_i} tend to $g_A = g$.

Formula (5.4) then gives the TFP growth rate of country i in the long run as

$$\begin{aligned} g_{\mathrm{TFP}_i} &\equiv g_{Y_i} - (\alpha_i^* g_{K_i} + (1-\alpha_i^*) g_{L_i}) = g_{Y_i} - g_{L_i} - \alpha_i^*(g_{K_i} - g_{L_i}) \\ &= g_{y_i} - \alpha_i^* g_{k_i} = (1-\alpha_i^*) g, \end{aligned} \tag{5.12}$$

where α_i^* is the output elasticity w.r.t. capital, $f^{(i)\prime}(\tilde{k}_i)\tilde{k}_i/f^{(i)}(\tilde{k}_i)$, evaluated at $\tilde{k}_i = \tilde{k}_i^*$. Under labor-augmenting technical progress, the TFP growth rate thus varies negatively with the output elasticity w.r.t. capital (the capital income share under perfect competition). Owing to differences in product and industry composition, the countries have different α_i^*'s. In view of (5.12), for two different countries, i and j, we get

$$\frac{TFP_i}{TFP_j} \rightarrow \begin{cases} \infty \text{ if } \alpha_i^* < \alpha_j^*, \\ 1 \text{ if } \alpha_i^* = \alpha_j^*, \\ 0 \text{ if } \alpha_i^* > \alpha_j^*, \end{cases} \tag{5.13}$$

for $t \rightarrow \infty$.[8] Thus, in spite of long-run growth in the essential variable, y, being the same across the countries, their TFP growth rates are very different. Countries with low α_i^*'s appear to be technologically very dynamic and countries with high α_i^*'s appear to be lagging behind. It is all due to the difference in α across countries; a higher α just means that a larger fraction of $g_{y_i} = g_{k_i} = g$ becomes "explained" by g_{k_i} in the growth accounting (5.12), leaving a smaller residual. And the level of α has nothing to do with technical progress.

We conclude that comparison of TFP levels across countries or time may misrepresent the intuitive meaning of productivity and technical progress when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral (even if technical progress were at the same time Hicks-neutral as is the case with a Cobb-Douglas specification). It may be more reasonable to just compare levels of Y/L across countries and time.

Warning 3 Growth accounting is - as the name says - just about accounting and measurement. So do not confuse growth *accounting* with *causality* in growth analysis. To talk about causality we need a theoretical model supported by the data. On the basis of such a model we can say that this or that set of exogenous factors through the propagation mechanisms of the model cause this or that phenomenon, including economic growth. In contrast, considering the growth accounting identity (5.3) in itself, none of the terms have

[8] If F is Cobb-Douglas with output elasticity w.r.t. capital equal to α_i, the result in (5.12) can be derived more directly by first defining $B_t = A_t^{1-\alpha_i}$, then writing the production function in the Hicks-neutral form (5.6), and finally use (5.7).

priority over the others w.r.t. a causal role. And there are important omitted variables. There are simple illustrations in Exercises III.1 and III.2.

In a complete model with exogenous technical progress, part of $g_{K,t}$ will be *induced* by this technical progress. If technical progress is endogenous through learning by investing, as in Arrow (1962), there is mutual causation between $g_{K,t}$ and technical progress. Yet another kind of model might explain both technical progress and capital accumulation through R&D, cf. the survey by Barro (1999).

5.6 References

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Chapter 6

Transitional dynamics. Barro-style growth regressions

In this chapter we discuss three issues, all of which are related to the transitional dynamics of a growth model:

- Do poor countries necessarily tend to approach their steady state from below?
- How fast (or rather how slow) are the transitional dynamics in a growth model?
- What exactly is the theoretical foundation for a Barro-style growth regression analysis?

The Solow growth model may serve as the analytical point of departure for the first two issues and to some extent also for the third.

6.1 Point of departure: the Solow model

As is well-known, the fundamental differential equation for the Solow model is

$$\dot{\tilde{k}}(t) = sf(\tilde{k}(t)) - (\delta + g + n)\tilde{k}(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \tag{6.1}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $f(\tilde{k}(t)) \equiv F(\tilde{k}(t), 1)$, $A(t) = A_0 e^{gt}$, and $L(t) = L_0 e^{nt}$ (standard notation). The production function F is neoclassical with CRS and the parameters satisfy $0 < s < 1$ and $\delta + g + n > 0$. The production function on intensive form, f, therefore satisfies $f(0) \geq 0$, $f' > 0$, $f'' < 0$, and

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) > \frac{\delta + g + n}{s} > \lim_{\tilde{k} \to \infty} f'(\tilde{k}). \tag{A1}$$

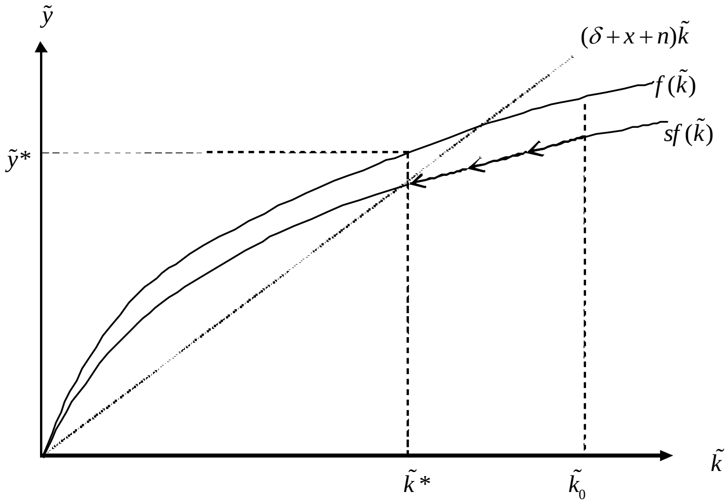

Figure 6.1: Note: x means g.

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, that is, a unique positive solution to the equation

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{6.2}$$

Furthermore, given an arbitrary $\tilde{k}_0 > 0$, we have for all $t \geq 0$,

$$\dot{\tilde{k}}(t) \gtreqless 0 \text{ for } \tilde{k}(t) \lesseqgtr \tilde{k}^*, \tag{6.3}$$

respectively. The steady state, $\tilde{k}^*$, is thus *globally asymptotically stable* in the sense that for all $\tilde{k}_0 > 0$, $\lim_{t\to\infty} \tilde{k}(t) = k^*$ and this convergence is *monotonic* (in the sense that $\tilde{k}(t) - \tilde{k}^*$ does not change sign during the adjustment process).

From now on the dating of $\tilde{k}$ is suppressed unless needed for clarity. Figure 6.1 illustrates the dynamics as seen from the perspective of (6.1) (in this and the two next figures, x should read g. Figure 6.2 illustrates the dynamics emerging when we rewrite (6.1) this way:

$$\dot{\tilde{k}} = s\left(f(\tilde{k}) - \frac{\delta + g + n}{s}\tilde{k}\right) \gtreqless 0 \text{ for } \tilde{k} \lesseqgtr \tilde{k}^*.$$

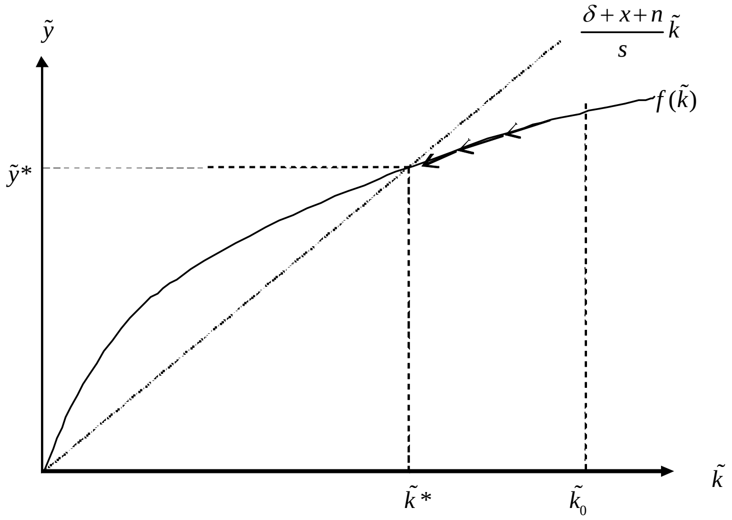

Figure 6.2: Note: x means g.

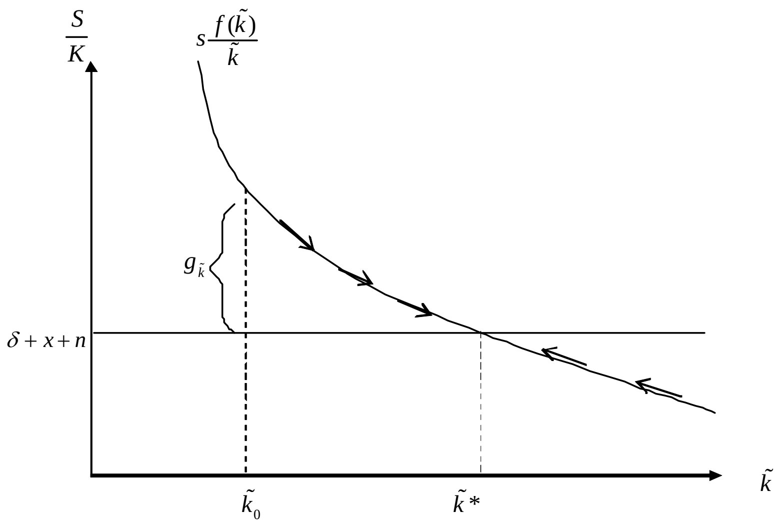

Figure 6.3: Note: x means g.

In Figure 6.3 yet another illustration is exhibited, based on rewriting (6.1) this way:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\frac{f(\tilde{k})}{\tilde{k}} - (\delta + g + n),$$

where $sf(\tilde{k})/\tilde{k}$ is gross saving per unit of capital, $S/K \equiv (Y - C)/K$.

An important variable in the analysis of the adjustment process towards steady state is the output elasticity w.r.t. capital:

$$\frac{K}{Y}\frac{\partial Y}{\partial K} = \frac{\tilde{k}}{f(\tilde{k})}f'(\tilde{k}) \equiv \varepsilon(\tilde{k}), \tag{6.4}$$

where $0 < \varepsilon(\tilde{k}) < 1$ for all $\tilde{k} > 0$.

6.2 Do poor countries tend to approach their steady state from below?

From some textbooks (for instance Barro and Sala-i-Martin, 2004) one gets the impression that poor countries tend to approach their steady state *from below*. But this is *not* what the Penn World Table data seems to indicate. And from a theoretical point of view the size of $\tilde{k}_0$ relative to $\tilde{k}^*$ is certainly ambiguous, whether the country is rich or poor. To see this, consider a poor country with initial effective capital intensity

$$\tilde{k}_0 \equiv \frac{K_0}{A_0 L_0}.$$

Here K_0/L_0 will typically be small for a poor country (the country has not yet accumulated much capital relative to its fast-growing population). The technology level, A_0, however, *also* tends to be small for a poor country. Hence, whether we should expect $\tilde{k}_0 < \tilde{k}^*$ or $\tilde{k}_0 > \tilde{k}^*$ is not obvious *apriori*. Or equivalently: whether we should expect that a poor country's GDP at an arbitrary point in time grows at a rate higher or lower than the country's steady-state growth rate, $g + n$, is not obvious *apriori*.

While Figure 6.3 illustrates the case where the inequality $\tilde{k}_0 < \tilde{k}^*$ holds, Figure 6.1 and 6.2 illustrate the opposite case. There *exists* some empirical evidence indicating that poor countries tend to approach their steady state *from above*. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

The prejudice that poor countries *apriori* should tend to approach their steady state from below seems to come from a confusion of conditional and unconditional β convergence. The Solow model predicts - and data supports - that within a group of countries with similar structural characteristics (approximately the same f, A_0, g, s, n, and δ), the initially poorer countries will grow faster than the richer countries. This is because the poorer countries (small $y(0) = f(\tilde{k}_0)A_0$) will be the countries with relatively small initial capital-labor ratio, k_0. As all the countries in the group have approximately the same A_0, the poorer countries thus have $\tilde{k}_0 \equiv k_0/A_0$ relatively small, i.e., $\tilde{k}_0 < \tilde{k}^*$. From $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k})A$ follows that the growth rate in output per worker of these poor countries tends to exceed g. Indeed, we have generally

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + g = \frac{f'(\tilde{k})\dot{\tilde{k}}}{f(\tilde{k})} + g \gtreqless g \text{ for } \dot{\tilde{k}} \gtreqless 0, \text{ i.e., for } \tilde{k} \lesseqgtr \tilde{k}^*.$$

So, *within* the group, the poor countries tend to approach the steady state, $\tilde{k}^*$, *from below*.

The countries in the world as a whole, however, differ a lot w.r.t. their structural characteristics, including their A_0. Unconditional β convergence is definitely rejected by the data. Then there is no reason to expect the poorer countries to have $\tilde{k}_0 < \tilde{k}^*$ rather than $\tilde{k}_0 > \tilde{k}^*$. Indeed, according to the mentioned study by Cho and Graham (1996), it turns out that the data for the relatively poor countries favors the latter inequality rather than the first.

6.3 Convergence speed and adjustment time

Our next issue is: How fast (or rather how slow) are the transitional dynamics in a growth model? To put it another way: according to a given growth model with convergence, how fast does the economy approach its steady state? The answer turns out to be: not very fast - to say the least. This is a rather general conclusion and is confirmed by the empirics: adjustment processes in a growth context are quite time consuming.

In Acemoglu's textbook we meet the concept of speed of convergence at p. 54 (under an alternative name, rate of adjustment) and p. 81 (in connection with Barro-style growth regressions). Here we shall go more into detail with the issue of speed of convergence.

Again the Solow model is our frame of reference. We search for a formula for the *speed of convergence* of $\tilde{k}(t)$ and $y(t)/y^*(t)$ in a closed economy described by the Solow model. So our analysis is concerned with *within-country* *convergence*: how fast do variables such as $\tilde{k}$ and y approach their steady state paths in a closed economy? The key adjustment mechanism is linked to diminishing returns to capital (falling marginal productivity of capital) in the process of capital accumulation. The problem of *cross-country convergence* (which is what "β convergence" and "σ convergence" are about) is in principle more complex because also such mechanisms as technological catching-up and cross-country factor movements are involved.

6.3.1 Convergence speed for $\tilde{k}(t)$

The ratio of $\dot{\tilde{k}}(t)$ to $(\tilde{k}(t) - \tilde{k}^*) \neq 0$ can be written

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t) - \tilde{k}^*} = \frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*}, \tag{6.5}$$

since $d\tilde{k}^*/dt = 0$. We define the *instantaneous speed of convergence* at time t as the (proportionate) rate of *decline* of the distance $\left|\tilde{k}(t) - \tilde{k}^*\right|$ at time t and we denote it $\text{SOC}_t(\tilde{k})$.[1] Thus,

$$\text{SOC}_t(\tilde{k}) \equiv -\frac{d\left(\left|\tilde{k}(t) - \tilde{k}^*\right|\right)/dt}{\left|\tilde{k}(t) - \tilde{k}^*\right|} = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*}, \tag{6.6}$$

where the equality sign is valid for monotonic convergence.

Generally, $\text{SOC}_t(\tilde{k})$ depends on both the absolute size of the difference $\tilde{k} - \tilde{k}^*$ at time t and its sign. But if the difference is already "small", $\text{SOC}_t(\tilde{k})$ will be "almost" constant for increasing t and we can find an approximate measure for it. Let the function $\varphi(\tilde{k})$ be defined by $\varphi(\tilde{k}) \equiv sf(\tilde{k}) - m\tilde{k}$, where $m \equiv \delta + g + n$. A first-order Taylor approximation of $\varphi(\tilde{k})$ around $\tilde{k} = \tilde{k}^*$ gives

$$\varphi(\tilde{k}) \approx \varphi(\tilde{k}^*) + \varphi'(\tilde{k}^*)(\tilde{k} - \tilde{k}^*) = 0 + (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*).$$

[1] Synonyms for speed of convergence are *rate of convergence, rate of adjustment* or *adjustment speed.*

For $\tilde{k}$ in a small neighborhood of the steady state, $\tilde{k}^*$, we thus have

$$\begin{aligned}
\dot{\tilde{k}} &= \varphi(\tilde{k}) \approx (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*) \\
&= (\frac{sf'(\tilde{k}^*)}{m} - 1)m(\tilde{k} - \tilde{k}^*) \\
&= (\frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (6.2))} \\
&\equiv (\varepsilon(\tilde{k}^*) - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (6.4))}.
\end{aligned}$$

Applying the definition (6.6) and the identity $m \equiv \delta + g + n$, we now get

$$\mathrm{SOC}_t(\tilde{k}) = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \approx (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*) > 0. \quad (6.7)$$

This result tells us how fast, approximately, the economy approaches its steady state if it starts "close" to it. If, for example, $\beta(\tilde{k}^*) = 0.02$ per year, then 2 percent of the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ vanishes per year. We also see that everything else equal, a higher output elasticity w.r.t. capital implies a lower speed of convergence.

In the limit, for $\left|\tilde{k} - \tilde{k}^*\right| \to 0$, the instantaneous speed of convergence coincides with what is called the *asymptotic speed of convergence*, defined as

$$\mathrm{SOC}^*(\tilde{k}) \equiv \lim_{\left|\tilde{k}-\tilde{k}^*\right| \to 0} \mathrm{SOC}_t(\tilde{k}) = \beta(\tilde{k}^*). \quad (6.8)$$

Multiplying through by $-(\tilde{k}(t) - \tilde{k}^*)$, the equation (6.7) takes the form of a homogeneous linear differential equation (with constant coefficient), $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. With $x(t) = \tilde{k}(t) - \tilde{k}^*$ and "=" replaced by "≈", we get in the present case

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t}. \quad (6.9)$$

This is the approximative time path for the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ and shows how the gap becomes smaller and smaller at the rate $\beta(\tilde{k}^*)$.

One of the reasons that the speed of convergence is important is that it indicates what weight should be placed on transitional dynamics of a growth model relative to the steady-state behavior. The speed of convergence matters for instance for the evaluation of growth-promoting policies. In growth models with diminishing marginal productivity of production factors, successful growth-promoting policies have transitory growth effects and permanent level effects. Slower convergence implies that the full benefits are slower to arrive.

6.3.2 Convergence speed for $\log \tilde{k}(t)$

We have found an approximate expression for the convergence speed of $\tilde{k}$. Since models in empirical analysis and applied theory are often based on log-linearization, we might ask what the speed of convergence of $\log \tilde{k}$ is. The answer is: approximately the same and asymptotically exactly the same as that of $\tilde{k}$ itself! Let us see why.

A first-order Taylor approximation of $\log \tilde{k}(t)$ around $\tilde{k} = \tilde{k}^*$ gives

$$\log \tilde{k}(t) \approx \log \tilde{k}^* + \frac{1}{\tilde{k}^*}(\tilde{k}(t) - \tilde{k}^*). \tag{6.10}$$

By definition

$$\begin{aligned} \mathrm{SOC}_t(\log \tilde{k}) &= -\frac{d(\log \tilde{k}(t) - \log \tilde{k}^*)/dt}{\log \tilde{k}(t) - \log \tilde{k}^*} = -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)(\log \tilde{k}(t) - \log \tilde{k}^*)} \\ &\approx -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)\frac{\tilde{k}(t)-\tilde{k}^*}{\tilde{k}^*}} = \frac{\tilde{k}^*}{\tilde{k}(t)}\mathrm{SOC}_t(\tilde{k}) \to \mathrm{SOC}^*(\tilde{k}) = \beta(\tilde{k}^*) \text{ for } \tilde{k}(t) \to \tilde{k}^* \end{aligned} \tag{6.11}$$

where in the second line we have used, first, the approximation (6.10), second, the definition in (6.7), and third, the definition in (6.8).

So, at least in a neighborhood of the steady state, the instantaneous rate of decline of the logarithmic distance of $\tilde{k}$ to the steady-state value of $\tilde{k}$ approximates the instantaneous rate of decline of the distance of $\tilde{k}$ itself to its steady-state value. The asymptotic speed of convergence of $\log \tilde{k}$ coincides with that of $\tilde{k}$ itself and is exactly $\beta(\tilde{k}^*)$.

In the Cobb-Douglas case (where $\varepsilon(\tilde{k}^*)$ is a constant, say α) it is possible to find an explicit solution to the Solow model, see Acemoglu p. 53 and Exercise II.2. It turns out that the instantaneous speed of convergence in a finite distance from the steady state is a constant and equals the asymptotic speed of convergence, $(1-\alpha)(\delta + g + n)$.

6.3.3 Convergence speed for $y(t)/y^*(t)$

The variable which we are interested in is usually not so much $\tilde{k}$ in itself, but rather labor productivity, $y(t) \equiv \tilde{y}(t)A(t)$. In the interesting case where $g > 0$, labor productivity does not converge towards a constant. We therefore focus on the ratio $y(t)/y^*(t)$, where $y^*(t)$ denotes the hypothetical value of labor productivity at time t, conditional on the economy being on its steady-state path, i.e.,

$$y^*(t) \equiv \tilde{y}^* A(t). \tag{6.12}$$

We have

$$\frac{y(t)}{y^*(t)} \equiv \frac{\tilde{y}(t)A(t)}{\tilde{y}^*A(t)} = \frac{\tilde{y}(t)}{\tilde{y}^*}. \tag{6.13}$$

As $\tilde{y}(t) \to \tilde{y}^*$ for $t \to \infty$, the ratio $y(t)/y^*(t)$ converges towards 1 for $t \to \infty$.

Taking logs on both sides of (6.13), we get

$$\begin{aligned}
\log \frac{y(t)}{y^*(t)} &= \log \frac{\tilde{y}(t)}{\tilde{y}^*} = \log \tilde{y}(t) - \log \tilde{y}^* \\
&\approx \log \tilde{y}^* + \frac{1}{\tilde{y}^*}(\tilde{y}(t) - y^*) - \log \tilde{y}^* \quad \text{(first-order Taylor approx. of } \log \tilde{y}\text{)} \\
&= \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}(t)) - f(\tilde{k}^*)) \\
&\approx \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}^*) + f'(\tilde{k}^*)(\tilde{k}(t) - \tilde{k}^*) - f(\tilde{k}^*)) \quad \text{(first-order approx. of } f(\tilde{k})) \\
&= \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \equiv \varepsilon(\tilde{k}^*) \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \\
&\approx \varepsilon(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \quad \text{(by (6.10))}.
\end{aligned} \tag{6.14}$$

Multiplying through by $-(\log \tilde{k}(t) - \log \tilde{k}^*)$ in (6.11) and carrying out the differentiation w.r.t. time, we find an approximate expression for the growth rate of $\tilde{k}$,

$$\begin{aligned}
\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)} &\equiv g_{\tilde{k}}(t) \approx -\frac{\tilde{k}^*}{\tilde{k}(t)} \text{SOC}_t(\tilde{k})(\log \tilde{k}(t) - \log \tilde{k}^*) \\
&\to -\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \quad \text{for } \tilde{k}(t) \to \tilde{k}^*,
\end{aligned} \tag{6.15}$$

where the convergence follows from the last part of (6.11). We now calculate the time derivative on both sides of (6.14) to get

$$\begin{aligned}
d(\log \frac{y(t)}{y^*(t)})/dt &= d(\log \frac{\tilde{y}(t)}{\tilde{y}^*})/dt = \frac{d\tilde{y}(t)/dt}{\tilde{y}(t)} \equiv g_{\tilde{y}}(t) \\
&\approx \varepsilon(\tilde{k}^*) g_{\tilde{k}}(t) \approx -\varepsilon(\tilde{k}^*)\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*).
\end{aligned} \tag{6.16}$$

from (6.15). Dividing through by $-\log(y(t)/y^*(t))$ in this expression, taking (6.14) into account, gives

$$-\frac{d(\log \frac{y(t)}{y^*(t)})/dt}{\log \frac{y(t)}{y^*(t)}} = -\frac{d(\log \frac{y(t)}{y^*(t)} - \log 1)/dt}{\log \frac{y(t)}{y^*(t)} - \log 1} \equiv \text{SOC}_t(\log \frac{y}{y^*}) \approx \beta(\tilde{k}^*), \tag{6.17}$$

in view of $\log 1 = 0$. So the logarithmic distance of y from its value on the steady-state path at time t has approximately the same rate of decline as the logarithmic distance of $\tilde{k}$ from $\tilde{k}$'s value on the steady-state path at time t. The asymptotic speed of convergence for $\log y(t)/y^*(t)$ is exactly the same as that for $\tilde{k}$, namely $\beta(\tilde{k}^*)$.

What about the speed of convergence of $y(t)/y^*(t)$ itself? Here the same principle as in (6.11) applies. The asymptotic speed of convergence for $\log(y(t)/y^*(t))$ is the same as that for $y(t)/y^*(t)$ (and vice versa), namely $\beta(\tilde{k}^*)$.

With one year as our time unit, standard parameter values are: $g = 0.02$, $n = 0.01$, $\delta = 0.05$, and $\varepsilon(\tilde{k}^*) = 1/3$. We then get $\beta(\tilde{k}^*) = (1-\varepsilon(\tilde{k}^*))(\delta+g+n)$ $= 0.053$ per year. In the empirical Chapter 11 of Barro and Sala-i-Martin (2004), it is argued that a lower value of $\beta(\tilde{k}^*)$, say 0.02 per year, fits the data better. This requires $\varepsilon(\tilde{k}^*) = 0.75$. Such a high value of $\varepsilon(\tilde{k}^*)$ ($\approx$ the income share of capital) may seem difficult to defend. But if we reinterpret K in the Solow model so as to include *human* capital (skills embodied in human beings and acquired through education and learning by doing), a value of $\varepsilon(\tilde{k}^*)$ at that level may not be far out.

6.3.4 Adjustment time

Let τ_ω be the time that it takes for the fraction $\omega \in (0,1)$ of the initial gap between $\tilde{k}$ and $\tilde{k}^*$ to be eliminated, i.e., τ_ω satisfies the equation

$$\frac{\left|\tilde{k}(\tau_\omega) - \tilde{k}^*\right|}{\left|\tilde{k}(0) - \tilde{k}^*\right|} = \frac{\tilde{k}(\tau_\omega) - \tilde{k}^*}{\tilde{k}(0) - \tilde{k}^*} = 1 - \omega, \tag{6.18}$$

where $1 - \omega$ is the fraction of the initial gap still remaining at time τ_ω. In (6.18) we have applied that $sign(\tilde{k}(t) - \tilde{k}^*) = sign(\tilde{k}(0) - \tilde{k}^*)$ in view of monotonic convergence.

By (6.9), we have

$$\tilde{k}(\tau_\omega) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

In view of (6.18), this implies

$$1 - \omega \approx e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

Taking logs on both sides and solving for τ_ω gives

$$\tau_\omega \approx -\frac{\log(1-\omega)}{\beta(\tilde{k}^*)}. \tag{6.19}$$

This is the approximate *adjustment time* required for $\tilde{k}$ to eliminate the fraction ω of the initial distance of $\tilde{k}$ to its steady-state value, $\tilde{k}^*$, when the adjustment speed (speed of convergence) is $\beta(\tilde{k}^*)$.

Often we consider the *half-life* of the adjustment, that is, the time it takes for half of the initial gap to be eliminated. To find the half-life of the adjustment of $\tilde{k}$, we put $\omega = \frac{1}{2}$ in (6.19). Again we use one year as our time unit. With the previous parameter values, we have $\beta(\tilde{k}^*) = 0.053$ per year and thus

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.053} \approx \frac{0.69}{0.053} = 13,1 \text{ years.}$$

As noted above, Barro and Sala-i-Martin (2004) estimate the asymptotic speed of convergence to be $\beta(\tilde{k}^*) = 0.02$ per year. With this value, the half-life is approximately

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.02} \approx \frac{0.69}{0.02} = 34.7 \text{ years.}$$

And the time needed to eliminate three quarters of the initial distance to steady state, $\tau_{3/4}$, will then be about 70 years ($= 2 \cdot 35$ years, since $1 - 3/4 = \frac{1}{2} \cdot \frac{1}{2}$).

Among empirical analysts there is not general agreement about the size of $\beta(\tilde{k}^*)$. Some authors, for example Islam (1995), using a panel data approach, find speeds of convergence considerably larger, between 0.05 and 0.09. McQuinne and Whelan (2007) get similar results. There is a growing realization that the speed of convergence differs across periods and groups of countries. Perhaps an empirically reasonable range is $0.02 < \beta(\tilde{k}^*) < 0.09$. Correspondingly, a reasonable range for the half-life of the adjustment will be 7.6 years $< \tau_{\frac{1}{2}} < 34.7$ years.

Most of the empirical studies of convergence use a variety of cross-country regression analysis of the kind described in the next section. Yet the theoretical frame of reference is often the Solow model - or its extension with human capital (Mankiw et al., 1992). These models are closed economy models with exogenous technical progress and deal with "within-country" convergence. It is not obvious that they constitute an appropriate framework for studying cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and where some countries are pushing the technological frontier further out, while others try to imitate and catch up. At least one should be aware that the empirical estimates obtained may reflect mechanisms in addition to the falling marginal productivity of capital in the process of capital accumulation.

6.4 Barro-style growth regressions

Barro-style growth regression analysis, which became very popular in the 1990s, draws upon transitional dynamics aspects (including the speed of convergence) as well as steady state aspects of neoclassical growth theory (for instance the Solow model or the Ramsey model).

In his Section 3.2 of Chapter 3 Acemoglu presents Barro's growth regression equations in an unconventional form, see Acemoglu's equations (3.12), (3.13), and (3.14). The left-hand side appears as if it is just the growth rate of y (output per unit of labor) from one year to the next. But the true left-hand side of a Barro equation is the average compound annual growth rate of y over many years. Moreover, since Acemoglu's text is very brief about the formal links to the underlying neoclassical theory of transitional dynamics, we will spell the details out here.

Most of the preparatory work has already been done above. The point of departure is a neoclassical one-sector growth model for a closed economy:

$$\dot{\tilde{k}}(t) = s(\tilde{k}(t))f(\tilde{k}(t)) - (\delta + g + n)k(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \text{ given}, \tag{6.20}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $A(t) = A_0e^{gt}$, and $L(t) = L_0e^{nt}$ as above. The Solow model is the special case where the saving-income ratio, $s(\tilde{k}(t))$, is a constant $s \in (0, 1)$.

It is assumed that the model, (6.20), generates monotonic convergence, i.e., $\tilde{k}(t) \to \tilde{k}^* > 0$ for $t \to \infty$. Applying again a first-order Taylor approximation, as in Section 3.1, and taking into account that $s(\tilde{k})$ now may depend on $\tilde{k}$, as for instance it generally does in the Ramsey model, we find the asymptotic speed of convergence for $\tilde{k}$ to be

$$\text{SOC}^*(\tilde{k}) = (1 - \varepsilon(\tilde{k}^*) - \eta(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*) > 0, \tag{*}$$

where $\eta(\tilde{k}^*) \equiv \tilde{k}^*s'(\tilde{k}^*)/s(\tilde{k}^*)$ is the elasticity of the saving-income ratio w.r.t. the effective capital intensity, evaluated at $\tilde{k} = \tilde{k}^*$. (In case of the Ramsey model, one can alternatively use the fact that $\text{SOC}^*(\tilde{k})$ equals the absolute value of the negative eigenvalue of the Jacobian matrix associated with the dynamic system of the model, evaluated in the steady state. For a fully specified Ramsey model this eigenvalue can be numerically calculated by an appropriate computer algorithm; in the Cobb-Douglas case there exists even an explicit algebraic formula for the eigenvalue, see Barro and Sala-i-Martin, 2004). In a neighborhood of the steady state, the previous formulas remain valid with $\beta(\tilde{k}^*)$ defined as in (*). The asymptotic speed of convergence of for example $y(t)/y^*(t)$ is thus $\beta(\tilde{k}^*)$ as given in (*). For notational convenience, we will just denote it β, interpreted as a derived parameter, i.e.,

$$\beta = (1 - \varepsilon(\tilde{k}^*) - \eta(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*). \tag{6.21}$$

In case of the Solow model, $\eta(\tilde{k}^*) = 0$ and we are back in Section 3.

In view of $y(t) \equiv \tilde{y}(t)A(t)$, we have $g_y(t) = g_{\tilde{y}}(t) + g$. By (6.16) and the definition of β,

$$g_y(t) \approx g - \varepsilon(\tilde{k}^*)\beta(\log \tilde{k}(t) - \log \tilde{k}^*) \approx g - \beta(\log y(t) - \log y^*(t)), \tag{6.22}$$

where the last approximation comes from (6.14). This generalizes Acemoglu's Equation (3.10) (recall that Acemoglu concentrates on the Solow model and that his k^* is the same as our $\tilde{k}^*$).

With the horizontal axis representing time, Figure 6.4 gives an illustration of these transitional dynamics. As $g_y(t) = d\log y(t)/dt$ and $g = d\log y^*(t)/dt$, (6.22) is equivalent with

$$\frac{d(\log y(t) - \log y^*(t))}{dt} \approx -\beta(\log y(t) - \log y^*(t)). \tag{6.23}$$

So again we have a simple differential equation of the form $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. The solution of (6.23) is thus

$$\log y(t) - \log y^*(t) \approx (\log y(0) - \log y^*(0))e^{-\beta t}.$$

As $y^*(t) = y^*(0)e^{gt}$, this can written

$$\log y(t) \approx \log y^*(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t}. \tag{6.24}$$

The solid curve in Figure 6.4 depicts the evolution of $\log y(t)$ in the case where $\tilde{k}_0 < \tilde{k}^*$ (note that $\log y^*(0) = \log f(\tilde{k}^*) + \log A_0$). The dotted curve exemplifies the case where $\tilde{k}_0 > \tilde{k}^*$. The figure illustrates per capita income convergence: low initial income is associated with a high subsequent growth rate which, however, diminishes along with the diminishing logarithmic distance of per capita income to its level on the steady state path.

For convenience, we will from now on treat (6.24) as an equality. Subtracting $\log y(0)$ on both sides, we get

$$\begin{aligned}\log y(t) - \log y(0) &= \log y^*(0) - \log y(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t} \\ &= gt - (1 - e^{-\beta t})(\log y(0) - \log y^*(0)).\end{aligned}$$

Dividing through by $t > 0$ gives

$$\frac{\log y(t) - \log y(0)}{t} = g - \frac{1 - e^{-\beta t}}{t}(\log y(0) - \log y^*(0)). \tag{6.25}$$

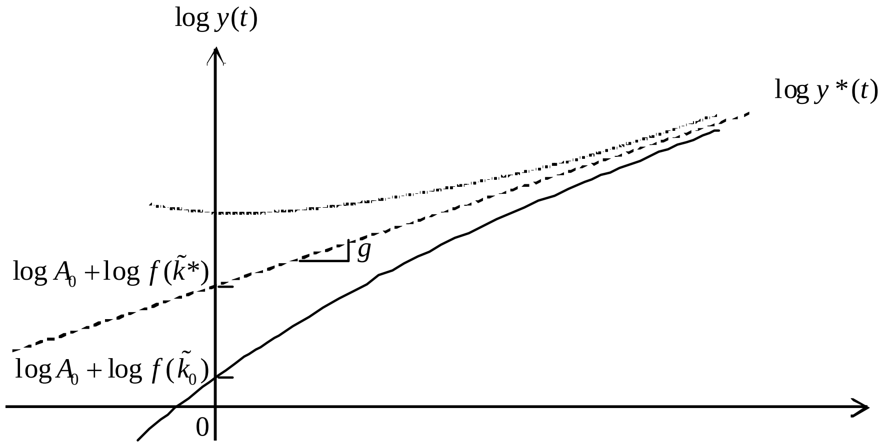

Figure 6.4

On the left-hand side appears the average compound annual growth rate of y from period 0 to period t, which we will denote $\bar{g}_y(0,t)$. On the right-hand side appears the initial distance of $\log y$ to its hypothetical level along the steady state path. The coefficient, $-(1-e^{-\beta t})/t$, to this distance is negative and approaches zero for $t \to \infty$. Thus (6.25) is a translation into growth form of the convergence of $\log y_t$ towards the steady-state path, $\log y_t^*$, in the theoretical model without shocks. Rearranging the right-hand side, we get

$$\bar{g}_y(0,t) = g + \frac{1-e^{-\beta t}}{t}\log y^*(0) - \frac{1-e^{-\beta t}}{t}\log y(0) \equiv b^0 + b^1 \log y(0),$$

where both the constant $b^0 \equiv g + \left[(1-e^{-\beta t})/t\right]\log y^*(0)$ and the coefficient $b^1 \equiv -(1-e^{-\beta t})/t$ are determined by "structural characteristics". Indeed, β is determined by δ, g, n, $\varepsilon(\tilde{k}^*)$, and $\eta(\tilde{k}^*)$ through (6.21), and $y^*(0)$ is determined by A_0 and $f(\tilde{k}^*)$ through (6.12), where, in turn, $\tilde{k}^*$ is determined by the steady state condition $s(\tilde{k}^*)f(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*$, $s(\tilde{k}^*)$ being the saving-income ratio in the steady state.

With data for N countries, $i = 1, 2, \ldots, N$, a test of the *unconditional convergence hypothesis* may be based on the regression equation

$$\bar{g}_{y_i}(0,t) = b^0 + b^1 \log y_i(0) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2), \tag{6.26}$$

where ϵ_i is the error term. This can be seen as a Barro growth regression equation in its simplest form. For countries in the entire world, the theoretical hypothesis $b^1 < 0$ is clearly not supported (or, to use the language of statistics, the null hypothesis, $b^1 = 0$, is not rejected).[2]

Allowing for the considered countries having different structural characteristics, the Barro growth regression equation takes the form

$$\bar{g}_{y_i}(0,t) = b_i^0 + b^1 \log y_i(0) + \epsilon_i, \qquad b^1 < 0, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2). \tag{6.27}$$

In this "fixed effects" form, the equation has been applied for a test of the *conditional convergence hypothesis,* $b^1 < 0$, often supporting this hypothesis.

From the estimate of b^1 the implied estimate of the asymptotic speed of convergence, β, is readily obtained through the formula $b^1 \equiv (1 - e^{-\beta t})/t$. Even β, and therefore also the slope, b^1, does depend, theoretically, on country-specific structural characteristics. But the sensitivity on these do not generally seem large enough to blur the analysis based on (6.27) which abstracts from this dependency.

With the aim of testing hypotheses about growth determinants, Barro (1991) and Barro and Sala-i-Martin (1992, 2004) decompose b_i^0 so as to reflect the role of a set of measurable potentially causal variables,

$$b_i^0 = \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \ldots + \alpha_m x_{im},$$

where the α's are the coefficients and the x's are the potentially causal variables.[3] These variables could be measurable Solow-type parameters among those appearing in (6.20) or a broader set of determinants, including for instance the educational level in the labor force, and institutional variables like rule of law and democracy. Some studies include the initial within-country inequality in income or wealth among the x's and extend the theoretical framework correspondingly.[4]

From an econometric point of view there are several problematic features in regressions of Barro's form (also called the β convergence approach). These problems are discussed in Acemoglu pp. 82-85.

6.5 References

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[2] Cf. Acemoglu, p. 16. For the OECD countries, however, b^1 is definitely estimated to be negative (cf. Acemoglu, p. 17).

[3] Note that our α vector is called β in Acemoglu, pp. 83-84. So Acemoglu's β is to be distinquished from our β which denotes the asymptotic speed of convergence.

[4] See, e.g., Alesina and Rodrik (1994) and Perotti (1996), who argue for a negative relationship between inequality and growth. Forbes (2000), however, rejects that there should be a robust negative correlation between the two.

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Chapter 7

Michael Kremer's population-breeds-ideas model

This chapter relates to Section 2 of Acemoglu's Chapter 4 and explains the details of what may also be called the *Simon-Kremer version* of the population-breeds-ideas model (cf. Acemoglu, p. 114).

7.1 The model

Suppose a pre-industrial economy can be described by:

$$\begin{aligned} Y_t &= A_t^\sigma L_t^\alpha Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, && (7.1)\\ \dot{A}_t &= \lambda A_t^\varepsilon L_t, \qquad \lambda > 0, 0 < \varepsilon \leq 1, \quad A_0 > 0 \text{ given}, && (7.2)\\ L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, && (7.3) \end{aligned}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), Z the amount of land (fixed), and $\bar{y}$ subsistence minimum (so the φ in Acemoglu's equation (4.2) is simply the inverse of the subsistence minimum). Both Z and $\bar{y}$ are considered as constant parameters. Time is continuous and it is understood that a kind of Malthusian population mechanism (see below) is operative behind the scene.

The exclusion of capital from the aggregate production function, (7.1), reflects the presumption that capital (tools etc.) is quantitatively of minor importance in a pre-industrial economy. In accordance with the replication argument, the production function has CRS w.r.t. the rival inputs, labor and land. The factor A_t^σ measures total factor productivity. In view of (7.2), the technology level, A_t, is rising over time. The increase in A_t per time unit is seen to be an increasing function of the size of the population. This reflects the hypothesis that population breeds ideas; these are non-rival and enter the pool of technical knowledge available for society as a whole. The rate per capita, λA^{ε}, by which population breeds ideas is an increasing function of the already existing level of technical knowledge. This reflects the hypothesis that the larger is the stock of ideas the easier do new ideas arise (perhaps by combination of existing ideas).

Equation (7.3) is a shortcut description of a Malthusian population mechanism. Suppose the true mechanism is

$$\dot{L}_t = \beta(y_t - \bar{y})L_t \gtreqless 0 \quad \text{for} \quad y_t \gtreqless \bar{y}, \tag{7.4}$$

where $\beta > 0$ is the speed of adjustment, $y_t \equiv Y_t/L_t$ is per capita income, and $\bar{y} > 0$ is subsistence minimum. A rise in y_t above $\bar{y}$ will lead to increases in L_t, thereby generating downward pressure on Y_t/L_t and perhaps end up pushing y_t below $\bar{y}$. When this happens, population will be decreasing for a while and so return towards its sustainable level, $Y_t/\bar{y}$. Equation (7.3) treats this mechanism as if the population instantaneously adjusts to its sustainable level (as if $\beta \to \infty$). The model hereby gives a long-run picture, ignoring the Malthusian ups and downs in population and per capita income about the subsistence minimum. The important feature is that the technology level and thereby Y_t as well as the sustainable population will be rising over time. This speeds up the arrival of new ideas and so raises Y_t even faster although per-capita income remains at its long-run level, $\bar{y}$.[1]

For simplicity, we now normalize the constant Z to be 1.

7.2 Law of motion

The dynamics of the model can be reduced to one differential equation, the law of motion of technical knowledge. By (7.3), $L_t = \varphi Y_t = \varphi A_t^{\sigma} L_t^{\alpha}$. Consequently $L_t^{1-\alpha} = \varphi A_t^{\sigma}$ so that

$$L_t = \varphi^{\frac{1}{1-\alpha}} A_t^{\frac{\sigma}{1-\alpha}}. \tag{7.5}$$

Substituting this into (7.2) gives the law of motion of technical knowledge:

$$\dot{A}_t = \lambda \varphi^{\frac{1}{1-\alpha}} A_t^{\varepsilon + \frac{\sigma}{1-\alpha}} \equiv \hat{\lambda} A_t^{\varepsilon + \frac{\sigma}{1-\alpha}}. \tag{7.6}$$

[1]Extending the model with the institution of private ownership and competitive markets, the absence of a growing standard of living corresponds to the doctrine from classical economics called the *iron law of wages.* This is the theory (from Malthus and Ricardo) that scarce natural resources and the pressure from population growth causes real wages to remain at subsistende level.

Define $\mu \equiv \varepsilon + \frac{\sigma}{1-\alpha}$ and assume $\mu > 1$. Then (7.6) can be written

$$\dot{A}_t = \hat{\lambda} A_t^{\mu}, \tag{7.7}$$

which is a nonlinear differential equation in A.[2] Let $x \equiv A^{1-\mu}$. Then

$$\dot{x}_t = (1-\mu) A_t^{-\mu} \hat{\lambda} A_t^{\mu} = (1-\mu)\hat{\lambda}, \tag{7.8}$$

a constant. To find x_t from this, we only need simple integration:

$$x_t = x_0 + \int_0^t \dot{x}_\tau d\tau = x_0 + (1-\mu)\hat{\lambda} t.$$

As $A = x^{\frac{1}{1-\mu}}$ and $x_0 = A_0^{1-\mu}$, this implies

$$A_t = x_t^{\frac{1}{1-\mu}} = \left[A_0^{1-\mu} + (1-\mu)\hat{\lambda} t\right]^{\frac{1}{1-\mu}} = \frac{1}{\left[A_0^{1-\mu} - (\mu-1)\hat{\lambda} t\right]^{\frac{1}{\mu-1}}}. \tag{7.9}$$

7.3 The inevitable ending of the Malthusian regime

The result (7.9) helps us in understanding why the Malthusian regime must come to an end (at least if the model is an acceptable description of the Malthusian regime).

Although to begin with, A_t may grow extremely slowly, the growth in A_t will be *accelerating* because of the *positive feedback* (visible in (7.2)) from both rising population and rising A_t. Indeed, since $\mu > 1$, the denominator in (7.9) will be decreasing over time and approach zero in finite time, namely as t approaches the finite value $t^* = A_0^{1-\mu}/((\mu-1)\hat{\lambda})$. Figure 7.1 illustrates. The evolution of technical knowledge becomes explosive as t approaches t^*.

It follows from (7.5) and (7.1) that explosive growth in A implies explosive growth in L and Y, respectively. The acceleration in the evolution of Y will sooner or later make Y move fast enough so that the Malthusian population mechanism (which for biological reasons has to be slow) can not catch up. Then, what was in the Malthusian population mechanism, equation (7.4), earlier only a transitory excess of y_t over $\bar{y}$, will sooner or later become a permanent excess and take the form of sustained growth in y_t. This is known as the *take-off.*

[2] The differential equation, (7.7), is a special case of what is known as the *Bernoulli equation.* In spite of being a non-linear differential equation, the Bernoulli equation always has an explicit solution.

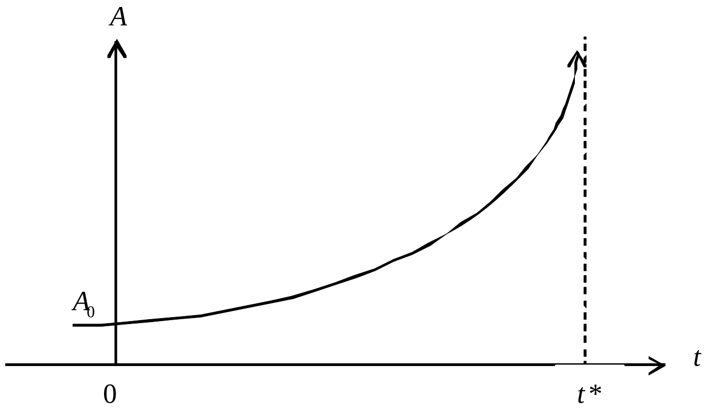

Figure 7.1

According to equation (7.4) the take-off should lead to a permanently rising population growth rate. As economic history has testified, however, along with the rising standard of living the demographics changed The *demographic transition* took place with fertility declining faster than mortality. This results in completely different dynamics about which the present model has nothing to say.[3] As to the demographic transition as such, explanations suggested by economists include: higher opportunity costs of raising children, the trade-off between "quality" (educational level) of the offspring and their "quantity" (Becker, Galor), skill-biased technical change, and improved contraception technology.

7.4 Closing remarks

The present model is about dynamics in the Malthusian regime of the pre-industrial epoch. The story told by the model is the following. When the feedback parameter, μ, is above one, the Malthusian regime has to come to an end because the battle between scarcity of land (or natural resources more generally) and technological progress will inevitably be won by the latter.[4]

The cases $\mu < 1$ and $\mu = 1$ are considered in Exercise III.3. The case $\mu = 1$ corresponds to Acemoglu's first version (p. 113) of the population-breeds-ideas model. In that version, σ has the value $1 - \alpha$ and $\varepsilon = 0$ (two

[3] Kremer (1993), however, also includes an extended model taking some of these changed dynamics into account.

[4] The mathematical background for the explosion result is explained in the appendix.

arbitrary knife-edge conditions). Then a constant growth rate in A, L, and Y is the result and y remains at $\bar{y}$ forever. Take-off never takes place.

On the basis of demographers' estimates of the growth in global population over most of human history, Kremer (1993) finds empirical support for $\mu > 1$. Indeed, in the opposite case, $\mu \leq 1$, there would *not* have been a rising world population growth rate since one million years B.C. to the industrial revolution. The data in Kremer (1993, p. 682) indicates that the population growth rate has been more or less proportional to the size of population until about the1960s.

7.5 Appendix

Mathematically, the background for the explosion result is that the solution to a first-order differential equation of the form $\dot{x}(t) = \alpha + bx(t)^c$, $c > 1$, $b \neq 0$, $x(0) = x_0$ given, is always explosive. Indeed, the solution, $x = x(t)$, will have the property that $x(t) \to \pm\infty$ for $t \to t^*$ for some fixed $t^* > 0$; and thereby the solution is defined only on a bounded time interval.

Take the differential equation $\dot{x}(t) = 1 + x(t)^2$ as an example. As is well-known, the solution is $x(t) = \tan t = \sin t/\cos t$, defined on the interval $(-\pi/2, \pi/2)$.

7.6 References

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List of contents to be continued.

Chapter 8

Choice of social discount rate

With an application to the climate change problem

A controversial issue within economists' debate on long-term public investment and the climate change problem is the choice of discount rate. This choice matters a lot for the present value of a project which involves costs that begin now and benefits that occur only after many years, say 75-100-200 years from now, as is the case with the measures against global warming.

Compare the present value of receiving 1000 inflation-corrected euros a hundred years from now under two alternative discount rates, $r = 0.07$ and $r = 0.01$ per year:

$$PV_0 = 1000 * e^{-r*100} = \begin{cases} 0.9 \text{ if } r = 0.07, \\ 368 \text{ if } r = 0.01. \end{cases}$$

So when evaluated at a 7 percent discount rate the 1000 inflation-corrected euros a hundred years from now are worth less than 1 euro today. But with a discount rate at 1 percent they are worth 368 euros today.

In this chapter we discuss different aspects of social discounting, that is, discounting from a policy maker's point of view. We shall set up the theoretical framework around the concept of *optimal capital accumulation* described in Acemoglu, Chapter 8, Section 8.3. In the final sections we apply the framework to an elementary discussion of the climate change problem from an economic perspective.

Unfortunately it is not always recognized that "discount rate" can mean several different things. This sometimes leads to serious confusion, even within academic debates about policies addressing climate change. We therefore start with the ABC of discounting.

A *discount rate* is an interest rate applied in the construction of a *discount factor.* The latter is a factor by which a project's future costs or benefits, measured in some unit of account, are converted into present equivalents. Applying a discount factor thus allows economic effects occurring at different times to be compared. The lower the discount factor the higher the associated discount rate.

Think of period t as running from date t to date $t+1$. More precisely, think of period t as the time interval $[t, t+1)$ on a continuous time axis with time unit equal to the period length. With time t thus referring to the beginning of period t, we speak of "date t" as synonymous with time t. This timing convention is common in discrete-time growth and business cycle theory and is convenient because it makes switching between discrete and continuous time analysis fairly easy.[1] Unless otherwise indicated, our period length, hence our time unit, will be one year.

8.1 Basic distinctions relating to discounting

A basic reason that we have to distinguish between different types of discount rates is that there is a variety of possible units of account.

To simplify matters, in this section we assume there is no uncertainty unless otherwise indicated. Future market interest rates will thus with probability one be equal to the ex ante expected future interest rates.

8.1.1 The unit of account

Money as the unit of account

When the unit of account is *money*, we talk about a *nominal discount rate*. More specifically, if the money unit is euro, we talk about an euro discount rate. Consider a one-period bond promising one euro at date one to the investor buying the bond at date 0. If the market interest rate is i_0, the present value at date 0 of the bond is

$$\frac{1}{1+i_0} \quad \text{euro.}$$

In this calculation the (nominal) *discount factor* is $1/(1+i_0)$ and tells how many euro need be invested in the bond at time 0 to obtain 1 euro at time 1. When the interest rate in this way appears as a constituent of a discount factor, it is called a (nominal) *discount rate*. Like any interest rate it tells

[1] Note, however, that this timing convention is different from that in the standard finance literature where, for example, K_t would denote the *end-of-period* t stock that begins to yield its services *next* period.

how many additional units of account (here euros) are returned after one period of unit length, if one unit of account (one euro) is invested in the asset at the beginning of the period.[2]

A payment stream, $z_0, z_1, \ldots, z_t, \ldots, z_T$, where z_t ($\gtrless 0$) is the net payment in euro due at the *end* of period t, has present value (in euro as seen from the beginning of period 0)

$$PV_0 = \frac{z_0}{1+i_0} + \frac{z_1}{(1+i_0)(1+i_1)} + \cdots + \frac{z_{T-1}}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}, \tag{8.1}$$

where i_t is the nominal interest rate in euro on a one-period bond from date t to date $t+1$, $t = 0, 1, \ldots, T-1$.

The *average nominal discount rate* from date T to date 0 is the number $\bar{i}_{0,T-1}$ satisfying

$$1 + \bar{i}_{0,T-1} = ((1+i_0)(1+i_1)\cdots(1+i_{T-1}))^{1/T}. \tag{8.2}$$

The corresponding nominal discount factor is

$$(1 + \bar{i}_{0,T-1})^{-T} = \frac{1}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}. \tag{8.3}$$

If i is constant, the average nominal discount rate is of course the same as i and the nominal discount factor is simply $1/(1+i)^T$.

If the stream of z's in (8.1) represents expected but uncertain dividends to an investor as seen from date 0, we may ask: What is the *relevant* discount rate to be applied on the stream by the investor? The answer is that the relevant discount rate is that rate of return the investor can obtain generally on investments with a similar risk profile. So the relevant discount rate is simply the *opportunity cost* faced by the investor.

In continuous time with continuous compounding the formulas corresponding to (8.1), (8.2), and (8.3) are

$$PV_0 = \int_0^T z(t)\, e^{-\int_0^t i(\tau)d\tau} dt, \tag{8.4}$$

$$\bar{i}(0,T) \equiv \frac{\int_0^T i(\tau)d\tau}{T}, \quad \text{and} \tag{8.5}$$

$$e^{-\bar{i}(0,T)T} = e^{-\int_0^T i(\tau)d\tau}. \tag{8.6}$$

And as above, if i is constant, the nominal discount factor takes the simple form e^{-iT}.

[2]A discount factor is by definition a non-negative number. Hence, a discount rate in discrete time is by definition greater than -1.

Consumption as the unit of account

When the unit of account is a basket of consumption goods or, for simplicity, just a homogeneous consumption good, we talk about a *consumption discount rate* (or a *real* discount rate). Let the consumption good's price in terms of euros be P_t, $t = 0, 1, \ldots, T$. A consumption stream $c_0, c_1, \ldots, c_t, \ldots, c_T$, where c_t is available at the end of period t, has present value (as seen from the beginning of period 0)

$$PV_0 = \frac{c_0}{1+r_0} + \frac{c_1}{(1+r_0)(1+r_1)} + \cdots + \frac{c_{T-1}}{(1+r_0)(1+r_1)\cdots(1+r_{T-1})}. \quad (8.7)$$

Instead of the nominal interest rate, the proper discount rate is now the *real* interest rate, r_t, on a one-period bond from date t to date $t+1$. Ignoring indexed bonds, the real interest rate is not directly observable, but can be calculated in the following way from the observable nominal interest rate i_t :

$$1 + r_t = \frac{P_{t-1}(1+i_t)}{P_t} = \frac{1+i_t}{1+\pi_t},$$

where P_{t-1} is the price (in terms of money) of a period-$(t-1)$ consumption good paid for at the *end* of period $t-1$ (= the beginning of period t) and $\pi_t \equiv P_t/P_{t-1} - 1$ is the inflation rate from period $t-1$ to period t.

The *consumption discount factor* (or *real* discount factor) from date $t+1$ to date t is $1/(1+r_t)$. This discount factor tells how many consumption goods' worth need be invested in the bond at time t to obtain one consumption good's worth at time $t+1$. The stream $c_0, c_1, \ldots, c_t, \ldots, c_T$ could alternatively represent an income stream measured in current consumption units. Then the real interest interest rate, r_t, would still be the relevant real discount rate and (8.7) would give the present real value of the income stream.

The *average consumption discount rate* and the corresponding consumption discount factor are defined in a way analogous to (8.2) and (8.3), respectively, but with i_t replaced by r_t. Similarly for the continuous time versions (8.4), (8.5), and (8.6).

Utility as the unit of account

Even though "utility" is not a measurable entity but just a convenient mathematical devise used to represent preferences, a utility discount rate is in many cases a meaningful concept.

Suppose intertemporal preferences can be represented by a sum of period utilities discounted by a constant rate, ρ :

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \cdots + \frac{u(c_{T-1})}{(1+\rho)^{T-1}}, \quad (8.8)$$

where $u(\cdot)$ is the period utility function. Here ρ appears as a *utility discount rate.* The associated *utility discount factor* from date T to date 0 is $1/(1+\rho)^{T-1}$. We may alternatively write the intertemporal utility function as $\tilde{U}(c_0, c_1, \cdots, c_{T-1}) \equiv (1+\rho)^{-1}U(c_0, c_1, \cdots, c_{T-1})$. Then the utility discount factor from date T to date 0 appears instead as $1/(1+\rho)^T$, which in form corresponds exactly to (8.3); this difference is, however, immaterial, since $\tilde{U}(\cdot)$ and $U(\cdot)$ represent the same preferences and will imply the same choices. In continuous time (with continuous compounding) the "sum" of discounted utility is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt,$$

where $e^{-\rho t}$ is the utility discount factor from time t to time 0.[3]

8.1.2 The economic context

Along with the unit of account the economic context of the investment project to be evaluated matters for the choice of discount rate. Here is a brief list of important distinctions:

1. It matters whether the circumstances of relevance for the investment project are endowed with *certainty, computable risk,* or *non-computable risk*, also called *fundamental uncertainty.* In the latter case, the probability distribution is unknown (or scientists deeply disagree about it) and, typically, the full range of possible outcomes is unknown.

2. *Length of the time horizon.* Recently several countries have decided to draw a line between less than vs. more than 30-50 years, choosing a lower discount rate for years on the other side of the line. This is in accordance with recommendations from economists and statisticians arguing that the further ahead in time the discount rate applies, the smaller should it be. With longer time horizons systematic risk and fundamental uncertainty, about both the socio-economic environment as such and the results of the specific project, play a larger role, thus motivating precautionary saving.

3. A *single* or several *different kinds consumption goods.* As we shall see below, the relevant consumption discount rate in a given context

[3]Note that a first-order Taylor approximation of e^x around $x = 0$ gives $e^x \approx e^0 + e^0(x-0) = 1+x$ for x "small"; hence, $x \approx \ln(1+x)$ for x "small". Replacing x by ρ and taking powers, we see the analogy between $e^{-\rho t}$ and $(1+\rho)^{-t}$. Because of the continuous compounding, we have $e^{-\rho t} < (1+\rho)^{-t}$ whenever $\rho > 0$ and $t > 0$ and the difference increases with rising t.

depends on several factors, including the growth rate of consumption. When fundamentally different consumption goods enter the utility function - for instance an ordinary produced commodity versus services from the eco-system - then a disaggregate setup is needed and the relevant consumption discount rate may become an intricate matter. Sterner and Persson (2008) give an introduction to this issue.

4. *Private vs. social.* Discounting from an individual household's or firm's point of view, as it occurs in private investment analysis, is one thing. Discounting from a government's point of view is another, and in connection with evaluation of government projects we speak of *social* cost-benefit analysis. Here externalities and other market failures should be taken into account. Whatever the unit of account, a discount rate applied in social cost-benefit analysis is called a *social discount rate.*

5. *Micro vs. macro.* Social cost-benefit analysis may be concerned with a *microeconomic* project and policy initiatives that involve only marginal changes. In this case a lot of circumstances are exogenous (like in partial equilibrium analysis). Alternatively social cost-benefit analysis may be concerned with a *macroeconomic* project and involve over-all changes. At this level more circumstances are *endogenous*, including possibly the rate of economic growth and the quality of the natural environment on a grand scale. In macroeconomic cost-benefit analysis intra- and intergenerational ethical issues are thus important.

8.2 Criteria for choice of a social discount rate

There has been some disagreement among both economists and policy makers about how to discount in *social* cost-benefit analysis, in particular when the economy as a whole and a long time horizon are involved. At one side we have the *descriptive approach* to social discounting, sometimes called the *opportunity cost* view:

> According to this view, even when considering climate change policy evaluation and caring seriously about future generations, the average market rate of return, before taxes, is the relevant discount rate. This is because funds used today to pay the cost of, say, mitigating greenhouse gas emissions, could be set aside and invested in other things and thereby accumulate at the market rate of return for the benefit of the future generations.

At the other side we find a series of opinions that are not easily lumped together apart from their scepticism about the descriptive approach (in its narrow sense as defined above). These "*other views*" are commonly grouped together under the label the *normative* or *prescriptive* approach. This terminology has become standard. With some hesitation we adopt it here (the reason for the hesitation should become clear below).

One reason that the descriptive approach is by some considered inappropriate is the presence of *market failures.*[4] Another is the presence of *conflicting interests*: those people who benefit may not be the same as those who bear the costs. And where as yet unborn generations are involved, difficult ethical and coordination issues arise.

Amartia Sen (1961) pointed at the *isolation paradox.* Suppose each old has an altruistic concern for *all* members of the next generation. Then a transfer from any member of the old generation to the heir entails an externality that benefits all other members of the old generation. A nation-wide coordination (political agreement) that internalizes these externalities would raise intergenerational transfers (bequests etc.) and this corresponds to a lowering of the intergenerational utility discount rate, ρ, cf. (8.8).

More generally, members of the present generations may be willing to join in a collective contract of more saving and investment by all, though unwilling to save more in isolation.

Other reasons for a relatively low social discount rate have been proposed. One is based on the *super-responsibility argument:* the government has responsibility over a longer time horizon than those currently alive. Another is based on the *dual-role argument*: the members of the currently alive generations may in their political or public role be more concerned about the welfare of the future generations than they are in their private economic decisions.

Critics of the descriptive approach may agree about the relevance of asking: "To what extent will investments made to reduce greenhouse gas emissions displace investments made elsewhere?". They may be inclined to add that there is no guarantee that the funds in question *are* set aside for investment benefitting generations alive two hundred years ahead, say.

Another point against the descriptive approach is that the future damages of global warming could easily be underestimated. If nothing is done now, the risk of the damage being irreparable at any cost becomes higher. Applying the current market rate of return as discount rate for damages occurring

[4] Intervening into the debate about the suitable discount rate for climate change projects, Heal (2008) asks ironically: "Is it appropriate to assume no market failure in evaluating a consumption discount rate for a model of climate change?".

say 200 years from now on may imply that these damages become almost imperceptible and so action tends to be postponed. This may be problematic if there is a positive albeit low probability that a tipping point with disastrous consequences is reached.

The reason for hesitation to lump together these "other views" under the labels *normative* or *prescriptive* approach is that the contraposition of "descriptive" versus "normative" in this context may be misleading. In the final analysis also the *descriptive approach* has a normative element namely the view that the social discount rate *ought to* be that implied by the market behavior of the current generations as reflected in the current market interest rate - the alternative is seen as paternalism.[5]

Anyway, in practice there seems to be a kind of convergence in the sense that elements from the descriptive and the prescriptive way of thinking tend to be combined. Nevertheless, there is considerable diversity across countries regarding the governments' official "social consumption discount rate" (sometimes just called the "social discount rate") to be applied for public investment projects. Even considering only West-European countries and Western Offshoots, including the U.S., the range is roughly from 8% to 2% per year. An increasing fraction of these countries prescribe a lower rate for benefits and costs accruing more than 30-50 years in the future (Harrison, 2010). The Danish Ministry of Finance recently (May 2013) reduced its social consumption discount rate from 5% per year to 4% per year for the first 35 years of the time horizon of the project, 3% for the years in the interval 36 to 69 years, and 2% for the remainder of the time horizon if relevant.[6] Among economists involved in climate change policy evaluation there is a wide range regarding what the recommended social discount rate should be (from 1.4% to 8.0%).[7] An evaluation of the net worth of the public involvement in the Danish wind energy sector in the 1990s gives opposite conclusions depending on whether the discount rate is 5-6% (until recently the official Danish discount rate) or 3-4% (Hansen, 2010).

This diversity notwithstanding, let us consider some examples of social cost-benefit problems of a macroeconomic nature and with a long time horizon. Our first example will be the standard neoclassical problem of optimal capital accumulation.

[5]Here the other side of the debate may respond that such "paternalism" need not be illigitimate but rather the responsibility of democratically elected governments.

[6]Finansministeriet (2013) .

[7]Harrison (2010).

8.3 Optimal capital accumulation

The perspective is that of an "all-knowing and all-powerful" social planner facing a basic intertemporal allocation problem in a closed economy: how much should society save? The point of departure for this problem is the prescriptive approach. The only discount rate which is decided in advance is the *utility* discount rate, ρ. No consumption discount rate is part of either the objective function or the constraints. Instead, a long-run consumption discount rate applicable to a class of public investment problems comes out as a *by-product* of the steady-state *solution* to the problem.

8.3.1 The setting

We place our social planner in the simplest neoclassical set-up with exogenous Harrod-neutral technical change. Uncertainty is ignored. Although time is continuous, for simplicity we date the variables by sub-indices, thus writing Y_t etc. The aggregate production function is neoclassical and has CRS:

$$Y_t = F(K_t, T_t L_t) \equiv T_t L_t f(\tilde{k}_t), \tag{8.9}$$

where Y_t is output, K_t physical capital input, and L_t labor input which equals the labor force which in turn equals the population and grows at the constant rate n. The argument in the production function on intensive form is defined by $\tilde{k}_t \equiv K_t/(T_t L_t)$. The factor T_t represents the economy-wide level of technology and grows exogenously according to

$$T_t = T_0 e^{gt}, \tag{8.10}$$

where $T_0 > 0$ and $g \geq 0$ are given constants. Population grows at the constant rate $n \geq 0$. Output is used for consumption and investment so that

$$\dot{K}_t = Y_t - c_t L_t - \delta K_t, \tag{8.11}$$

where c_t is per capita consumption and $\delta \geq 0$ a constant capital depreciation rate.

The social planner's objective is to maximize a *social welfare function*, W. We assume that this function is time separable with (i) an instantaneous utility function $u(c)$ with $u' > 0$ and $u'' < 0$ and where c is per capita consumption; (ii) a constant utility discount rate $\rho \geq 0$, often named "the pure rate of time preference"; and (iii) an infinite time horizon. The social planner's optimization problem is to choose a plan $(c_t)_{t=0}^{\infty}$ so as to maximize

$$W = \int_0^{\infty} u(c_t) L_t e^{-\rho t} dt \qquad \text{s.t.} \tag{8.12}$$

$$c_t \geq 0, \tag{8.13}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{8.14}$$

$$\tilde{k}_t \geq 0 \qquad \text{for all } t \geq 0. \tag{8.15}$$

Comments

1. If there are technically feasible paths along which the improper integral W goes to $+\infty$, a maximum of W does not exist (in the CRRA case, $u(c) = c^{1-\theta}/(1-\theta)$, $\theta > 0$, this will happen if and only if the parameter condition $\rho - n > (1-\theta)g$ is *not* satisfied). By "optimizing" we then mean finding an "overtaking optimal" solution or a "catching-up optimal" solution, assuming one of either exists (cf. Sydsæter et al. 2008).

2. The long time horizon should be seen as involving many successive and as yet unborn generations. Comparisons across time should primarily be interpreted as comparisons across generations.

3. The model abstracts from inequality within generations.

4. By weighting per capita utility by L_t and thereby effectively taking population growth, n, into account, the social welfare function (8.12) respects the principle of *discounted classical utilitarianism.* A positive *pure* rate of time preference, ρ, implies discounting the utility of future people just because they belong to the future. Some analysts defend this discounting of the future by the argument that it is a typical characteristic of an individual's preferences. Others find that this is not a valid argument for long-horizon evaluations because these involve different persons and even as yet unborn generations. For example Stern (2007) argues that the only ethically defensible reason for choosing a positive ρ is that there is always a small risk of extinction of the human race due to for example a devastating meteorite or nuclear war. This issue aside, in (8.12) the *effective* utility discount rate will be $\rho - n$. This implies that the larger is n, the more weight is assigned to the future because more people will be available.[8] We shall throughout assume that the size of population is exogenous although this may not accord entirely well with large public investment projects, like climate change mitigation, that have implication for health and mortality. With endoge-

[8]In contrast, the principle of discounted *average* utilitarianism is characterized by population growth *not* affecting the effective utility discount rate. This corresponds to eliminating the factor L_t in the integrand in (8.12).

nous population very difficult ethical issues arise (Dasgupta (2001), Broome (2005)).

8.3.2 First-order conditions and their economic interpretation

To characterize the solution to the problem, we use the Maximum Principle. The current-value Hamiltonian is

$$H(\tilde{k}, c, \lambda, t) = u(c) + \lambda \left[f(\tilde{k}) - \frac{c}{T} - (\delta + g + n)\tilde{k} \right],$$

where λ is the adjoint variable associated with the dynamic constraint (8.14). An interior optimal path $(\tilde{k}_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda = \lambda_t$ such that, for all $t \geq 0$,

$$\frac{\partial H}{\partial c} = 0, \text{ i.e., } u'(c) = \frac{\lambda}{T}, \quad \text{and} \tag{8.16}$$

$$\frac{\partial H}{\partial \tilde{k}} = \lambda(f'(\tilde{k}) - \delta - g - n) = (\rho - n)\lambda - \dot{\lambda} \tag{8.17}$$

hold along the path and the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t \lambda_t e^{-(\rho-n)t} = 0, \tag{8.18}$$

is satisfied.

By taking logs on both sides of (8.16) and differentiating w.r.t. t we get

$$\frac{du'(c_t)/dt}{u'(c_t)} = \frac{u''(c_t)}{u'(c_t)}\dot{c}_t = \frac{\dot{\lambda}_t}{\lambda_t} - g = \rho - (f'(\tilde{k}_t) - \delta),$$

where the last equality comes from (8.17). Reordering gives

$$f'(\tilde{k}_t) - \delta = \rho + \left(-\frac{u''(c_t)}{u'(c_t)} \right) \dot{c}_t, \tag{8.19}$$

where the term $(-u''(c_t)/u'(c_t)) > 0$ indicates the rate of decline in marginal utility when consumption is increased by one unit. So the right-hand side of (8.19) exceeds ρ when $\dot{c}_t > 0$.

A technically feasible path satisfying both (8.19) and the transversality condition (12.33) with $\lambda_t = T_t u'(c_t)$ will be an optimal path and there are no other optimal paths.[9]

[9] This follows from *Mangasarian's sufficiency theorem* and the fact that the Hamiltonian is *strictly concave* in $(\tilde{k}, \tilde{c})$. The implied resource allocation will be the same as that of a competitive conomy with the same technology as that given in (8.9) and with a representative household that has the same intertemporal preferences as those of the social planner given in (8.12) (this is the *Equivalence theorem*).

The optimality condition (8.19) could of course be written on the standard Keynes-Ramsey rule form, where $\dot{c}_t/c_t$ is isolated on one side of the equation. But from the perspective of rates of return, and therefore discount rates, the form (8.19) is more useful, however. The condition expresses the general principle that in the optimal plan the marginal unit of per capita output is equally valuable whether used for investment or current consumption. When used for investment, it gives a rate of return equal to the net marginal productivity of capital indicated on the left-hand side of (8.19). When used for current consumption, it raises current utility. Doing this to an extent just enough so that no further postponement of consumption is justified, the *required* rate of return is exactly obtained. The condition (8.19) says that in the optimal plan the *actual* marginal rate of return (the left-hand side) equals the *required* marginal rate of return (the right-hand side).

Reading the optimality condition (8.19) from the right to the left, there is an analogy between this condition and the general microeconomic principle that the consumer equates the marginal rate of substitution, MRS, between any two consumption goods with the price ratio given from the market. In the present context the two goods refer to the same consumption good delivered in two successive time intervals. And instead of a price ratio we have the marginal rate of transformation, MRT, between consumption in the two time intervals as given by technology. The analogy is only partial, however, because this MRT is, from the perspective of the optimizing agent (the social planner) not a given but is endogenous just as much as the MRS is endogenous.

8.3.3 The social consumption discount rate

More specifically, (8.19) says that the social planner will sacrifice per capita consumption today for more per capita consumption tomorrow only up to the point where this saving for the next generations is compensated by a rate of return sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines with rising consumption, i.e., the larger is $(-u''/u')\dot{c}$. Indeed, every extra unit of consumption handed over to future generations delivers a smaller and smaller marginal utility to these future generations. So the marginal unit of investment today is only warranted if the marginal rate of return is sufficiently above ρ, as indicated by (8.19).

Letting the required marginal rate of return be denoted r_t^{SP} and letting the values of the variables along the optimal time path be marked by a bar, we can write the right-hand side of (8.19) as

$$r_t^{SP} = \rho + \theta(\bar{c}_t)\frac{\dot{\bar{c}}_t}{\bar{c}_t}, \tag{8.20}$$

where $\theta(c) \equiv -cu''(c)/u'(c) > 0$ (the absolute elasticity of marginal utility of consumption). For a given $\theta(\bar{c}_t)$, a higher per capita consumption growth rate implies a higher required rate of return on marginal saving. In other words, the higher the standard of living of future generations compared with current generations, the higher is the required rate of return on current marginal saving. Indeed, less should be saved for the future generations. Similarly, for a given per capita consumption growth rate, $\dot{\bar{c}}_t/\bar{c}_t > 0$, the required rate of return on marginal saving is higher, the larger is $\theta(\bar{c}_t)$. This is because $\theta(\bar{c}_t)$ reflects *aversion towards consumption inequality across time and generations* (in a context with uncertainty $\theta(\bar{c}_t)$ also reflects what is known as the *relative risk aversion,* see below). Indeed, $\theta(\bar{c}_t)$ indicates the percentage fall in marginal utility when per capita consumption is raised by one percent. So a higher $\theta(\bar{c}_t)$ contributes to more consumption smoothing over time.

So far these remarks are only various ways of interpreting an optimality condition. Worth emphasizing is:

- The *required* marginal rate of return (the right-hand side of (8.20)) at time t is not something given in advance, but an endogenous and time-dependent variable which along the optimal path must equal the *actual* marginal rate of return (the endogenous rate of return on investment represented by the left-hand side of equation (8.20)). Indeed, both the required and the actual marginal rates of return are endogenous because they depend on the endogenous variables c_t and $\dot{c}_t$ and on what has been decided up to time t and is reflected in the current value of the state variable, $\tilde{k}_t$. As we know from phase diagram analysis in the $(\tilde{k}, c/T)$ plane, there are infinitely many technically feasible paths satisfying the inverted Keynes-Ramsey rule in (8.20) for all $t \geq 0$. What is lacking up to now is to select among these paths one that satisfies the transversality condition (12.33).

- In the present problem the only discount rate which is decided in advance is the *utility* discount rate, ρ. No consumption discount rate is part of either the objective function or the constraints. We shall now see, however, that a long-run consumption discount rate applicable to (less-inclusive) public investment problems comes out as a *by-product* of the steady-state *solution* to the problem.

Steady state

To help existence of a steady state we now assume that the instantaneous utility, $u(c)$, belong to the CRRA family so that $\theta(c) = \theta$, a positive constant. Then

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{8.21}$$

We know that if the parameter condition $\rho - n > (1 - \theta)g$ holds and f satisfies the Inada conditions, then there *exists* a *unique* path satisfying the necessary and sufficient optimality conditions, including the transversality condition (12.33). Moreover, this path *converges* to a balanced growth path with a constant effective capital-labor ratio, $\tilde{k}^*$, satisfying $f'(\tilde{k}^*) - \delta = \rho + \theta g$. So, at least for the *long run,* we may replace $\dot{\bar{c}}_t/\bar{c}_t$ in (8.20) with the constant rate of exogenous technical progress, g. Then (8.20) reduces to a required consumption rate of return that is now *constant* and *given* by the parameters in the problem:

$$r^{SP} = \rho + \theta g. \tag{8.22}$$

This r^{SP} is the prevalent suggestion for the choice of a social consumption discount rate. Note that as long as $g > 0$, r^{SP} will be positive even if $\rho = 0$. A higher θ will imply stronger discounting of additional consumption in the future because higher θ means faster decline in the marginal utility of consumption in response to a given rise in consumption. So with g equal to, say, 1.5% per year, the social discount rate r_{SP} is in fact more sensitive to the value of θ than to the value of ρ. Note also that a higher g raises r_{sp} and thereby reduces the incentive to save and invest.

Now consider a potential public investment project with time horizon T ($\leq \infty$) which comes at the expense of an investment in capital in the "usual" way as described above. Suppose the project is "minor" or "local" in the sense of not affecting the structure of the economy as a whole, like for instance the long-run per capita growth rate, g. Let the project involve an initial investment outlay of k_0 and a stream of real net revenues, $(z_t)_{t=0}^T$, assuming that both costs and benefits are measurable in terms of current consumption equivalents.[10] Letting r^{SP} serve to convert future consumption into current consumption equivalents, we calculate the present value of the project,

$$PV_0 = -k_0 + \int_0^T z_t e^{-r^{SP}t} dt.$$

[10] We bypass all the difficult issues involved in converting non-marketed goods like environmental qualities, biodiversity, health, and mortality risk etc. into consumption equivalents.

The project is worth undertaking if $PV_0 > 0$.

Limitations of the Ramsey formula $r^{SP} = \rho + \theta g$

For a closed economy, reasonably well described by the model, it makes sense to choose the r^{SP} given in (8.22) as discount rate for public investment projects if the economy is not "far" from its steady state. Yet there are several cases where modification is needed:

1. Assuming the model still describes the economy reasonably well, if the actual economy is initially "far" from its steady state and T is of moderate size, g in (8.22) should be replaced by a somewhat larger value if $\tilde{k}_0 < \tilde{k}$ (since in that case $\dot{c}/c > g$) and somewhat smaller value if $\tilde{k}_0 > \tilde{k}$ (since in that case $\dot{c}/c < g$).

2. The role of natural resources, especially non-renewable natural resources, has been ignored. If they are essential inputs, the parameter g needs reinterpretation and a negative value can not be ruled out *apriori*. In that case the social discount rate can in principle be negative.

3. Global problems like the climate change problem has an important international dimension. As there is great variation in the standard of living, c, and to some extent also in g across developed and developing countries, it might be relevant to include not only a parameter, θ_1, reflecting aversion towards consumption inequality over time and generations but also a parameter, θ_2, reflecting aversion towards *spatial* consumption inequality, i.e., inequality across countries.

4. Another limitation of the Ramsey formula (8.22), as it stands, is that it ignores *uncertainty*. In particular with a long planning horizon uncertainty both concerning the results of the investment project and concerning the socio-economic environment are important and should of course be incorporated in the analysis.

5. Finally, for "large" macroeconomic projects, the long-run technology growth rate may not be given, but dependent on the chosen policy. In that case, neither g nor r^{SP} are given. This is in fact the typical situation within the macroeconomic theory of *endogenous productivity growth.* Then formulation of a "broader" optimization problem is necessary and only parameters like the utility discount rate, ρ, and the elasticity of marginal utility of consumption, θ, will in this case serve as points of departure.

In connection with the climate change problem we shall in the next section apply a brief article by Arrow (2007)[11] to illustrate at least one way to deal with the problems 4 and 5.

8.4 The climate change problem from an economic point of view

There is now overwhelming agreement among scientists that man-made global warming is a reality. Mankind faces a truly large-scale and global economic problem with potentially dramatic consequences for economic and social development in centuries. Future economic evolution is *uncertain* and *depends* on policies chosen now. A series of possible "act now" measures has been described in detail in the voluminous *The economics of climate change. The Stern Review,* made by a team of researchers lead by the prominent British economist Nicholas Stern (Stern 2007).

The mentioned article by Arrow is essentially a comment on the *Stern Review* and on the debate about discount rates it provoked among climate economists as well as in the general public. It is Arrow's view that taking *risk aversion* properly into account implies that the conclusion of the Stern Review goes through: Mankind is better off to act *now* to reduce CO_2 emissions substantially rather than to risk the consequences of failing to meet this challenge. In many areas of life, high insurance premia are willingly paid to reduce risks. It is in such a perspective that part of the costs of mitigation should be seen.

8.4.1 Damage projections

As asserted by the Stern Review, the CO_2 problem is "the greatest and widest-ranging market failure ever seen" (Stern 2007, p.). The current level of CO_2 (including other greenhouse gases, in CO_2 equivalents) is today (i.e., in 2007) about 430 parts per million (ppm), compared with 280 ppm before the industrial revolution. Under a "business as usual" assumption the level will likely be around 550 ppm by 2035 and will continue to increase. The level 550 ppm is almost twice the pre-industrial level, and a level that has not been reached for several million years.

Most climate change models predict this would be associated with a rise in temperature of at least two degrees Centigrade, probably more. A continuation of "business as usual" is likely to lead to a trebling of CO_2 by the end

[11] Arrow won the Nobel Prize in Economics in 1972.

of the century and to a 50% likelihood of a rise in temperature of more than five degrees Centigrade. Five degrees Centigrade are about the same as the increase from the last ice age to the present.

The full consequences of such rises are not known. But drastic negative effects on agriculture in the heavily populated tropical regions due to changes in rainfall patterns are certain. The rise in the sea level will wipe out small island countries, and for example Bangladesh will loose much of its land area. A reversal of the Gulf Stream is possible, which could cause climate in Europe to resemble that of Greenland. Tropical storms and other kinds of extreme weather events will become severe and many glaciers will disappear and with them, valuable water supplies.

The challenging factors are that the emissions of CO_2 and other gases are *almost irreversible.* They constitute a global *negative externality at a grand scale*. The Stern Review assesses that avoiding such an outcome is possible by a series of concrete measures (carbon taxes, technology policy, international collective action) aimed at stopping or at least reducing the emission of green-house gases and mitigate their consequences. Out of the Stern Review's suggested range of the estimated costs associated with this, in his evaluation of an "act now" policy Arrow chooses a cost level of 1% of GNP every year forever (see below).

According to many observers, postponing action is likely to increase both risks and costs. The Stern Review suggests that the costs of action *now* are less than the costs of inaction because the marginal damages of rising temperature increase strongly as temperatures rise. In the words by Nobel Laureate, Joseph Stiglitz: "[The Stern Review] makes clear that the question is not whether we can afford to act, but whether we can afford not to act" (Stiglitz, 2007).

8.4.2 Uncertainty, risk aversion, and the certainty-equivalent loss

Since there is *uncertainty* about the size of the future damages, we follow Arrow's attempt to convert this uncertainty into the *certainty-equivalent* damage.

Given preferences involving risk aversion, an uncertain gain can be evaluated as being equivalent to a single gain *smaller* than the expected value (the "average") of the possible outcomes. With the green-house gas effect mankind is facing an uncertain *damage* which should be evaluated as being equivalent to a single loss *greater* than the expected value of the possible damages. For the so-called High-climate Scenario (considered by Arrow to

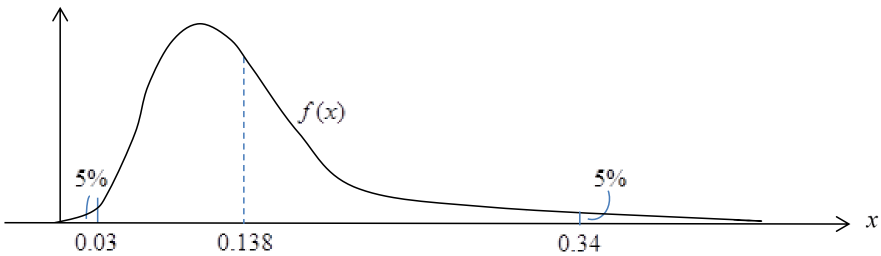

Figure 8.1: The density function of the per capita consumption loss X in year 2200.

be the best-substantiated scenario) the Stern Review estimates that by year 2200 the losses in global GNP per capita, by following a "business as usual" policy compared with , have an *expected* value of 13.8% of what global GNP per capita would be if green-house gas concentration is prevented from exceeding 550 ppm. The estimated loss distribution has a 0.05 percentile of about 3% and a 0.95 percentile of about 34%.

Assuming consumption per capita, c, in year 2200 is proportional to GNP per capita in year 2200, let us recapitulate:

$$\begin{aligned} \text{under "mitigation now" policy (MNP):} \qquad & c = c_1, \\ \text{under "business as usual" (BAU):} \qquad & c = (1 - X)c_1 \equiv c_0, \end{aligned}$$

where c_1 is considered given while X is a stochastic variable measuring the fraction of c_1 lost in year 2200 due to the damage occurring under BAU. A probability density function of X according to the High-climate Scenario is represented by $f(x)$ in Figure 8.1. The expected loss of $EX = \int_0^1 xf(x)dx$ $= 0.138$ is indicated and so are the 5th and 95th percentiles of 0.03 and 0.34, respectively.[12] The distribution is right-skew.

Let x_0 denote the *certainty-equivalent loss*, that is, the number x_0 satisfying

$$u((1 - x_0)c_1) = Eu((1 - X)c_1) = Eu(c_0). \tag{8.23}$$

This means that an agent with preferences expressed by u is indifferent between facing a certain loss of size x_0 or an uncertain loss, X, that has density function f.

The condition (8.23) is illustrated in Figure 8.2. The density function for the stochastic BAU consumption level, c_0, is indicated in the lower panel of

[12] The Stern Review estimates that $X < 0$ has zero probability.

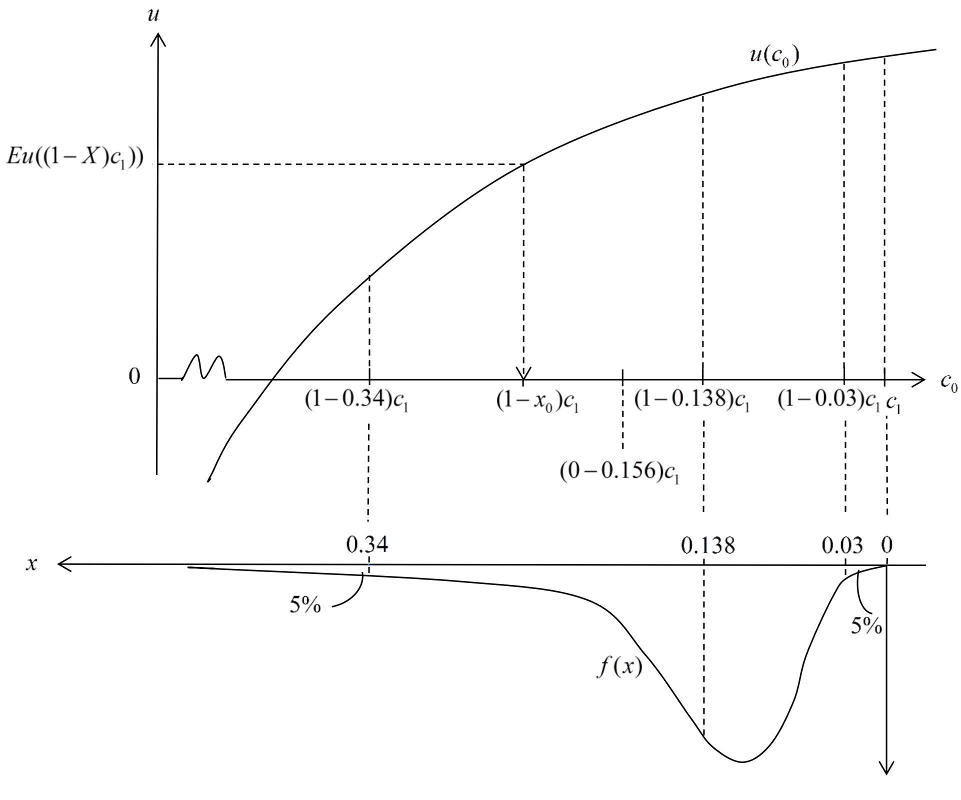

Figure 8.2: The certanty-equivalent loss, x_0, assuming expected utility is known.

the figure by a reversed coordinate system. If the utility function is specified and one knows the complete density function, then $Eu((1-X)c_1)$ is known and the certainty-equivalent BAU consumption level $(1-x_0)c_1$, can be read off the diagram.

The instantaneous utility function chosen by Arrow as well as Stern is of the CRRA form (8.21). Arrow proposes the value 2 for θ, while the Stern Review relies on $\theta = 1$ which by many critics was considered "too low" from a descriptive-empirical point of view. As mentioned above, in a context of uncertainty, θ not only measures the aversion towards consumption inequality across time and generations but also the level of *relative risk aversion*.

The problem now is that the loss density function $f(x)$ is *not* known. The Stern Review only reports an estimated mean of 0.138 together with estimated 5th and 95th percentiles of 0.03 and 0.34, respectively. This does not suffice for calculation of a good estimate of expected utility, $Eu((1-X)c_1)$. At best one can give a rough approximation. Arrow's approach to this problem is to split the probability mass into two halves and place them on the 5th and 95th percentiles, respectively, assuming this gives a reasonable approximation:

$$Eu((1-X)c_1) \approx u((1-0.03)c_1)0.5 + u((1-0.34)c_1)0.5. \tag{8.24}$$

With $u(c)$ given as CRRA, by (8.23) and (8.24) we thus have

$$\frac{[(1-x_0)c_1]^{1-\theta}}{1-\theta} \approx \frac{[(1-0.03)c_1]^{1-\theta}}{1-\theta}0.5 + \frac{[(1-0.34)c_1]^{1-\theta}}{1-\theta}0.5,$$

since the additive constant $-1/(1-\theta)$ cancels out on both sides. We see that also $c_1^{1-\theta}/(1-\theta)$ cancels out on both sides so that we are left with

$$(1-x_0)^{1-\theta} \approx (1-0.03)^{1-\theta}0.5 + (1-0.34)^{1-\theta}0.5.$$

With $\theta = 2$ the approximative estimate of the certainty-equivalent loss is $\hat{x}_0 = 0.21$, that is "about 20%" (of GNP per capita in year 2200) as Arrow says (Arrow 2007, p. 5).[13]

Here we shall proceed with this estimate of the certainty-equivalent loss while in the appendix we briefly discuss the quality of the estimate. On average the estimated certainty-equivalent loss corresponds to a decrease of the expected growth rate per year of GNP per capita between year 2001 and year 2200 from $g_1 = 1.3\%$ (the base rate of GNP per capita growth before the damages by further "business as usual") to $g_0 = 1.2\%$ per year.

[13] Although the calculation behind these "about 20%" is not directly reported in Arrow's brief article, he has in an e-mail to me confirmed that (8.24) *is* the applied method.

8.4.3 Comparing benefit and costs

Avoiding the projected fall in average per capita consumption growth is thus the *benefit* of the "mitigation now" policy while the *costs* amount to the above-mentioned 1% of GNP every year forever.

The criterion for assessing whether the "mitigation now" policy is worth the costs is the social (in fact "global") welfare function presented in (8.12) above with instantaneous utility being of CRRA form.[14] Following Arrow we let θ equal 2 (while Stern has $\theta = 1$).

The Stern Review has been criticized by several economic analysts for adopting "too low" values of both the two intergenerational preference parameters, θ and ρ. As to the rate of time preference, ρ, following the "descriptive approach", these critics argue that a level about 1-3% per year is better in line with a backward calculation from observed market rates of return. Anticipating such criticism, the Stern Review fights back by claiming that such high values are not ethically defensible since they amount to discriminating future generations for the only reason that they belong to the future. As mentioned in Section 8.3.1, Stern argues that the only ethically defensible reason for choosing a positive ρ is that there always is a small risk of extinction of the human race due to for example a devastating meteorite or nuclear war. Based on this view, Stern chooses a value of ρ close to zero, namely $\rho = 0.001$.[15] As Arrow argues and as we shall see in a moment, this disagreement as to the size of ρ is not really crucial given the involved benefit and costs.

The break-even utility discount rate

Assuming balanced growth with some constant productivity growth rate, g, consumption per capita will also grow at the rate g, i.e., $c_t = c(0)e^{gt}$ for all $t \geq 0$.[16] Then

$$u(c_t) = \frac{(c(0)e^{gt})^{1-\theta}}{1-\theta} - \frac{1}{1-\theta},$$

[14] We ignore the minor difference vis-a-vis the Stern Review that it brings in a so-called scrap value function subsuming discounted utility from year 2200 to infinity.

[15] This is in fact a relatively high value of ρ in the sense that it suggests that the probability of extinction within one hundred years from now is as high as 9.5% ($1-P(X < x)) = 1 - e^{-0.1} = 0.095$). But as the Stern Review (p. 53) indicates, the term "extinction" is meant to include "partial extinction by som exogenous or man-made force which has little to do with climate change".

[16] To avoid confusion with the above c_0, we write initial per capita consumption $c(0)$ rather than c_0.

along the balanced growth path. As adding or subtracting a constant from the utility function changes neither the preferences nor the economic behavior, from now we skip the constant $(1-\theta)^{-1}$. Under the BAU policy the social welfare function then takes the value[17]

$$\begin{aligned} W_0 &= \frac{c(0)^{1-\theta}}{1-\theta}\int_0^\infty (e^{g_0 t})^{1-\theta} e^{-(\rho-n)t} dt = \frac{c(0)^{1-\theta}}{1-\theta}\int_0^\infty e^{[(1-\theta)g_0-(\rho-n)]t} dt \\ &= \frac{c(0)^{1-\theta}}{1-\theta}\frac{1}{\rho-n-(1-\theta)g_0}. \end{aligned}$$

Let the value of the welfare outcome under the "mitigation now" policy be denoted W_1. According to the numbers mentioned above, the latter policy involves a *cost* whereby $c(0)$ is replaced by $c(0)' = 0.99c(0)$ and a *benefit* whereby $g_0 = 0.012$ is replaced by $g_1 = 0.013$.[18] We get

$$W_1 = \frac{(0.99c(0))^{1-\theta}}{1-\theta}\frac{1}{\rho-n-(1-\theta)g_1}.$$

Since the benefits of the "mitigation now" policy come in the future and the costs are there from date zero, we have $W_1 > W_0$ only if the effective utility discount rate, $\rho - n$, is below some upper bound. Let us calculate the least upper bound. With $\theta = 2$, we have

$$\begin{aligned} W_1 &= -(0.99c(0))^{-1}\frac{1}{\rho-n+g_1} > W_0 = -(c(0))^{-1}\frac{1}{\rho-n+g_0} \\ &\Rightarrow \frac{1}{0.99(\rho-n+g_1)} < \frac{1}{\rho-n+g_0} \\ &\Rightarrow 0.99(\rho-n+g_1) > \rho-n+g_0 \\ &\Rightarrow 0.01(\rho-n) < 0.99g_1 - g_0 = 0.00087 \\ &\Rightarrow \rho-n < 0.087 \text{ or } \rho-n < 8.7\% \text{ per year.} \end{aligned}$$

The *break-even level* for $\rho - n$ at which $W_1 = W_0$ is thus 8.7% per year.

As Arrow remarks, "no estimate of the pure rate of time preference even by those who believe in relatively strong discounting of the future has ever approached 8.5%".[19] The conclusion is that given the estimated certainty-

[17] The transversality condition holds and the utility integral W_0 is convergent if $\rho - n > (1-\theta)g_0$. In the present case where $\rho = 0.001$, $\theta = 2$ and $g_0 = 0.012$, W_0 is thus convergent for $n < \rho - (1-\theta)g_0 = \rho + g_0 = 0.013$. This inequality seems likely to hold.

[18] By taking $g_1 = 0.013 > g_0$ also after year 2200, we deviate a little from both Arrow and Stern in a direction favoring the Stern conclusion slightly.

[19] Possibly the difference between Arrow's 8.5% and our result is due to the point mentioned in the previous footnote. Another minor difference is that Arrow seemingly takes n to be zero since he speaks of the "pure rate of time preference" rather than the "effective rate of time preference", $\rho - n$.

equivalent loss, the "mitigation now" policy passes the cost-benefit test for *any* reasonable value of the pure rate of time preference.

It should be mentioned that there has been considerably disagreement also about other aspects of the Stern Review's investigation, not the least the time profiles for the projected benefits and costs.[20] So it is fair to say that "further sensitivity analysis is called for", as Arrow remarks. He adds: "Still, I believe there can be little serious argument over the importance of a policy of avoiding major further increases in combustion by-products" (Arrow 2007, p. 5)

8.5 Conclusion

In his brief analysis of the economics of the climate change problem Arrow (2007) finds the fundamental conclusion of the Stern Review justified even if one, unlike the Stern Review, heavily discounts the utility of future generations. In addition to discounting, risk aversion plays a key role in the argument. A significant part of the costs of mitigation is like an insurance premium society should be ready to pay.

The analysis above took a *computable risk approach*. For more elaborate accounts about uncertainty issues, also involving situations with systematic uncertainty, about c_1 for instance, increasing with the length of the time horizon as well as fundamental uncertainty, see the list of references, in particular the papers by Gollier and Weitzman.

We have been tacit concerning the difficult political economy problems about how to obtain coordinated international action vis-a-vis global warming. About this, see, e.g., Gersbach (2008) and Roemer (2010).

8.6 Appendix: A closer look at Arrow's estimate of the certainty loss

In this appendix we briefly discuss Arrow's estimate of the certainty-equivalent loss based on (8.24). The applied procedure would be accurate if the density function $f(x)$ were *symmetric* and the utility function $u(c)$ were *linear.*

So let us first consider the case of a linear utility function, $\tilde{u}(c)$, cf. the stippled positively sloped line in Figure 8.3. With $f(x)$ symmetric, EX coincides with the median of the distribution. Given the estimated 5th and 95th percentiles of 0.03 and 0.34, respectively, we would thus have EX

[20] See for example: http://en.wikipedia.org/wiki/Stern_Review#cite_ref-5

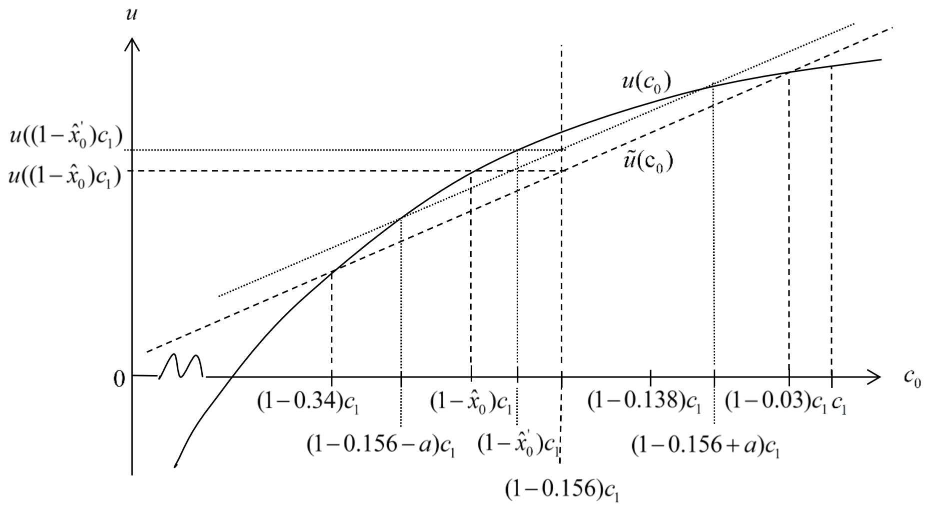

Figure 8.3: The case of symmetric density. Comparison of linear and strictly concave utility function.

$= (0.34 - 0.03)/2 = 0.156$. So $E((1-X)c_1) = (1-0.156)c_1$. In view of $\tilde{u}(c)$ being linear, we then get $\tilde{u}((1-0.156)c_1) = E\tilde{u}((1-X)c_1)$. And for this case an estimate of the certainty-equivalent loss, x_0, of course equals $EX = 0.156$.

The "true" density function, $f(x)$, is right-skew, however, and has $EX = 0.138$. In combination with the linear utility function, $\tilde{u}(c)$, this implies an estimate of x_0 equal to 0.138, that is, we get a *lower* value for the certainty-equivalent loss than with a symmetric density function.

Now let us consider the "true" utility function, $u(c)$. In Figure 8.3 it is represented by the solid strictly concave curve $u(c_0)$. Let us again imagine for a while that the density function is symmetric. As before, half of the probability mass would then be to the right of the mean of c_0, $(1-0.156)c_1$, and the other half to the left. The density function *might* happen to be such that the expected utility is just the average of utility at the 5th percentile and utility at the 95th percentile, that is, as if the two halves of the probability mass were placed at the 5th and 95th percentiles of 0.03 and 0.34, respectively; if so, the estimated certainty-equivalent loss is the $\hat{x}_0$ shown in Figure 8.3.

This would just be a peculiar coincidence, however. The probability mass of the symmetric density function could be more, or less, concentrated close to $EX = 0.156$. In case it is more concentrated, it is as if the two halves of the probability mass are placed at the consumption levels $(1 - 0.156 + a)c_1$ and $(1 - 0.156 - a)c_1$ for some "small" positive a, cf. Figure 8.3. The corresponding estimate of the certainty-equivalent loss is denoted $\hat{x}'_0$ in the figure and is smaller than $\hat{x}_0$ so that the associated c_0 is larger than before.

Finally, we may conjecture that allowing for the actual right-skewness of the density function will generally tend to *diminish* the estimate of the certainty-equivalent loss.

The conclusion seems to be that Arrow's procedure, as it stands, is questionable. Or the procedure is based on assumptions about the properties of the density function not spelled out in the article. Anyway, sensitivity analysis is called for. This could be part of an interesting master's thesis by someone better equipped in mathematical statistics than the present author is.

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Chapter 9

Human capital, learning technology, and the Mincer equation

This chapter is meant as a supplement to Acemoglu, §10.1-2 and §11.2. First an overview of different approaches to human capital formation in macroeconomics is given. Next we go into detail with one of these approaches, the life-cycle approach. In Section 9.3 a simple model of the choice of schooling length is considered. Finally, Section 9.4 presents the theory behind the empirical relationship named the Mincer equation.[1] In this connection it is emphasized that the Mincer equation should be seen as an equilibrium relationship for relative wages at a given point in time rather than as a production function for human capital.

9.1 Conceptual issues

We define *human capital* as the stock of productive skills embodied in an individual. Increases in this stock occurs through formal education and on-the-job-training. By contributing to the maintenance of life and well-being, also health care is of importance for the stock of human capital and the incentive to invest in human capital.

Since human capital is embodied in individuals and can only be used one place at a time, it is a *rival* and *excludable* good. Human capital is thus very different from technical knowledge. We think of *technical knowledge* as a list of instructions about how different inputs can be combined to produce a certain output. A principle of chemical engineering is an example of a piece

[1] After Mincer (1958, 1974).

of technical knowledge. In contrast to human capital, technical knowledge is a *non-rival* and only partially excludable good. Competence in *applying* technical knowledge is one of the skills that to a larger or smaller extent is part of human capital.

9.1.1 Macroeconomic approaches to human capital

In the macroeconomic literature there are different theoretical approaches to the modelling of human capital. Broadly speaking we may distinguish these approaches along two "dimensions": 1) What characteristics of human capital are emphasized? 2) What characteristics of the decision maker investing in human capital are emphasized? Combining these two "dimensions", we get Table 1.

Table 1. Macroeconomic approaches to the modelling of human capital

The character of the decision maker	*The character of human capital (hc):* Is hc treated as essentially different from physical capit	
	No	Yes
Solow-type rule-of-thumb households	Mankiw et al. (1992)	
Infinitely-lived family "dynasties" (the representative agent approach)	Barro&Sala-i-Martin (2004)	Lucas (1988)
Finitely-lived individuals going through a life cycle (the life cycle approach)		Ben-Porath (1967) Heijdra&Romp (2009)

My personal opinion is that for most issues the approach in the lower-right corner of Table 1 is preferable, that is, the approach treating human capital as a distinct capital good in a life cycle perspective. The viewpoint is:

First, by being *embodied* in a person and being lost upon death of this person, human capital is very different from physical capital. In addition, investment in human capital is *irreversible* (can not be recovered). Human capital is also distinct in view of the limited extend to which it can be used as a collateral, at least in non-slave societies. Financing an investment in physical capital, a house for example, by credit is comparatively easy because the house can serve as a collateral. A creditor can not gain title to a person, however. At most a creditor can gain title to a part of that person's future earnings in excess of a certain level required for a "normal" or "minimum" standard of living.

Second, educational investment is closely related to life expectancy and the life cycle of human beings: school - work - retirement. So a life cycle perspective seems the natural approach. Fortunately, convenient macroeconomic frameworks incorporating life cycle aspects exist in the form of overlapping generations models (for example Diamond's OLG model or Blanchard's continuous time OLG model).

9.1.2 Human capital and the efficiency of labor

Generally we tend to think of human capital as a combination of different skills. Macroeconomics, however, often tries (justified or not) to boil down the notion of human capital to a one-dimensional entity. So let us imagine that the current stock of human capital in society is measured by the one-dimensional index H. With L denoting the size of the labor force, we define $h \equiv H/L$, that is, h is the average stock of human capital in the labor force. Further, let the "quality" (or "efficiency") of this stock in production be denoted q (under certain conditions this quality might be proxied by the average real wage per man-hour). Then it is reasonable to link q and h by some increasing *quality function*

$$q = q(h), \qquad \text{where } q(0) \geq 0, q' > 0. \tag{9.1}$$

Consider an aggregate production function, $\tilde{F}$, giving output per time unit at time t as

$$Y = \tilde{F}(K, q(h)L; t), \qquad \frac{\partial \tilde{F}}{\partial t} > 0, \tag{9.2}$$

where K is input of physical capital. The third argument of $\tilde{F}$ is time, t, indicating that the production function is time-dependent due to technical progress.

Generally the analyst would prefer a measure of human capital such that the quality of human capital is proportional to the stock of human capital, allowing us to write $q(h) = h$ by normalizing the factor of proportionality to be 1. The main reason is that an expedient variable representing human capital in a model requires that the analyst can decompose the real wage per working hour of a given person multiplicatively into two factors, the real wage per unit of human capital per working hour and the stock of human capital, h. That is, an expedient human capital concept requires that we can write

$$w = w_h \cdot h, \tag{9.3}$$

where w_h is the real wage per unit of human capital per working hour. Indeed, if we can write

$$Y = \tilde{F}(K, hL; t), \tag{9.4}$$

under perfect competition we can write

$$w = \frac{\partial Y}{\partial L} = \tilde{F}_2(K, hL; t)h = w_h \cdot h.$$

Under Harrod-neutral technical progress, (9.4) would take the form

$$Y = \tilde{F}(K, hL; t) = F(K, AhL) \equiv F(K, EL), \tag{9.5}$$

where $E \equiv A \cdot h$ is the "effective" labor input. The proportionality between E and h will under perfect competition allow us to write

$$w = \frac{\partial Y}{\partial L} = \tilde{F}_2(K, AhL; t)h = w_E \cdot E = w_E \cdot A \cdot h = w_h \cdot h.$$

So with the introduction of the technology level, A, an additional decomposition, $w_h = w_E \cdot A$ comes in, while the original decomposition in (9.3) remains valid.

Whether or not the desired proportionality $q(h) = h$ can be obtained depends on how we model the formation of the "stuff" h. Empirically it turns out that treating the formation of human capital as similar to that of physical capital does *not* lead to the desired proportionality.

Treating the formation of human capital as similar to that of physical capital

Consider a model where human capital is formed in a way similar to physical capital. The Mankiw-Romer-Weil (1992) extension of the Solow growth model with human capital is a case in point. Non-consumed aggregate output is split into one part generating additional physical capital one-to-one, while the other part generates additional human capital one-to-one. Then for a closed economy in continuous time we can write:

$$\begin{aligned} Y &= C + I_K + I_H, \\ \dot{K} &= I_K - \delta_K K, \qquad \delta_K > 0, \\ \dot{H} &= I_H - \delta_H H, \qquad \delta_H > 0, \end{aligned} \tag{9.6}$$

where I_K and I_H denote gross investment in physical and human capital, respectively. This approach essentially assumes that human capital is produced by the same technology as consumption and investment goods.

Suppose the huge practical measurement problems concerning I_H have been somehow overcome. Then from long time series for I_H an index for H_t can be constructed by the *perpetual inventory method* in a way similar to the way an index for K_t is constructed from long time series for I_K. Indeed, in discrete time, with $0 < \delta_H < 1$, we get, by backward substitution,

$$\begin{aligned} H_{t+1} &= I_{H,t} + (1-\delta_H)H_t = I_{H,t} + (1-\delta_H)\left[I_{H,t-1} + (1-\delta_H)H_{t-1}\right] \\ &= \sum_{i=0}^{T}(1-\delta_H)^i I_{H,t-i} + (1-\delta_H)^{T+1}H_{t-T}. \end{aligned} \quad (9.7)$$

From the time series for I_H, an estimate of δ_H, and a rough conjecture about the initial value, H_{t-T}, we can calculate H_{t+1}. The result will not be very sensitive to the conjectured value of H_{t-T} since for large T the last term in (9.7) becomes very small.

In principle there need not be anything wrong with this approach. A snag arises, however, if, without further notice, the approach is combined with an explicit or implicit postulate that $q(h)$ is proportional to the "stuff", h, brought into being in the way described by (9.6). The snag is that the empirical evidence does not support this when the formation of human capital is modelled as in (9.6). This is what, for instance, Mankiw, Romer, and Weil (1992) find in their cross-country regression analysis based on the approach in equation (9.6). One of their conclusions is that the following production function for a country's GDP is an acceptable approximation:

$$Y = BK^{1/3}H^{1/3}L^{1/3}, \quad (9.8)$$

where B stands for the total factor productivity of the country and is generally growing over time.[2] Defining $A = B^{3/2}$ and applying that $H = hL$, we can write (9.8) on the form

$$Y = BK^{1/3}(hL)^{1/3}L^{1/3} = K^{1/3}(h^{1/2}AL)^{2/3}.$$

That is, we end up with the form $Y = F(K, q(h)AL)$ where $q(h) = h^{1/2}$, not $q(h) = h$. We should thus not expect the real wage to rise in proportion to h, when h is considered as some "stuff" formed in a way similar to the way physical capital is formed.

Before proceeding, a terminological point is in place. Why do we call $q(h)$ in (9.2) a "quality function" rather than simply a "productivity function"? The reason is the following. With perfect competition *and CRS*, in equilibrium the real wage per man-hour would be $w = \partial Y/\partial L = F_2'(K, Aq(h)L))Aq(h)$

[2] The way Mankiw-Romer-Weil measure I_H is indirect and questionable. In addition, the way they let their measure enter the regression equation has been criticized for confounding the effects of the human capital *stock* and human capital *investment*, cf. Gemmel (1996) and Sianesi and Van Reenen (2003). It will take us too far to go into detail with these problems here.

$= \left[f(\tilde{k}) - \tilde{k}f'(\tilde{k})\right] Aq(h)$, where $\tilde{k} \equiv K/(Aq(h)L)$. So, with a converging $\tilde{k}$, the long-run growth rate of the real wage would in continuous time tend to be

$$g_w = g_A + g_q.$$

In this context we are inclined to identify "labor productivity" with $Aq(h)$ rather than just $q(h)$ and "growth in labor productivity" with $g_A + g_q$ rather than just g_q. So a distinct name for q seems appropriate and an often used name is "quality". The latter name might seem more straightforward. Nevertheless we avoid it because it implies a risk of confusion with the standard meaning of terms like "labor productivity" and "growth in labor productivity". Indeed, with a converging $\tilde{k}$, the long-run growth rate of the real wage would in continuous time be

$$g_w = g_A + g_q.$$

We are in this context inclined to identify "labor productivity" with $Aq(h)$ rather than just $q(h)$ and "growth in labor productivity" with $g_A + g_q$ rather than just g_q. So a distinct name for q seems appropriate and an often used name is "quality".

The conclusion so far is that specifying human capital formation as in (9.6) does not generally lead to a linear quality function. To obtain the desired linearity we have to specify the formation of human capital in a way different from the equation (9.6). This dissociation with the approach (9.6) applies, of course, also to its equivalent form on a per capita basis,

$$\dot{h} = (\frac{\dot{H}}{H} - n)h = \frac{I_H}{L} - (\delta_H + n)h. \tag{9.9}$$

(In the derivation of (9.9) we have first calculated the growth rate of $h \equiv H/L$, then inserted (9.6), and finally multiplied through by h.)

9.2 The life-cycle approach to human capital

In the life-cycle approach to human capital formation we perceive h as the human capital embodied in a single individual and lost upon death of this individual. We study how h evolves over the lifetime of the individual as a result of both educational investment (say time spent in school) and work experience. In this way the life-cycle approach recognizes that human capital is different from physical capital. By seeing human capital formation as the result of individual learning, the life-cycle approach opens up for distinguishing between the production technologies for human and physical capital.

Thereby the life-cycle approach offers a better chance for obtaining the linear relationship, $q(h) = h$.

Let the human capital at date t of an individual "born" (i.e., entering life beyond childhood) at date 0 be denoted h_x, where x stands for the age of this individual. Let the total time available per time unit for study, work, and leisure be normalized to 1. Let s_x denote the fraction of time the individual spends in school at age x. This allows the individual to go to school only part-time and spend the remainder of non-leisure time working. If ℓ_x denotes the fraction of time spent at work, we have

$$0 \leq s_x + \ell_x \leq 1.$$

The fraction of time used as leisure (or child rearing, say) at age x is $1-s_x-\ell_x$. If full retirement occurs at age $\bar{x}$, we have $s_x = \ell_x = 0$ for $x \geq \bar{x}$.

As a slight generalization of equation (10.2) in Acemoglu (2009, p. 360, where leisure is not considered), we assume that the increase in h_x per time unit generally depends on four variables: current time in school, current time at work, human capital already obtained, and current calendar time itself, that is,

$$\dot{h}_x \equiv \frac{dh_x}{dx} = G(s_x, \ell_x, h_x, t), \qquad h_0 \ \geq 0 \text{ given.} \tag{9.10}$$

The function G can be seen as a production function for human capital - in brief a *learning technology*. The first argument of G reflects the role of formal education. Empirically, the primary input in formal education is the time spent by the students studying; this time is not used in work or leisure and it thereby gives rise to an opportunity cost of studying.[3] The second argument of G takes work experience into account and the third argument allows for the already obtained level of human capital to affect the strength of the influence from s_x and ℓ_x. Finally, the fourth argument, current calendar time allows for changes over time in the learning technology (organization of the learning process). If we maintain our starting point that time of birth is 0, we can replace t by x.

More generally, consider an individual "born" at date $v \leq t$ (v for vintage). If still alive at time t, the age of this individual is $x \equiv t - v$. Let us assume that (9.10) is valid for *general* time of birth. Then obtained stock of human capital at age x will be

$$h_x = h_0 + \int_0^x G(s_u, \ell_u, h_u, v + u)du.$$

[3] We may perceive the costs associated with teachers' time and educational buildings and equipment as being either quantitatively negligible or implicit in the function symbol G.

A basic supposition in the life-cycle approach is that it is possible to specify the function G such that a person's time-t human capital involves a time-t labor productivity proportional to this amount of human capital and thereby, under perfect competition, a real wage proportional to this human capital.

Below we consider four specifications of the learning technology that one may encounter in the literature.

EXAMPLE 1 In a path-breaking model by the Israeli economist Ben-Porath (1967) the learning technology is specified this way:

$$\dot{h}_x = g(s_x h_x) - \delta h_x, \qquad g' > 0, g'' < 0,\ \delta > 0, \qquad h_0 > 0.$$

Here time spent in school is more efficient in building human capital the more human capital the individual has already. Work experience does not add to human capital formation. The parameter δ enters to reflect obsolescence (due to technical change) of skills learnt in school and/or mortality. □

EXAMPLE 2 Growiec (2010) and Growiec and Groth (2013) study the aggregate implications of a learning technology specified this way:

$$\dot{h}_x = (\lambda s_x + \xi \ell_x) h_x, \qquad \lambda > 0, \xi \geq 0, \qquad h_0 > 0. \tag{9.11}$$

Here λ measures the efficiency of schooling and ξ the efficiency of work experience. The effects of schooling and (if $\xi > 0$) work experience are here proportional to the level of human capital already obtained by the individual (a strong assumption which may be questioned). The linear differential equation (9.11) allows an explicit solution,

$$h_x = h_0 e^{\int_0^x (\lambda s_u + \xi \ell_u) du}, \tag{9.12}$$

a formula valid as long as the person is alive. This result has some affinity with the approach by Lucas (1988) and with the "Mincer equation", to be considered below. □

EXAMPLE 3 Here we consider an individual with exogenous and constant leisure. Hence time available for study and work is constant and conveniently normalized to 1 (as if there were no leisure at all). In the beginning of life beyond childhood the individual goes to school full-time in S time units (years) and thereafter works full-time until death (no retirement). Thus

$$s_x = \begin{cases} 1 & \text{for } 0 \leq x < S, \\ 0 & \text{for } x \geq S. \end{cases} \tag{9.13}$$

We further simplify by ignoring the effect of work experience (or we may say that work experience just offsets obsolescence of skills learnt in school). The learning technology is specified as

$$\dot{h}_x = \eta x^{\eta-1} s_x, \qquad \eta > 0, \quad h_0 = 0, \tag{9.14}$$

If $\eta < 1$, it becomes more difficult to learn more the longer you have already been to school. If $\eta > 1$, it becomes easier to learn more the longer you have already been under education.

The specification (9.13) implies that throughout working life the individual has constant human capital equal to S^η. Indeed, integrating (9.9), we have for $t \geq S$ and until time of death,

$$h_x = h_0 + \int_0^x \dot{h}_u du = 0 + \int_0^S \eta u^{\eta-1} du = u^\eta\big|_0^S = S^\eta. \tag{9.15}$$

So the parameter η measures the elasticity of human capital w.r.t. the number of years in school. As briefly commented on in the concluding section, there is some empirical support for the power function specification in (9.15) and even the hypothesis $\eta = 1$ may not be rejected. □

In Example 1 there is no explicit solution for the level of human capital. Then the solution can be characterized by phase diagram analysis (as in Acemoglu, §10.3). In the examples 2 and 3 we can find an explicit solution for the level of human capital. In this case the term "learning technology" is used not only in connection with the original differential form as in (9.10), but also for the integrated form, as in (9.12) and (9.15), respectively. Sometimes the integrated form, like (9.15), is called a *schooling technology*.

EXAMPLE 4 Here we still assume the setup in (9.13) of Example 3, including the absence of both after-school learning and gradual depreciation. But the right-hand side of (9.14) is generalized to $\varphi(x)s_x$, where $\varphi(x)$ is some positively valued function of age. Then we end up with human capital after leaving school equal to some increasing function of S :

$$h = h(S), \qquad \text{where } h(0) \geq 0, \ h' > 0. \tag{9.16}$$

In cross-section or time series analysis it may be relevant to extend this by writing $h = ah(S)$, $a > 0$; the parameter a would then reflect quality of schooling. In the next section we shall focus on the form (9.16). □

Before proceeding, let us briefly comment on the problem of aggregation over the different members of the labor force at a given point in time. In the aggregate framework of Section 9.1 multiplicity of skill types and job types is ignored. Human capital is treated as a one-dimensional and additive production factor. In production functions like (9.4) only aggregate human capital, H, matters. So output is thought to be the same whether the input is 2 million workers, each with one unit of human capital, or 1 million workers, each with 2 units of human capital. In human capital theory this questionable assumption is called the *perfect substitutability assumption* or the *efficiency unit assumption* (Sattinger, 1980). If we are willing to impose this assumption going from micro to macro at a given point in time is conceptually simple. With h denoting individual human capital and $f(h)$ being the density function (so that $\int_0^\infty f(h)dh = 1$), we find average human capital in the labor force as $\bar{h} = \int_0^\infty hf(h)dh$ and aggregate human capital as $H = \bar{h}L$, where L is the size of the labor force. To build a theory of the evolution over time of the density function, $f(h)$, is, however, a complicated matter. This is because heterogeneity within the different cohorts regarding both schooling and retirement and changing fertility and mortality patterns are involved.

If we want to open up for a distinction between different types of jobs and different types of labor, say, skilled and unskilled labor, we may replace the production function (9.4) with

$$Y = \tilde{F}(K, h_1 L_1, h_2 L_2; t), \tag{9.17}$$

where L_1 and L_2 indicate man-hours delivered by the two types of workers, respectively, and h_1 and h_2 are the associated human capital levels (measured in efficiency units for each of the two kinds of jobs), respectively. This could be the basis for studying skill-biased technical change. In passing we note that if and only if it is possible to rewrite this production function (9.17) as $Y = F(K, H; t)$, where $H = h_1 L_1 + h_2 L_2$, are the two types of labor *perfectly substitutable.*

9.3 Choosing length of education

9.3.1 Human wealth

Whereas human capital is a production factor, *human wealth* is the present value of expected future labor earnings generated by this production factor.

We assume (realistically!) that expected lifetime is finite while the age at death, X, is stochastic (uncertain). Let $\ell_{t-v}(S)$ denote the supply of labor to the labor market at time t by a person born at time v who at birth decides to attend school full-time in the first S years of life and after that work full time until death. As $\ell_{t-v}(S)$ depends on the stochastic variable, X, $\ell_{t-v}(S)$ is itself a stochastic variable with two possible outcomes:

$$\ell_{t-v}(S) = \begin{cases} 0 & \text{when } t \leq v+S \text{ or } t > v+X, \\ \ell & \text{when } v+S < t \leq v+X, \end{cases}$$

where $\ell > 0$ is an exogenous constant ("full-time" working). We further assume that the probability for a newborn to survive at least until age x is $P(X > x) = e^{-mx}$, where $m > 0$. Although far from realistic, for simplicity we assume that the involved *mortality rate*, m, is independent of x (and also independent of calendar time).[4]

Let $w_t(S)$ denote the real wage received *per working hour* delivered at time t by a person who after S years in school works ℓ hours per time unit, say per year, until death. This allows us to write the present value as seen from time v of expected earned lifetime earnings, i.e., the human wealth, for a person "born" at time v as

$$\begin{aligned} HW(v,S) &= 0 + E_v\left(\int_{v+S}^{v+X} w_t(S)\ell e^{-r(t-v)}dt\right) \\ &= 0 + E_v\left(\int_{v+S}^{\infty} w_t(S)\ell_{t-v}(S) e^{-r(t-v)}dt\right) \\ &= \int_{v+S}^{\infty} E_v\left(w_t(S)\ell_{t-v}(S)\right) e^{-r(t-v)}dt, \end{aligned}$$

as in this context the integration operator $\int_{v+S}^{\infty}(\cdot)dt$ acts like a discrete-time summation operator $\sum_{t=v}^{\infty}$. Hence,

$$\begin{aligned} HW(v,S) &= \int_{v+S}^{\infty} w_t(S)e^{-r(t-v)}\left(\ell \cdot P(X > t-v) + 0 \cdot P(X \leq t-v)\right)dt \\ &= \int_{v+S}^{\infty} w_t(S)e^{-r(t-v)}\ell e^{-m(t-v)}dt \\ &= \int_{v+S}^{\infty} w_t(S)\ell e^{-(r+m)(t-v)}dt. \qquad (9.18) \end{aligned}$$

In writing the present value of the expected stream of labor income this way, we have assumed that:

[4] If X denotes the uncertain age at death (a stochastic variable) and x is a nonnegative number, the mortality rate (or "hazard rate" of death) at the age x is defined as $\lim_{\Delta x \to 0} \frac{1}{\Delta x} P\left(X \leq x + \Delta x | \, X > x\right)$. In the present model this is assumed equal to a constant, m. The unconditional probability of not reaching age x is then $P(X \leq x) = 1 - e^{-mx}$. Hence the density function is $f(x) = me^{-mx}$ and life expectancy is $E(X) = \int_0^{\infty} xme^{-mx}dx = 1/m$. This is like in the "perpetual youth" overlapping generations model by Blanchard (1985).

1. There is no educational fee.

2. The risk-free interest rate, r, is constant over time.

We now introduce some additional assumptions:

3. Labor efficiency is proportional to human capital so that the real wage per working hour for a person with human capital h is $w_t = \hat{w}_t h$. Here $\hat{w}_t$ is the real wage *per unit of human capital* per working hour at time t faced by the individual.[5]

4. Human capital is formed in accordance with the conditions in Example 4 of the previous section so that a person with S years of schooling has $h = h(S)$, $h' > 0$.

5. Owing to technical progress at a constant rate $g \in [0, r+m) \geq 0$, $\hat{w}_t = \hat{w}_0 e^{gt}$. So technical progress makes a given h more and more productive (direct complementarity between technology level and human capital as in (9.5) above).

By the assumption 3, 4, and 5 (9.18) gives

$$\begin{aligned} HW(v,S) &= \int_S^\infty \hat{w}_t h(S)\ell e^{-(r+m)(t-v)} dt \qquad (9.19) \\ &= \hat{w}_0 e^{gv} h(S)\ell \int_S^\infty e^{[g-(r+m)](t-v)} dt \\ &= \hat{w}_0 e^{gv} h(S)\ell \left(\frac{e^{[g-(r+m)](t-\nu)}}{g-(r+m)} \bigg|_{\nu+S}^\infty \right) = \hat{w}_0 e^{g\nu} h(S)\ell \frac{e^{[g-(r+m)]S}}{r+m-g}. \end{aligned}$$

From now on we chose measurement units such that the "normal" working time per year is 1 rather than ℓ.

9.3.2 A perfect credit and life annuity market

Assuming the students themselves have to finance their costs of living, the question is: how do students make a living while studying? While studying, they borrow and later in life, when they have an income, they repay the loans with interest.

In this context we shall introduce the simplifying assumption of a *perfect credit and life annuity market*. The financial sector will be unwilling to offer the students loans at the going risk-free interest rate, r. Indeed, a

[5] In the previous section this variable was called w_h.

creditor faces the risk that the student dies before having paid off the debt including the compound interest. Given the described constant mortality rate and existence of a perfect credit and life insurance market, it can be shown that the equilibrium interest rate on student loans is the "actuarial rate", $r + m$. (This result presupposes that the insurance companies have negligible administration costs.)

If the individual later in life, after having paid off the debt and obtained a positive net financial position, places the savings on life annuity accounts in life insurance companies, the *actuarial rate*, $\hat{r} = r + m$, will also be the equilibrium rate of return received (until death) on these deposits. At death the liability of the insurance company is cancelled.

The advantage of saving in life annuities (at least for people without a bequest motive) is that life annuities imply a transfer of income from after time of death to before time of death by offering a higher rate of return than risk-free bonds, but only until the depositor dies. At that time the total deposit is automatically transferred to the insurance company in return for the high annuity payouts while the depositor was alive.[6]

9.3.3 Maximizing human wealth

Suppose that neither the educational process itself nor the resulting stock of human capital enter the utility function (no "joy of going to school", no "joy of being a learned person"). In this perspective human capital is only an investment good (not also a consumption good).[7]

If moreover there is no utility from leisure, the educational decision can be separated from whatever plan for the time path of consumption and saving through life the individual may decide (cf. the Separation Theorem in Acemoglu, §10.1). That is, the only incentive for acquiring human capital is to increase the human wealth $HW(\nu, S)$ given in (9.19).

An interior solution to the problem $\max_S HW(v, S)$ satisfies the first-order condition:

$$\begin{aligned} \frac{\partial HW}{\partial S}(v, S) &= \frac{\hat{w}_0}{r+m-g}\left[h'(S)e^{[g-(r+m)]S} - h(S)e^{[g-(r+m)]S}(r+m-g)\right] \\ &= HW(S)\left[\frac{h'(S)}{h(S)} - (r+m-g)\right] = 0, \end{aligned} \tag{9.20}$$

[6] Whatever name is in practice used for the real world's private pension arrangements, including labor market pension arrangements, many of them have such life annuity ingredients.

[7] For a broader conception of human capital, see for instance Sen (1997).

from which follows

$$\frac{h'(S)}{h(S)} = r + m - g \equiv \tilde{r}. \tag{9.21}$$

We call (9.21) the *schooling first-order condition* and $\tilde{r}$ the *effective discount rate* for the schooling decision. In the optimal plan this equals the effective discount rate appearing on the right-hand side of (9.21), namely the interest rate adjusted for (a) the approximate probability of dying within a year from "now", $1 - e^{-m} \approx m$) and (b) wage growth due to technical progress. The trade-off faced by the individual is the following: increasing S by one year results in a higher level of human capital (higher future earning power) but postpones by one year the time when earning an income begins. The effective interest cost is diminished by g, reflecting the fact that the real wage per unit of human capital will grow by the rate g from the current year to the next year.

The intuition behind the first-order condition (9.21) is perhaps easier to grasp if we put g on the left-hand-side and multiply by $\hat{w}_t$ in the numerator as well as the denominator. Then the condition reads:

$$\frac{\hat{w}_t h'(S) + \hat{w}_t h(S) g}{\hat{w}_t h(S)} = r + m.$$

On the the left-hand side we now have the actual net rate of return obtained by investing one more year in education. In the numerator we have the direct increase in wage income by increasing S by one unit plus the gain arising from the fact that human capital, $h(S)$, is worth more in earnings capacity one year later due to technical progress. In the denominator we have the educational investment made by letting the obtained human capital, $h(S)$, "stay" one more year in school instead of at the labor market. In an optimal plan the actual net rate of return on the marginal investment equals the required rate of return, $r + m$. This is what could be obtained by the alternative strategy, which is to leave school already after S years and then invest the first years's labor income in life annuities paying the net rate of return, $r + m$, per year until death. That is, the first-order condition can be seen as a no-arbitrage equation. (As is quite usual, our interpretation treats marginal changes as if they were discrete. Thereby our interpretation is, of course, only approximative.)

Suppose $S = S^* > 0$ satisfies the first-order condition (9.21). To check the second-order condition, we consider

$$\begin{aligned}
&\frac{\partial^2 HW}{\partial S^2}(v, S^*) \\
= \ & HW'(S^*)\left[\frac{h'(S^*)}{h(S^*)} - (r+m-g)\right] + HW(S^*)\frac{h(S^*)h''(S^*) - h'(S^*)^2}{h(S^*)^2} \\
= \ & HW(S^*)\frac{\frac{S^*}{h'(S^*)}h''(S^*) - \frac{S^*}{h(S^*)}h'(S^*)}{S^* h(S^*)}h'(S^*), \qquad (9.22)
\end{aligned}$$

since the first term on the right-hand side in the second row vanishes due to (9.21) being satisfied at $S = S^*$. The second-order condition, $\partial^2 HW/\partial S^2 < 0$ at $S = S^*$ holds if and only if the elasticity of h w.r.t. S exceeds that of h' w.r.t. S at $S = S^*$. A sufficient but not necessary condition for this is that $h'' \leq 0$. Anyway, since $HW(v, S)$ is a continuous function of S, if there is a unique $S^* > 0$ satisfying (9.21), and if $\partial^2 HW/\partial S^2 < 0$ holds for this S^*, then this S^* is the unique optimal length of education for the individual. If individuals are alike in the sense of having the same innate abilities and facing the same schooling technology $h(\cdot)$, they will all choose S^*.

EXAMPLE 5 Suppose $h(S) = S^\eta$, $\eta > 0$, as in Example 3. Then the first-order condition (9.21) gives a unique solution $S^* = \eta/(r + m - g)$; and the second-order condition (9.22) holds for all $\eta > 0$. More sharply decreasing returns to schooling (smaller η) shortens the optimal time spent in school as does of course a higher effective discount rate, $r + m - g$.

Consider two countries, one rich (industrialized) and one poor (agricultural). With one year as the time unit, let the parameter values be as in the first four columns in the table below. The resulting optimal S for each of the countries is given in the last column.

	η	r	m	g	S^*
rich country	0.6	0.06	0.01	0.02	12.0
poor country	0.6	0.12	0.02	0.00	4.3

The difference in S^* is due to r and m being higher and g lower in the poor country. □

The above example follows a short note by Jones (2007) entitled “A simple Mincerian approach to endogenizing schooling”. The term “Mincerian approach” should here be interpreted in a broad sense as more or less synonymous with “life-cycle approach”.

Often in the macroeconomic literature, however, the term "Mincerian approach" is identified with an *exponential* specification of the learning technology:

$$h(S) = h(0)e^{\psi S}, \qquad \psi > 0. \tag{9.23}$$

This exponential form can at the formal level be seen as resulting from a combination of equation (9.11) from Example 2 and equation (9.13) from Example 3. One should be aware, however, that the present simple framework does not really embrace an exponential specification of h. Indeed, the second-order condition (9.22) implied by the "perpetual youth" assumption of age-independent mortality and no retirement, is incompatible with the strong convexity implied by the exponential function. Of course, this must be seen as a limitation of the "perpetual youth" setup (where there is no conclusive upper bound for anyone's lifetime) rather than a reason for rejecting apriori the exponential specification (9.23).

Anyway, the sole basis for a Mincerian exponential relationship is empirical cross-sectional evidence on relative wages at a given point in time, cf. Figure 9.1. As briefly commented in the concluding section, there seems to be little empirical support for an exponential production function for human capital. Moreover, as we shall now see, Mincer's microeconomic explanation of the exponential relationship (cf. Mincer, 1958, 1974) has nothing to do with a specific production function for human capital.

9.4 Explaining the Mincer equation

In Mincer's theory behind the observed exponential relationship called the Mincer equation, there is no role at all for any specific schooling technology, $h(\cdot)$, leading to a unique solution, S^*. The essential point is that the empirical Mincer equation is based on *heterogeneity in the jobs* offered to people (different educational levels not being perfectly substitutable). An exponential relationship where people, in spite of being alike ex ante, choose different educational levels ex post can then arise through the equilibrium forces of *supply and demand in the job markets.*

Imagine, first, a case where all individuals have in fact chosen the same educational level, S^*, because they are ex ante alike and all face the same arbitrary human capital production function, $h(S)$, satisfying (9.22). Then jobs that require other educational levels will go unfilled and so the job markets will not clear. The forces of excess demand and excess supply will then tend to generate an educational wage profile different from the one presumed in (9.19), that is, different from $\hat{w}_t h(S)$. Sooner or later an *equilibrium* educational wage profile tends to arise such that people are indifferent as to how

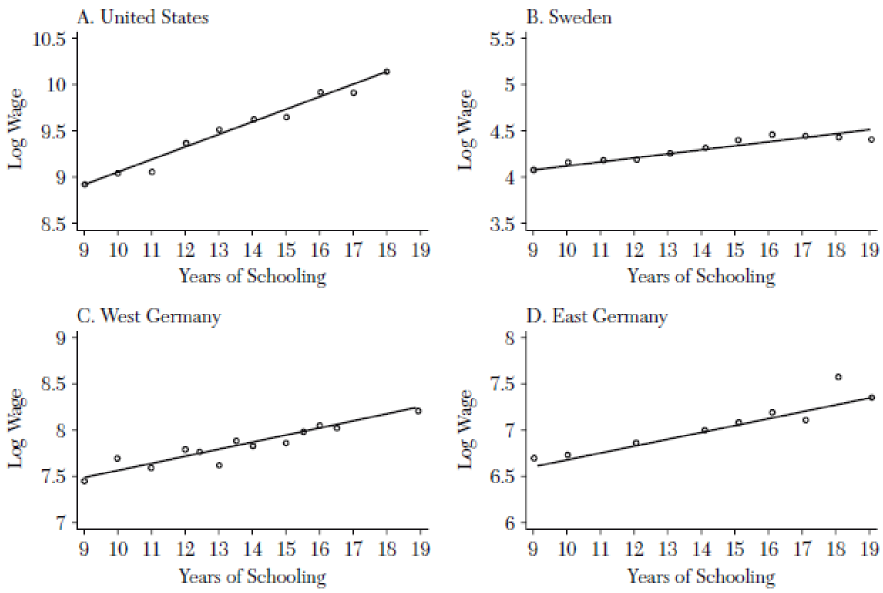

Figure 9.1: The semi-log schooling-wage relationship for different countries. Source: Krueger and Lindahl (2001).

much schooling they choose, thereby allowing market clearing. This requires a wage profile, $w_t(S)$, such that a marginal condition analogue to (9.21) holds for *all* S for which there is a positive amount of labor traded in equilibrium, say all $S \in [0, \bar{S}]$:

$$\frac{dw_t(S)/dS}{w_t(S)} = r + m - g \equiv \tilde{r} \qquad \text{for all } S \in [0, \bar{S}] . \tag{9.24}$$

It is here assumed, in the spirit of assumption 7 above, that technical progress implies that $w_t(S)$ *for fixed* S grows at the rate g, i.e., $w_t(S) = w_0(S)e^{gt}$, for all $S \in [0, \bar{S}]$. The equation (9.24) is a linear *differential equation* for w_t w.r.t. S, defined in the interval $0 \leq S \leq \bar{S}$. And the function $w_t(S)$, where t is fixed, is then unknown solution to this differential equation. That is, we have a differential equation of the form $dx(S)/dS = \tilde{r}x(S)$. This is a differential equation where the unknown function $x(S)$ is a function of schooling length rather than calendar time. The solution is $x(S) = x(0)e^{\tilde{r}S}$. Replacing the function $x(\cdot)$ with the function $w_t(\cdot)$, we thus have the solution

$$w_t(S) = w_t(0)e^{\tilde{r}S}. \tag{9.25}$$

Note that in the previous section, in the context of (9.21), we required the proportionate marginal return to schooling to equal $\tilde{r}$ only for a specific S, i.e.,

$$\frac{d(\hat{w}_t h(S))/dS}{\hat{w}_t h(S)} = \frac{h'(S)}{h(S)} = r + m - g \equiv \tilde{r} \qquad \text{for } \; S = S^*. \tag{9.26}$$

This is only a first-order condition assumed to hold at some point, S^*. It will generally not be a differential equation the solution of which gives a Mincerian exponential relationship. A differential equation requires a derivative relationship to hold not only at one point, but in an interval for the independent variable (S in (9.24)). Indeed, in (9.24) we require the proportionate marginal return to schooling to equal $\tilde{r}$ in a whole interval of schooling levels. Otherwise, with heterogeneity in the jobs offered there could not be equilibrium.[8]

Returning to (9.25), by taking logs on both sides, we get

$$\log w_t(S) = \log w_t(0) + \tilde{r}S, \tag{9.27}$$

which is the *Mincer equation* on log-linear form.

[8] As I see it, Acemoglu (2009, p. 362) makes the logical error of identifying a first-order condition, (9.26), with a differential equation.

Empirically, the Mincer equation does surprisingly well, cf. Figure 9.1.[9] Note that (9.25) also yields a theory of how the "Mincerian slope", ψ, in (9.23) is *determined*, namely as the mortality- and growth-corrected real interest rate, $\tilde{r}$. The evidence for this part of the theory is more scarce.

Given the equilibrium educational wage profile, $w_t(S)$, the human wealth of an individual "born" at time 0 can be written

$$\begin{aligned} HW_0 &= \int_S^\infty w_t(0)e^{\tilde{r}S}e^{-(r+m)t}dt = e^{\tilde{r}S}\int_S^\infty w_0(0)e^{\bar{g}t}e^{-(r+m)t}dt \\ &= w_0(0)e^{\tilde{r}S}\int_S^\infty e^{[g-(r+m)]t}dt = w_0(0)e^{\tilde{r}S}\left[\frac{e^{[g-(r+m)]t}}{g-(r+m)}\right]_S^\infty \\ &= \frac{w_0(0)}{r+m-g}, \end{aligned} \tag{9.28}$$

since $\tilde{r} \equiv r+m-g$. In view of the adjustment of the S-dependent wage levels, in equilibrium the human wealth of the individual is thus *independent of* S (within an interval) according to the Mincerian theory. Indeed, the essence of Mincer's theory is that if one level of schooling implies a higher human wealth than the other levels of schooling, the number of individuals choosing that level of schooling will rise until the associated wage has been brought down so as to be in line with the human wealth associated with the other levels of schooling. Of course, such adjustment processes must in practice be quite time consuming and can only be approximative.[10]

In this context, the original schooling technology, $h(\cdot)$, for human capital formation has lost any importance. It does not enter human wealth in a long-run equilibrium in the disaggregate model where human wealth is simply given by (9.28). In this equilibrium people have different S's and the received wage of an individual per unit of work has no relationship with the human capital production function, $h(\cdot)$, by which we started in this section.

Although there thus exists a microeconomic theory behind a Mincerian relationship, this theory gives us a relationship for relative wages in a cross-section at a given point in time. It leaves open what an intertemporal production function for human capital, relating educational investment, S, to a resulting level, h, of labor efficiency in a macroeconomic setting, looks like. Besides, the Mincerian slope, $\tilde{r}$, is a market price, not an aspect of schooling technology.

[9] The slopes are in the interval $(0.05, 0.15)$.

[10] Who among the ex ante similar individuals ends up with what schooling level is indeterminate.

9.5 Some empirics

In their cross-country regression analysis de la Fuente and Domenech (2006) find a relationship essentially like that in Example 3 with $\eta = 1$.[11]

Similarly, the cross-country study, based on calibration, by Bills and Klenow (2000) as well as the time series study by Cervelatti and Sunde (2010) favor the hypothesis of diminishing returns to schooling. According to this, the linear term, $\tilde{r}S$, in the exponent in (9.23) should be replaced by a strictly concave function of S. These findings are in accordance with the results by Psacharopoulus (1994). For $S > 0$, the power function in Example 5 can be written $h = S^{\eta} = e^{\eta \ln S}$ and thus in better harmony with the data than the exponential function (9.23). A parameter indicating the quality of schooling may be added: $h = ae^{\eta S}$, where $a > 0$ may be a function of the teacher-pupil ratio, teaching materials per student etc. See Caselli (2005).

Outlook

Models based on the life-cycle approach to human capital typically conclude that education is productivity enhancing, i.e., education has a *level* effect on income per capita but is not a factor which in itself can explain sustained per capita growth, cf. Exercise V.7 and V.8. A more plausible main driving factor behind growth seems rather to be technological innovations. A higher level of per capita human capital may temporarily raise the speed of innovations, however.

Final remark

This chapter considered human capital as a productivity-enhancing factor. There is a complementary perspective on human capital, namely the Nelson-Phelps hypothesis about the key role of human capital for technology adoption and technological catching up, see Acemoglu, §10.8, and Exercise Problem V.3.

9.6 Literature

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[11] The authors find that the elasticity of GDP w.r.t. average years in school in the labor force is at least 0.60. The empirical macroeconomic literature typically measures S as the average number of years of schooling in the working-age population, taken for instance from the Barro and Lee (2001) dataset.

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Chapter 10

Human capital and knowledge creation in a growing economy

As a follow-up on the concept of a schooling technology presented in Chapter 9, Section 9.2, the present chapter considers aspects of the interplay between physical capital, human capital, and knowledge creation in a simple balanced growth framework.

10.1 The model

We consider a closed economy with education and two production sectors, manufacturing and R&D. Time is continuous. Postponing the modeling of education a little, at the aggregate level we have:

$$\begin{aligned}
Y_t &= T_t K_t^{\alpha}(\bar{h}_t L_{Yt})^{1-\alpha}, \quad 0<\alpha<1, && (10.1)\\
\dot{K}_t &= Y_t - c_t N_t - \delta K_t, \quad \delta \geq 0, \quad K_0 > 0 \text{ given}, && (10.2)\\
T_t &= A_t^{\sigma}, \quad \sigma > 0, && (10.3)\\
\dot{A}_t &= \gamma A_t^{\varphi} \bar{h}_t L_{At}, \quad \gamma > 0, \varphi < 1, \quad A_0 > 0 \text{ given}, && (10.4)\\
0 &< L_{Yt} \leq L_{Yt} + L_{At} = L_t, && (10.5)
\end{aligned}$$

where Y_t is manufacturing output (the value of which is less than GNP when $L_{At} > 0$), T_t is total factor productivity (TFP), K_t is physical capital input, $\bar{h}_t$ is average human capital in the labor force, L_{Yt} and L_{At} are inputs of labor in manufacturing and R&D, respectively, C_t is aggregate consumption, A_t is the stock of technical knowledge, and L_t is aggregate labor input, all at time t. The size of population is denoted N_t and so per capita consumption is $c_t \equiv C_t/N_t$.

Comments: As to (10.5), $\bar{h}_t L_{At}$ is the total input of human capital per time unit in R&D and γA_t^{φ} is the productivity of this input at the aggregate level. The parameter φ measures the elasticity of research productivity w.r.t. the level of the available stock of technical knowledge. The case $0 < \varphi < 1$ represents the "standing on the shoulders" case where knowledge creation becomes easier the more knowledge there is already. In contrast, the case $\varphi < 0$ represents the "fishing out" case, also called the "easiest inventions are made first" case. This would reflect that it becomes more and more difficult to create the next advance in technical knowledge. As to (10.5), the strict and weak inequalities are motivated by the view that for the system to be economically viable, there must be activity in the Y-sector whereas it is of interest to allow for – and compare – the cases $L_{At} > 0$ and $L_{At} = 0$ (active versus passive R&D sector).

The population growth rate is assumed constant:

$$N_t = N_0 e^{nt}, \qquad n \geq 0, \quad N_0 > 0 \text{ given.} \tag{10.6}$$

We assume a stationary age distribution in the population. Although details about schooling are postponed, we already here assume that schooling and retirement are consistent with the labor force being a constant fraction of the population:

$$L_t = (1 - \beta)N_t, \tag{10.7}$$

where $\beta \in (0, 1)$. Then, by (10.6) follows

$$L_t = L_0 e^{nt}, \qquad n \geq 0, \quad L_0 > 0. \tag{10.8}$$

We let the growth rate at time t of a variable $x > 0$ be denoted g_{xt}. When writing just g_x, without the time index t, it is understood that the growth rate of x is constant over time.

10.2 Productivity growth along a BGP with R&D

Let us first find an expression for the TFP growth rate. By log-differentiation w.r.t. t in (10.1), we have

$$g_{Yt} = g_{Tt} + \alpha g_{Kt} + (1 - \alpha)(g_{\bar{h}t} + g_{L_{Yt}}). \tag{10.9}$$

The current TFP growth rate is thus

$$g_{TFPt} \equiv g_{Yt} - (\alpha g_{Kt} + (1 - \alpha)(g_{\bar{h}t} + g_{L_{Yt}})) = g_{Tt} = \sigma g_{At}, \tag{10.10}$$

where the last equality follows from (10.3). By (10.4), we get

$$g_{At} \equiv \frac{\dot{A}_t}{A_t} = \gamma A_t^{\varphi - 1} \bar{h}_t L_{At} \geqq 0, \text{ with } > \text{ if and only if } L_{At} > 0. \tag{10.11}$$

We shall first consider the case of active R&D:

ASSUMPTION (A1): $L_{At} > 0$ for all $t \geq 0$.

This assumption implies $g_{At} > 0$ and so the growth rate of g_{At} is well-defined. By log-differentiation w.r.t. t in (10.11) we have

$$\frac{\dot{g}_{At}}{g_{At}} = (\varphi - 1)g_{At} + g_{\bar{h}t} + g_{L_A t}. \tag{10.12}$$

10.2.1 Balanced growth with R&D

In the present context we define a *balanced growth path* (BGP) as a path along which g_{Yt}, g_{Ct}, g_{Kt}, g_{At}, and $g_{\bar{h}t}$ are constant (not necessarily equal and not necessarily positive). With y denoting per capita manufacturing output, i.e., $y \equiv Y/L$, let us find the growth rate of y in balanced growth with active R&D. We introduce the following additional assumptions:

ASSUMPTION (A2): The economy follows a BGP.

ASSUMPTION (A3): $Y_t - c_t N_t > 0$ for all $t \geq 0$.

By imposing (A3), we rule out the degenerate case where $g_K = -\delta$.

Along a BGP, by definition, g_{At} is a constant, g_A. Since thereby $\dot{g}_A = 0$, solving for g_A in (10.12) gives

$$g_A = \frac{g_{\bar{h}t} + g_{L_A t}}{1 - \varphi} > 0, \tag{10.13}$$

where the positivity is due to the assumption (A1). For the formula (10.13) to be consistent with balanced growth, $g_{L_A t}$ must be a constant, g_{L_A}, since otherwise g_A and $g_{\bar{h}t}$ could *not* both be constant as they must in balanced growth, by definition. Moreover, we must have $g_{L_A} = n$. To see this, imagine that $g_{L_A} < n$. Then, in order for the growth rate of the sum $L_{Yt} + L_{At}$ to accord with (10.8), we would need $g_{L_Y t} > n$ *forever*, which would imply $L_{Yt} + L_{At} > L_t$ sooner or later. This is a contradiction. And if instead we imagine that $g_{L_A} > n$ while still being constant, we would, at least after some time, have $L_{At} > L_t$, again a contradiction. We conclude that $g_{L_A} = n$. For $L_{Yt} + L_{At}$ to accord with (10.8), it then follows that also $g_{L_Y t}$ must be a constant, g_{L_Y}, and equal to n. We have hereby proved that along a BGP with R&D,

$$g_{L_A} = g_{L_Y} = n. \tag{10.14}$$

It follows that L_A/L is constant along a BGP with R&D.

Given the accumulation equation (10.2) and the assumption (A3), it follows by the Balanced Growth Equivalence Theorem of Chapter 4 that

$$g_C = g_Y = g_K$$

along a BGP. From (10.9), together with (10.7) and the definition $c \equiv C/N$, then follows that along a BGP,

$$\begin{aligned} g_c &= g_y = g_Y - n = g_T + \alpha g_K + (1-\alpha)(g_{\bar{h}} + n) - n \\ &= g_T + \alpha(g_K - n) + (1-\alpha)g_{\bar{h}} = g_T + \alpha g_k + (1-\alpha)g_{\bar{h}}, \quad (10.15) \end{aligned}$$

where the last equality comes from $k \equiv K/L_Y$ and $g_{L_Y} = n$. As $g_K = g_Y$ and $g_{L_Y} = g_L$, we have $g_k = g_y$. Then (10.15) gives

$$g_y = \frac{g_T}{1-\alpha} + g_{\bar{h}} = \frac{\sigma g_A}{1-\alpha} + g_{\bar{h}}, \quad (10.16)$$

in view of (10.10).

Education

Let the time unit be one year. Suppose an individual "born" at time v (v for "vintage") spends the first S years of life in school and then enters the labor market with a human capital equal to $h(S)$, where $h' > 0$. We ignore the role of teachers and schooling equipment in the formation of human capital. The role of work experience for human capital later in life is likewise ignored. Moreover, we assume that S is the same for all members of a given cohort and also – until further notice – the same across cohorts. So

$$\bar{h} = h(S), \qquad h' > 0. \quad (10.17)$$

After leaving school, individuals work full-time until either death before age R or retirement at age R where $R > S$, of course; life expectancy is assumed the same for all cohorts. Assuming a stationary age distribution in the population, we see that β in (10.7) represents the constant fraction of the population consisting of people either below age S, i.e., under education, or above age R, i.e., retired people (β will be an increasing function of S and a decreasing function of R).[1]

[1] A complete model would treat S as endogenous in general equilibrium. In a partial equilibrium analysis one could possibly use an approach similar to the one in Chapter 9, Section 9.3. We shall not enter into that, however, because the next step, determination of the real rate of interest in general equilibrium, is a complex problem and requires a lot of additional specifications of households' characteristica and market structure. Fortunately, it is not necessary to determine S as long as the focus is only on determining the productivity growth rate along a BGP.

Sustained productivity growth along a BGP

It follows that average human capital is constant. Thus $g_{\bar{h}} = 0$ and (10.16) reduces to

$$g_y = \frac{\sigma g_A}{1-\alpha} > 0, \tag{10.18}$$

In equation (10.15) productivity growth, g_y, is decomposed into a contribution from technical change, a contribution from "capital deepening" (growth in k), and a contribution from human capital growth if any. As long as S in (10.17) is assumed constant over time, there is no human capital growth. So we can re-write (10.15):

$$g_y = g_T + \alpha g_k = g_{TFP} + \alpha g_k, \tag{10.19}$$

in view of $g_{TFP} = g_T = \sigma g_A$ from (10.10). This equation decomposes the productivity growth rate into a direct contribution from technical change and a direct contribution from capital deepening. Digging deeper, (10.18) tells us that both these direct contributions rest on sustained knowledge growth. The correct interpretation of (10.19) is that it just displays the two factors behind the *current* increase in y, while (10.18) takes into account that both TFP growth and capital deepening are in a long-run perspective themselves driven by knowledge growth.

10.2.2 A precondition for sustained productivity growth when $g_{\bar{h}} = 0$: population growth

We saw that along a BGP with R&D, $g_{L_A} = n$. By (10.13) and $g_{\bar{h}} = 0$ then follows that along a BGP with R&D,

$$g_A = \frac{n}{1-\varphi} > 0. \tag{10.20}$$

From this inequality we see that existence of a BGP with R&D requires

ASSUMPTION (A4): $n > 0$

to hold.

On the basis of (A4) and (10.18) we finally conclude that

$$g_y = \frac{\sigma n}{(1-\varphi)(1-\alpha)} > 0. \tag{10.21}$$

Here we have taken into account that also knowledge growth is endogenous in that it is determined by allocation of resources (research workers) to R&D

activity. The result (10.21) tells us that not only is population growth necessary for sustained productivity growth but productivity growth is faster the faster is population growth.

Why does population growth ultimately help productivity growth (at least in this model)? The explanation is that productivity growth is driven by knowledge creation. Knowledge is a *nonrival* good − its use by one agent does not, in itself, limit its simultaneous use by other agents. Considering the producible T in (10.1) as an additional production factor along with capital and labor, (10.1) displays *increasing returns to scale* in manufacturing w.r.t. these three production factors. Although there are diminishing marginal returns to capital, there are increasing returns to scale w.r.t. capital, labor, *and* the accumulative technology level. For the increasing returns to unfold in the long run, growth in the labor force (hence in population) is needed. Growth in the labor force and T not only counterbalances the falling marginal productivity of capital,[2] but actually upholds sustained per capita growth − the more so the faster is population growth.

The growth-promoting role of the exogenous rate of population growth reflects the presence of what is called a *weak scale effect* in the model. A *scale effect* is said to be present in an economic system if there is an advantage of scale measured by population size. This advantage of scale is in the present case due to the productivity-enhancing role of a nonrival good, technical knowledge, that is produced by the research workers in the idea-creating R&D sector. Thereby higher population growth results in higher per capita growth in the long run. On the other hand, a large population is not in itself, when $\varphi < 1$, sufficient to generate sustained positive per capita growth. This is why we talk of a *weak* scale effect. In contrast, what is known as a *strong* scale effect (associated with the case $\varphi \geq 1$) is present if a larger population as such (without population growth) would be enough to generate higher per capita growth in the long run.

In view of cross-border diffusion of ideas and technology, the result (10.21) should not be seen as a prediction about individual countries. It should rather be seen as pertaining to larger regions, nowadays probably the total industrialized part of the world. So the single country is not the relevant unit of observation and cross-country regression analysis thereby not the right framework for testing such a link from n to g_y.

The reason that in (10.21), a higher σ promotes productivity growth is that σ indicates the sensitivity (elasticity) of TFP w.r.t. accumulative knowledge. Indeed, the larger is σ, the larger is the percentage increase

[2] This counter-balancing role reflects the direct complementarity between the production factors in (10.1).

in manufacturing output that results from a one-percentage increase in the stock of knowledge.

The intuition behind the growth-enhancing role of α in (10.21) follows from (10.1) which indicates that α measures the elasticity of manufacturing output w.r.t. another accumulative input, physical capital. The larger is α, the larger is the percentage increase in manufacturing output resulting from a one-percentage increase in the stock of capital.

Finally, the intuition behind the growth-enhancing role of φ in (10.21) can be obtained from the equation (10.4) which describes the creation of new knowledge. The equation shows that the larger is φ, the larger is the percentage increase in the time-derivative of technical knowledge resulting from a one-percentage increase in the stock of knowledge.

10.2.3 The concept of endogenous growth

The above analysis provides an example of *endogenous growth* in the sense that the positive sustained per capita growth rate is generated through an economic mechanism within the model, allocation of resources to R&D. This is in contrast to the Solow or standard Ramsey model where technical progress is exogenous, given as manna from heaven.

There are basically two types of endogenous growth. One is called *semi-endogenous* growth and is present if growth is endogenous but a positive per capita growth rate can not be sustained in the long run without the support from growth in some exogenous factor (for example growth in the labor force). As $n > 0$ is needed for sustained per capita growth in the above model, growth is here driven by R&D in a semi-endogenous way.

The other type of endogenous growth is called *fully endogenous* growth and occurs if the long-run growth rate of Y/L is positive without the support from growth in any exogenous factor (for example growth in the labor force).

10.3 Permanent level effects

In the result (10.21), there is no trace of the *size* of the fraction, L_A/L, of the labor force allocated to R&D. This is due to the assumption that $\varphi < 1$. This assumption implies diminishing *marginal* productivity of knowledge in the creation of new knowledge. Indeed, when $\varphi < 1$, $\partial \dot{A}/\partial A = \gamma\varphi A^{\varphi-1}\bar{h}L_A$ is a decreasing function of the stock of knowledge already obtained. A shift of L_A/L to a higher level can *temporarily* generate faster knowledge growth and thereby faster productivity growth, but due to the diminishing *marginal* productivity of knowledge in the creation of new knowledge, in the long run

g_A and g_y will be back at their balanced-growth level given in (10.20) and (10.21), respectively.

It can be shown, however, that a marginally higher L_A/L generally has a permanent *level* effect, that is, a permanent effect on y along a BGP. If initially L_A/L is "small", this level effect tends to be positive. This is like in the Solow growth model where a shift to a higher saving-income ratio, s, has a temporary positive growth effect and a permanent positive level effect on y. In contrast to the Solow model, however, if L_A/L is already "large", the level effect on y of a marginal increase in L_A/L may be negative. This is because Y is produced by $L_Y = (1 - L_A/L)L$, not L.[3]

As mentioned we treat the number of years in school and average human capital, $\bar{h}$,, as exogenous. Then it is simple to study the comparative-dynamic effect of a higher level of average human capital, $\bar{h}$ In the present model there will be a permanent level effect on y but no permanent growth effect.

A complicating aspect here is that, given the model, a higher value of $\bar{h}$ will be at the cost of a higher number of years in school, i.e., a higher S. A higher S implies that a smaller fraction of the population will be in the labor force, cf. (10.7) where β is an increasing function of S. This implies that there is no longer a one-to-one relationship between a positive level effect on $y \equiv Y/L$ and a positive level effect on per capita consumption, $c \equiv C/N$ $= (C/Y)\cdot(Y/L)\cdot(L/N)$. We will not go into detail with this kind of trade-off here.

10.4 The case of no R&D

As an alternative to (A1) we now consider the case of no R&D:

ASSUMPTION (A5): $L_{At} = 0$ for all $t \geq 0$.

Under this assumption the whole labor force is employed in manufacturing, i.e., $L_{Yt} = L_t$ for all $t \geq 0$. There is no growth in knowledge and therefore no TFP growth. Whether $n > 0$ or $n = 0$, along a BGP satisfying (A3), (10.19) is still valid but reduces to $g_y = \alpha g_k$. At the same time, however, the Balanced Growth Equivalence Theorem of Chapter 4 says that along a BGP satisfying (A3), $g_Y = g_K$, which implies $g_y = g_k$. As $\alpha \in 0, 1)$, we have thus reached a contradiction unless $g_y = g_k = 0$.

So, as expected, without technological progress there can not exist sustained per capita growth. To put it differently, along a BGP we necessarily have $g_Y = g_K = g_C = n$, where $C \equiv cN$.

[3] In Exercise VII.7 you are asked to analyze this kind of problems in a more precise way.

10.5 Outlook

Given the prospect of non-increasing population in the world economy already within a century from now (United Nations, 2013), the prospect of sustained per capita growth in the world economy in the very long run may seem bleak according to the model. Let us take a closer look at the issue.

10.5.1 The case $n = 0$

Suppose $n = 0$ in the above model and return to the assumption (A1). As g_{L_A} can no longer be a positive constant, g_A and g_y can no longer be positive constants. Hence balanced growth with $g_y > 0$ is impossible. Does this imply that there need be economic stagnation in the sense of $g_y = 0$? No, what is ruled out is that $y_t = y_0 e^{g_y t}$ is impossible for any constant $g_y > 0$. So it is *exponential* growth that is impossible.

Still paths along which $y_t \to \infty$ and $c_t \to \infty$ for $t \to \infty$ are technically feasible. Along such paths, g_y and g_c will be positive forever, but with $\lim_{t\to\infty} g_y = 0$ and $\lim_{t\to\infty} g_c = 0$. To see this, suppose $L_{At} = L_A$, a positive constant less than L, where L is the constant labor force which is proportional to the constant population. Suppose further, for simplicity, that $\bar{h}$ is can be considered exogenous. Then, from (10.4) and (10.17) follows

$$\dot{A}_t = \gamma A_t^{\varphi} \bar{h} L_A \equiv \xi A_t^{\varphi}, \qquad \xi \equiv \gamma \bar{h} L_A.$$

This Bernoulli differential equation has the solution[4]

$$A_t = \left[A_0^{1-\varphi} + (1-\varphi)\xi \cdot t\right]^{\frac{1}{1-\varphi}} \equiv \left[A_0^{1-\varphi} + (1-\varphi)\gamma \bar{h} L_A \cdot t\right]^{\frac{1}{1-\varphi}} \to \infty \text{ for } t \to \infty. \tag{10.22}$$

The stock of knowledge thus follows what is known as a *quasi-arithmetic growth* path – a form of less-than-exponential growth. The special case $\varphi = 0$ leads to simple arithmetic growth: $A_t = A_0 + (1-\varphi)\eta \bar{h} L_A \cdot t$. In case $0 < \varphi < 1$, A_t features more-than-arithmetic growth and in case $\varphi < 0$, A_t features less-than-arithmetic growth. It can be shown that with a social welfare function of the standard Ramsey type, cf. Chapter 8, the social planner's solution converges, for $t \to \infty$, toward a path where also K_t, Y_t, y_t, and c_t feature quasi-arithmetic growth.[5]

[4] See Section 7.2 of Chapter 7.
[5] Groth et al. (2010).

10.5.2 The case of rising life expectancy

There is another demographic aspect of potential importance for future productivity growth, namely the prospect of increasing schooling length in the wake of an increasing life expectancy.

Over the past 30-40 years average years of schooling have tended to grow arithmetically at a rate of about 0.8 years per decade in the EU as a whole, compared to 0.7 years in the US (Montanino et al. 2004). A central factor behind this development is the rising life expectancy due to improved income, salubrity, nutrition, sanitation, and medicine. Increased life expectancy heightens the returns to education. In the first half of the twentieth century life expectancy in the US improved at a rate of four years per decade. In the second half the rate has been smaller, but still close to two years per decade (Arias, 2004). Oeppen and Vaupel (2002) report that since 1840 female life expectancy in the record-holding country in the world has steadily increased by almost a quarter of a year per year. To what extent such developments may continue is not clear. But at least for a long time to come we may expect growth in life expectancy and thereby also in educational investment because of the lengthening of the recovery period for that investment.

Increasing schooling length introduces heterogeneity w.r.t. individual human capital into the model. In a cross-section of workers at a given point in time the workers' h becomes a decreasing function of age. And increasing life expectancy changes the aggregate growth process for population and labor force. This takes us somewhat outside the above analytical framework with a stationary age structure and no schooling heterogeneity. Yet let us speculate a little.

Suppose the schooling technology can be presented by a power function:[6]

$$h = h(S) = S^{\eta}, \qquad \eta > 0. \tag{10.23}$$

Let every member of cohort $v \geq 0$ spend $S(v)$ years in school, thereby leaving school with human capital $h(v) = S(v)^{\eta}$. Then the growth rate of h of the cohort just leaving school is

$$\frac{dh(v)/dv}{h(v)} = \frac{\eta S(v)^{\eta-1}S'(v)}{S(v)^{\eta}} = \eta\frac{S'(v)}{S(v)}.$$

Assume sustained arithmetic growth in schooling length takes place due to a steadily rising life expectancy. Then

$$S(v) = S_0 + \mu v, \qquad S_0 \geq 0, \mu > 0. \tag{10.24}$$

[6] There is some empirical support for this hypothesis, cf. Section 9.5 of Chapter 9.

Hence,

$$\eta \frac{S'(v)}{S(v)} = \frac{\eta\mu}{S_0 + \mu v} \to 0 \text{ for } v \to \infty.$$

On this background the projection will be that also average human capital, $\bar{h}_t$, will be growing over time but at a rate, $g_{\bar{h}}$, approaching 0 for $t \to \infty$. This gives no chance that the $g_{\bar{h}}$ in the formula (10.13) can avoid approaching nil. So our model rules out exponential per capita growth in the long run when $n = 0$ and $h(v) = S(v)^{\eta}$.

As a thought experiment, suppose instead that the schooling technology is exponential:

$$h = h(S) = e^{\psi S(v)}, \qquad \psi > 0. \tag{10.25}$$

Then the growth rate of the human capital of the cohort just leaving school is

$$\frac{dh(v)/dv}{h(v)} = \frac{e^{\psi S(v)}\psi s'(v)}{e^{\psi S(v)}} = \psi S'(v) > 0. \tag{10.26}$$

Assume arithmetic growth in life expectancy as well as schooling length, the latter following (10.24). Then $\psi S'(v) = \psi\mu$, a positive constant. My conjecture is that also average human capital, $\bar{h}_t$, will in this case under certain conditions grow at the constant rate, $\psi\mu$, at least approximately (I have not made the required demographic calculus).

Let us try some numbers. Suppose life expectancy in modern times steadily increases by λ years per year and let schooling time and retirement age be constant fractions of life expectancy. Let the schooling time fraction be denoted ω. Then $S'(v) = \mu = \omega\lambda$ and $g_{\bar{h}} = \psi S'(v) = \psi\omega\lambda$. With $\lambda = 0.2$, $\omega = 0.2$, and $\psi = 0.10$, we get $g_{\bar{h}} = 0.004$.[7] Suppose $n = 0.005$ and $\varphi = 0.5$ (as suggested by Jones, 1995).[8]

Along a BGP with R&D we then have, by (10.13),

$$g_A = \frac{g_{\bar{h}} + n}{1 - \varphi} = \frac{0.004 + 0.005}{0.5} = 0.018.$$

In case $\sigma = 1 - \alpha$, (10.16) thus yields

$$g_y = g_A + g_{\bar{h}} = 0.018 + 0.004 = 0.022.$$

[7] As reported by Krueger and Lindahl (2001), ψ is usually estimated to be in the range $(0.05, 0.15)$.

[8] $n = 0.005$ per year may seem a low number for the empirical growth rate of research labor (scientists and engineers) in the US and other countries over the last century. On the other hand, for simplicity our model has ignored the likely duplication externality due to overlap in R&D at the economy-wide level. Taking that overlap into account, we should replace $\bar{h}L_A$ in (10.4) by $(\bar{h}L_A)^{1-\pi}$, and n in (10.20) by $(1-\pi)n$, where $\pi \in (0,1)$ measures the extent of duplication. Jones (1995) suggests $\pi = 0.5$.

If instead $n = 0$ (in accordance with the long-run projection) and $L_{At} = L_A \in (0, L)$, we get along a BGP with R&D

$$g_y = \frac{g_{\bar{h}}}{1-\varphi} + g_{\bar{h}} = \frac{0.004}{1-\varphi} + 0.004 = 0.012.$$

In spite of $n = 0$, the thought experiment (10.25) thus leads to a non-negligible level of exponential growth. With the compounding effects of exponential growth it is certainly substantial. I call it a "thought experiment" because the empirical support for the exponential human capital production function (10.25) is weak if not non-existing.

10.6 Concluding remarks

In a semi-endogenous growth setting we have considered human capital formation and knowledge creating R&D. The latter is ultimately the factor driving productivity growth unless one is willing to make very strong assumptions about the human capital production function. Technical knowledge is capable of performing this role because it is a nonrival good and is "infinitely expansible", as emphasized by Paul Romer (1990) and Danny Quah (1996). Contrary to this, in Lucas (1988) the distinction between technical knowledge and human capital is not emphasized and it is the accumulation of human capital that is driving long-run productivity growth.[9]

In the above analysis we have ignored the role of scarce natural resources for limits to growth. We will come back to this issue in chapters 12 and 16.

We have ruled $\varphi = 1$ out because in combination with $n > 0$ it would tend to generate a forever growing productivity growth rate. We have ruled $\varphi > 1$ out because in combination even with $n = 0$, it would tend to generate infinite output in finite time. Jones (2005) argues that the empirical evidence speaks for $\varphi < 1$ in modern times.

The above analysis simply tells us what the growth rate *must* be in the long run provided that the system converges to balanced growth. On the other hand, specification of the market structure and the household sector, including demography and preferences, will be needed if we want to study the adjustment processes outside balanced growth, determine an equilibrium real interest rate etc.

It is due to the semi-endogenous growth setting (the $\varphi < 1$ assumption) that one can find the long-run per capita growth rate from knowledge of

[9] Although distinguishing between human capital and knowledge creation, the approach by Dalgaard and Kreiner (2001) is very different from the one we have followed above and has more in common with Lucas.

technology parameters and the rate of population growth alone. How the market structure and the household sector are described, is immaterial for the long-run growth rate. These things will in the long run have "only" *level* effects.

Only if economic policy affects the technology parameters or the population growth rate, will it be able to affect the long-run growth rate. Still, economic policy can *temporarily* affect economic growth and in this way affect the *level* of the long-run growth path.

10.7 References

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Chapter 11

AK and reduced-form AK models. Consumption taxation.

In his Chapter 11 Acemoglu discusses simple fully-endogenous growth models in the form of Ramsey-style *AK* and *reduced-form AK models*, respectively. The name "AK" refers to a special feature of the aggregate production function, namely the absence of diminishing returns to capital. We present the AK story within a Ramsey (i.e., representative agent) framework. A characteristic result from AK models is that they have *no* transitional dynamics.

With the aim of synthesizing the formal characteristics of such models, this lecture note gives an account of the common formal features of AK models (Section 11.1) and reduced-form AK models (Section 11.2), respectively. Finally, in Section 11.3 we discuss conditions under which consumption taxation is not distortionary.

11.1 General equilibrium dynamics in the simple AK model

In the simple AK model (Acemoglu, Ch. 11.1) we consider a fully automatized economy where the aggregate production function is

$$Y(t) = AK(t), \qquad A > 0. \tag{11.1}$$

Thus there are constant returns to capital, not diminishing returns, and labor is no longer a production factor. This section provides a detailed proof that when we embed this technology in a Ramsey framework with perfect competition, the model generates balanced growth *from the beginning.* So there will be no transitional dynamics.

We consider a closed economy with perfect competition and no government sector. The dynamic resource constraint for the economy is

$$\dot{K}(t) = Y(t) - c(t)L(t) - \delta K(t) = AK(t) - c(t)L(t) - \delta K(t), \qquad K(0) > 0 \text{ given,} \tag{11.2}$$

where $L(t)$ is the population size. After having found the equilibrium interest rate to be $r = A - \delta$, we find the equilibrium growth rate of per capita consumption to be

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r - \rho) \equiv \frac{1}{\theta}(A - \delta - \rho) \equiv g_c, \tag{11.3}$$

a constant. To ensure positive growth we impose the parameter restriction

$$A - \delta > \rho. \tag{A1}$$

And to ensure boundedness of discounted utility (as well as existence of an equilibrium path) we impose the additional parameter restriction:

$$\rho - n > (1 - \theta)g_c. \tag{A2}$$

Reordering gives

$$r = \theta g_c + \rho > g_c + n, \tag{11.4}$$

where the equality is due to (11.3).

Solving the linear differential equation (11.3) gives

$$c(t) = c(0)e^{g_c t}, \tag{11.5}$$

where $c(0)$ is unknown so far (because c is not a predetermined variable). We shall find $c(0)$ by appealing to the household's transversality condition,

$$\lim_{t \to \infty} a(t)e^{-(r-n)t} = 0, \tag{TVC}$$

where $a(t)$ is per capita financial wealth at time t. Recalling the No-Ponzi-Game condition,

$$\lim_{t \to \infty} a(t)e^{-(r-n)t} \geq 0, \tag{NPG}$$

we see that the transversality condition is equivalent to the No-Ponzi-Game condition being not over-satisfied.

Defining $k(t) \equiv K(t)/L(t)$, the dynamic resource constraint, (11.2), is in per-capita terms

$$\dot{k}(t) = (A - \delta - n)k(t) - c(0)e^{g_c t}, \qquad k(0) > 0 \text{ given,} \tag{11.6}$$

where we have inserted (12.24). The solution to this linear differential equation is (cf. Appendix to Chapter 3)

$$k(t) = \left(k(0) - \frac{c(0)}{r-n-g_c}\right)e^{(r-n)t} + \frac{c(0)}{r-n-g_c}e^{g_c t}, \qquad r \equiv A - \delta. \quad (11.7)$$

In our closed-economy framework with no public debt, $a(t) = k(t)$. So the question is: When will the time path (11.7) satisfy (TVC) with $a(t) = k(t)$? To find out, we multiply by the discount factor $e^{-(r-n)t}$ on both sides of (11.7) to get

$$k(t)e^{-(r-n)t} = k(0) - \frac{c(0)}{r-n-g_c} + \frac{c(0)}{r-n-g_c}e^{-(r-g_c-n)t}.$$

Thus, in view of the assumption (A2), (11.4) holds and thereby the last term on the right-hand side vanishes for $t \to \infty$. Hence

$$\lim_{t\to\infty} k(t)e^{-(r-n)t} = k(0) - \frac{c(0)}{r-n-g_c}.$$

From this we see that the representative household satisfies (TVC) if and only if it chooses

$$c(0) = (r-n-g_c)k(0). \quad (11.8)$$

This is the equilibrium solution for the household's chosen per capita consumption at time $t = 0$. If the household instead had chosen $c(0) < (r-n-g_c)k(0)$, then $\lim_{t\to\infty} k(t)e^{-(r-n)t} > 0$ and so the household would not satisfy (TVC) but instead be over-saving. And if it had chosen $c(0) > (r-n-g_c)k(0)$, then $\lim_{t\to\infty} k(t)e^{-(r-n)t} < 0$ and so the household would be over-consuming and violate (NPG) (hence also (TVC)).

Substituting the solution for $c(0)$ into (11.7) gives the evolution of $k(t)$ in equilibrium,

$$k(t) = \frac{c(0)}{r-n-g_c}e^{g_c t} = k(0)e^{g_c t}.$$

So from the beginning k grows at the same constant rate as c. Since per capita output is $y \equiv Y/L = Ak$, the same is true for per capita output. Hence, from start the system is in balanced growth (there is no transitional dynamics).

The AK model features one of the simplest kinds of *endogenous growth* one can think of. Growth is endogenous in the model in the sense that there is positive per capita growth in the long run, generated by an internal mechanism in the model (not by exogenous technology growth). The endogenously determined capital accumulation constitutes the mechanism through which sustained per capita growth is generated. It is because the net marginal productivity of capital is assumed constant and, according to (A1), higher than the rate of impatience, ρ, that capital accumulation itself is so powerful.

11.2 Reduced-form AK models

The models known as reduced-form AK models are a generalization of the simple AK model considered above. In contrast to the simple AK model, where only physical capital is an input, a reduced-form AK model assumes a technology involving at least two different inputs. Yet it is possible that in general equilibrium the aggregate production function ends up implying proportionality between output and some measure of "broad capital", i.e.,

$$Y(t) = B\tilde{K}(t),$$

where B is some endogenously determined positive constant and $\tilde{K}(t)$ is "broad capital".[1] If in addition the real interest rate in general equilibrium ends up being a constant, the model is called a *reduced-form AK model.* In the simple AK model constancy of average productivity of capital is postulated from the beginning. In the reduced-form AK models the average productivity of capital becomes and remains *endogenously* constant over time.

Thus, we end up with quite similar aggregate relations as those in the simple AK model. Hence the solution procedure to find the equilibrium path (see Chapter 12) is quite similar to that in the simple AK model above. Again there will be no transitional dynamics.

The nice feature of AK models is that they provide very simple theoretical examples of endogenous growth. The problematic feature is that they simplify the technology description *too* much and at best constitute knife-edge cases. More about this in Chapter 13.

11.3 On consumption taxation

As a preparation for the discussion later in this course of fiscal policy in relation to economic growth, we shall here try to clarify an aspect of consumption taxation. This is the question: is a consumption tax distortionary - always? never? sometimes?

The answer is the following.

1. Suppose labor supply is *elastic* (due to leisure entering the utility function). Then a consumption tax (whether constant or time-dependent) is generally distortionary (not neutral). This is because it reduces the effective opportunity cost of leisure by reducing the amount of consumption forgone by working one hour less. Indeed, the tax makes consumption goods more

[1] Theoretically, $\tilde{K}(t)$ could be the sum of physical and human capital. Empirically, however, this does not seem to be a realistic example, cf. Exercises V.4 and V.5.

expensive and so the amount of consumption that the agent can buy for the hourly wage becomes smaller. The substitution effect on leisure of a consumption tax is thus positive, while the income and wealth effects will be negative. Generally, the net effect will not be zero, but it can be of any sign; it may be small in absolute terms.

2. Suppose labor supply is *inelastic* (no trade-off between consumption and leisure). Then, at least in the type of growth models we consider in this course, a constant (time-independent) consumption tax acts as a lump-sum tax and is thus non-distortionary. If the consumption tax is *time-dependent*, however, a distortion of the *intertemporal* aspect of household decisions tends to arise.

To understand answer 2, consider a Ramsey household with inelastic labor supply. Suppose the household faces a time-varying consumption tax rate $\tau_t > 0$. To obtain a consumption level per time unit equal to c_t per capita, the household has to spend

$$\bar{c}_t = (1 + \tau_t)c_t$$

units of account (in real terms) per capita. Thus, spending $\bar{c}_t$ per capita per time unit results in the per capita consumption level

$$c_t = (1 + \tau_t)^{-1}\bar{c}_t. \tag{11.9}$$

In order to concentrate on the consumption tax as such, we assume the tax revenue is simply given back as lump-sum transfers and that there are no other government activities. Then, with a balanced government budget, we have

$$x_t L_t = \tau_t c_t L_t,$$

where x_t is the per capita lump-sum transfer, exogenous to the household, and L_t is the size of the representative household.

Assuming CRRA utility with parameter $\theta > 0$, the instantaneous per capita utility can be written

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} = \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta}.$$

In our standard notation the household's intertemporal optimization problem is then to choose $(\bar{c}_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\ \bar{c}_t &\geq 0, \\ \dot{a}_t &= (r_t - n)a_t + w_t + x_t - \bar{c}_t, \qquad a_0 \text{ given,} \\ \lim_{t\to\infty} a_t e^{-\int_0^{\infty}(r_s-n)ds} &\geq 0. \end{aligned}$$

From now, we let the timing of the variables be implicit unless needed for clarity. The current-value Hamiltonian is

$$H = \frac{(1+\tau)^{\theta-1}\bar{c}^{1-\theta} - 1}{1-\theta} + \lambda\left[(r-n)a + w + x - \bar{c}\right],$$

where λ is the co-state variable associated with financial per capita wealth, a. An interior optimal solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial \bar{c}} = (1+\tau)^{\theta-1}\bar{c}^{-\theta} - \lambda = 0, \text{ so that } (1+\tau)^{\theta-1}\bar{c}^{-\theta} = \lambda, \quad (11.10)$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} + (\rho - n)\lambda, \quad (11.11)$$

and a transversality condition which amounts to

$$\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s - n)ds} = 0. \quad (11.12)$$

We take logs in (11.10) to get

$$(\theta - 1)\log(1+\tau) - \theta \log \bar{c} = \log \lambda.$$

Differentiating w.r.t. time, taking into account that $\tau = \tau_t$, gives

$$(\theta-1)\frac{\dot{\tau}}{1+\tau} - \theta\frac{\dot{\bar{c}}}{\bar{c}} = \frac{\dot{\lambda}}{\lambda} = \rho - r.$$

By ordering, we find the growth rate of consumption spending,

$$\frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\theta}\left[r + (\theta-1)\frac{\dot{\tau}}{1+\tau} - \rho\right].$$

Using (11.9), this gives the growth rate of consumption,

$$\frac{\dot{c}}{c} = \frac{\dot{\bar{c}}}{\bar{c}} - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}\left[r + (\theta-1)\frac{\dot{\tau}}{1+\tau} - \rho\right] - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}(r - \frac{\dot{\tau}}{1+\tau} - \rho).$$

Assuming firms maximize profit under perfect competition, in equilibrium the real interest rate will satisfy

$$r = \frac{\partial Y}{\partial K} - \delta. \quad (11.13)$$

But the *effective* real interest rate, $\hat{r}$, faced by the consuming household, is

$$\hat{r} = r - \frac{\dot{\tau}}{1+\tau} \lesseqgtr r \text{ for } \dot{\tau} \gtreqless 0,$$

respectively. If for example the consumption tax is increasing, then the effective real interest rate faced by the consumer is smaller than the market real interest rate, given in (11.13), because saving implies postponing consumption and future consumption is more expensive due to the higher consumption tax rate.

The conclusion is that a time-varying consumption tax rate is distortionary. It implies a wedge between the intertemporal rate of transformation faced by the consumer, reflected by $\hat{r}$, and the intertemporal rate of transformation available in the technology of society, indicated by r in (11.13). On the other hand, *if* the consumption tax rate is constant, the consumption tax is non-distortionary when there is no utility from leisure.

A remark on tax smoothing

In models with transitional dynamics it is often so that maintaining constant tax rates is inconsistent with maintaining a balanced government budget. Is the implication of this that we should recommend the government to let tax rates be continually adjusted so as to maintain a forever balanced budget? No! As the above example as well as business cycle theory suggest, maintaining tax rates constant ("tax smoothing"), and thereby allowing government deficits and surpluses to arise, will generally make more sense. In itself, a budget deficit is not worrisome. It only becomes worrisome if it is not accompanied later by sufficient budget surpluses to avoid an exploding government debt/GDP ratio to arise. This requires that the tax rates taken together have a *level* which in the long run matches the level of government expenses.

Chapter 12

Learning by investing: two versions

This lecture note is a supplement to Acemoglu, §11.4-5, where only Paul Romer's version of the learning-by-investing hypothesis is presented.

The *learning-by-investing model,* sometimes called the *learning-by-doing model,* is one of the basic complete endogenous growth models. By "complete" is meant that the model specifies not only the technological aspects of the economy but also the market structure and the household sector, including household preferences. As in much other endogenous growth theory the modeling of the household sector follows Ramsey and assumes the existence of a representative infinitely-lived household. Since this results in a simple determination of the long-run interest rate (the modified golden rule), the analyst can in a first approach concentrate on the main issue, technological change, without being detracted by aspects secondary to this issue.

In the present model learning from investment experience and diffusion across firms of the resulting new technical knowledge (positive externalities) play an important role.

There are two popular alternative versions of the model. The distinguishing feature is whether the learning parameter (see below) is less than one or equal to one. The first case corresponds to (a simplified version of) a famous model by Nobel laureate Kenneth Arrow (1962). The second case has been drawn attention to by Paul Romer (1986) who assumes that the learning parameter equals one. These two contributions start out from a common framework which we now consider.

12.1 The common framework

We consider a closed economy with firms and households interacting under conditions of perfect competition. Later, a government attempting to internalize the positive investment externality is introduced.

Let there be N firms in the economy (N "large"). Suppose they all have the same neoclassical production function, F, with CRS. Firm no. i faces the technology

$$Y_{it} = F(K_{it}, A_t L_{it}), \qquad i = 1, 2, ..., N, \tag{12.1}$$

where the economy-wide technology level A_t is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$A_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \qquad 0 < \lambda \leq 1, \tag{12.2}$$

where I_s^n is aggregate net investment and $K_t = \sum_i K_{it}$.[1]

The idea is that investment – the production of capital goods – as an unintended *by-product* results in *experience* or what we may call on-the-job *learning.* Experience allows producers to recognize opportunities for process and quality improvements. In this way knowledge is achieved about how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive and better satisfy the needs of the users. Moreover, as emphasized by Arrow,

> "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).[2]

The learning is assumed to benefit essentially all firms in the economy. There are knowledge spillovers across firms and these spillovers are reasonably fast relative to the time horizon relevant for growth theory. In our macroeconomic approach both F and A are in fact assumed to be exactly

[1] With arbitrary units of measurement for labor and output the hypothesis is $A_t = BK_t^{\lambda}$, $B > 0$. In (12.2) measurement units are chosen such that $B = 1$.

[2] Concerning empirical evidence of learning-by-doing and learning-by-investing, see Chapter 13. The citation of Arrow indicates that it was rather experience from cumulative *gross* investment he had in mind as the basis for learning. Yet the hypothesis in (12.2) is the more popular one - seemingly for no better reason than that it leads to simpler dynamics. Another way in which (12.2) deviates from Arrow's original ideas is by assuming that technical progress is disembodied rather than embodied, an important distinction to which we return in Chapter 13.

the same for all firms in the economy. That is, in this specification the firms producing consumption-goods benefit from the learning just as much as the firms producing capital-goods.

The parameter λ indicates the elasticity of the general technology level, A, with respect to cumulative aggregate net investment and is named the "learning parameter". Whereas Arrow assumes $\lambda < 1$, Romer focuses on the case $\lambda = 1$. The case of $\lambda > 1$ is ruled out since it would lead to explosive growth (infinite output in finite time) and is therefore not plausible.

12.1.1 The individual firm

In the simple Ramsey model we assumed that households directly own the capital goods in the economy and rent them out to the firms. When discussing learning-by-investment, it somehow fits the intuition better if we (realistically) assume that the firms generally own the capital goods they use. They then finance their capital investment by issuing shares and bonds. Households' financial wealth then consists of these shares and bonds.

Consider firm i. There is perfect competition in all markets. So the firm is a price taker. Its problem is to choose a production and investment plan which maximizes the present value, V_i, of expected future cash-flows. Thus the firm chooses $(L_{it}, I_{it})_{t=0}^{\infty}$ to maximize

$$V_{i0} = \int_0^{\infty} \left[F(K_{it}, A_t L_{it}) - w_t L_{it} - I_{it}\right] e^{-\int_0^t r_s ds} dt$$

subject to $\dot{K}_{it} = I_{it} - \delta K_{it}$. Here w_t and I_t are the real wage and gross investment, respectively, at time t, r_s is the real interest rate at time s, and $\delta \geq 0$ is the capital depreciation rate. Rising marginal capital installation costs and other kinds of adjustment costs are assumed minor and can be ignored. It can be shown that in this case the firm's problem is equivalent to maximization of current pure profits in every short time interval. So, as hitherto, we can describe the firm as just solving a series of static profit maximization problems.

We suppress the time index when not needed for clarity. At any date firm i maximizes current pure profits, $\Pi_i = F(K_i, AL_i) - (r + \delta)K_i - wL_i$. This leads to the first-order conditions for an interior solution:

$$\begin{aligned} \partial\Pi_i/\partial K_i &= F_1(K_i, AL_i) - (r + \delta) = 0, \\ \partial\Pi_i/\partial L_i &= F_2(K_i, AL_i)A - w = 0. \end{aligned} \qquad (12.3)$$

Behind (12.3) is the presumption that each firm is small relative to the economy as a whole, so that each firm's investment has a negligible effect on the economy-wide technology level A_t. Since F is homogeneous of degree one, by Euler's theorem,[3] the first-order partial derivatives, F_1 and F_2, are homogeneous of degree 0. Thus, we can write (12.3) as

$$F_1(k_i, A) = r + \delta, \tag{12.4}$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (12.4) determines k_i uniquely. From (12.4) follows that the chosen capital-labor ratio, k_i, will be the same for all firms, say $\bar{k}$.

12.1.2 The individual household

The household sector is described by our standard Ramsey framework with inelastic labor supply and a constant population growth rate $n \geq 0$. The households have CRRA instantaneous utility with parameter $\theta > 0$. The pure rate of time preference is a constant, ρ. The flow budget identity in per capita terms is

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given},$$

where a is per capita financial wealth. The NPG condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0.$$

The resulting consumption-saving plan implies that per capita consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho),$$

and the transversality condition that the NPG condition is satisfied with strict equality. In general equilibrium of our closed economy without natural resources and government debt, a_t will equal K_t/L_t.

12.1.3 Equilibrium in factor markets

In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $\sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital intensity, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{12.5}$$

[3] Recall that a function $f(x, y)$ defined in a domain D is homogeneous of degree h if for all (x, y) in D, $f(\lambda x, \lambda y) = \lambda^h f(x, y)$ for all $\lambda > 0$. If a differentiable function $f(x, y)$ is homogeneous of degree h, then (i) $xf_1'(x, y) + yf_2'(x, y) = hf(x, y)$, and (ii) the first-order partial derivatives, $f_1'(x, y)$ and $f_2'(x, y)$, are homogeneous of degree $h - 1$.

As a consequence we can use (12.4) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, A_t) - \delta. \tag{12.6}$$

That is, whereas in the firm's first-order condition (12.4) causality goes from r_t to k_{it}, in (12.6) causality goes from k_t to r_t. Note also that in our closed economy with no natural resources and no government debt, a_t will equal k_t.

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, A)L_i = \sum_i F(k, A)L_i \quad \text{(by (12.1) and (12.5))} \\ &= F(k, A)\sum_i L_i = F(k, A)L = F(K, AL) = F(K, K^{\lambda}L) \quad \text{(by (12.2))}, \end{aligned} \tag{12.7}$$

where we have several times used that F is homogeneous of degree one.

12.2 The arrow case: $\lambda < 1$

The Arrow case is the robust case where the learning parameter satisfies $0 < \lambda < 1$. The method for analyzing the Arrow case is analogue to that used in the study of the Ramsey model with exogenous technical progress. In particular, aggregate capital per unit of effective labor, $\tilde{k} \equiv K/(AL)$, is a key variable. Let $\tilde{y} \equiv Y/(AL)$. Then

$$\tilde{y} = \frac{F(K, AL)}{AL} = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{12.8}$$

We can now write (12.6) as

$$r_t = f'(\tilde{k}_t) - \delta, \tag{12.9}$$

where $\tilde{k}_t$ is pre-determined.

12.2.1 Dynamics

From the definition $\tilde{k} \equiv K/(AL)$ follows

$$\begin{aligned} \frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - \lambda\frac{\dot{K}}{K} - n \quad \text{(by (12.2))} \\ &= (1-\lambda)\frac{Y - C - \delta K}{K} - n = (1-\lambda)\frac{\tilde{y} - \tilde{c} - \delta\tilde{k}}{\tilde{k}} - n, \quad \text{where } \tilde{c} \equiv \frac{C}{AL} \equiv \frac{c}{A}. \end{aligned}$$

Multiplying through by $\tilde{k}$ we have

$$\dot{\tilde{k}} = (1-\lambda)(f(\tilde{k}) - \tilde{c}) - [(1-\lambda)\delta + n]\,\tilde{k}. \tag{12.10}$$

In view of (12.9), the Keynes-Ramsey rule implies

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left(f'(\tilde{k}) - \delta - \rho\right). \tag{12.11}$$

Defining $\tilde{c} \equiv c/A$, now follows

$$\begin{aligned}\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{A}}{A} = \frac{\dot{c}}{c} - \lambda\frac{\dot{K}}{K} = \frac{\dot{c}}{c} - \lambda\frac{Y - cL - \delta K}{K} = \frac{\dot{c}}{c} - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}) \\ &= \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}).\end{aligned}$$

Multiplying through by $\tilde{c}$ we have

$$\dot{\tilde{c}} = \left[\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(f(\tilde{k}) - \tilde{c} - \delta\tilde{k})\right]\tilde{c}. \tag{12.12}$$

The two coupled differential equations, (12.10) and (12.12), determine the evolution over time of the economy.

Phase diagram

Figure 12.1 depicts the phase diagram. The $\dot{\tilde{k}} = 0$ locus comes from (12.10), which gives

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}, \tag{12.13}$$

where we realistically may assume that $\delta + n/(1-\lambda) > 0$. As to the $\dot{\tilde{c}} = 0$ locus, we have

$$\begin{aligned}\dot{\tilde{c}} &= 0 \text{ for } \tilde{c} = f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda\theta}(f'(\tilde{k}) - \delta - \rho) \\ &= f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda}g_c \equiv c(\tilde{k}) \quad \text{(from (12.11))}.\end{aligned} \tag{12.14}$$

Before determining the slope of the $\dot{\tilde{c}} = 0$ locus, it is convenient to consider the steady state, $(\tilde{k}^*, \tilde{c}^*)$.

Steady state

In a steady state $\tilde{c}$ and $\tilde{k}$ are constant so that the growth rate of C as well as K equals $\dot{A}/A + n$, i.e.,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{A}}{A} + n = \lambda\frac{\dot{K}}{K} + n.$$

Solving gives

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda}.$$

Thence, in a steady state

$$g_c = \frac{\dot{C}}{C} - n = \frac{n}{1-\lambda} - n = \frac{\lambda n}{1-\lambda} \equiv g_c^*, \quad \text{and} \tag{12.15}$$

$$\frac{\dot{A}}{A} = \lambda\frac{\dot{K}}{K} = \frac{\lambda n}{1-\lambda} = g_c^*. \tag{12.16}$$

The steady-state values of r and $\tilde{k}$, respectively, will therefore satisfy, by (12.11),

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda}. \tag{12.17}$$

To ensure existence of a steady state we assume that the private marginal product of capital is sufficiently sensitive to capital per unit of effective labor, from now called the "capital intensity":

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \delta + \rho + \theta\frac{\lambda n}{1-\lambda} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

The transversality condition of the representative household is that $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$, where a_t is per capita financial wealth. In general equilibrium $a_t = k_t \equiv \tilde{k}_t A_t$, where A_t in steady state grows according to (12.16). Thus, in steady state the transversality condition can be written

$$\lim_{t\to\infty} \tilde{k}^* e^{(g_c^* - r^* + n)t} = 0. \tag{TVC}$$

For this to hold, we need

$$r^* > g_c^* + n = \frac{n}{1-\lambda}, \tag{12.18}$$

by (12.15). In view of (12.17), this is equivalent to

$$\rho - n > (1-\theta)\frac{\lambda n}{1-\lambda}, \tag{A2}$$

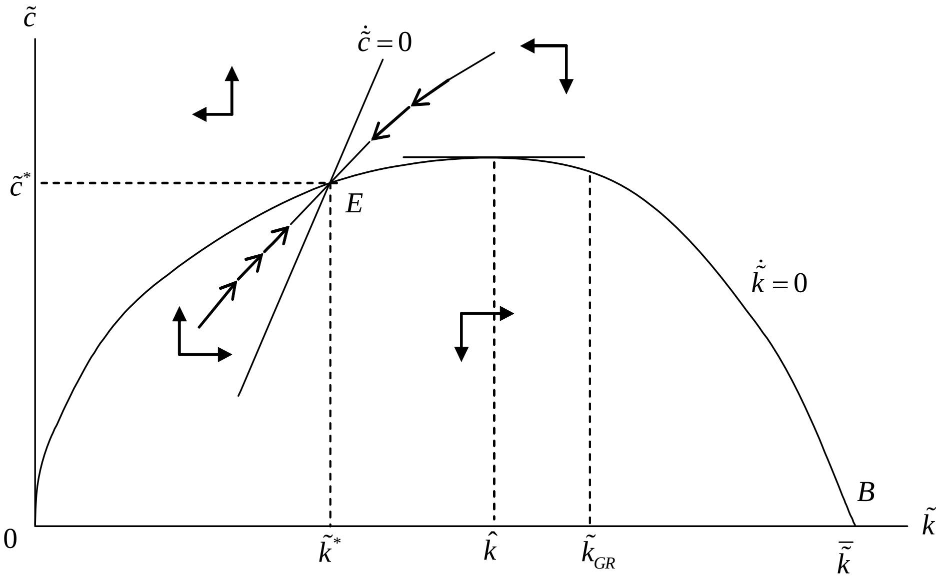

Figure 12.1: Phase diagram for the Arrow model.

which we assume satisfied.

As to the slope of the $\dot{\tilde{c}} = 0$ locus we have, from (12.14),

$$c'(\tilde{k}) = f'(\tilde{k}) - \delta - \frac{1}{\lambda}(\tilde{k}\frac{f''(\tilde{k})}{\theta} + g_c) > f'(\tilde{k}) - \delta - \frac{1}{\lambda}g_c, \tag{12.19}$$

since $f'' < 0$. At least in a small neighborhood of the steady state we can sign the right-hand side of this expression. Indeed,

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* = \rho + \theta g_c^* - \frac{1}{\lambda}g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} = \rho - n - (1-\theta)\frac{\lambda n}{1-\lambda} > 0, \tag{12.20}$$

by (12.15) and (A2). So, combining with (12.19), we conclude that $c'(\tilde{k}^*) > 0$. By continuity, in a small neighborhood of the steady state, $c'(\tilde{k}) \approx c'(\tilde{k}^*) > 0$. Therefore, close to the steady state, the $\dot{\tilde{c}} = 0$ locus is positively sloped, as indicated in Figure 12.1.

Still, we have to check the following question: In a neighborhood of the steady state, which is steeper, the $\dot{\tilde{c}} = 0$ locus or the $\dot{\tilde{k}} = 0$ locus? The slope of the latter is $f'(\tilde{k}) - \delta - n/(1-\lambda)$, from (12.13). At the steady state this slope is

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* \in (0, c'(\tilde{k}^*)),$$

in view of (12.20) and (12.19). The $\dot{\tilde{c}} = 0$ locus is thus steeper. So, the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus from below and can only cross once.

The assumption (A1) ensures existence of a $\tilde{k}^* > 0$ satisfying (12.17). As Figure 12.1 is drawn, a little more is implicitly assumed namely that there exists a $\hat{k} > 0$ such that the *private* net marginal product of capital equals the steady-state growth rate of output, i.e.,

$$f'(\hat{k}) - \delta = (\frac{\dot{Y}}{Y})^* = (\frac{\dot{A}}{A})^* + \frac{\dot{L}}{L} = \frac{\lambda n}{1-\lambda} + n = \frac{n}{1-\lambda}, \tag{12.21}$$

where we have used (12.16). Thus, the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \hat{k}$ is horizontal and $\hat{k} > \tilde{k}^*$ as indicated in the figure.

Note, however, that $\hat{k}$ is not the golden-rule capital intensity. The latter is the capital intensity, $\tilde{k}_{GR}$, at which the *social* net marginal product of capital equals the steady-state growth rate of output (see Appendix). If $\tilde{k}_{GR}$ exists, it will be larger than $\hat{k}$ as indicated in Figure 12.1. To see this, we now derive a convenient expression for the social marginal product of capital. From (12.7) we have

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= F_1(\cdot) + F_2(\cdot)\lambda K^{\lambda-1}L = f'(\tilde{k}) + F_2(\cdot)K^{\lambda}L(\lambda K^{-1}) \quad \text{(by (12.8))} \\
&= f'(\tilde{k}) + (F(\cdot) - F_1(\cdot)K)\,\lambda K^{-1} \quad \text{(by Euler's theorem)} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda}L - f'(\tilde{k})K)\lambda K^{-1} \quad \text{(by (12.8) and (12.2))} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda-1}L - f'(\tilde{k}))\lambda = f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} > f'(\tilde{k}).
\end{aligned}$$

in view of $\tilde{k} = K/(K^{\lambda}L) = K^{1-\lambda}L^{-1}$ and $f(\tilde{k})/\tilde{k} - f'(\tilde{k}) > 0$. As expected, the positive externality makes the social marginal product of capital larger than the private one. Since we can also write $\partial Y/\partial K = (1-\lambda)f'(\tilde{k}) + \lambda f(\tilde{k})/\tilde{k}$, we see that $\partial Y/\partial K$ is (still) a decreasing function of $\tilde{k}$ since both $f'(\tilde{k})$ and $f(\tilde{k})/\tilde{k}$ are decreasing in $\tilde{k}$. So the golden rule capital intensity, $\tilde{k}_{GR}$, will be that capital intensity which satisfies

$$f'(\tilde{k}_{GR}) + \lambda\frac{f(\tilde{k}_{GR}) - \tilde{k}_{GR}f'(\tilde{k}_{GR})}{\tilde{k}_{GR}} - \delta = \left(\frac{\dot{Y}}{Y}\right)^* = \frac{n}{1-\lambda}.$$

To ensure there exists such a $\tilde{k}_{GR}$, we strengthen the right-hand side inequality in (A1) by the assumption

$$\lim_{\tilde{k}\to\infty}\left(f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}}\right) < \delta + \frac{n}{1-\lambda}. \tag{A3}$$

This, together with (A1) and $f'' < 0$, implies existence of a unique $\tilde{k}_{GR}$, and in view of our additional assumption (A2), we have $0 < \tilde{k}^* < \hat{k} < \tilde{k}_{GR}$, as displayed in Figure 12.1.

Stability

The arrows in Figure 12.1 indicate the direction of movement, as determined by (12.10) and (12.12)). We see that the steady state is a saddle point. The dynamic system has one pre-determined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$. The saddle path is not parallel to the jump variable axis. We claim that for a given $\tilde{k}_0 > 0$, (i) the initial value of $\tilde{c}_0$ will be the ordinate to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path; (ii) over time the economy will move along the saddle path towards the steady state. Indeed, this time path is consistent with all conditions of general equilibrium, including the transversality condition (TVC). And the path is the *only* technically feasible path with this property. Indeed, all the divergent paths in Figure 12.1 can be ruled out as equilibrium paths because they can be shown to violate the transversality condition of the household.

In the long run c and $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k}^*)A$ grow at the rate $\lambda n/(1-\lambda)$, which is positive if and only if $n > 0$. This is an example of *endogenous growth* in the sense that the positive long-run per capita growth rate is generated through an internal mechanism (learning) in the model (in contrast to exogenous technology growth as in the Ramsey model with exogenous technical progress).

12.2.2 Two types of endogenous growth

As also touched upon in Chapter 10, it is useful to distinguish between two types of endogenous growth. *Fully endogenous* growth occurs when the long-run growth rate of c is positive without the support from growth in any exogenous factor (for example exogenous growth in the labor force); the Romer case, to be considered in the next section, provides an example. *Semi-endogenous growth* occurs if growth is endogenous but a positive per capita growth rate can not be maintained in the long run without the support from growth in some exogenous factor (for example growth in the labor force). Clearly, in the Arrow version of learning by investing, growth is "only" semi-endogenous. The technical reason for this is the assumption that the learning parameter, λ, is below 1, which implies diminishing marginal returns to capital at the aggregate level. As a consequence, if and only if $n > 0$, do we have $\dot{c}/c > 0$ in the long run.[4] In line with this, $\partial g_y^*/\partial n > 0$.

The key role of population growth derives from the fact that although there are diminishing marginal returns to capital at the aggregate level, there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. To put it differently: when there are increasing returns to K and L together, growth in the labor force not only counterbalances the falling marginal product of aggregate capital (this counter-balancing role reflects the direct complementarity between K and L), but also upholds sustained productivity growth via the learning mechanism.

Note that in the semi-endogenous growth case, $\partial g_y^*/\partial \lambda = n/(1-\lambda)^2 > 0$ for $n > 0$. That is, a higher value of the learning parameter implies higher per capita growth in the long run, when $n > 0$. Note also that $\partial g_y^*/\partial \rho = 0$ $= \partial g_y^*/\partial \theta$, that is, in the semi-endogenous growth case, preference parameters do not matter for the long-run per capita growth rate. As indicated by (12.15), the long-run growth rate is tied down by the learning parameter, λ, and the rate of population growth, n. Like in the simple Ramsey model, however, it can be shown that preference parameters matter for the *level* of the growth path. For instance (12.17) shows that $\partial \tilde{k}^*/\partial \rho < 0$ so that more patience (lower ρ) imply a higher $\tilde{k}^*$ and thereby a higher $y_t = f(\tilde{k}^*)A_t$.

This suggests that although taxes and subsidies do not have long-run growth effects, they can have *level* effects.

12.3 Romer's limiting case: $\lambda = 1$, $n = 0$

We now consider the limiting case $\lambda = 1$. We should think of it as a thought experiment because, by most observers, the value 1 is considered an unrealistically high value for the learning parameter. Moreover, in combination with $n > 0$, the value 1 will lead to a forever rising per capita growth rate which does not accord the economic history of the industrialized world over more than a century. To avoid a forever rising growth rate, we therefore introduce the parameter restriction $n = 0$.

The resulting model turns out to be extremely simple and at the same time it gives striking results (both circumstances have probably contributed to its popularity).

First, with $\lambda = 1$ we get $A = K$ and so the equilibrium interest rate is,

[4] Note, however, that the model, and therefore (12.15), presupposes $n \geq 0$. If $n < 0$, then K would tend to be decreasing and so, by (12.2), the level of technical knowledge would be decreasing, which is implausible, at least for a modern industrialized economy.

by (12.6),

$$r = F_1(k, K) - \delta = F_1(1, L) - \delta \equiv \bar{r},$$

where we have divided the two arguments of $F_1(k, K)$ by $k \equiv K/L$ and again used Euler's theorem. Note that the interest rate is constant "from the beginning" and independent of the historically given initial value of K, K_0. The aggregate production function is now

$$Y = F(K, KL) = F(1, L)K, \qquad L \text{ constant}, \tag{12.22}$$

and is thus *linear* in the aggregate capital stock.[5] In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. Thereby the Romer model belongs to the class of *reduced-form AK models*, that is, models where in general equilibrium the interest rate and the aggregate output-capital ratio are necessarily constant over time whatever the initial conditions.

The method for analyzing an AK model is different from the one used for a diminishing returns model as above.

12.3.1 Dynamics

The Keynes-Ramsey rule now takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv \gamma, \tag{12.23}$$

which is also constant "from the beginning". To ensure positive growth, we assume

$$F_1(1, L) - \delta > \rho. \tag{A1'}$$

And to ensure bounded intertemporal utility (and existence of equilibrium), it is assumed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A2'}$$

Solving the linear differential equation (12.23) gives

$$c_t = c_0 e^{\gamma t}, \tag{12.24}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by applying the households' transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

[5] Acemoglu, p. 400, writes this as $Y = \tilde{f}(L)K$.

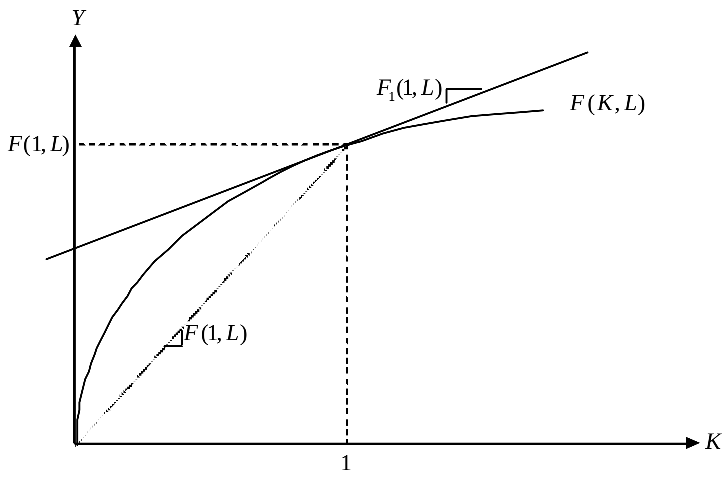

Figure 12.2: Illustration of the fact that for L given, $F(1, L) > F_1(1, L)$.

First, note that the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1, L)K - cL - \delta K,$$

or, in per-capita terms,

$$\dot{k} = [F(1, L) - \delta]\, k - c_0 e^{\gamma t}. \tag{12.25}$$

In this equation it is important that $F(1, L) - \delta - \gamma > 0$. To understand this inequality, note that, by (A2'), $F(1, L) - \delta - \gamma > F(1, L) - \delta - \bar{r} = F(1, L) - F_1(1, L) = F_2(1, L)L > 0$, where the first equality is due to $\bar{r} = F_1(1, L) - \delta$ and the second is due to the fact that since F is homogeneous of degree 1, we have, by Euler's theorem, $F(1, L) = F_1(1, L) \cdot 1 + F_2(1, L)L > F_1(1, L) > \delta$, in view of (A1'). The key property $F(1, L) - F_1(1, L) > 0$ is illustrated in Figure 12.2.

The solution of a general linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$, is

$$x(t) = (x(0) - \frac{c}{a + h})e^{-at} + \frac{c}{a + h}e^{ht}. \tag{12.26}$$

Thus the solution to (12.25) is

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma}e^{\gamma t}. \tag{12.27}$$

To check whether (TVC) is satisfied we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= (k_0 - \frac{c_0}{F(1,L)-\delta-\gamma})e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L)-\delta-\gamma}e^{(\gamma-\bar{r})t} \\ &\rightarrow (k_0 - \frac{c_0}{F(1,L)-\delta-\gamma})e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \rightarrow \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A2'). But $\bar{r} = F_1(1,L) - \delta < F(1,L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma)k_0. \qquad (12.28)$$

If c_0 is less than this, there will be over-saving and (TVC) is violated ($a_t e^{-\bar{r}t} \rightarrow \infty$ for $t \rightarrow \infty$, since $a_t = k_t$). If c_0 is higher than this, both the NPG and (TVC) are violated ($a_t e^{-\bar{r}t} \rightarrow -\infty$ for $t \rightarrow \infty$).

Inserting the solution for c_0 into (12.27), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional dynamics).

This is a case of *fully endogenous growth* in the sense that the long-run growth rate of c is positive without the support by growth in any exogenous factor. This outcome is due to the absence of diminishing returns to aggregate capital, which is implied by the assumed high value of the learning parameter. But the empirical foundation for this high value is weak, to say the least, cf. Chapter 13. A further drawback of this special version of the learning model is that the results are *non-robust.* With λ slightly less than 1, we are back in the Arrow case and growth peters out, since $n = 0$. With λ slightly above 1, it can be shown that growth becomes explosive: infinite output in finite time![6]

The Romer case, $\lambda = 1$, is thus a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which *apriori* can take any value within an interval. Second, the imposed value leads to non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system.

Note that the *causal structure* in the long run in the diminishing returns case is different than in the AK-case of Romer. In the diminishing returns case the steady-state growth rate is determined first, as g_c^* in (12.15), then r^* is determined through the Keynes-Ramsey rule and, finally, Y/K is determined by the technology, given r^*. In contrast, the Romer case has Y/K

[6] See Appendix B in Chapter 13.

and r directly given as $F(1, L)$ and $\bar{r}$, respectively. In turn, $\bar{r}$ determines the (constant) equilibrium growth rate through the Keynes-Ramsey rule.

12.3.2 Economic policy in the Romer case

In the AK case, that is, the fully endogenous growth case, we have $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$. Thus, preference parameters *matter* for the long-run growth rate and not "only" for the *level* of the upward-sloping time path for per capita output. This suggests that taxes and subsidies can have *long-run* growth effects. In any case, in this model there is a motivation for government intervention due to the positive externality of private investment. This motivation is present whether $\lambda < 1$ or $\lambda = 1$. Here we concentrate on the latter case, for no better reason than that it is simpler. We first find the social planner's solution.

The social planner

The social planner faces the aggregate production function $Y_t = F(1, L)K_t$ or, in per capita terms, $y_t = F(1, L)k_t$. The social planner's problem is to choose $(c_t)_{=0}^{\infty}$ to maximize

$$\begin{aligned}
U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\
c_t &\geq 0, \\
\dot{k}_t &= F(1, L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, && (12.29) \\
k_t &\geq 0 \text{ for all } t > 0. && (12.30)
\end{aligned}$$

The current-value Hamiltonian is

$$H(k, c, \eta, t) = \frac{c^{1-\theta}}{1-\theta} + \eta \left(F(1, L)k - c - \delta k \right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\begin{aligned}
\frac{\partial H}{\partial c} &= c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, && (12.31) \\
\frac{\partial H}{\partial k} &= \eta(F(1, L) - \delta) = -\dot{\eta} + \rho\eta. && (12.32)
\end{aligned}$$

We guess that also the transversality condition,

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \qquad (12.33)$$

must be satisfied by an optimal solution.[7] This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really an optimal solution.

Log-differentiating w.r.t. t in (12.31) and combining with (12.32) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1,L) - \delta - \rho) \equiv \gamma_{SP}. \tag{12.34}$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal product of capital is $\partial y_t/\partial k_t = F(1,L) > F_1(1,L)$. To ensure bounded intertemporal utility we sharpen (A2') to

$$\rho > (1-\theta)\gamma_{SP}. \tag{A2''}$$

To find the time path of k_t, note that the dynamic resource constraint (12.29) can be written

$$\dot{k}_t = (F(1,L) - \delta)k_t - c_0 e^{\gamma_{SP} t},$$

in view of (12.34). By the general solution formula (12.26) this has the solution

$$k_t = (k_0 - \frac{c_0}{F(1,L) - \delta - \gamma_{SP}})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1,L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t}. \tag{12.35}$$

In view of (12.32), in an interior optimal solution the time path of the adjoint variable η is

$$\eta_t = \eta_0 e^{-[(F(1,L)-\delta-\rho]t},$$

where $\eta_0 = c_0^{-\theta} > 0$, by (12.31). Thus, the conjectured transversality condition (12.33) implies

$$\lim_{t\to\infty} k_t e^{-(F(1,L)-\delta)t} = 0, \tag{12.36}$$

where we have eliminated η_0. To ensure that this is satisfied, we multiply k_t from (12.35) by $e^{-(F(1,L)-\delta)t}$ to get

$$\begin{aligned} k_t e^{-(F(1,L)-\delta)t} &= k_0 - \frac{c_0}{F(1,L) - \delta - \gamma_{SP}} + \frac{c_0}{F(1,L) - \delta - \gamma_{SP}} e^{[\gamma_{SP} - (F(1,L)-\delta)]t} \\ &\to k_0 - \frac{c_0}{F(1,L) - \delta - \gamma_{SP}} \text{ for } t \to \infty, \end{aligned}$$

[7] The proviso implied by saying word "guess" is due to the fact that optimal control theory does not guarantee that this "standard" transversality condition is necessary for optimality in *all* infinite horizon optimization problems.

since, by (A2"), $\gamma_{SP} < \rho + \theta\gamma_{SP} = F(1, L) - \delta$ in view of (12.34). Thus, (12.36) is only satisfied if

$$c_0 = (F(1, L) - \delta - \gamma_{SP})k_0. \tag{12.37}$$

Inserting this solution for c_0 into (12.35), we get

$$k_t = \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t} = k_0 e^{\gamma_{SP} t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1, L)k$, the same is true for y. Hence, our candidate for the social planner's solution is from start in balanced growth (there is no transitional dynamics).

The next step is to check whether our candidate solution satisfies a set of *sufficient* conditions for an optimal solution. Here we can use *Mangasarian's theorem* which, applied to a problem like this, with one control variable and one state variable, says that the following conditions are sufficient:

(a) Concavity: The Hamiltonian is jointly concave in the control and state variables, here c and k.

(b) Non-negativity: There is for all $t \geq 0$ a non-negativity constraint on the state variable; and the co-state variable, η, is non-negative for all $t \geq 0$.

(c) TVC: The candidate solution satisfies the transversality condition $\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0$, where $\eta_t e^{-\rho t}$ is the discounted co-state variable.

In the present case we see that the Hamiltonian is a sum of concave functions and therefore is itself concave in (k, c). Further, from (12.30) we see that condition (b) is satisfied. Finally, our candidate solution is constructed so as to satisfy condition (c). The conclusion is that our candidate solution *is* an optimal solution. We call it the SP allocation.

Implementing the SP allocation in the market economy

Returning to the market economy, we assume there is a policy maker, say the government, with only two activities. These are (i) paying an investment subsidy, s, to the firms so that their capital costs are reduced to

$$(1 - s)(r + \delta)$$

per unit of capital per time unit; (ii) financing this subsidy by a constant consumption tax rate τ.

Let us first find the size of s needed to establish the SP allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-s)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-s)(r+\delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-s)(r+\delta),$$

where $k \equiv K/L$, which is pre-determined from the supply side. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k, K)}{1-s} - \delta = \frac{F_1(1, L)}{1-s} - \delta, \tag{12.38}$$

again using Euler's theorem.

It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1, L) - \delta$. If we can obtain $r = F(1, L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required s thus satisfies

$$r = \frac{F_1(1, L)}{1-s} - \delta = F(1, L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1, L)}{F(1, L)} = \frac{F(1, L) - F_1(1, L)}{F(1, L)} = \frac{F_2(1, L)L}{F(1, L)}.$$

In case $Y_i = K_i^{\alpha}(AL_i)^{1-\alpha}$, $0 < \alpha < 1$, $i = 1,\ldots, N$, this gives $s = 1 - \alpha$.

It remains to find the required consumption tax rate τ. The tax revenue will be τcL, and the *required* tax revenue is

$$\mathcal{T} = s(r+\delta)K = \left(F(1, L) - F_1(1, L)\right)K = \tau cL.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{\mathcal{T}}{cL} = \frac{F(1, L) - F_1(1, L)}{c/k} = \frac{F(1, L) - F_1(1, L)}{F(1, L) - \delta - \gamma_{SP}} > 0, \tag{12.39}$$

where we have used that the proportionality in (12.37) between c and k holds for all $t \geq 0$. Substituting (12.34) into (12.39), the solution for τ can be written

$$\tau = \frac{\theta\left[F(1,L) - F_1(1,L)\right]}{(\theta-1)(F(1,L)-\delta)+\rho} = \frac{\theta F_2(1,L)L}{(\theta-1)(F(1,L)-\delta)+\rho}.$$

The required tax rate on consumption is thus a constant. It therefore does not distort the consumption/saving decision on the margin, cf. Chapter 11.

It follows that the allocation obtained by this subsidy-tax policy *is* the SP allocation. A policy, here the policy (s, τ), which in a decentralized system induces the SP allocation, is called a *first-best policy.*

12.4 Appendix: The golden-rule capital intensity in the Arrow case

In our discussion of the Arrow model in Section 12.2 (where $0 < \lambda < 1$), we claimed that the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that effective capital-labor ratio at which the social net marginal product of capital equals the steady-state growth rate of output. In this respect the Arrow model with endogenous technical progress is similar to the standard neoclassical growth model with exogenous technical progress.

The claim corresponds to a very general theorem, valid also for models with many capital goods and non-existence of an aggregate production function. This theorem says that the highest sustainable path for consumption per unit of labor in the economy will be that path which results from those techniques which profit maximizing firms choose under perfect competition when the real interest rate equals the steady-state growth rate of GNP (see Gale and Rockwell, 1975).

To prove our claim, note that in steady state, (12.14) holds whereby consumption per unit of labor (here the same as per capita consumption in view of L = labor force = population) can be written

$$\begin{aligned}
c_t &\equiv \tilde{c}_t A_t = \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] K_t^{\lambda} \\
&= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \left(K_0 e^{\frac{n}{1-\lambda}t}\right)^{\lambda} \quad (\text{by } g_K^* = \frac{n}{1-\lambda}) \\
&= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \left((\tilde{k}L_0)^{\frac{1}{1-\lambda}} e^{\frac{n}{1-\lambda}t}\right)^{\lambda} \quad (\text{from } \tilde{k} = \frac{K_t}{K_t^{\lambda}L_t} = \frac{K_t^{1-\lambda}}{L_t} \text{ also for } t=0) \\
&= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \tilde{k}^{\frac{\lambda}{1-\lambda}} L_0{}^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t} \equiv \varphi(\tilde{k}) L_0{}^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t},
\end{aligned}$$

defining $\varphi(\tilde{k})$ in the obvious way.

We look for that value of $\tilde{k}$ at which this steady-state path for c_t is at the highest technically feasible level. The positive coefficient, $L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}$, is the only time dependent factor and can be ignored since it is exogenous. The problem is thereby reduced to the static problem of maximizing $\varphi(\tilde{k})$ with respect to $\tilde{k} > 0$. We find

$$\begin{aligned} \varphi'(\tilde{k}) &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\right] \tilde{k}^{\frac{\lambda}{1-\lambda}} + \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right] \frac{\lambda}{1-\lambda} \tilde{k}^{\frac{\lambda}{1-\lambda}-1} \\ &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda}) + \left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \frac{n}{1-\lambda})\right) \frac{\lambda}{1-\lambda}\right] \tilde{k}^{\frac{\lambda}{1-\lambda}} \\ &= \left[(1-\lambda)f'(\tilde{k}) - (1-\lambda)\delta - n + \lambda\frac{f(\tilde{k})}{\tilde{k}} - \lambda(\delta + \frac{n}{1-\lambda})\right] \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \\ &= \left[(1-\lambda)f'(\tilde{k}) - \delta + \lambda\frac{f(\tilde{k})}{\tilde{k}} - \frac{n}{1-\lambda}\right] \frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \equiv \psi(\tilde{k})\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda}, \quad (12.40) \end{aligned}$$

defining $\psi(\tilde{k})$ in the obvious way. The first-order condition for the problem, $\varphi'(\tilde{k}) = 0$, is equivalent to $\psi(\tilde{k}) = 0$. After ordering this gives

$$f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} - \delta = \frac{n}{1-\lambda}. \quad (12.41)$$

We see that

$$\varphi'(\tilde{k}) \gtreqless 0 \quad \text{for} \quad \psi(\tilde{k}) \gtreqless 0,$$

respectively. Moreover,

$$\psi'(\tilde{k}) = (1-\lambda)f''(\tilde{k}) - \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}^2} < 0,$$

in view of $f'' < 0$ and $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$. So a $\tilde{k} > 0$ satisfying $\psi(\tilde{k}) = 0$ is the unique maximizer of $\varphi(\tilde{k})$. By (A1) and (A3) in Section 12.2 such a $\tilde{k}$ exists and is thereby the same as the $\tilde{k}_{GR}$ we were looking for.

The left-hand side of (12.41) equals the social marginal product of capital and the right-hand side equals the steady-state growth rate of output. At $\tilde{k} = \tilde{k}_{GR}$ it therefore holds that

$$\frac{\partial Y}{\partial K} - \delta = \left(\frac{\dot{Y}}{Y}\right)^*.$$

This confirms our claim in Section 12.2 about $\tilde{k}_{GR}$.

Remark about the absence of a golden rule in the Romer model. In the Romer moldel the golden rule is not a well-defined concept for the following reason. Along any balanced growth path we have from (12.29),

$$g_k \equiv \frac{\dot{k}_t}{k_t} = F(1, L) - \delta - \frac{c_t}{k_t} = F(1, L) - \delta - \frac{c_0}{k_0},$$

because g_k $(= g_K)$ is by definition constant along a balanced growth path, whereby also c_t/k_t must be constant. We see that g_k is decreasing linearly from $F(1, L) - \delta$ to $-\delta$ when c_0/k_0 rises from nil to $F(1, L)$. So choosing among alternative technically feasible balanced growth paths is inevitably a choice between starting with low consumption to get high growth forever or starting with high consumption to get low growth forever. Given any $k_0 > 0$, the alternative possible balanced growth paths will therefore sooner or later cross each other in the $(t, \ln c)$ plane. Hence, for the given k_0, there exists no balanced growth path which for all $t \geq 0$ has c_t higher than along any other technically feasible balanced growth path. So no golden rule path exists. This is a general property of AK and reduced-form AK models.

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Chapter 13

Perspectives on learning by doing and learning by investing

This chapter adds some theoretical and empirical perspectives to the discussion in Chapter 12 and in Acemoglu, Chapter 11 and 12. The contents are:

1. Learning by doing*
2. Disembodied learning by investing*
3. Disembodied vs. embodied technical change*
4. Static comparative advantage vs. dynamics of learning by doing*
5. Robustness and scale effects
   - (a) On terminology
   - (b) Robustness of simple endogenous growth models
   - (c) Weak and strong scale effects
   - (d) Discussion

Sections marked by an asterisk are only cursory reading.

The growth rate of any time-dependent variable $z > 0$ is written $g_z \equiv \dot{z}/z$. In this chapter the economy-wide technology level at time t is denoted T_t rather than A_t.

13.1 Learning by doing*

The term *learning by doing* refers to the hypothesis that accumulated work experience, especially repetition of the same type of action, improves workers' productivity and adds to technical knowledge.

A learning-by-doing model typically combines an aggregate CRS production function,

$$Y_t = F(K_t, T_t L_t), \tag{13.1}$$

with a learning function, for example,

$$\dot{T}_t = BY_t^{\lambda}, \qquad B > 0, 0 < \lambda \leq 1, \tag{13.2}$$

where λ is a learning parameter and B is a constant that, depending on the value of λ and the complete model in which (13.2) is embedded, is either an unimportant constant that depends only on measuring units or a parameter of importance for the productivity level or even the productivity growth rate. In Section 13.4 below, on the resource curse problem, we consider a two-sector model where each sector's productivity growth is governed by such a relationship.

Another learning hypothesis is of the form

$$\dot{T}_t = BT_t^{\lambda} L_t^{\mu}, \qquad T_0 > 0 \text{ given}, B > 0, \lambda \leq 1, \mu > 0. \tag{13.3}$$

Here both λ and μ are learning parameters, reflecting the elasticities of learning w.r.t. the technology level and labor hours, respectively. The higher the number of human beings involved in production and the more time they spend in production, the more experience is accumulated. Sub-optimal ingredients in the production processes are identified and eliminated. The experience and knowledge arising in one firm or one sector is speedily diffused to other firms and other sectors in the economy (knowledge spillovers or "learning by watching"), and as a result the aggregate productivity level is increased.[1]

Since hours spent, L_t, is perhaps a better indicator for "new experience" than output, Y_t, specification (13.3) may seem more appealing than specification (13.2). So this section concentrates on (13.3).

If the labor force is growing, λ should be assumed strictly less than one, because with $\lambda = 1$ there would be a built-in tendency to forever faster growth, which does not seem plausible. In fact, $\lambda < 0$ can not be ruled out; that would reflect that learning becomes more and more difficult ("the easiest ideas are found first"). On the other hand, the case of "standing on

[1] Diffusion of proficiency also occurs via apprentice-master relationships.

the shoulders" is also possible, that is, the case $0 < \lambda \leq 1$, which is the case where new learning becomes easier, the more is learnt already.

In "very-long-run" growth theory concerned with human development in an economic history perspective, the L in (13.3) has been replaced simply by the size of population in the relevant region (which may be considerably larger than a single country). This is the "population breeds ideas" view, cf. Kremer (1993). Anyway, many simple models consider the labor force to be proportional to population size, and then it does not matter whether we use the learning-by-doing interpretation or the population-breeds-ideas interpretation.

The so-called Horndal effect (reported by Lundberg, 1961) was one of the empirical observations motivating the learning-by-doing idea in growth theory:

> "The Horndal-iron works in Sweden had no new investment (and therefore presumably no significant change in its methods of production) for a period of 15 years, yet productivity (output per man-hour) rose on the average close to 2 % per annum. We find again steadily increasing performance which can only be imputed to learning from experience" (here cited after Arrow, 1962).

Similar patterns of on-the-job productivity improvements have been observed in ship-building, airframe construction, and chemical industries. On the other hand, within a single production line there seems to be a tendency for this kind of productivity increases to gradually peter out, which suggests $\lambda < 0$ in (13.3). We may call this phenomenon "diminishing returns in the learning process": the potential for new learning gradually evens out as more and more learning has already taken place. But new products are continuously invented and the accumulated knowledge is transmitted, more or less, to the production of these new products that start on a "new learning curve", along which there is initially "a large amount to be learned".[2] This combination of qualitative innovation and continuous productivity improvement through learning *may* at the aggregate level end up in a $\lambda \geq 0$ in (13.3).

In any case, whatever the sign of λ at the aggregate level, with $\lambda < 1$, this model is capable of generating sustained endogenous per capita growth (without "growth explosion") if the labor force is growing at a rate $n > 0$. Indeed, as in Chapter 12, there are two cases that are consistent with a balanced growth path (BGP for short) with positive per capita growth,

[2] A *learning curve* is a graph of estimated productivity (or its inverse, cf. Fig. 13.1 or Fig. 13.2 below) as a function of cumulative output or of time passed since production of the new product began at some plant.

namely the case $\lambda < 1$ combined with $n > 0$, and the case $\lambda = 1$ combined with $n = 0$.

We will show this for a closed economy with $L_t = L_0 e^{nt}$, $n \geq 0$, and with capital accumulation according to

$$\dot{K}_t = I_t - \delta K_t = Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ given.} \tag{13.4}$$

13.1.1 The case: $\lambda < 1$ in (13.3)

Let us first consider the growth rate of $y \equiv Y/L$ along a BGP. There are two steps in the calculation of this growth rate.

Step 1. Given (13.4), from basic balanced growth theory (Chapter 4) we know that along a BGP with positive gross saving, not only are, by definition, g_Y and g_K constant, but they are also the same, so that Y_t/K_t is constant over time. Owing to the CRS assumption, (13.1) implies that

$$1 = F(\frac{K_t}{Y_t}, \frac{T_t L_t}{Y_t}). \tag{13.5}$$

Since Y_t/K_t is constant, $T_t L_t / Y_t \equiv T_t / y_t$ must be constant. This implies that

$$g_T = g_y = g_Y - n, \tag{13.6}$$

a constant.

Step 2. Dividing through by T_t in (13.3), we get

$$g_T \equiv \frac{\dot{T}_t}{T_t} = A T_t^{\lambda - 1} L_t^{\mu}.$$

Taking logs gives $\log g_T = \log A + (\lambda - 1) \log T + \mu \log L$. And taking the time derivative on both sides of this equation leads to

$$\frac{\dot{g}_T}{g_T} = (\lambda - 1) g_T + \mu n. \tag{13.7}$$

In view of g_T being constant along a BGP, we have $\dot{g}_T = 0$, and so (13.7) gives

$$g_T = \frac{\mu n}{1 - \lambda},$$

presupposing $\lambda < 1$. Hence, by (13.6),

$$g_y = \frac{\mu n}{1 - \lambda}.$$

Under the assumption that $n > 0$, this per capita growth rate is positive, whatever the sign of λ. Given n, the growth rate is an increasing function of *both* learning parameters. Since a positive per capita growth rate can in the long run be maintained only if supported by $n > 0$, this is an example of *semi-endogenous growth* (as long as n is exogenous).

This model thus gives growth results somewhat similar to the results in Arrow's learning-by-investing model, cf. Chapter 12. In both models the learning is an unintended by-product of the work process and construction of investment goods, respectively. And both models assume that knowledge is non-appropriable (non-exclusive) and that knowledge spillovers across firms are fast (in the time perspective of growth theory). So there are positive externalities which may motivate government intervention.

Methodological remark: Different approaches to the calculation of long-run growth rates Even within this semi-endogenous growth case, depending on the situation, different approaches to the calculation of long-run growth rates may be available. In Chapter 12, in the analysis of the Arrow case $\lambda < 1$, the point of departure in the calculation was the steady state property of Arrow's model that $\tilde{k} \equiv K/(TL)$ is a constant. But this point of departure presupposes that we have established a well-defined steady state in the sense of a stationary point of a complete dynamic system (which in the Arrow model consists of two first-order differential equations in $\tilde{k}$ and $\tilde{c}$, respectively), usually involving also a description of the household sector. In the present case we are not in this situation because we have not specified how the saving in (13.4) is determined. This explains why above (as well as in Chapter 10) we have taken another approach to the calculation of the long-run growth rate. We simply assume balanced growth and ask what the growth rate must then be. If the technologies in the economy are such that per capita growth in the long run can only be due to either exogenous productivity growth or semi-endogenous productivity growth, this approach is usually sufficient to determine a unique growth rate.

Note also, however, that this latter feature is in itself an interesting and useful result (as exemplified in Chapter 10). It tells us what the growth rate *must* be in the long run provided that the system converges to balanced growth. The growth rate will be the same, independently of the market structure and the specification of the household sector, that is, it will be the same whether, for example, there is a Ramsey-style household sector or an overlapping generations set-up.[3] And at least in the first case the growth

[3]Specification of these things is needed if we want to study the transitional dynamics: the adjustment processes outside balanced growth/steady state, including the question of

rate will be the same whatever the size of the preference parameters (the rate of time preference and the elasticity of marginal utility of consumption). Moreover, only if economic policy affects the learning parameters or the population growth rate (two things that are often ruled out inherently by the setup), will the long-run growth rate be affected. Still, economic policy can *temporarily* affect economic growth and in this way affect the *level* of the long-run growth path.

13.1.2 The case $\lambda = 1$ in (13.3)

With $\lambda = 1$ in (13.3), the above growth rate formulas are no longer valid. But returning to (13.3), we have $g_T = BL_t^{\mu}$. Then, unless $n = 0$, the growth rate of y will tend to rise forever, since we have $g_T = BL_0^{\mu}e^{\mu nt} \rightarrow \infty$ for $n > 0$.

So we will assume $n = 0$. Then $L_t = L_0$ for all t, implying $g_T = BL_0^{\mu}$ for all t. Since both B and L_0 are exogenous, it is *as if* the rate of technical progress, g_T, were exogenous. Yet, technical progress is generated by an internal mechanism. If the government by economic policy could affect B or L_0, also g_T would be affected. In any case, under balanced growth, (13.5) holds again and so $T_tL_t/Y_t = T_t/y_t$ must be constant. This implies $g_y = g_T = BL_0^{\mu} > 0$. Consequently, positive per capita growth can be maintained forever without support of growth in any exogenous factor, that is, growth is *fully endogenous.*

As in the semi-endogenous growth case we can here determine the growth rate along a BGP independently of how the household sector is described. And preference parameters do *not* affect the growth rate. The fact that this is so even in the fully endogenous growth case is due to the "law of motion" of technology making up a subsystem that is independent of the remainder of the economic system. This is a special feature of the "growth engine" (13.3). Although it is not a typical ingredient of endogenous growth models, this growth engine can not be ruled out *apriori.* The simple alternative, (13.2), is very different in that the endogenous aggregate output, Y_t, is involved. We return to (13.2) in Section 13.4 below.

Before proceeding, a brief remark on the explosive case $\lambda > 1$ in (13.2) or (13.3) is in place. If we imagine $\lambda > 1$, growth becomes explosive in the extreme sense that output as well as productivity, hence also per capita consumption, will tend to *infinity in finite time.* This is so even if $n = 0$. The argument is based on the mathematical fact that, given a differential equation $\dot{x} = x^a$, where $a > 1$ and $x_0 > 0$, the solution x_t has the property

convergence to balanced growth/steady state.

that there exists a $t_1 > 0$ such that $x_t \to \infty$ for $t \to t_1$. For details, see Appendix B.

13.2 Disembodied learning by investing*

In the above framework the work process is a source of learning whether it takes place in the consumption or capital goods sector. This is *learning by doing* in a broad sense. If the source of learning is specifically associated with the construction of capital goods, the learning by doing is often said to be of the form of *learning by investing.* Why in the headline of this section we have added the qualification "disembodied", will be made clear in Section 13.3. Another name for learning by investing is *investment-specific learning by doing.*

The prevalent view in the empirical literature seems to be that learning by investing is the most important form of learning by doing; ship-building and airframe construction are prominent examples. To the extent that the construction of capital equipment is based on more complex and involved technologies than is the production of consumer goods, we are also, intuitively, inclined to expect that the greatest potential for productivity increases through learning is in the investment goods sector.[4]

In the simplest version of the learning-by-investment hypothesis, (13.3) above is replaced by

$$T_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{13.8}$$

where I_s^n is aggregate *net* investment. This is the hypothesis that the economy-wide technology level T_t is an increasing function of society's previous experience, proxied by cumulative aggregate net investment.[5] The Arrow and Romer models, as described in Chapter 12, correspond to the cases $0 < \lambda < 1$ and $\lambda = 1$, respectively.

In this framework, where the "growth engine" depends on capital accumulation, it is only in the Arrow case that we can calculate the per-capita growth rate along a BGP without specifying anything about the household sector.

[4] After the information-and-communication technology (ICT) revolution, where a lot of technically advanced consumer goods have entered the scene, this traditional presumption may be less compelling.

[5] Contrary to the dynamic learning-by-doing specification (13.3), there is here no good reason for allowing $\lambda < 0$.

13.2.1 The Arrow case: $\lambda < 1$ and $n \geq 0$

We may apply the same two steps as in Section 13.1.1. Step 1 is then an exact replication of step 1 above. Step 2 turns out to be even simpler than above, because (13.8) immediately gives $\log T = \lambda \log K$ so that $g_T = \lambda g_K$, which substituted into (13.6) yields

$$g_T = \lambda g_K = g_y = g_Y - n = g_K - n.$$

From this follows, first,

$$g_K = \frac{n}{1-\lambda}, \tag{13.9}$$

and, second,

$$g_y = \frac{\lambda n}{1-\lambda}.$$

Alternatively, we may in this case condense the two steps into one by rewriting (13.5) in the form

$$\frac{Y_t}{K_t} = F(1, \frac{T_t L_t}{K_t}) = F(1, K_t^{\lambda-1} L_t),$$

by (13.8). Along the BGP, since Y/K is constant, so must the second argument, $K_t^{\lambda-1} L_t$, be. It follows that

$$(\lambda - 1) g_K + n = 0,$$

thus confirming (13.9).

Whatever the approach to the calculation, the per capita growth rate is here tied down by the size of the learning parameter and the growth rate of the labor force.

13.2.2 The Romer case: $\lambda = 1$ and $n = 0$

In the Romer case, however, the growth rate along a BGP cannot be determined until the saving behavior in the economy is modeled. Indeed, the knife-edge case $\lambda = 1$ opens up for many different per capita growth rates under balanced growth. Which one is "selected" by the economy depends on how the household sector is described.

For a Ramsey setup with $n = 0$ the last part of Chapter 12 showed how the growth rate generated by the economy depends on the rate of time preference and the elasticity of marginal utility of consumption of the representative household. Growth is here *fully endogenous* in the sense that a positive per capita growth rate can be maintained forever without the support by growth in any exogenous factor. Moreover, according to this model, economic policy that internalizes the positive externality in the system can raise not only the productivity level, but also the long-run productivity growth rate.

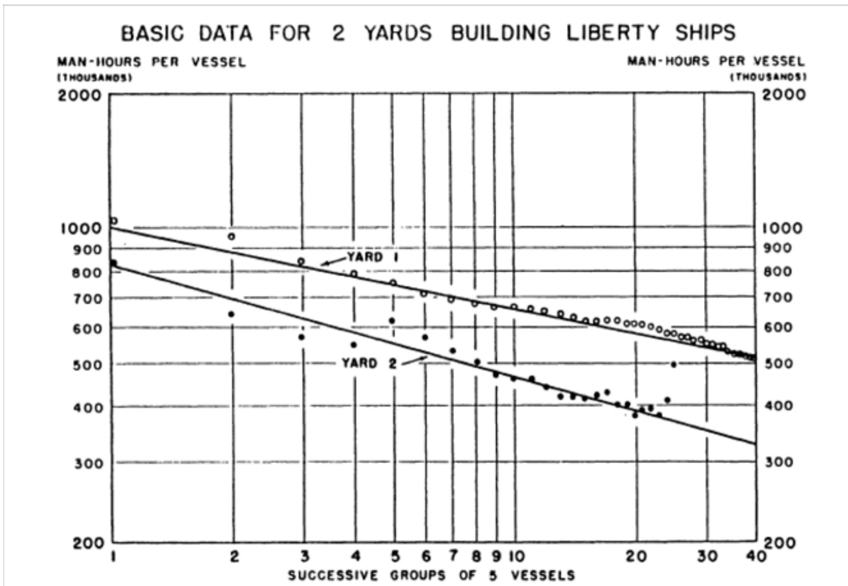

FIGURE 1.—Reductions in man-hours per vessel with increasing production. Merchant shipyards.

Figure 13.1

13.2.3 The size of the learning parameter

What is from an empirical point of view a plausible value for the learning parameter, λ? This question is important because quite different results emerge depending on whether λ is close to 1 or considerably lower (fully endogenous growth versus semi-endogenous growth). At the same time the question is not easy to answer because λ in the models is a parameter that is meant to reflect the aggregate effect of the learning going on in single firms and spreading across firms and industries.

Like Lucas (1993), we will consider the empirical studies of on-the-job productivity increases in ship-building by Searle (1945) and Rapping (1965). Both studies used data on the production of different types of cargo vessels during the second world war. Figures 1 and 2 are taken from Lucas' review article. For the vessel type called "Liberty Ships" Lucas cites the observation by Searle (1945):

> "the reduction in man-hours per ship with each doubling of cumulative output ranged from 12 to 24 percent."

Let us try to connect this observation to the learning parameter λ in Arrow's and Romer's framework. We begin by considering firm i which

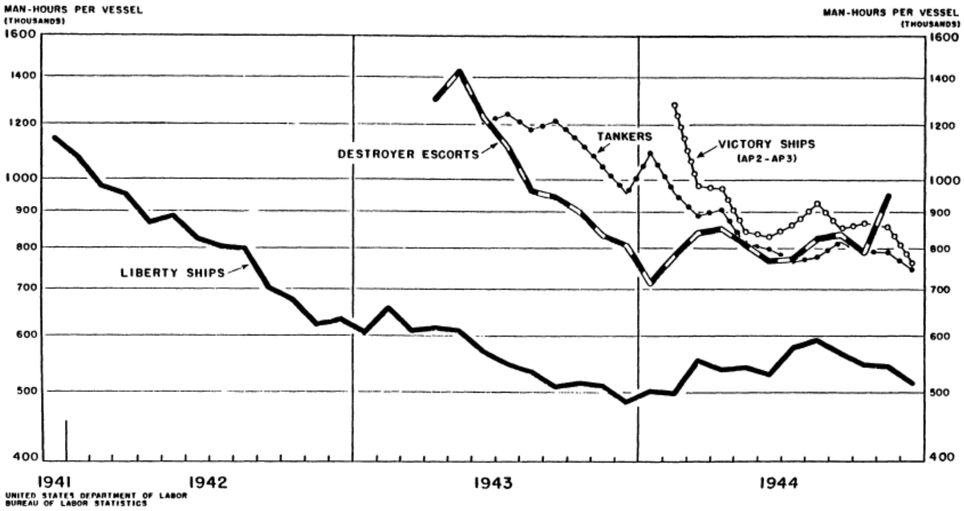

FIGURE 2.—Unit man-hour requirements for selected shipbuilding programs. Vessels delivered December 1941–December 1944.

Figure 13.2

operates in the investment goods sector. We imagine that firm i's equipment is unchanged during the observation period (as is understood in the above citation as well as the citation from Arrow (1962) in Section 13.1). Let firm i's current output and employment be Y_{it} and L_{it}, respectively. The current labor productivity is then $a_{it} = Y_{it}/L_{it}$. Let the firm's *cumulative* output be denoted Q_{it}. This cumulative output is a part of cumulative investment in society. At the micro-level the learning-by-investing hypothesis is the hypothesis that labor productivity is an increasing function of the firm's cumulative output, Q_{it}.

In figures 1 and 2 the dependent variable is not directly labor productivity, but its inverse, namely the required man-hours per unit of output, $m_{it} = L_{it}/Y_{it} = 1/a_{it}$. Figure 13.1 suggests a log-linear relationship between this variable and the cumulative output:

$$\log m_{it} = \alpha - \beta \log Q_{it}. \tag{13.10}$$

That is, as cumulative output rises, the required man-hours per unit of output declines over time in this way:

$$m_{it} = \frac{e^{\alpha}}{Q_{it}^{\beta}}.$$

Equivalently, labor productivity rises over time in this way:

$$a_{it} = \frac{1}{m_{it}} = e^{-\alpha} Q_{it}^{\beta}.$$

So, specifying the relationship by a power function, as in (13.8), makes sense.

Now, let $t = t_1$ be a fixed point in time. Then, (13.10) becomes

$$\log m_{it_1} = \alpha - \beta \log Q_{it_1}.$$

Let t_2 be the later point in time where cumulative output has been doubled. Then at time t_2 the required man-hours per unit of output has declined to

$$\log m_{it_2} = \alpha - \beta \log Q_{it_2} = \alpha - \beta \log(2Q_{it_1}).$$

Hence,

$$\log m_{it_1} - \log m_{it_2} = -\beta \log Q_{it_1} + \beta \log(2Q_{it_1}) = \beta \log 2. \tag{13.11}$$

Lucas' citation above from Searle amounts to a claim that

$$0.12 < \frac{m_{it_1} - m_{it_2}}{m_{it_1}} < 0.24. \tag{13.12}$$

By a first-order Taylor approximation we have $\log m_{it_2} \approx \log m_{it_1} + (m_{it_2} - m_{it_1})/m_{it_1}$. Hence, $(m_{it_1} - m_{it_2})/m_{it_1} \approx \log m_{it_1} - \log m_{it_2}$. Substituting this into (13.12) gives, approximately,

$$0.12 < \log m_{it_1} - \log m_{it_2} < 0.24.$$

Combining this with (13.11) gives $0.12 < \beta \log 2 < 0.24$ so that

$$0.17 = \frac{0.12}{\log 2} < \beta < \frac{0.24}{\log 2} = 0.35.$$

Rapping (1965) finds by a more rigorous econometric approach β to be in the vicinity of 0.26 (still ship building). Arrow (1962) and Solow (1997) refer to data on airframe building. This data roughly suggests $\beta = 1/3$.

How can this be translated into a guess on the "aggregate" learning parameter λ in (13.8)? This is a complicated question and the subsequent remarks are very tentative. First of all, the potential for both internal and external learning seems to vary a lot across different industries. Second, the amount of spillovers can not simply be added to the β above, since they are already partly included in the estimate of β. Even theoretically, the role of experience in different industries cannot simply be added up because to some extent there is redundancy due to *overlapping* experience and sometimes the learning in other industries is of limited relevance. Given that we are interested in an upper bound for λ, a "guestimate" is that the spillovers matter for the final λ at most the same as β from ship building so that $\lambda \leq 2\beta$.[6]

On the basis of these casual considerations we claim that a λ much higher than about 2/3 may be considered fairly implausible. This speaks for the Arrow case of semi-endogenous growth rather than the Romer case of fully endogenous growth, at least as long as we think of learning by investing as the sole source of productivity growth. Another point is that to the extent learning is internal and at least temporarily appropriable, we should expect at least some firms to internalize the phenomenon in its optimizing behavior (Thornton and Thompson, 2001). Although the learning is far from fully excludable, it takes time for others to discover and imitate technical and organizational improvements. Many simple growth models ignore this and treat all learning by doing and learning by investing as a 100 percent externality, which seems an exaggeration.

A further issue is to what extent learning by investing takes the form of *disembodied* versus *embodied* technical change. This is the topic of the next section.

13.3 Disembodied vs. embodied technical change*

Arrow's and Romer's models build on the idea that the *source* of learning is primarily experience in the investment goods sector. Both models assume that the learning, via knowledge spillovers across firms, provides an engine of productivity growth in essentially *all* sectors of the economy. And both models (Arrow's, however, only in its simplified version, which we considered in Chapter 12, not in its original version) assume that a firm can benefit from recent technical advances irrespective of whether it buys new equipment or just uses old equipment. That is, the models assume that technical change is *disembodied*.

[6] For more elaborate studies of empirical aspects of learning by doing and learning by investing, see Irwin and Klenow (1994), Jovanovic and Nyarko (1995), and Greenwood and Jovanovic (2001). Caballero and Lyons (1992) find clear evidence of positive externalities across US manufacturing industries. Studies finding that the quantitative importance of spillovers is significantly smaller than required by the Romer case include Englander and Mittelstadt (1988) and Benhabib and Jovanovic (1991). See also the surveys by Syverson (2011) and Thompson (2012).

Although in this lecture note we focus on learning as an externality, there exists studies focusing on *internal* learning by doing, see, e.g., Gunn and Johri, 2011.

13.3.1 Disembodied technical change

Disembodied technical change occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again (13.1) and (13.3). When the K_t appearing in (13.1) refers to the total, historically accumulated capital stock, then the interpretation is that the higher technology level generated in (13.3) or (13.8) results in higher productivity of *all* labor, independently of the vintage of the capital equipment with which this labor is combined. Thus also firms with old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.

Examples of this kind of productivity increases include improvement in management and work practices/organization and improvement in accounting.

13.3.2 Embodied technical change

In contrast, we say that technical change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technical progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.[7]

One way to formally represent embodied technical progress is to write capital accumulation in the following way,

$$\dot{K}_t = q_t I_t - \delta K_t, \tag{13.13}$$

where I_t is gross investment at time t and q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising q_t so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in efficiency units. Even

[7] The concept of embodied technical change was introduced by Solow (1960). The notion of Solow-neutral technical change is related to embodied technical change and capital of different vintages.

if technical change does not directly appear in the production function, that is, even if for instance (13.1) is replaced by $Y_t = F(K_t, L_t)$, the economy may in this manner still experience a rising standard of living.

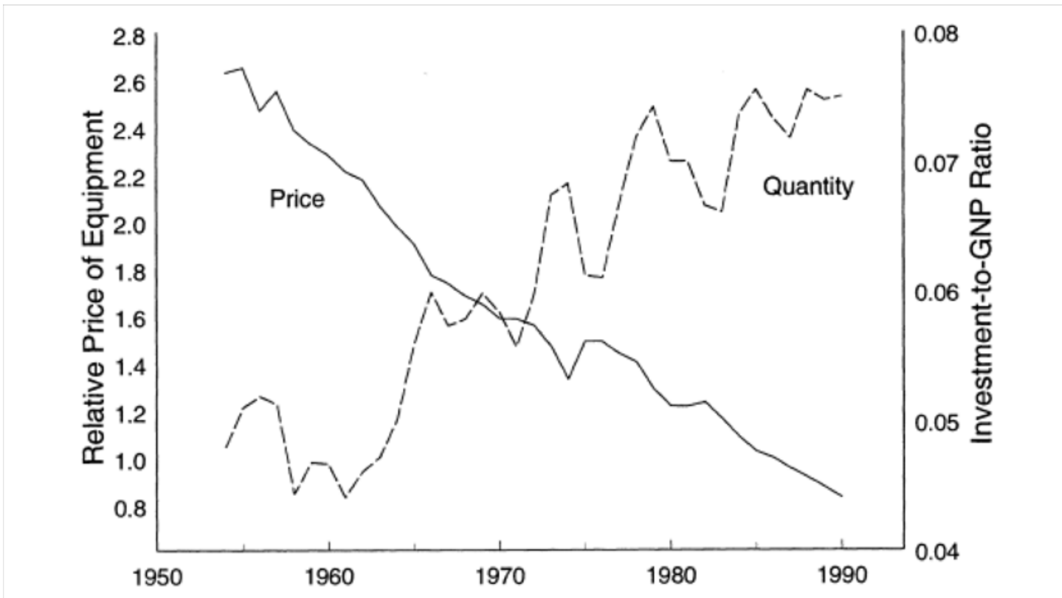

Figure 13.3: Relative price of equipment and quality-adjusted equipment investment-to-GNP ratio. Source: Greenwood, Hercowitz, and Krusell (1997).

Embodied technical progress is likely to result in a steady decline in the price of capital equipment relative to the price of consumption goods. This prediction is confirmed by the data. Greenwood et al. (1997) find for the U.S. that the relative price, p, of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990, cf. the "Price" curve in Figure 13.3.[8] As the "Quantity" curve in Figure 13.3 shows, over the same period there has been a secular rise in the ratio of new equipment investment (in efficiency units) to GNP; note that what in the figure is called the "investment-to-GNP Ratio" is really "quality-adjusted investment-to-GNP Ratio", qI/GNP, not the usual investment-income ratio, I/GNP.

Moreover, the correlation between de-trended p and de-trended qI/GNP is -0.46. Greenwood et al. interpret this as evidence that technical advances have made equipment less expensive, triggering increases in the accumulation of equipment both in the short and the long run. The authors also estimate

[8]The relative price index in Fig. 13.3 is based on the book by R. Gordon (1990), which is an attempt to correct previous price indices for equipment by better taking into account quality improvements in new equipment.

that embodied technical change explains 60% of the growth in output per man hour.

13.3.3 Embodied technical change and learning by investing

Whether technological progress is disembodied or embodied says nothing about whether its *source* is exogenous or endogenous. Indeed, the increases of q in (13.13) may be modeled as exogenous or endogenous. In the latter case, a popular hypothesis is that the source is learning by investing. This learning may take the form (13.8) above. In that case the experience that matter for learning is cumulative *net* investment.

An alternative hypothesis is:

$$q_t = \left(\int_{-\infty}^{t} I_s ds \right)^{\lambda}, \qquad 0 < \lambda \leq \bar{\lambda}, \tag{13.14}$$

where I_s is *gross* investment at time s. Here the experience that matter has its basis in cumulative *gross* investment. An upper bound, $\bar{\lambda}$, for the learning parameter is introduced to avoid explosive growth. The hypothesis (13.14) seems closer to both intuition and the original ideas of Arrow:

> "Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).

Contrary to the integral based on net investment in (13.8), the integral in the learning hypothesis (13.14) does not allow an immediate translation into an expression in terms of the accumulated capital stock. Instead a new state variable, cumulative gross investment, enter the system and opens up for richer dynamics.

We may combine (13.14) with an aggregate Cobb-Douglas production function,

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}. \tag{13.15}$$

Then the upper bound for the learning parameter in (13.14) is $\bar{\lambda} = (1-\alpha)/\alpha$.[9]

[9] An alternative to the specification of embodied learning by gross investment in (13.14) is

$$q_t = \left(\int_{-\infty}^{t} q_s I_s ds \right)^{\tilde{\lambda}}, \qquad 0 < \tilde{\lambda} \leq \bar{\tilde{\lambda}},$$

The case $\lambda < (1-\alpha)/\alpha$

Suppose $\lambda < (1-\alpha)/\alpha$. Using (13.14) together with (13.13), (13.15), and $I = Y - C$, one finds, under balanced growth with $s = I/Y$ constant and $0 < s < 1$,

$$\begin{aligned} g_K &= \frac{(1-\alpha)(1+\lambda)n}{1-\alpha(1+\lambda)}, && (13.16)\\ g_q &= \frac{\lambda}{1+\lambda}g_K, && (13.17)\\ g_Y &= \frac{1}{1+\lambda}g_K, && (13.18)\\ g_c &= g_y = g_Y - n = \frac{\alpha\lambda n}{1-\alpha(1+\lambda)}, && (13.19) \end{aligned}$$

cf. Appendix A. We see that $g_y > 0$ if and only if $n > 0$. So growth is here semi-endogenous.

Let us assume there is perfect competition in all markets. Since q capital goods can be produced at the same minimum cost as one consumption good, the equilibrium price, p, of capital goods in terms of the consumption good must equal the inverse of q, that is, $p = 1/q$. With the consumption good being the numeraire, let the rental rate in the market for capital services be denoted R and the real interest rate in the market for loans be denoted r. Ignoring uncertainty, we have the no-arbitrage condition

$$\frac{R_t - (\delta p_t - \dot{p}_t)}{p_t} = r_t, \qquad (13.20)$$

where $\delta p_t - \dot{p}_t$ is the true economic depreciation of the capital good per time unit. Since $p = 1/q$, (13.17) and (13.16) indicate that along a BGP the relative price of capital goods will be declining according to

$$g_p = -\frac{(1-\alpha)\lambda n}{1-\alpha(1+\lambda)} < 0.$$

Note that $g_K > g_Y$ along the BGP. Is this a violation of Proposition 1 of Chapter 4? No, that proposition presupposes that capital accumulation occurs according to the standard equation (13.4), not (13.13). And although g_K differs from g_Y, the output-capital ratio in *value* terms, $Y/(pK)$, is constant

implying that it is cumulative quality-adjusted gross investment that matters, cf. Greenwood and Jovanovic (2001). If combined with the production function (13.15) the appropriate upper bound on the learning parameter, $\tilde{\lambda}$, is $\bar{\tilde{\lambda}} = 1 - a$.

along the BGP. In fact, the BGP complies entirely with Kaldor's stylized facts if we interpret "capital" as the value of capital, pK.

The formulas (13.16) and (13.19) display that $\alpha(1+\lambda) < 1$ is needed to avoid a forever rising growth rate if $n > 0$. This inequality is equivalent to $\lambda < (1-\alpha)/\alpha$ and confirms that the upper bound, $\bar{\lambda}$, in (13.14) equals $(1-\alpha)/\alpha$. With $\alpha = 1/3$, this upper bound is 2. The bound is thus no longer 1 as in the simple learning-by-investing model of Section 13.2. The reason is twofold, namely partly that now q is formed via cumulative gross investment instead of net investment, partly that the role of q is to strengthen capital formation rather than the efficiency of production factors in aggregate final goods produce.

When $n = 0$, the system can no longer generate a constant positive per capita growth rate (exponential growth). Groth et al. (2010) show, however, that the system is capable of generating *quasi-arithmetic growth.* This class of growth processes, which fill the whole range between exponential growth and complete stagnation, was briefly commented on in Section 10.5 of Chapter 10.

The case $\lambda = (1-\alpha)/\alpha$ and $n = 0$

When $\lambda = (1-\alpha)/\alpha$, we have $\alpha(1+\lambda) = 1$ and so the growth formulas (13.16) and (13.19) no longer hold. But the way that (13.17) and (13.18) are derived (see Appendix A) ensures that these two equations remain valid along a BGP. Given $\lambda = (1-\alpha)/\alpha$, (13.17) can be written $g_q = (1-\alpha)g_K$, which is equivalent to

$$q_t = BK_t^{1-\alpha}$$

along a BGP (B is some positive constant to be determined).

To see whether a BGP exists, note that (**??**) implies

$$g_q = \frac{\dot{q}_t}{q_t} = \lambda q_t^{-1/\lambda} I_t = \lambda q_t^{-\alpha/(1-\alpha)} I_t = \lambda B^{-\alpha/(1-\alpha)} K_t^{-\alpha} I_t = \lambda B^{-\alpha/(1-\alpha)} K_t^{-\alpha} s Y_t,$$

considering a BGP with $s = I/Y$ constant. Substituting (13.15) into this, we get

$$g_q = \lambda B^{-\alpha/(1-\alpha)} K_t^{-\alpha} s K_t^{\alpha} L^{1-\alpha} = \lambda B^{-\alpha/(1-\alpha)} s L^{1-\alpha}. \tag{13.21}$$

If $n = 0$, the right-hand side of (13.21) is constant and so is $g_K = g_q/(1-\alpha)$, by (13.17), and $g_Y = \alpha g_K = \alpha g_q/(1-\alpha)$, by (13.19).

If $n > 0$ at the same time as $\lambda = (1-\alpha)/\alpha$, however, there is a tendency to a forever rising growth rate in q, hence also in K and Y. No BGP exists in this case.

Returning to the case where a BGP exists, a striking feature revealed by (13.21) is that the saving rate, s, matters for the growth rate of q, hence also for the growth rate of K and Y, respectively, along a BGP. As in the Romer case of the disembodied learning-by-investing model, the growth rates along a BGP cannot be determined until the saving behavior in the economy is modeled.

So the considered knife-edge case, $\lambda = (1 - \alpha)/\alpha$ combined with $n = 0$, opens up for many different per capita growth rates under balanced growth. Which one is "selected" by the economy depends on how the household sector is described. In a Ramsey setup with $n = 0$ one can show that the growth rate under balanced growth depends negatively on the rate of time preference and the elasticity of marginal utility of consumption of the representative household. And not only is growth in this case *fully endogenous* in the sense that a positive per capita growth rate can be maintained forever without the support by growth in any exogenous factor. An economic policy that subsidizes investment can generate not only a transitory rise in the productivity growth rate, but also a permanently higher productivity growth rate.

In contrast to the Romer (1986) model, cf. Section 13.2.2 above, we do not here end up with a reduced-form AK model. Indeed, we end up with a model with transitional dynamics, as a consequence of the presence of *two* state variables, K and q.

If instead $\alpha > 1/(1+\lambda)$, we get a tendency to explosive growth − infinite output in finite time − a not plausible scenario, cf. Appendix B.

13.4 Static comparative advantage vs. dynamics of learning by doing*

In this section we will briefly discuss a development economics perspective of the above learning-based growth models.

More specifically we will take a look at the possible "conflict" between static comparative advantage and economic growth. The background to this possible "conflict" is the dynamic externalities inherent in learning by doing and learning by investing.

13.4.1 A simple two-sector learning-by-doing model[10]

We consider an isolated economy with two production sectors, *sector 1* and *sector 2*, each producing its specific consumption good. Labor is the only input and aggregate labor supply L is constant. There are many small firms in the two sectors. Aggregate output in the sectors are:

$$Y_{1t} = T_{1t}L_{1t}, \tag{13.22}$$
$$Y_{2t} = T_{2t}L_{2t}, \tag{13.23}$$

where

$$L_{1t} + L_{2t} = L.$$

There are *sector-specific* learning-by-doing externalities in the following form:

$$\dot{T}_{1t} = B_1 Y_{1t}, \qquad B_1 \geq 0, \tag{13.24}$$
$$\dot{T}_{2t} = B_2 Y_{2t}, \qquad B_2 \geq 0. \tag{13.25}$$

Although not visible in our aggregate formulation, there are substantial knowledge spillovers across firms within the sectors. Across sectors, spillovers are assumed negligible.

Assume firms maximize profits and that there is perfect competition in the goods and labor markets. Then, prices are equal to the (constant) marginal costs. Let the relative price of sector 2-goods in terms of sector-1 goods be called p_t (i.e., we use sector-1 goods as numeraire). Let the hourly wage in terms of sector-1 goods be w_t. In general equilibrium with production in both sectors we then have

$$T_{1t} = p_t T_{2t} = w_t,$$

saying that the value of the (constant) marginal productivity of labor in each sector equals the wage. Hence,

$$p_t \frac{T_{2t}}{T_{1t}} = 1 \quad \text{or} \quad p_t = \frac{T_{1t}}{T_{2t}}, \tag{13.26}$$

saying that the relative price of the two goods is inversely proportional to the relative labor productivities in the two sectors. The demand side, which is not modelled here, will of course play a role for the final allocation of labor to the two sectors.

Taking logs in (13.26) and differentiating w.r.t. t gives

$$\frac{\dot{p}_t}{p_t} = \frac{\dot{T}_{1t}}{T_{1t}} - \frac{\dot{T}_{2t}}{T_{2t}} = \frac{B_1 Y_{1t}}{T_{1t}} - \frac{B_2 Y_{2t}}{T_{2t}} = B_1 L_{1t} - B_2 L_{2t},$$

[10] Krugman (1987), Lucas (1988, Section 5).

using (13.24) and (13.25). Thus,

$$\dot{p}_t = (B_1 L_{1t} - B_2 L_{2t})p_t.$$

Assume sector 2 (say some industrial activity) is more disposed to learning-by-doing than sector 1 (say mining) so that $B_2 > B_1$. Consider for simplicity the case where at time 0 there is symmetry in the sense that $L_{10} = L_{20}$. Then, the relative price p_t of sector-2 goods in terms of sector-1 goods will, at least initially, tend to diminish over time. The resulting substitution effect is likely to stimulate demand for sector-2 goods. Suppose this effect is large enough to ensure that $L_2 = Y_2/T_2$ never becomes lower than B_1L_1/B_2, that is, $B_2L_2 \geq B_1L_1$ for all t. Then the scenario with $\dot{p} \leq 0$ is sustained over time and the sector with highest growth potential remains a substantial constituent of the economy. This implies sustained economic growth in the aggregate economy.

Now, suppose the country considered is a rather backward, developing country which until time t_0 has been a closed economy (very high tariffs etc.). Then the country decides to open up for free foreign trade. Let the relative world market price of sector 2-goods be $\bar{p}$, which we for simplicity assume is constant. At time t_0 there are two alternative possibilities to consider:

Case 1: $\bar{p} > \frac{T_{1t_0}}{T_{2t_0}}$ (world-market price of good 2 higher than the opportunity cost of producing good 2). Then the country specializes fully in sector-2 goods. Since this is the sector with a high growth potential, economic growth is stimulated. The relative productivity level T_{1t}/T_{2t} decreases so that the scenario with $\bar{p} > T_{1t}/T_{2t}$ remains. A virtuous circle of dynamics of learning by doing is unfolded and high economic growth is sustained.

Case 2: $\bar{p} < \frac{T_{1t_0}}{T_{2t_0}}$ (world-market price of good 2 lower than the opportunity cost of producing good 2). Then the country specializes fully in sector-1 goods. Since this is the sector with a low growth potential, economic growth is impeded or completely halted. The relative productivity level T_{1t}/T_{2t} does not decrease. Hence, the scenario with $\bar{p} < T_{1t}/T_{2t}$ sustains itself and persists. Low or zero economic growth is sustained. The static comparative advantage in sector-1 goods remains and the country is locked in low growth.

If instead $\bar{p}$ is time-dependent, suppose $\dot{\bar{p}}_t < 0$ (by similar arguments as for the closed economy). Then the case 2 scenario is again self-sustaining.

The point is that there may be circumstances (like in case 2), where temporary protection for a backward country is growth promoting (this is a specific kind of "infant industry" argument).

13.4.2 A more robust specification

The way (13.24) and (13.25) are formulated, we have

$$\frac{\dot{T}_{1t}}{T_{1t}} = B_1 L_{1t}, \tag{13.27}$$

$$\frac{\dot{T}_{2t}}{T_{2t}} = B_2 L_{2t}, \tag{13.28}$$

by (13.22) and (13.23). Thus, the model implies scale effects on growth, that is, *strong* scale effects.

An alternative specification introduces limits to learning-by-doing in the following way:

$$\dot{T}_{1t} = B_1 Y_{1t}^{\lambda_1}, \qquad \lambda_1 < 1,$$

$$\dot{T}_{2t} = B_2 Y_{2t}^{\lambda_2}, \qquad \lambda_2 < 1.$$

Then (13.27) and (13.28) are replaced by

$$\frac{\dot{T}_{1t}}{T_{1t}} = B_1 T_{1t}^{\lambda_1 - 1} L_{1t}^{\lambda_1}, \tag{13.29}$$

$$\frac{\dot{T}_{2t}}{T_{2t}} = B_2 T_{2t}^{\lambda_2 - 1} L_{2t}^{\lambda_2}. \tag{13.30}$$

Now the problematic strong scale effect has disappeared. At the same time, since $\lambda_1 - 1 < 0$ and $\lambda_2 - 1 < 0$, (13.29) and (13.30) show that growth peters out as long as the "diminishing returns" to learning-by-doing are not offset by an increasing labor force or an additional source (outside the model) of technical progress. If $n > 0$, we get sustained growth of the semi-endogenous type as in the Arrow model of learning-by-investing.

Yet the analysis may still be a basis for an "infant industry" argument. If the circumstances are like in case 2, temporary protection may help a backward country to enter a higher long-run path of evolution. Stiglitz underlines South Korea as an example:

> What matters is *dynamic* comparative advantage, or comparative advantage in the long run, which can be shaped. Forty years ago, South Korea had a comparative advantage in growing rice. Had it stuck to that strength, it would not be the industrial giant that it is today. It might be the world's most efficient rice grower, but it would still be poor (Stiglitz, 2012, p. 2).

13.4.3 Resource curse?

The analysis also suggests a mechanism that, along with others, may help explaining what is known as the *resource curse* problem. This problem refers to the paradox that being abundant in natural resources may sometimes seem a curse for a country rather than a blessing. At least quite many empirical studies have shown a negative correlation between resource abundance and economic growth (see, e.g., Sachs and Warner 1995, Gylfason et al., 1999).

The mechanism behind this phenomenon could be the following. Consider a mining country with an abundance of natural resources in the ground. Empirically, growth in total factor productivity in mining activity is relatively low. Interpreting this as reflecting a relatively low learning potential, the mining sector may be represented by sector 1 above. Given the abundance of natural resources, T_{1t_0} is likely to be high relative to the productivity in the manufacturing sector, T_{2t_0}. So the country is likely to be in the situation described as case 2. As a result, economic growth may never get started.

The basic problem here is, however, not of an economic nature in a narrow sense, but rather of an institutional character. Taxation on the natural resource and use of the tax revenue for public investment in growth promoting factors (infrastructure, health care, education, R&D) or directly in the sector with high learning potential can from an economic point of view circumvent the curse to a blessing. It is not the natural resources as such, but rather barriers of a political character, conflicts of interest among groups and social classes, even civil war over the right to exploit the resources, or dominance by foreign superpowers, that may be the obstacles to a sound economic development.[11]

Summing up: Discovery of a valuable mineral in the ground in a country with weak institutions may, through corruption etc. have adverse effects on resource allocation and economic growth in the country. But: "Resources should be a blessing, not a curse. They can be, but it will not happen on its own. And it will not happen easily" (Stiglitz, 2012, p. 2).

13.5 Robustness issues and scale effects

First some words about terminology.

[11] An additional potential obstacle is related to the possible response of a country's real exchange rate, and therefore its competitiveness, to a new discovery of natural resources in a country.

Ploeg (2011) provides a survey over different theories related to the resource curse problem. See also Ploeg and Venables (2012) and Stiglitz (2012).

13.5.1 On terminology

How terms like "endogenous growth" and "semi-endogenous growth" are defined varies in the literature. Recalling the notation $y \equiv Y/L$ and $g_y \equiv \dot{y}/y$, in this course we use the definitions:

Endogenous growth is present if there is a positive long-run per capita growth rate (i.e., $g_y > 0$) and the source of this is some internal mechanism in the model (so that exogenous technology growth is not deeded).

Fully endogenous growth (sometimes called *strictly endogenous growth*) is present if there is a positive long-run per capita growth rate and this occurs without the support by growth in any exogenous factor (for example exogenous growth in the labor force).

An example: the Romer version of the model of learning by investing features fully endogenous growth. The technical reason for this is the assumption that the learning parameter, λ, is such that there are constant returns to capital at the aggregate level. We get $g_y > 0$, constant, and, in a Ramsey set-up, results like $\partial g_y/\partial \rho < 0$ and $\partial g_y/\partial \theta < 0$, that is, preference parameters matter for long-run growth. This suggests, at least at the theoretical level, that taxes and subsidies, by affecting incentives, may have effects on long-run growth (cf. Chapter 12). On the other hand, a fully-endogenous growth model *need* not have this implication. We saw an example of this in Section 13.1, where the "law of motion" of technology makes up a subsystem that is independent of the remainder of the economic system.

In any case, fully endogenous growth is technologically possible if and only if there are non-diminishing returns (at least asymptotically) to the producible inputs in the growth-generating sector(s), also called the "growth engine". The growth engine in an endogenous growth model is defined as the set of input-producing sectors or activities using their own output as input. This set may consist of only one sector such as the manufacturing sector in the simple AK model, the educational sector in the Lucas (1988) model, or the R&D sector in the Romer (1990) model. A model is capable of generating fully endogenous growth if the growth engine has *CRS w.r.t. producible inputs*.

No argument like the replication argument for CRS w.r.t. the *rival* inputs exists regarding CRS w.r.t. the *producible inputs.* This is one of the reasons that also another kind of endogenous growth is often considered in the literature. This takes us to "semi-endogenous growth".

Semi-endogenous growth is present if growth is endogenous but a positive long-run per capita growth rate can not be sustained without the support by growth in some exogenous factor (for example exogenous growth in the labor force).

For example, the Arrow model of learning by investing features semi-endogenous growth. The technical reason for this is the assumption that the learning parameter, λ, is less than 1, which implies diminishing marginal returns to capital at the aggregate level. Along a BGP we get

$$g_y = g_k = g_c = \frac{\lambda n}{1-\lambda}. \tag{13.31}$$

If and only if $n > 0$, can a positive g_y be maintained forever. When the learning mechanism is *assisted* by population growth, it is strong enough to over time endogenously maintain a constant average productivity of capital. The key role of population growth derives from the fact that at the aggregate level there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be sufficiently exploited to generate exponential growth, population growth is needed.[12] Note that in this case $\partial g_y/\partial \rho = 0 = \partial g_y/\partial \theta$, that is, preference parameters do not matter for *long-run* growth (only for the *level* of the growth path). This suggests that taxes and subsidies do not have *long-run* growth effects. Yet, in Arrow's model and similar semi-endogenous growth models economic policy can have important long-run *level* effects.

Strangely enough, some textbooks (for example Barro and Sala-i-Martin, 2004) do not call much attention to the distinction between fully endogenous growth and semi-endogenous growth. Rather, they tend to use the term endogenous growth as synonymous with what we here call fully endogenous growth. But there is certainly no reason to rule out *apriori* the parameter cases corresponding to semi-endogenous growth.

In the Acemoglu textbook (Acemoglu, 2009, p. 448) "semi-endogenous growth" is defined or characterized as endogenous growth where the long-run per capita growth rate of the economy "does not respond to taxes or other policies". As an implication, endogenous growth which is not semi-endogenous is in Acemoglu's text implicitly defined as endogenous growth where the long-run per capita growth rate of the economy *does* respond to taxes or other policies.

We have defined the distinction between "semi-endogenous growth" and "fully endogenous growth" differently. In our terminology, this distinction does not coincide with the distinction between policy-dependent and policy-invariant growth. Indeed, in our terminology positive per capita growth

[12] Of course the model shifts from featuring "semi-" to featuring "fully endogenous" growth if the model is extended with an internal mechanism *determining* the population growth rate.

may rest on an "exogenous source" in the sense of deriving from exogenous technical progress and yet the long-run per capita growth rate may be policy-dependent. In Chapter 16 we will see an example in connection with the so-called DHSS model.

There also exist models that according to our definition feature *semi*-endogenous growth and yet the long-run per capita growth rate is policy-*dependent* (Cozzi, 1997; Sorger, 2010). Similarly, there exist models that according to our definition feature *fully* endogenous growth and yet the long-run per capita growth rate is policy-*invariant* (Section 13.1.2 above shows an example).

13.5.2 Robustness of simple endogenous growth models

The series of learning-based growth models considered above illustrate the fact that endogenous growth models with exogenous population typically exist in two varieties or cases. One is the fully endogenous growth case where a particular value is imposed on a key parameter. This value is such that there are constant returns (at least asymptotically) to *producible* inputs in the growth engine of the economy.[13] In the "corresponding" semi-endogenous growth case, the key parameter is allowed to take any value in an open interval. The endpoint of this interval appears as the "knife-edge" value assumed in the fully endogenous growth case.

Although the two varieties build on qualitatively the same mathematical model of a certain growth mechanism (say, learning by doing or research and development), the long-run results turn out to be very sensitive to which of the two cases is assumed. In the fully endogenous growth case a positive per-capita growth rate is maintained forever without support of growth in any exogenous factor. In the semi-endogenous growth case, the growth process needs "support" by some growing exogenous factor in order for sustained growth to be possible. The established terminology is somewhat seductive here. "Fully endogenous" sounds as something going much deeper than "semi-endogenous". But nothing of that sort should be implied. It is just a matter of different parameter values.

[13] Suppose our CRS aggregate production function is $Y = AK + BK^{\alpha}L^{1-\alpha}$, $A > 0, B > 0, 0 < \alpha < 1$, we have $y \equiv Y/L = Ak + Bk^{\alpha}$, where $k \equiv K/L$. We then get $y/k = A + Bk^{\alpha-1} \to A$ for $k \to \infty$, that is, the output-capital ratio converges to a positive constant when the capital-labor ratio goes to infinity. We then say that *asymptotically* there are *CRS* w.r.t. the *producible* inputs, here just K. In this kind of "asymptotic" AK models the force of diminishing returns to capital ultimately becomes negligible.

As Solow (1997, pp. 7-8) emphasizes in connection with learning-by-investing models (with constant population), the Romer case with $\lambda = 1$ is a very special case, indeed an "extreme case, not something intermediate". A value of λ slightly above 1 leads to explosive growth: infinite output in finite time even when $n = 0$.[14] And a value of λ slightly below 1 leads to growth petering out in the long run even when $n = 0$.

Whereas the strength of the semi-endogenous growth case is its theoretical and empirical robustness, the convenience of the fully endogenous growth case is that it has much simpler dynamics. Then the question arises to what extent a fully endogenous growth model can be seen as a useful approximation to its semi-endogenous growth "counterpart". Imagine that we contemplate applying the fully endogenous growth case as a basis for making forecasts or for policy evaluation in a situation where the "true" case is the semi-endogenous growth case. Then we would like to know: Are the impulse-response functions generated by a shock in the fully endogenous growth case an acceptable approximation to those generated by the same shock in the corresponding semi-endogenous growth case for *a sufficiently long time horizon to be of interest*?[15] The answer is "yes" if the critical parameter has a value "close" to the knife edge value and "no" otherwise. How close it need be, depends on circumstances. My own tentative impression is that usually it is "closer" than what the empirical evidence warrants.

Even if a single growth-generating mechanism, like learning by doing, does not in itself seem strong enough to generate a reduced-form AK model (the fully endogenous growth case), there might exist complementary factors and mechanisms that in total could generate something close to a reduced-form AK model. The time-series test by Jones (1995b), however, rejects this.[16]

Comment on "petering out" when $n = 0$ The above-mentioned "petering out" of long-run growth in the semi-endogenous case when $n = 0$ takes different forms in different models. When exponential growth cannot be sustained in a model, sometimes it remains true that $y \to \infty$ for $t \to \infty$, and sometimes instead complete stagnation results. In the present context, where we focus on learning, it is the *source* of learning that matters. Suppose that, as in the simple Arrow version ($\lambda < 1$) of learning-by-investing in Section 13.2.1 above (and in Chapter 12), learning is associated with *net* investment, then $n = 0$ will lead to complete stagnation in the sense that there is an

[14] A demonstration is in Appendix B.

[15] Obviously, the ultimate effects of the shock tend to be very different in the two models.

[16] There is an ongoing debate about this and similar empirical issues, see the course website under Supplementary Material.

upper bound on y that is never transcended. The productivity-driving factor, net investment, dries out. Even if there is an incentive to maintain the capital stock, this does not require positive net investment and so learning tends to stop. The productivity-driving factor, net investment, dries out.

When learning is associated with *gross* investment, however, learning continues because even when net investment is vanishing, gross investment remains positive because there is generally an incentive to maintain the capital stock. Thereby sustained learning is generated. In turn, this tends to induce more investment than needed to replace wear and tear and so capital accumulates, although at a declining rate. The diminishing marginal returns to capital are countervailed by the rising productivity of investment goods due to learning. We get permanent though diminishing growth, that is, $y_t \to \infty$ for $t \to \infty$ at the same time as $g_y \to 0$, but $g_y > 0$ remains true. Arithmetic growth, $y_t = y_0 + \gamma t$ with $\gamma > 0$, is an example. More generally, as mentioned in Section 13.3.3, quasi-arithmetic growth tends to arise.

It is similar in the learning-by-doing examples of sections 13.1 and 13.4, where learning is simply associated with producing. Learning continues even if the capital stock is just upheld.

Another issue is whether there exist factors that in spite of $n = 0$ (or, to be more precise, in spite of $n \to 0$ as projected by the United Nations to happen within a century from now (United Nations, 2013)) may *replace* the growth-supporting role of population growth under semi-endogenous parameter conditions like $\lambda < 1$. In Section 10.5 of Chapter 10 we indicated scepticism that human capital accumulation would be able to do that. But both urbanization and the evolution of information and communication technologies seem likely for a long time to at least help in that direction.

13.5.3 Weak and strong scale effects

Romer's learning-by-investing hypothesis (where the learning parameter equals 1) implies a problematic (strong) scale effect. When embedded in a Ramsey set-up, the model generates a time path along which

$$g_y = g_k = g_c = \frac{1}{\theta}(F_1(1, L) - \delta - \rho).$$

From this follows not only standard results for fully endogenous growth models, such as

$$\frac{\partial g_y}{\partial \rho} < 0, \quad \frac{\partial g_y}{\partial \theta} < 0,$$

but also[17]

$$\frac{\partial g_y}{\partial L} = \frac{1}{\theta} F_{12}(1, L) > 0. \tag{13.32}$$

This is because in this model the rate of return, $F_1(1, L) - \delta$, depends (positively) on L. Interpreting the size ("scale") of the economy as measured by the size, L, of the labor force, we call such an effect a *scale effect.* To distinguish it from another kind of scale effect, it is useful to name it a *scale effect on growth* or a *strong scale effect.*

Scale effects can be of a less dramatic form. In this case we speak of a *scale effect on levels* or a *weak scale effect.* This form arises when the learning parameter is less than 1. We thus see from (13.31) that in Arrow's model of learning-by-investing, the steady state growth rate is independent of the *size* of the economy. Consequently, in Arrow's model there is no strong scale effect. There is, however, a (positive) scale effect on *levels* in the sense that along a steady state growth path,

$$\frac{\partial y_t}{\partial L_0} > 0. \tag{13.33}$$

This says the following. Suppose we consider two closed economies characterized by the same parameters, including the same n.[18] The economies differ only w.r.t. initial size of the labor force. Suppose both economies are in steady state. Then, according to (13.33), the economy with the larger labor force has, for all t, larger output per unit of labor. The background is the positive relationship between the labor efficiency index, T_t, and *aggregate* cumulative (net) investment,

$$T_t = K_t^{\lambda},$$

which is due to *learning* and *knowledge spillovers* across firms. Thus, a given level of per capita investment increases labor productivity more in a larger economy (where $\dot{K}_t$ will be larger) than in a smaller economy.

More generally, the fundamental background is that *technical knowledge is a non-rival good* – its use by one firm does not (in itself) limit the amount of knowledge available to other firms.[19] In a large economic system, say an integrated set of open economies, *more* people benefit from a given increase in knowledge than in a small economic system. At the same time the per

[17] Here we use that a neoclassical production function $F(K, TL)$ with CRS satisfies the "direct complementarity condition" $F_{12} > 0$.

[18] Remember that in contrast to the Romer model, Arrow's model allows $n > 0$.

[19] By patent protection, secrecy, and copyright some aspects of technical knowledge are sometimes *partially excludable,* but that is another matter. Excludability is ignored in our simple learning-by-doing and learning-by-investing models.

capita cost of creating the increase in knowledge is less in the large system than in the small system.

To prove (13.33), note that along a steady state path

$$y_t \equiv \tilde{y}_t T_t = \tilde{y}^* T_t = f(\tilde{k}^*) T_t = f(\tilde{k}^*) K_t^{\lambda}, \tag{13.34}$$

where

$$K_t \equiv \tilde{k}_t T_t L_t = \tilde{k}^* T_t L_t = \tilde{k}^* K_t^{\lambda} L_t.$$

Solving this equation for K_t gives

$$K_t = (\tilde{k}^* L_t)^{1/(1-\lambda)} = (\tilde{k}^* L_0 e^{nt})^{1/(1-\lambda)}.$$

Substituting this into (13.34), we get

$$y_t = f(\tilde{k}^*)(\tilde{k}^* L_0 e^{nt})^{\lambda/(1-\lambda)}, \tag{13.35}$$

from which follows

$$\frac{\partial y_t}{\partial L_0} = \frac{\lambda}{1-\lambda} f(\tilde{k}^*)(\tilde{k}^* e^{nt})^{\lambda/(1-\lambda)} L_0^{[\lambda/(1-\lambda)]-1} = \frac{\lambda}{1-\lambda} \frac{y_t}{L_0} > 0, \tag{13.36}$$

since $\tilde{k}^*$ is independent of L_0. This confirms (13.33). The scale effect on y_t also gives scope for higher per capita consumption the higher is L_0.

The scale effect on levels displayed by (13.36) is increasing in the learning parameter λ, everything else equal. When $\lambda = 1$, the scale effect is so powerful that it is transformed into a scale effect on the growth rate.

13.5.4 Discussion

Are there good theoretical and/or empirical reasons to believe in the existence of (positive) scale effects on levels or perhaps even on growth in the long run?

Let us start with some theoretical considerations.

Theoretical aspects

From the point of view of theory, we should recognize the likelihood that offsetting forces are in play. On the one hand, there is the problem of *limited natural resources*. For a given level of technology, if there are CRS w.r.t. capital, labor, *and* land (or other natural resources), there are diminishing returns to capital and labor taken together. In this *Malthusian* perspective, an increased scale (increased population) results, everything else equal, in lower rather than higher per capita output, that is, a negative scale effect should be expected.

On the other hand, there is the *anti-Mathusian* view that repeated improvements in technology tend to overcome, or rather *more* than overcome, this Malthusian force, if appropriate socio-economic conditions are present. Here the theory of endogenous technical change comes in by telling us that a large population may be good for technical progress if the institutions in society are growth-friendly. A larger population breeds more ideas, the more so the better its education is; a larger population also promotes division of labor and larger markets. This helps the creation of new technologies or, from the perspective of an open economy, it helps the local adoption of already existing technologies outside the country. In a less spectacular way it helps by furthering day-by-day productivity increases due to learning by doing and learning by watching. The non-rival character of technical knowledge is an important feature behind all this. It implies that output per capita depends on the *total* stock of ideas, not on the stock per person. This implies − everything else equal − an advantage of scale.

In the models considered so far in this course, natural resources and the environment have been more or less ignored. Here only a few remarks about this limitation. The approach we have followed is intended to clarify certain *mechanisms* − in abstraction from numerous things. The models in focus have primarily been about aspects of an industrialized economy. Yet the natural environment is always a precondition. A tendency to positive scale effects on levels *may* be more or less counteracted by *congestion* and aggravated *environmental problems* ultimately caused by increased population and a population density above some threshold.

What can we say from an *empirical* point of view?

Empirical aspects

First of all we should remember that in view of cross-border diffusion of ideas and technology, a positive scale effect (whether weak or strong) should not be seen as a prediction about individual countries, but rather as pertaining to larger regions, nowadays probably the total industrialized part of the world. So cross-country regression analysis is not the right framework for testing for scale effects, whether on levels or the growth rate. The relevant scale variable is not the size of the country, but the size of a larger region to which the country belongs, perhaps the whole world; and multivariate time series analysis seems the most relevant approach.

Since in the last century there has been no clear upward trend in per capita growth rates in spite of a growing world population (and also a growing population in the industrialized part of the world separately), most economists do not believe in *strong* scale effects. But on the issue of *weak* scale effects the opinion is definitely more divided.

Considering the *very*-long run *history* of population and per capita income of different regions of the world, there clearly exists evidence in favour of scale effects (Kremer, 1993). Whether advantages of scale are present also in a contemporary context is more debated. Recent econometric studies supporting the hypothesis of positive scale effects on levels include Antweiler and Trefler (2002) and Alcalá and Ciccone (2004). Finally, considering the economic growth in China and India since the 1980s, we must acknowledge that this impressive performance at least does not speak *against* the existence of positive scale effects on levels.

Acemoglu seems to find positive scale effects on levels plausible at the theoretical level (pp. 113-114). At the same time, however, later in his book he seems somewhat skeptical as to the existence of empirical support for this. Indeed, with regard to the fact that R&D-based theoretical growth models tend to generate at least weak scale effects, Acemoglu claims: "It is not clear whether data support these types of scale effects" (Acemoglu, 2009, p. 448).

My personal view on the matter is that although we should, of course, recognize that offsetting forces, coming from our finite natural environment, and a lot of uncertainty are in play, it seems likely that at least up to a certain point there are positive scale effects on levels.

Policy implications If this holds true, it supports the view that international economic integration is generally a good idea. The concern about congestion and environmental problems, in particular global warming, should probably, however, preclude recommending governments and the United Nations to try to *promote* population growth.

Moreover, it is important to remember the distinction between the global and the local level. The n in the formula (13.31) refers to a much larger region than a single country. No recommendation of higher population growth in a single country is implied by this theoretical formula. When discussing economic policy from the perspective of a single country, all aspects of relevance in the given local context should be incorporated. For a developing country with limited infrastructure and weak educational system, family-planning programs and similar may in many cases make sense from both a social and a productivity point of view (cf. Dasgupta, 1995).

13.6 Appendix

A. Balanced growth in the embodied technical change model with investment-specific learning

In this appendix the results (13.16), (13.17), (13.18), and (13.19) are derived. The model is:

$$
\begin{aligned}
Y &= K^{\alpha}L^{1-\alpha}, \qquad 0<\alpha<1, && (13.37)\\
I &= Y - C, && (13.38)\\
\dot{K} &= qI - \delta K, && (13.39)\\
q_t &= \left(\int_{-\infty}^{t} I_s ds\right)^{\lambda}, \qquad 0<\lambda\leq\bar{\lambda}, && (13.40)\\
L &= L_0 e^{nt}, \qquad n\geq 0. && (13.41)
\end{aligned}
$$

Consider a BGP. By definition, Y, K, and C then grow at constant rates, not necessarily positive. With $s = I/Y$ constant and $0 < s < 1$, (13.37) gives

$$g_I = g_Y = \alpha g_K + (1-\alpha)n, \qquad (13.42)$$

a constant. By (13.39), $g_K = q\frac{I}{K} - \delta$, showing that qI/K is constant along a BGP. Hence,

$$g_q + g_I = g_K, \qquad (13.43)$$

and so also g_q must be constant. From (13.40) follows that $g_q = \lambda q^{-1/\lambda} I$. Taking logs in this equation and differentiating w.r.t. t gives

$$\frac{\dot{g}_q}{g_q} = -\frac{1}{\lambda}g_q + g_I = 0,$$

in view of constancy of g_q. Substituting into (13.43) yields $(1+\lambda)g_I = g_K$, which combined with (13.42) gives

$$g_K = \frac{(1-\alpha)(1+\lambda)n}{1-\alpha(1+\lambda)},$$

which is (13.16). In view of $g_q = \lambda g_I = \lambda g_Y = \lambda(g_y + n) = \lambda g_K/(1+\lambda)$, the results (13.17), (13.18), and (13.19) immediately follow.

B. Big bang a hair's breadth from the AK

Here we shall prove the statement in Section 13.5.1: a hair's breadth from the AK assumption the technology is so productive as to generate infinite output in finite time.

The simple AK model as well as reduced-form AK models end up in an aggregate production function

$$Y = AK.$$

We ask the question: what happens if the exponent on K is not exactly 1, but slightly above. For simplicity, let $A = 1$ and consider

$$Y = K^{\alpha}, \qquad \alpha = 1 + \varepsilon, \ \varepsilon \gtrapprox 0.$$

Our claim is that when $\alpha > 1$, a constant saving rate, s, will generate infinite Y and C in finite time.

We embed the technology in a Solow-style model with $\delta = n = 0$ and get:

$$\dot{K} \equiv \frac{dK}{dt} = sK^{\alpha}, \qquad 0 < s < 1, \qquad K(0) = K_0 > 0 \text{ given.} \tag{13.44}$$

We see that not only is $\dot{K} > 0$ for all $t \geq 0$, but $\dot{K}$ is increasing over time since K is increasing. So, for sure, $K \to \infty$, but how fast?

One way of answering this question exploits the fact that $\dot{x} = x^a$ is a Bernouilli equation and can be solved by considering the transformation $z = x^{1-\alpha}$ as we do in Chapter 7 and Exercise III.3. Closely related to that method is the approach below, which may have the advantage of being somewhat more transparent.

To find out, note that (13.44) is a separable differential equation which implies

$$K^{-\alpha}dK = sdt.$$

By integration,

$$\begin{aligned} \int K^{-\alpha}dK &= \int sdt + \mathcal{C} \ \Rightarrow \\ \frac{K^{-\alpha+1}}{1-\alpha} &= st + \mathcal{C}, \end{aligned} \tag{13.45}$$

where $\mathcal{C}$ is some constant, determined by the initial condition $K(0) = K_0$. For $t = 0$ (13.45) gives $\mathcal{C} = K_0^{-\alpha+1}/(1-\alpha)$. Consequently, the solution $K = K(t)$ satisfies

$$\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} = st. \tag{13.46}$$

As t increases, the left-hand side of this equation follows suit since $K(t)$ increases and $\alpha > 1$. There is a $\bar{t} < \infty$ such that when $t \to \bar{t}$ from below, $K(t) \to \infty$. Indeed, by (13.46) we see that such a $\bar{t}$ must be the solution to the equation

$$\lim_{K(t)\to\infty} \left(\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} \right) = s\bar{t}.$$

Since

$$\lim_{K(t)\to\infty}\left(\frac{K_0{}^{1-\alpha}}{\alpha-1}-\frac{K(t)^{1-\alpha}}{\alpha-1}\right)=\frac{K_0{}^{1-\alpha}}{\alpha-1},$$

we find

$$\bar{t}=\frac{1}{s}\frac{K_0{}^{1-\alpha}}{\alpha-1}.$$

To get an idea about the implied order of magnitude, let the time unit be one year and $s = 0.1$, $K_0/Y_0 = K_0^{1-\alpha} = 2$, and $\alpha = 1.05$. Then $\bar{t} = 400$ years. So the Big Bang ($Y = \infty$) would occur in 400 years from now if $\alpha = 1.05$.

As Solow remarks (Solow 1994), this arrival to the Land of Cockaigne would imply the "end of scarcity", a very optimistic perspective.

In a discrete time setup we get an analogue conclusion. With airframe construction in mind let us imagine that the learning parameter λ is slightly above 1. Then we must accept the implication that it takes only a finite number of labor hours to produce an infinite number of airframes. This is because, given the (direct) labor input required to produce the q'th in a sequence of identical airframes is proportional to $q^{-\lambda}$, the total labor input required to produce the first q airframes is proportional to $1/1 + 1/2^{\lambda} + 1/3^{\lambda}$ + ... + $1/k^{\lambda}$. Now, the infinite series $\sum_{k=1}^{\infty} 1/q^{\lambda}$ converges if $\lambda > 1$. As a consequence only a finite amount of labor is needed to produce an infinite number of airframes. "This seems to contradict the whole idea of scarcity", Solow observes (Solow 1997, p. 8).

13.7 References

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Chapter 14

The lab-equipment model

In innovation-based endogenous growth models, technical knowledge and its intentional creation is at the center of attention. Recall the definition of *technical knowledge* as a list of instructions about how different inputs can be combined to produce a certain output. For example it could be a principle of chemical engineering. Such a list or principle can be copied on the blackboard, in books, in journals, on floppy disks etc. and can, by its nature, be available and used over and over again at arbitrarily many places at the same time. Thus, technical knowledge is a *non-rival* good.[1] At least temporarily, however, new technical knowledge may be *temporarily excludable* by patents, secrecy, or copyright so that the innovator can maintain a monopoly on the commercial use of new technical knowledge for some time.

The lab-equipment model (based on Paul Romer, AER 1987) is the simplest model within the class of models focusing on *horizontal* innovations. This term refers to inventions of *new* types of goods, i.e., new "technical designs" in the language of Romer. The present model considers invention of new technical designs for input goods, but a more general framework would include new types of consumption goods as well.[2] The rising number of varieties of goods contributes to productivity via *increased division of labor and specialization* in society. Thus this class of models is known as "increasing-variety models".

In Acemoglu's Chapter 13, Section 13.1, the lab-equipment model is presented in a version containing two arbitrary parameter links. In the present chapter we present the lab-equipment model without these parameter links.

[1] Even though a particular *medium* on which a copy of a list of inctructions is placed is a rival good, it can usually be reproduced at very low cost in comparison with the cost of making additions to the stock of technical knowledge.

[2] For a model where the new goods are new consumption goods, see Acemoglu, Chapter 13, Section 13.4.

In addition, the presentation below goes into detail with the national income aspects of the model and with the interaction between the financing needs of R&D labs and the saving by the households.

14.1 Overview of the economy

We consider a closed market economy. The activities in the economy can be subdivided into three sectors:

1. The *basic-goods* sector which operates under conditions of perfect competition and free entry.
2. The specialized *intermediate goods* sector which operates under conditions of monopolistic competition.
3. The R&D sector inventing *new technical designs* and operating under conditions of perfect competition and free entry.

All produced goods are non-durable goods. There is no physical capital (durable produced means of production) in the economy. All firms are profit maximizers.

14.1.1 The sectorial production functions

In the *basic-goods sector,* sector 1, firms combine labor and N_t different intermediate goods to produce a homogeneous output good. The representative firm in the sector has the production function

$$Y_t = A\left(\sum_{i=1}^{N_t} {x_{it}}^{1-\beta}\right) L_t^{\beta}, \qquad A > 0,\ 0 < \beta < 1, \tag{14.1}$$

where Y_t is output in the sector, A is a positive constant, x_{it} is input of intermediate good i ($i = 1, 2, ..., N_t$), N_t is the number of different types of intermediate goods available at time t, and L_t is labor input. To avoid arbitrary parameter links, we do not introduce Acemoglu's assumption that the technical coefficient A happens to equal $1/(1-\beta)$. Labor is not used in the two other sectors.

Basic goods have three alternative uses. They can be used a) for consumption, C_t; b) as raw material, X_t, to be converted into specialized intermediate goods (in Danish "halvfabrikata"); and c) as investment, Z_t, in R&D. Hence,

$$Y_t = C_t + X_t + Z_t. \tag{14.2}$$

In the *specialized intermediate-goods sector*, sector 2, at time t there are N_t monopoly firms, each of which supplies a particular already invented intermediate good. Once the technical design for intermediate good i has been invented in sector 3 (see below), the inventor takes out (free of charge) a perpetual patent on the commercial use of this design and enters sector 2 as an innovator. Given the technical design, the innovator can instantly transform a certain number of basic goods into a proportional number of intermediate goods of the invented specialized kind. Specifically, at every time t it takes ψx_i units of the basic good to supply x_i units of intermediate good i :

$$\psi x_i \text{ units of the basic good } \curvearrowright \ x_i \text{ units of intermediate good } i, \qquad (14.3)$$

where ψ is a positive constant. We may think of the new technical design as a computer code which, once in place, just requires pressing a key on a computer in order activate the desired number of transformations. The computer cost is negligible and the transformation requires no labor.

Thus, ψ is both the marginal and the average cost of supplying the intermediate good i. This transformation technology applies to all intermediate goods, $i = 1, 2, ..., N_t$, and all t. Hence, the X_t in (14.2) satisfies

$$X_t \equiv \psi \sum_{i=1}^{N_t} x_{it} \equiv \psi Q_t, \qquad (14.4)$$

where Q_t is the total supply of intermediate goods, all of which are used up in the production of basic goods. Apart from introducing a specific symbol, Q_t, for this total supply of intermediate goods, our notation is the same as Acemoglu's, Chapter 13. Yet, to help intuition, we think of variety as something discrete rather than a continuum and use summation across varieties as in (14.1) and (14.4) whereas Acemoglu's uses integrals.

The model gives a "truncated" picture of the R&D sector, sector 3, as fictional research labs that transform incoming basic goods (now considered as R&D "equipment") into a random stream of research successes. A research success is an invention of a technical design (blueprint) for making a new specialized intermediate good. There is free entry to R&D activity. The uncertainty associated with R&D is "ideosyncratic" (unsystematic, diversifiable) and the economy is "large". On average it takes an input flow of $1/\eta$ units of the basic good, and *nothing else*, to obtain one successful R&D outcome (an invention) per time unit. By the law of large numbers, the aggregate number of new technical designs (inventions) in the economy per time unit equals the expected number. Ignoring indivisibilities, we can therefore write

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \eta Z_t, \qquad \eta > 0, \ \eta \text{ constant}, \tag{14.5}$$

where, as noted above, Z_t is the aggregate research input per time unit and η is "research productivity". Since the payoff to the outlay, Z_t, on R&D comes in the future, this outlay makes up an *investment*. Although the invested basic goods are non-durable goods, the resulting new technical knowledge is durable.

At first sight this whole production setup may seem peculiar. In sector 2 as well as sector 3, parts of the output from sector 1 is used as input to be transformed into specialized intermediate goods and new technical designs, respectively. But there is no labor input in sector 2 and sector 3. Formulating the three kinds of production in the economy in this manner is a convenient way of saving notation and is typical in this type of models.[3] A more realistic full-fledged description of the production structure would start with a production function, with both labor and intermediate goods as inputs, in each sector. Then an assumption could be imposed that the production functions are the same, apart from allowing the total factor productivity to vary across the sectors (only if $1/\psi = \eta = 1$, would the total factor productivities be the same). Setting the model up that way would fit intuition better but would also require a more cumbersome notation. Anyway, the conclusions would not be changed.

Before considering agents' behavior, it may be clarifying to do a little national income accounting.

14.1.2 National income accounting

The production side Using the basic good as our unit of account, all the specialized intermediate goods will in equilibrium have the same price p_t (see Section 14.3.2). We therefore have:

$$\begin{aligned} \text{value added in sector 1} &= Y_t - p_t Q_t, \\ \text{value added in sector 2} &= p_t Q_t - X_t, \\ \text{value added in sector 3} &= V_t \dot{N}_t - Z_t, \end{aligned} \tag{14.6}$$

[3] At the same time it is the lack of direct research labor in sector 3 that motivates the term "lab-equipment model". And it is the multi-faceted use of output from sector 1 that motivates the term "basic goods".

where V_t is the market value of an innovation and turns out to be independent of time. The aggregate value added, or net national product, is

$$\begin{aligned} NNP_t &= Y_t - p_t Q_t + p_t Q_t - X_t + V_t \dot{N}_t - Z_t \\ &= Y_t - p_t Q_t + p_t Q_t - \psi Q_t + V_t \dot{N}_t - Z_t = Y_t - \psi Q_t, \quad (14.7) \end{aligned}$$

where the last equality comes from $V_t \dot{N}_t - Z_t = 0$ in equilibrium due to CRS and perfect competition in sector 3. Since there is no capital that depreciates in the economy, gross national product and net national product are the same.

Notice that the production function for Y is a production function neither for NNP nor even for value added in sector 1, but simply for the quantity of produced goods in that sector. It is typical for a multi-sector model with non-durable intermediate goods that the production functions in the different sectors do not describe value added in the sector but the produced quantity.

The income side There are two kinds of income in the economy, wage income and profits. The time-t real wage per unit of labor is denoted w_t and the profit per time unit earned by each monopoly firm in sector 2 is denoted π_t (in equilibrium it turns out to be the same for all the monopoly firms). Profits are immediately paid out to the share owners. Owing to perfect competition and CRS in both sector 1 and sector 3, there is no profit generated in these sectors. The income side of NNP is thereby

$$NNP_t = w_t L_t + \pi_t N_t,$$

since the number of monopoly firms is N_t. Aggregate income is used for consumption and saving,

$$w_t L + \pi N_t = C_t + S_t.$$

The uses of NNP By (14.7) and (14.4), final output can be written

$$NNP_t = Y_t - \psi Q_t = Y_t - X_t = C_t + Z_t, \quad (14.8)$$

that is, as the sum of aggregate consumption and investment. Aggregate saving is

$$S_t = w_t L + \pi N_t - C_t = NNP_t - C_t = Z_t,$$

by (14.8), reflecting that aggregate saving in a closed economy equals aggregate investment, the R&D expense, Z_t.

14.1.3 The potential for sustained productivity growth

Already the production function (14.1) conveys the basic idea of an "increasing-variety model". In equilibrium we get $x_{it} = x_t$ for all i since the intermediate goods enter symmetrically in this production function and end up having the same price (see below). Thereby, (14.1) becomes

$$Y = AN_t x_t^{\beta} L_t^{1-\beta} = A(N_t x_t)^{\beta} N_t^{1-\beta} L_t^{1-\beta} \equiv f(N_t x_t, N_t, L_t),$$

where $N_t x_t$ is the total input of intermediate goods. We see that

$$\frac{\partial Y_t}{\partial N_t}|_{N_t x_t = \text{const.}} = f_2(N_t x_t, N_t, L_t) > 0.$$

This says that for a given total input, $N_t x_t$, of intermediate goods, and a given L_t, the higher the number of varieties (with which follows a lower x_t of each intermediate since $N_t x_t$ is given), the more productive is this total input of intermediate goods. "Variety is productive". There are "gains to division of labor and specialization in society". The number of input varieties, N_t, can thus be interpreted as a measure of the level of productivity-enhancing knowledge.[4] Note also that the function f displays a form of increasing returns to scale with respect to *three* "inputs": intermediate goods, $N_t x_t$, variety, N_t, and labor, L_t.

14.2 Households and the labor market

There are L households, all alike, with infinite horizon and preference parameters $\theta > 0$ and ρ. Each household supplies inelastically one unit of labor per time unit. Let c_t denote per capita consumption C_t/L. A household chooses a plan $(c_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{c_t{}^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= r_t a_t + w_t - c_t, \qquad a_0 \text{ given}, \\ \lim_{t\to\infty} a_t e^{-\int_0^t r_s ds} &\geq 0, \end{aligned} \tag{14.9}$$

[4]There exists a related class of models where growth (measured in terms of produced economic value) is driven by increasing variety of *consumption* goods rather than increasing variety of input goods. These models are sometimes called "love of variety" models. See Acemoglu, Section 13.4.

where a_t equals per capita financial wealth. In equilibrium

$$a_t = \frac{V_t N_t}{L},$$

because the only asset with market value in the economy is equity shares in the monopoly firms the value of which equals the market value per technical designs multiplied by the number of technical designs available. As accounted for in Section 14.3.3, the households can fully diversify any risk so as to obtain the rate of return, r_t, with certainty on all their saving.

The first-order conditions for the consumption-saving problem lead to the Keynes-Ramsey rule

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{14.10}$$

The necessary transversality condition is that the No-Ponzi-Game condition (14.9) is satisfied with equality.

The labor market

There is perfect competition in the labor market. For every t, the supply of labor is L, a constant. The demand for labor, L_t, comes from the basic-goods sector (as the two other sectors do not use labor). In equilibrium,

$$L_t = L. \tag{14.11}$$

14.3 Firms' behavior

To save notation, in the description below, we take (14.11) for granted.

14.3.1 The competitive producers of basic goods

The representative firm in the basic-goods sector maximizes profit under perfect competition:

$$\max_{L, x_1, x_2, \ldots, x_N} \Pi = A\left(\sum_{i=1}^{N_t} x_{it}^{1-\beta}\right) L^\beta - \sum_{i=1}^{N} p_i x_i - wL. \tag{14.12}$$

The first-order conditions are, for every t,

$$\partial\Pi/\partial L = \partial Y/\partial L - w = \beta Y/L - w = 0 \tag{14.13}$$

and

$$\partial\Pi/\partial x_i = \partial Y/\partial x_i - p_i = A(1-\beta)x_i^{-\beta}L^\beta - p_i = 0, \qquad i = 1, 2, \ldots, N.$$

This gives the demand for intermediate good i:

$$x_i = \left(\frac{p_i}{A(1-\beta)L^\beta}\right)^{-1/\beta} = [A(1-\beta)]^{1/\beta} L p_i^{-1/\beta}, \qquad i = 1, 2, \ldots, N. \tag{14.14}$$

The price elasticity of demand, $\mathrm{E}\ell_{p_i} x_i$, for intermediate good i is thus $-1/\beta$. This reflects that the elasticity of substitution between14.17n the specialized intermediate goods in (14.12) $1/(1-(1-\beta)) = 1/\beta$. This elasticity is above 1. Hence, while the specialized intermediate goods are not perfect substitutes, they are sufficiently substitutable for a monopolistic competition equilibrium in sector 2 to exist, as we shall now see.

14.3.2 The monopolist suppliers of intermediate goods

In principle the decision problem of monopolist i is the following. Subject to the demand function (14.14), a price and quantity path $(p_{i\tau}, x_{i\tau})_{\tau=t}^{\infty}$ should be chosen so as to maximize the value of the firm (the present value of future cash flows):

$$V_{it} = \int_t^\infty \pi_{i\tau} e^{-\int_t^\tau r_s ds} d\tau, \tag{14.15}$$

where $\pi_{i\tau}$ is the profit at time τ,

$$\pi_{i\tau} = (p_{i\tau} - 1)x_{i\tau}, \tag{14.16}$$

and where the discount discount rate is r_s, the risk-free interest rate.

Since there is in this intertemporal problem no interdependence across time, the problem reduces to a series of static problems, one for each τ:

$$\begin{aligned} \max_{p_i} \pi_i &= (p_i - 1)x_i \\ & \text{s.t. } (14.14). \end{aligned}$$

To solve for p_i, we could substitute the constraint into the expression for π_i, take the derivative w.r.t. p_i, and then equalize the result to zero.

Alternatively, we may use the rule that the profit maximizing price of a monopolist is the price at which marginal revenue equals marginal cost, $MR = MC$. This is the more intuitive route we will take. We have

$$TR \text{ (= total revenue)} = p_i x_i = p_i(x_i)x_i,$$

where $p_i(x_i)$ denotes the maximum price at which the amount x_i can be sold.

Thus, by the product rule,

$$\begin{aligned} MR &= \frac{dTR}{dx_i} = p_i(x_i) + x_i p_i'(x_i) = p_i \left(1 + \frac{x_i}{p_i} \mathrm{E}\ell_{x_i} p_i\right) \\ &\equiv p_i \left(1 + \frac{1}{\mathrm{E}\ell_{p_i} x_i}\right) = p_i \left(1 + \frac{1}{-1/\beta}\right) = p_i (1 - \beta), \end{aligned}$$

from (14.14). Marginal cost is $MC = \psi$. So the profit maximizing price is

$$p_i = \frac{\psi}{1 - \beta} \equiv p > \psi. \tag{14.17}$$

Owing to monopoly power, the price is above MC; the mark-up (or "degree of monopoly") is $1/(1 - \beta)$. As expected, a lower absolute price elasticity of demand, $1/\beta$, results in a higher mark-up.

Since the elasticity of demand w.r.t. the price is independent of the quantity demanded and since MC is constant, the chosen price is time independent. Moreover the price is the same for all $i = 1, 2, \ldots, N$. Substitution into (14.14), (14.16), and (14.15), gives

$$x_{it} = \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv x, \text{ for all } i, \tag{14.18}$$

$$\pi_{it} = (p_{it} - \psi)x_{it} = \left(\frac{\psi}{1-\beta} - \psi\right)x = \frac{\beta}{1-\beta}\psi x \equiv \pi \text{ for all } i, \text{ and} \tag{14.19}$$

$$V_{it} = \int_t^\infty \pi_{is} e^{-\int_t^s r_\tau d\tau} ds = \pi \int_t^\infty e^{-\int_t^s r_\tau d\tau} ds \equiv V_t \text{ for all } i, \tag{14.20}$$

respectively. x and π are constant over time. We see that all the monopoly firms sell the same quantity x, earn the same profit, π, and have the same market value, V_t. In addition, (14.18) and (14.19) show that x and π are constant over time. We will soon see that so is V_t.

The reduced-form aggregate production function in the economy

Note that although we have skipped the two arbitrary parameter links, $A = 1/(1 - \beta)$ and $\psi = 1 - \beta$, applied by Acemoglu, the resulting expressions for p, x, π, and V_t are tractable anyway.[5] So is the implied result for gross

[5] With his two parameter links Acemoglu obtains $A/\psi = (1 - \beta)^{-2}$ from which follows the simple formulas $x_{it} = L$ and $\pi_{it} = \beta L$ for all i and all t. Although these formulas are, of course, simpler, they are "dangerous" when one wants to calculate, for instance, $\partial\pi/\partial\beta$, in order to assess the effect of a rise in β (the output elasticity w.r.t. labor) on the monopoly profit π.

output in the basic-goods sector:

$$Y_t = AN_t x^{1-\beta} L^{\beta} = AN_t \left(\frac{A(1-\beta)^2}{\psi} \right)^{\frac{1-\beta}{\beta}} L \equiv \hat{A} N_t L, \tag{14.21}$$

where we have inserted (14.18) into (14.1) and defined

$$\hat{A} \equiv A \left(\frac{A(1-\beta)^2}{\psi} \right)^{\frac{1-\beta}{\beta}}.$$

The value added in the sector is

$$Y_t - pQ_t = \hat{A} N_t L - pN_t x = (\hat{A}L - px) N_t,$$

where p and x are constants given in (14.17) and (14.18), respectively.

So both gross and net output in the basic-goods sector are proportional to the number of intermediate-goods varieties (in some sense an index of the endogenous level of technical knowledge in society). Moreover, a similar proportionality will hold for the net national product, NNP. Indeed, according to (14.8),

$$NNP_t = Y_t - \psi Q_t = \hat{A} N_t L - \psi N_t x = (\hat{A}L - \psi x) N_t. \tag{14.22}$$

This is a first signal that the model is likely to end up as a reduced-form AK model with N ("knowledge capital") acting as the capital variable.

Now to the R&D firms of sector 3.

14.3.3 R&D firms

In Section 1.1 we expressed the aggregate number of new technical designs (inventions) per time unit this way:

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \eta Z_t, \qquad \eta > 0, \ \eta \text{ constant}, \tag{*}$$

where Z_t is the R&D investment (in terms of basic goods) and η is "research productivity". What is the microeconomic story behind this?

There is a "large" number of R&D labs and free entry and exit. All R&D labs operate under the same conditions with regard to "research technology". The following simplifying assumptions are made. The random R&D outcomes are:

(i) uncorrelated across time (*no memory*),

(ii) uncorrelated across the R&D labs,

(iii) uncorrelated with any variable in the economy, and

(iv) there is *no overlap* in research.

The "no memory" assumption, (i), ignores learning over time within the lab which seems a quite drastic assumption. Assumption (ii) seems drastic as well, since some learning across R&D labs is likely. In combination, the assumptions (i), (ii), and (iii) sum up to what is called "ideosyncratic" uncertainty. The "no overlap" assumption, (iv), amounts to assuming that inventions can go in so many directions that the likelihood of different research labs chasing and making the same invention is negligible. So we can find the aggregate increase in "knowledge" simply by summing the contributions by the individual research labs.

The "research technology"

The "research technology" faced by the individual R&D labs can be described as a *Poisson process*. The expected number of successful research outcomes (inventions) per time unit is proportional to the flow input of basic goods into the lab.

Consider an arbitrary R&D lab, j, at time t, $j = 1, 2, \ldots, J_t$, where J_t is "large". Let z_{jt} be the amount of basic goods the lab devotes to research per time unit. There is an instantaneous *success arrival rate*, η, per unit invested such that, given the research flow z_{jt}, the success arrival rate (= expected number of inventions per time unit) at time t, is

$$\eta_{jt} = \eta z_{jt}, \qquad \eta > 0. \tag{14.23}$$

The Poisson parameter, η, measures "research productivity". The interpretation is that if a_{jt} denotes the number of success arrivals in the time interval $(t, t + \Delta t]$, then

$$\eta_{jt} = \lim_{\Delta t \to 0} \frac{E_t(a_{jt} \,|z_{jt}, \Delta t)}{\Delta t}, \tag{14.24}$$

where E_t is the conditional expectation operator at time t.

At the aggregate level, since, by assumption, there is no overlap in research,

$$\frac{\Delta N_t}{\Delta t} = \frac{\sum_j (a_{jt})}{\Delta t} \approx \frac{E_t\left(\sum_j a_{jt} \,|(z_{jt})_{j=1}^{J_t}, \Delta t\right)}{\Delta t} = \sum_j \frac{E_t(a_{jt} \,|z_{jt}, \Delta t)}{\Delta t}.$$

Appealing to the law of large numbers, we replace "≈" by "=", ignore indivisibilities, and take limits:

$$\dot{N}_t = \lim_{\Delta t \to 0} \frac{\Delta N_t}{\Delta t} = \sum_j \lim_{\Delta t \to 0} \frac{E_t(a_{jt}\,|z_{jt}, \Delta t)}{\Delta t} = \sum_j \eta_{jt} = \eta \sum_j z_{jt} = \eta Z_t, \tag{14.25}$$

which is (*). The third equality in (14.25) comes from (14.24), the fifth from (14.23), and the last from the definition of aggregate R&D input, Z_t.

The financing of R&D

There is a time lag of *random* length between a research lab's outlay on R&D and the arrival of a successful research outcome, an invention. During this period, which in principle has no upper bound, the R&D lab is incurring sunk costs and has no revenue at all. R&D is thus risky and continuous refinancing is needed until the research is successful.

Under certain conditions, the required financing of R&D will nevertheless be available. To clarify this, we first consider the situation *after* a successful research outcome.

When a successful research outcome arrives, the inventor takes out (free of charge) a perpetual patent on the commercial use of the invention. This gives the invention the market value, V_t, the same for all research labs, cf. (14.20). The inventor can realize this market value either by licensing the right to use the invention commercially or by directly herself entering sector 2 as a monopolist supplier of the new good made possible by the invention. To fix ideas, we assume the latter always takes place.

We make two claims, one relating to a single R&D lab, the other relating to the "loanable funds" market.

CLAIM 1 Given the market value, V_t, of an invention, the expected payoff per time unit per unit of basic goods invested in R&D is $V_t\eta$.

Proof Consider an arbitrary R&D lab j. The probability of a successful research outcome in a "small" time interval $(t, t + \Delta t]$ is approximately $\eta_{jt}\Delta t$. And the probability that more than one successful research outcome arrives in the time interval is negligible. We thus have

$$E_t\,(\text{R\&D payoff}\,|z_{jt}, \Delta t) \approx V_t\eta_{jt}\Delta t + 0 \cdot (1 - \eta_{jt}\Delta t) = V_t\eta_{jt}\Delta t. \tag{14.26}$$

Substituting (14.23) into this and dividing through by $z_{jt}\Delta t$ gives

$$\frac{E_t\,(\text{R\&D payoff}\,|z_{jt}, \Delta t)}{z_{jt}\Delta t} \approx \frac{V_t\eta z_{jt}\Delta t}{z_{jt}\Delta t} = V_t\eta.$$

Letting $\Delta t \to 0$, "$\approx$" can in the limit be replaced by "$=$", thus confirming the claim. □

Now consider the demand and supply in the "loanable funds" market.

CLAIM 2 Let $\sum_j z_{jt} = Z_t$. (i) In any equilibrium in the "loanable funds" market, whether with $Z_t = 0$ or $Z_t > 0$, we have

$$V_t \eta \leq 1. \tag{14.27}$$

(ii) In any equilibrium in the "loanable funds" market where $Z_t > 0$, we have

$$V_t \eta = 1. \tag{14.28}$$

Proof. (i) Suppose that, contrary to (14.27), we have $V_t \eta > 1$. By Claim 1, the expected R&D payoff per time unit per unit cost of R&D is then higher than the R&D cost and so expected pure profit by doing R&D is positive. The flow demand for finance to R&D firms will therefore be unbounded. The flow supply of finance, ultimately coming from household saving, is, however, bounded and thus there is excess demand for funds and thereby not equilibrium.[6] Thus $V_t \eta > 1$ can be ruled out as an equilibrium and this leaves (14.27) as the only possible state in an equilibrium.

(ii) Consider an equilibrium with $Z_t > 0$. Since it is an equilibrium, (14.27) must hold. By way of contradiction, let us imagine there is strict inequality in (14.27). Then all R&D firms will choose $z_{jt} = 0$ and we reach the conclusion that $Z_t = 0$, thus contradicting that $Z_t > 0$. So there can not be strict inequality in (14.27) and we are left with (14.28) as the only possible state in an equilibrium with $Z_t > 0$. □

It follows from Claim 2 that when the market value of inventions satisfy (14.28), the cost of doing R&D is on average exactly covered by the expected payoff. In return for putting one unit of account at the disposal of a research lab, the household gets a payoff of V_t if the research turns out to be successful and zero otherwise. In expected value the payoff is one unit of account. It is as if the household buys a lottery ticket offered by the R&D lab to finance it current R&D costs. The lottery prize consists of shares of stock giving the right to the future monopoly profits if the current research is successful within one time unit. The lottery is "fair" because the cost of participating equals the expected payoff. In spite of being risk averse ($u''(c) < 0$),

[6] For the sake of intuition, allow disequilibrium to exist in the very short run. Then the excess demand for funds drives share prices down and the rate of return, r_t, up, thus lowering V_t (cf. (14.20)) until $V_t \eta = 1$.

the households are willing to participate because the uncertainty is "ideosyncratic" and the economy is "large". This allows the households to avoid the risk by spreading their investment over a variety of R&D labs, i.e., by diversifying their investment.

What is the size of the equilibrium real interest rate, r_t, coming out of this? This rate must satisfy the following no-arbitrage relation vis-a-vis the instantaneous rate of return on shares in sector-2 firms supplying specialized intermediate goods:

$$r_t = \frac{\pi + dV_t/dt}{V_t}, \tag{14.29}$$

where π is the constant dividend (assuming all profit is paid out to the share owners) and dV_t/dt is the capital gain (positive or negative) on holding shares. As an implication of Claim 2, in an equilibrium with $Z_t > 0$, the market value of any invention is

$$V_t = 1/\eta \equiv V,$$

a constant. So $dV_t/dt = 0$, and (14.29) simplifies to

$$r_t = \frac{\pi}{1/\eta} = \eta\pi \equiv r, \tag{14.30}$$

where π is determined by (14.19). That is, along an equilibrium path with $Z_t > 0$, the interest rate is *constant* and *determined* by (14.30).

To ensure that $Z_t > 0$ and thereby positive growth is present in the economy, we need that the parameters are such that households *do* save. In view of the Keynes-Ramsey rule, this requires $r > \rho$ which in turn, by (14.30), requires a sufficiently high research productivity

$$\eta > \rho/\pi. \tag{A1}$$

What ensures that household saving and R&D investment match each other? Let aggregate financial wealth at time t be denoted $\mathcal{A}_t$. Then, in an equilibrium with $Z_t > 0$,

$$\mathcal{A}_t \equiv a_t L = V N_t = \frac{1}{\eta} N_t.$$

In view of $\dot{N}_t = \eta Z_t$, we therefore have

$$\dot{\mathcal{A}}_t = V\dot{N}_t = \frac{1}{\eta}\dot{N}_t = \frac{1}{\eta}\eta Z_t = Z_t. \tag{14.31}$$

By definition, households' aggregate saving, S_t, equals the increase in financial wealth per time unit, i.e., $S_t = \dot{A}_t$.[7] Substituting this into (14.31), we see that the investment, Z_t, and saving, S_t, are two sides of the same coin.

To understand that there are neither losers nor winners in this saving-investment process, it may help intuition to imagine that all the saving, $S_t\Delta t$, in a short time interval $(t, t+\Delta t]$ first goes to large mutual funds which (without administrative costs). These mutual funds instantly use the receipts to buy lottery tickets offered by R&D labs to cover current R&D costs. For the mutual funds taken together this involves an exchange of the outlay $S_t\Delta t$ for shares giving the right to the future monopoly profits associated with those research labs that turn out to be successful in the time interval considered. By the law of large numbers the inventions by these labs have exactly the same value as the outlay. Indeed, by (14.31), we have

$$V\dot{N}_t\Delta t = S_t\Delta t.$$

From then on, holding shares in the monopolies supplying the newly invented intermediate goods gives the normal rate of return in the economy, r. A fraction of the R&D labs have not been successful in the time interval considered (and the financing to them has thereby been lost). But others have been successful and made an invention. The unequal occurrence of failures and successes across the many different R&D labs is neutralized when it comes to the payout to the customers, i.e., the households who have deposits in the mutual funds.

As an alternative financing setup, suppose that the R&D labs offer project contracts of the following form. A contract stipulates that the investor pays the lab $1/\eta$ units of account per time unit until a successful research outcome arrives. The corresponding liability of the lab, now an entrepreneur in sector 2, is that the permanent profit stream obtained on the invention goes to the investor. By Claim 1, such R&D contracts have no market value. But after a successful R&D outcome there is a capital gain in the sense that the contracts become shares in the hands of the investors giving permanent dividends equal to π per time unit and thus having a market value equal to $V = 1/\eta$ forever.

Note that as the model is formulated, there is *no value added* in the R&D sector, as was also mentioned in connection with (14.7) in Section 14.1.2. Instead, the value that at the aggregate level comes out as $V\dot{N}_t$ is just a costless one-to-one instantaneous transformation of Z_t which is a part of the value added created *in the basic-goods sector*. It is ultimately this value added that households' saving pays for.

[7] In this model households' gross saving equals their net saving since there are no assets that depreciate.

14.4 General equilibrium of an economy satisfying (A1)

The assumption (A1) ensures a research productivity high enough to provide a rate of return exceeding the rate of time preference and thereby induce the household saving needed for R&D investment, Z_t, to be positive. And from (14.30) we know that along an equilibrium path with $Z_t > 0$, and therefore $\dot{N} > 0$, the interest rate is a constant, r. Then the Keynes-Ramsey rule, (14.10), yields

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}(\eta\pi - \rho) \equiv g_c, \tag{14.32}$$

where π is given (14.19). To ensure that the path considered with $\dot{N} > 0$ is really capable of being an equilibrium path, we need the parameter restriction

$$\rho > (1 - \theta)g_c, \tag{A2}$$

since otherwise the transversality condition of the household could not be satisfied.[8]

From (14.21) and (14.22) we know that along an equilibrium path, gross as well as net output in the basic-goods sector are proportional to the stock of "knowledge capital", N_t. Moreover, the analysis of the previous section shows that the preliminary national income accounting sketched in Section 14.1.2 is correct. Hence, by (14.22), also the aggregate value added in the economy as a whole, NNP, is proportional to N_t. Indeed,

$$NNP_t = Y_t - \psi Q_t = \hat{A}N_t L - \psi N_t x = (\hat{A}L - \psi x)N_t \equiv \bar{A}N_t.$$

So the model does indeed belong to the class of reduced-form AK models.

14.4.1 The balanced growth path

From the general theory of reduced-form AK models with Ramsey households, we know that the "capital" variable of the model, here "knowledge capital", N_t, will grow at the same constant rate as per capita consumption already from the beginning. In the present case the latter growth rate is given by (14.32). And

$$\dot{N}_t = \eta Z_t = \eta(NNP_t - C_t) = \eta(\bar{A}N_t - c_t L), \tag{14.33}$$

[8] Another aspect of this is that (A2) ensures that the utility integral U_0 is bounded and thereby allows maximization in the first place.

so that

$$g_N \equiv \frac{\dot{N}_t}{N_t} = \eta\left(\bar{A} - \frac{c_t L}{N_t}\right).$$

As $g_N = g_c$, this implies

$$c_t L = (\bar{A} - \frac{g_c}{\eta})N_t,$$

for all $t \geq 0$. Hence, the so far unknown initial per capita consumption is

$$c_0 = (\bar{A} - \frac{g_c}{\eta})\frac{N_0}{L}.$$

Labour productivity can be defined as

$$y_t \equiv NNP_t/L = \bar{A}N_t/L. \tag{14.34}$$

hence $g_y = g_N = g_c \equiv g^*$.

Thus the model generates fully endogenous balanced growth and there are no transitional dynamics.

14.4.2 Comparative analysis

$\partial g^*/\partial\rho = -1/\theta < 0$. Higher impatience $\Rightarrow$ lower propensity to save $\Rightarrow$ less investment in R&D.

$\partial g^*/\partial\theta < 0$. Higher desire for consumption smoothing $\Rightarrow$ attempt to transform some of the higher future consumption possibility into higher consumption today $\Rightarrow$ lower saving $\Rightarrow$ less investment in R&D.

$\partial g^*/\partial A > 0$. Higher factor productivity $\Rightarrow$ higher return on saving $\Rightarrow$ more saving at the aggregate level (the negative substitution effect and wealth effect on consumption dominates the positive income effect) $\Rightarrow$ more investment in R&D. As usual the constant A need not have a narrow technical interpretation. It can reflect the quality of the institutions in society (rule of law etc.) and the level of "social capital".[9]

$\partial g^*/\partial\eta > 0$. Higher R&D productivity results in more R&D investment and higher growth.

$\partial g^*/\partial L > 0$. A larger population L implies lower per capita cost, η/L, associated with producing a given amount of new technical knowledge which in turn improves productivity for *all* members of society. This is an implication of knowledge being a nonrival good. In a larger society, with larger markets, the incentive to do R&D is therefore higher. In the present version

[9]By social capital is meant society's stock of social networks and shared norms that support and maintain confidence, credibility, trust, and trustworthiness.

of the R&D model the result is a higher growth rate permanently. This is a manifestation of the controversial *strong* scale effect (scale effect on growth), typical for the "first-generation" innovation-based growth models with fully endogenous growth. This strong scale effect, as well as the fully endogenous growth property, is due to a "hidden" knife-edge condition in the specification of the "growth engine", essentially a knife-edge condition in the production function for basic goods, cf. the general discussion in Chapter 13 and Exercise VII.5.

Chapter 15

Stochastic erosion of innovator's monopoly power

In this chapter we extend the lab-equipment model of Chapter 14 by adding stochastic erosion of innovators' monopoly power. The motivation is the following.

The model of Chapter 14 assumed that the innovator had perpetual monopoly over the production and sale of the new intermediate good. In practice, by legislation patents are of limited duration, say 15 years. Moreover, it may be difficult to codify exactly the technical aspects of inventions, hence not even within such a limited period do patents give 100% effective protection. While the pharmaceutical industry rely quite much on patents, in many other branches innovative firms use other protection strategies such as concealment of the new technical design. In ICT industries copyright to new software plays a significant role. Still, whatever the protection strategy used, imitators sooner or later find out how to make very close substitutes.

To better accommodate these facts, the present chapter sets up a lab-equipment model where competition in the supply of specialized intermediate goods is more intense than in Chapter 14. For convenience we name the model of Chapter 14 *Model I.* Compared with that model the only difference in the new model is that the duration of monopoly power over the commercial use of an invention is *limited* and *uncertain*. We name the resulting model *Model II.* The notation is the same as in Model I. The analysis is related to the brief discussion of the issues in Acemoglu's Section 13.1.6 of his Chapter 13.

First a recapitulation of the technological aspects of the economy.

15.1 The three production sectors

The technology of the economy is the same as in Model I. In the *basic-goods sector* (sector 1) firms combine labor and N_t different intermediate goods to produce a homogeneous output good. The representative firm in the sector has the production function

$$Y_t = A\left(\sum_{i=1}^{N_t} {x_{it}}^{1-\beta}\right) L_t^{\beta}, \qquad A > 0,\ 0 < \beta < 1, \tag{15.1}$$

where Y_t, L_t, and x_{it} denote output of the firm, labor input, and input of intermediate good i, respectively, where $i = 1, 2, ..., N_t$. This sector, as well as the labor market, operate under perfect competition.

The aggregate output of basic goods is used partly for replacing the basic goods, X_t, used in the production of intermediate goods used up in the production of basic goods, partly for consumption, C_t, and partly for investment in R&D, Z_t. Hence, we have

$$Y_t = X_t + C_t + Z_t. \tag{15.2}$$

In the *intermediate-goods sector*, sector 2, at time t there are N_t monopoly firms, each of which supplies a particular already invented intermediate good. Once the technical design for intermediate good i has been invented in sector 3, the inventor enters sector 2 as an innovator. Given the technical design, the innovator can instantly transform a certain number of basic goods into a proportional number of intermediate goods of the invented specialized kind. That is,

it takes ψx_i units of the basic good to supply x_i units of intermediate good i, (15.3)

where ψ is a positive constant. The transformation requires no labor. Thus, ψ is both the marginal and the average cost of supplying the intermediate good i. This transformation technology applies to all intermediate goods, $i = 1, 2, ..., N_t$, and all t. Hence, the X_t in (15.2) satisfies

$$X_t \equiv \psi \sum_{i=1}^{N_t} x_{it} \equiv \psi Q_t, \tag{15.4}$$

where Q_t is the total supply of intermediate goods, all of which are used up in the production of basic goods.

For a limited period after the invention has been made, through secrecy or imperfect patenting the inventor maintains monopoly power over the commercial use of the invention. The length of this period is uncertain, see below.

In the R&D sector, sector 3, new "technical designs" (blueprints) for making new specialized intermediate goods are invented. The uncertainty associated with R&D is "ideosyncratic". On average it takes an input of $1/\eta$ units of the basic good, and *nothing else*, to obtain one successful R&D outcome (an invention) per time unit. There is free entry to the R&D activity. Ignoring indivisibilities, the aggregate number of new technical designs (inventions) in the economy per time unit is

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \eta Z_t, \qquad \eta > 0, \ \eta \text{ constant}, \tag{15.5}$$

where, as noted above, Z_t is the aggregate R&D investment in terms of basic goods delivered to sector 3 per time unit. As also noted above, after an invention has been made, the inventor enters sector 2 as an innovator and begins supplying the new intermediate good to firms in sector 1.

15.2 Temporary monopoly

To begin with the innovator has a monopoly over the production and sale of the new intermediate good. This may be in the form of a more or less effective patent (free of charge) or copyright to software or simply by secrecy and concealment of the new technical design. But sooner or later imitators find out how to make very close substitutes. There is *uncertainty* as to how long the monopoly position of an innovator lasts.

We assume the cessation of monopoly power can be described by a Poisson process with an exogenous Poisson arrival rate $\lambda > 0$, the same for all monopolies.[1] The "event" which "arrives" sooner or later is "exposure to unbounded competition". Independently of how long the monopoly position for firm i has been maintained, the probability that it breaks down in the next time interval of length Δt is approximately $\lambda \cdot \Delta t$ for Δt "small". Equivalently, if T denotes the remaining lifetime of the monopoly status of intermediate good i, then the probability that $T > \tau$ is $e^{-\lambda\tau}$ for all $\tau > 0$. Further, the cessations of the different monopolies are stochastically independent. The expected duration of a monopoly is $1/\lambda$.

A prospective innovative "entrepreneur" who invests 1 unit of account (the basic good) per time unit with the purpose of making an invention now faces a double risk, namely first the risk that the R&D activity is unsuccessful for a long time and second the risk that, when finally it is successful, the monopoly profits on the R&D investment will only last for a short time. The model assumes, however, that all uncertainty is *idiosyncratic,* that is,

[1] This approach builds on Barro and Sala-i-Martin (1995).

the stochastic events that an R&D lab is successful in a given time interval and that an innovator looses her monopoly position a given time interval are uncorrelated across R&D labs, innovators, and time and are in fact not correlated with anything in the economy. Assuming a "large" number of intermediate goods still have a monopoly, investors can eliminate any risk by diversifying their investment as described in Chapter 14. Of course, this whole setup is an abstraction and can at best be considered a benchmark case.

As labor supply is a constant, L, clearing in the labor market implies $L_t = L$. We insert this into the production function (15.1) of the representative firm in sector 1. Maximizing profit, this firm then demands, at time t, $x_i(p_{it}) = (A(1-\beta))^{1/\beta} Lp_{it}^{-1/\beta}$ units of intermediate good i per time unit, $i = 1, 2, ..., N_t$. As long as innovator i is still a monopolist, she faces this downward-sloping demand curve with price elasticity $-1/\beta$ and sets the price, p_{it}, such that $MR = MC$ (marginal revenue = marginal cost), that is,

$$(1 - \frac{1}{1/\beta})p_{it} = \psi,$$

when the basic good is the numeraire. Solving for p_{it}, we get

$$p_{it} = (1 + \text{markup}) \cdot MC = \frac{\psi}{1-\beta} \equiv p.$$

Thereby, as long as innovator i is still a monopolist, the sales of intermediate good i is

$$x_i(p_{it}) = x_i(p) = (A(1-\beta))^{1/\beta} Lp^{-1/\beta} = \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv x^{(m)}, \quad (15.6)$$

for all $i = 1, 2, \ldots, N_t$. We shall call $x^{(m)}$ the *monopoly supply* of a specific intermediate good. The corresponding total revenue per time unit is $(\psi/(1-\beta)) \cdot x^{(m)}$ and the total cost is $\psi \cdot x^{(m)}$. The earned profit per time unit is thus

$$\pi_{it} = (p - \psi)x_i(p) = (\frac{\psi}{1-\beta} - \psi)x^{(m)} = \frac{\beta}{1-\beta}\psi x^{(m)} \equiv \pi^{(m)}. \qquad (15.7)$$

The formulas for $x^{(m)}$ and $\pi^{(m)}$ are the same as those for x and π, respectively, in Model I, cf. Chapter 14.

As described above, however, sooner or later innovator i loses the monopoly. When this happens, intermediate good i faces perfect competition and its price, p_{it}, is driven down to the competitive market price = marginal cost =

ψ. Since also *average* cost is ψ, the profit vanishes. The aggregate sales of intermediate good i (now supplied by many competitors) are

$$x_i(p_{it}) = x_i(\psi) = \left(\frac{A(1-\beta)}{\psi}\right)^{1/\beta} L \equiv (1-\beta)^{-1/\beta} x^{(m)} \equiv x^{(c)} > x^{(m)}, \quad (15.8)$$

where $x^{(c)}$ will be called the *competitive supply* of a specific intermediate good. The inequality in (15.8) follows from $0 < \beta < 1$. Economically, the inequality in (15.8) reflects that the demand depends negatively on the price, which is lower under competition.

To summarize: In view of production and cost symmetry, each intermediate good supplied under monopolistic conditions is supplied in the same amount, $x^{(m)}$, and each intermediate good supplied under competitive conditions is supplied in the same but larger amount, $x^{(c)}$. That is,

$$x_i = \begin{cases} x^{(m)} \text{ if } i \text{ is still a monopoly,} \\ x^{(c)} \text{ if } i \text{ is no longer a monopoly,} \end{cases} \quad (15.9)$$

where $x^{(m)}$ and $x^{(c)}$ are given in (15.6) and (15.8), respectively.

15.3 The reduced-form aggregate production function

Substituting (15.9) into (15.1), we can write output in sector 1 as

$$Y_t = A\left[N_t^{(m)}(x^{(m)})^{1-\beta} + N_t^{(c)}(x^{(c)})^{1-\beta}\right] L^\beta, \quad (15.10)$$

where $N_t^{(m)}$ is the number of intermediate goods that at time t are still supplied under monopolistic conditions and $N_t^{(c)}$ is the number of intermediate goods that have become competitive. For each t we have

$$N_t = N_t^{(m)} + N_t^{(c)}. \quad (15.11)$$

There are now *two* state variables in the model. There is therefore scope for transitional dynamics, as we will soon see.

It is convenient to rewrite (15.10) this way:

$$\begin{aligned} Y_t &= A\left[(N_t - N_t^{(c)})(x^{(m)})^{1-\beta} + N_t^{(c)}(1-\beta)^{-(1-\beta)/\beta}(x^{(m)})^{1-\beta}\right] L^\beta \\ &= A\left[N_t - N_t^{(c)} + N_t^{(c)}(1-\beta)^{-(1-\beta)/\beta}\right](x^{(m)})^{1-\beta} L^\beta \\ &= A\left[N_t + ((1-\beta)^{-(1-\beta)/\beta} - 1)N_t^{(c)}\right](x^{(m)})^{1-\beta} L^\beta \\ &= \left[1 + ((1-\beta)^{-(1-\beta)/\beta} - 1)\frac{N_t^{(c)}}{N_t}\right] A(x^{(m)})^{1-\beta} L^\beta N_t. \quad (15.12) \end{aligned}$$

Aggregate output is seen to depend on $N_t^{(c)}/N_t$. If the dynamics are such that $N_t^{(c)}/N_t$ tends to a constant, then Y_t will tend to be proportional to the produced input, N_t, since $A(x^{(m)})^{1-\beta}L^{\beta}$ is a constant, cf. (15.6). Therefore, the model is likely capable of generating fully endogenous growth, driven by R&D. We come back to this below.

The result in (15.12) can be written

$$Y_t = \left[1 + ((1-\beta)^{-(1-\beta)/\beta} - 1)\frac{N_t^{(c)}}{N_t}\right] Y_t^{(m)} > Y_t^{(m)}, \tag{15.13}$$

where $Y_t^m \equiv A(x^{(m)})^{1-\beta}L^{\beta}N_t$ is the equilibrium output of basic goods in case of universal and perpetual monopoly power. So Y_t^m is the same as the equilibrium output in Model I. In that model we have $\lambda = 0$, hence $N_t^{(c)} = 0$ for all t. But with erosion of monopoly power, we have $N_t^{(c)} > 0$ and so a fraction of the intermediate goods are supplied at a price equal to marginal cost thus inducing efficient use of these. Thereby productivity is enhanced and we get $Y_t > Y_t^{(m)}$ in (15.12) (indeed, $(1-\beta)^{-(1-\beta)/\beta} > 1$ in view of $0 < \beta < 1$).

15.4 The no-arbitrage condition under uncertainty

All uncertainty is assumed to be ideosyncratic. By holding shares in many monopoly firms (portfolio diversification), the risk-averse households can therefore eliminate any risk and obtain the risk-free rate of return, r_t, with certainty.[2] The appropriate discount rate for calculating the present value of expected future profits in any monopoly i is therefore r_t. Consequently, ruling out speculative bubbles, the market value of monopoly i at time t is

$$V_{it} = \int_t^{\infty} E_t(\pi_{i\tau})e^{-\int_t^{\tau} r_s ds}d\tau, \tag{15.14}$$

where $\pi_{i\tau}$ is the profit obtained at time τ, now a stochastic variable as seen from time $t < \tau$:

$$\pi_{i\tau} = \begin{cases} \pi^{(m)} \text{ if firm } j \text{ is still a monopolist at time } \tau, \\ 0 \text{ otherwise.} \end{cases}$$

[2] It may help the intuition to imagine that there is a market for safe loans. Then r_t will be the interest rate on that market.

Expected profit at time τ, as seen from time t, is

$$E_t(\pi_{i\tau}) = \pi^{(m)} e^{-\lambda(\tau-t)} + 0 \cdot (1 - e^{-\lambda(\tau-t)}) = \pi^{(m)} e^{-\lambda(\tau-t)}. \qquad (15.15)$$

Substituting into (15.14) we get

$$V_{it} = \pi^{(m)} \int_t^\infty e^{-\int_t^\tau (r_s+\lambda)ds} d\tau \equiv V_t, \qquad (15.16)$$

the same for all intermediate goods i that at time t still retain monopoly. This expression gives the market value of a monopoly firm in a certainty-equivalent form. On the one hand the integral in (15.16) "treats" the monopoly profit stream as if it were perpetual, on the other hand this future potential profit is discounted at an effective discount rate, $r_\tau + \lambda$, taking into account the probability, $e^{-\lambda(\tau-t)}$, that at time τ the ability to earn this profit has disappeared.[3]

At this point we face the question: how is the risk-free rate of return, r_t, determined? To approach an answer, it is useful to derive a no-arbitrage condition which is implicit in (15.14). It may help intuition to think of r_t as the interest rate on a market for safe loans.

By differentiating (15.16) w.r.t. t, using Leibniz's' formula,[4] we get

$$\frac{\pi^{(m)} + \dot{V}_t}{V_t} = r_t + \lambda, \qquad (15.17)$$

where $\dot{V}_t$ is the conditional capital gain, that is, the increase per time unit in the market value of the monopoly firm at time t, conditional on its monopoly position remaining in place also in the next moment. This formula equalizes the instantaneous *conditional* rate of return per time unit on shares in monopoly firms to the risk-free interest rate plus a premium reflecting the risk that the monopoly position expires within the next instant.

Alternatively we may derive the condition (15.17) without appealing to Leibniz's' formula (which may not be part of the reader's standard math toolbox). This alternative approach has the advantage of being more intuitive.

[3]So V_t in (15.16) has the same meaning as in the certainty case (Model I), in the sense that V_t equals the current market value of a monopoly firm, that is, V_t is an observable variable given that the firm is still a monopoly (otherwise, $V_t = 0$). The uncertainty is about profits in the future and the discount rate for these equals the risk-free interest rate plus a risk premium, here equal to λ, which is the approximate conditional probability that the monopoly status breaks down in the time interval $(\tau, \tau + 1]$, given it is retained up to time τ.

[4]See Appendix A.

Let

$u_t \equiv$ the firm's earnings in the time interval $(t, t + \Delta t)$, given that the firm is still a monopolist at time t.

There will be no opportunities for arbitrage if the expected instantaneous rate of return per time unit on shares in the monopoly firm equals the required rate of return which is the risk-free interest rate, r_t. This amounts to the condition

$$\frac{\lim_{\Delta t \to 0} \frac{E_t u(t)}{\Delta t}}{V_t} = r_t. \tag{15.18}$$

The firm's earnings, u_t, is a stochastic variable and its expected value as seen from time t is

$$E_t u_t \approx \lambda \Delta t(-V_t) + (1 - \lambda \Delta t)(\pi^{(m)} + \dot{V}_t)\Delta t. \tag{15.19}$$

Indeed, V_t is the capital loss in case the monopoly position ceases and $\lambda \Delta t$ is the approximate probability that this event occurs within the time interval $(t, t + \Delta t]$, given that at time t it has not yet occurred. Similarly, $1 - \lambda \Delta t$ is the approximate probability that a monopoly position retained up to time t remains in force at least up to time $t + \Delta t$. And $\pi^{(m)} + \dot{V}_t$ is the total return in that case. Now, (15.19) can be written:

$$\begin{aligned} E_t u(t) &\approx -\lambda \Delta t V_t + (\pi^{(m)} + \dot{V}_t)\Delta t - \lambda(\pi^{(m)} + \dot{V}_t)(\Delta t)^2 \qquad (15.20) \\ &= (\pi^{(m)} + \dot{V}_t - \lambda V_t)\Delta t - \lambda(\pi^{(m)} + \dot{V}_t)(\Delta t)^2 \Rightarrow \\ \frac{E_t u\,(t)}{\Delta t} &\approx \pi^{(m)} + \dot{V}_t - \lambda V_t - \lambda(\pi^{(m)} + \dot{V}_t)\Delta t \\ &\to \pi^{(m)} + \dot{V}_t - \lambda V_t \text{ for } \Delta t \to 0. \end{aligned}$$

Hence, the condition (15.18) implies the no-arbitrage condition

$$\frac{\pi^{(m)} + \dot{V}_t - \lambda V_t}{V_t} = r_t. \tag{15.21}$$

Reordering, we see that this is the same condition as (15.17).

15.5 The equilibrium rate of return when R&D is active

The cost of making $\dot{N}_t$ inventions per time unit in the aggregate at time t is $Z_t = \dot{N}_t/\eta$. The expected cost per invention is thus $1/\eta$. An equilibrium with active R&D therefore requires[5]

$$V_t = 1/\eta \equiv V. \tag{15.22}$$

So the market value of a monopoly firm is constant as long as the monopoly position is upheld. The conditional capital gain, $\dot{V}_t$, is therefore nil, whereby substituting (15.22) into (15.21) and applying (15.7) yields

$$r_t = \eta\pi^{(m)} - \lambda = \eta\frac{\beta}{1-\beta}\psi x^{(m)} - \lambda \equiv r^* \equiv r^{(m)} - \lambda < r^{(m)}, \tag{15.23}$$

where $r^{(m)}$ is the equilibrium interest rate in Model I, that is, the case of perpetual monopoly.

We see, first, that like in Model I, the equilibrium interest rate is a constant, r^*, from the beginning. Second, in view of $\lambda > 0$, we have $r^* < r^{(m)}$. Because of the limited duration of monopoly power in our present model, the expected rate of return on investing in R&D is smaller than in the case of no erosion of monopoly power.

The description of the household sector is as in Model I, but now per capita financial wealth is

$$a_t = \frac{N_t^{(m)}V_t}{L},$$

because there are only $N_t^{(m)} = N_t - N_t^{(c)}$ firms with positive market value, namely the number of firms that supply intermediate goods under monopolistic conditions. The households' first-order conditions lead to the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r^* - \rho) = \frac{1}{\theta}(r^{(m)} - \lambda - \rho) \equiv g_c^* < g_c^{(m)}, \tag{15.24}$$

where $g_c^{(m)}$ is the per capita consumption growth rate from Model I, the case of perpetual monopoly.

In order to have a growth model, we assume parameters are such that $g_c^* > 0$. In addition, to avoid unbounded utility and help fulfillment of the households' transversality condition, we assume $\rho > (1-\theta)g_c^*$. These two conditions amount to the parameter restrictions

$$r^{(m)} > \theta(\lambda + \rho), \quad \text{and} \tag{A1}$$

$$\rho > (1-\theta)g_c^* = (1-\theta)\frac{1}{\theta}(r^{(m)} - \lambda - \rho), \tag{A2}$$

[5]A more detailed argument is given in Chapter 14.

where $r^{(m)} \equiv \eta\beta(1-\beta)^{-1}\psi x^{(m)}$ (from (15.23)) with $x^{(m)} = ((A(1-\beta)^2)/\psi)^{1/\beta} L$ (from (15.6)). So (A1) requires that the "growth engine" of the economic system, as determined in particular by η, A, and L, is "powerful enough". Below we return to what exactly is meant by such a statement. Suffice it to say here that increases in η, A, and L augment the strength of the growth engine (makes (A1) more likely to hold) while a rise in ψ reduces its strength (makes (A1) less likely to hold).[6]

15.6 Transitional dynamics*

Given that cessations of individual monopolies follow the assumed independent Poisson processes with arrival rate λ, the aggregate number of transitions per time unit from monopoly to competitive status follow a Poisson process with arrival rate $\lambda N_t^{(m)}$. The expected number of transitions per time unit from monopoly to competitive status is then

$$E_t\dot{N}_t^{(c)} = \lambda N_t^{(m)}.$$

Assuming $N_t^{(m)}$ is "large", the difference between actual and expected transitions per time unit will be negligible (by the law of large numbers), and we simply write

$$\dot{N}_t^{(c)} = \lambda N_t^{(m)} = \lambda(N_t - N_t^{(c)}). \tag{15.25}$$

Let the fraction of intermediate goods supplied under competitive conditions be denoted $s_t \equiv N_t^{(c)}/N_t$ and let $g_x \equiv \dot{x}_t/x_t$ for any positively-valued variable x_t. Then, by (15.25),

$$\begin{aligned} g_s &= g_{N^{(c)}} - g_N = \lambda\frac{N_t - N_t^{(c)}}{N_t^{(c)}} - g_N = \lambda(s_t^{-1} - 1) - g_N \\ &= \lambda s_t^{-1} - (\lambda + g_N) \gtreqless 0 \quad \text{for} \quad s_t \lesseqgtr \frac{\lambda}{\lambda + g_N}. \end{aligned} \tag{15.26}$$

The general law of movement of N_t is given by (15.5), which, together

[6] In view of $r^{(m)} \equiv \eta\beta(1-\beta)^{-1}\psi(A(1-\beta)^2)^{1/\beta}\psi^{-1/\beta}L$, this role of ψ is due to $1-1/\beta < 0$.

with (15.2) and (15.13) and the definition $\tilde{c}_t \equiv \frac{c_t}{N_t}$, implies that

$$\begin{aligned}
\dot{N}_t &= \eta Z_t = \eta(Y_t - X_t - C_t) = \eta \left\{ Y_t - \psi(N_t^{(m)} x^{(m)} + N_t^{(c)} x^{(c)}) - C_t \right\} \\
&= \eta \left\{ \left(1 + ((1-\beta)^{-(1-\beta)/\beta} - 1) \frac{N_t^{(c)}}{N_t} \right) Y_t^{(m)} - \psi((N_t - N_t^{(c)}) x^{(m)} + N_t^{(c)} x^{(c)}) - c_t L \right\} \\
&= \eta \left\{ (1 + ((1-\beta)^{-(1-\beta)/\beta} - 1) s_t) \frac{Y_t^{(m)}}{N_t} - \psi x^{(m)} + \psi(x^{(m)} - x^{(c)}) s_t) - \tilde{c}_t L \right\} N_t \\
&= \eta \left\{ \left(1 + ((1-\beta)^{-(1-\beta)/\beta} - 1) s_t\right) A (x^{(m)})^{1-\beta} L^\beta - \psi x^{(m)} \right. \\
&\quad \left. + \psi \left[x^{(m)} - (1-\beta)^{-1/\beta} x^{(m)} \right] s_t) - \tilde{c}_t L \right\} N_t \qquad \text{(by (15.8))} \\
&= \eta \left\{ A (x^{(m)})^{1-\beta} L^\beta - \psi x^{(m)} \right. \\
&\quad \left. + \left[((1-\beta)^{-(1-\beta)/\beta} - 1) A (x^{(m)})^{1-\beta} L^\beta - \psi((1-\beta)^{-1/\beta} - 1) x^{(m)} \right] s_t - \tilde{c}_t L \right\} N_t \\
&\equiv \eta \left\{ B_1 + B_2 s_t - \tilde{c}_t L \right\} N_t,
\end{aligned}$$

where the constants B_1 and B_2 are implicitly defined. The growth rate of N_t can thus be written

$$g_N = \eta \left(B_1 + B_2 s_t - \tilde{c}_t L \right). \tag{15.27}$$

We now construct the implied dynamic system in the endogenous variables s_t and $\tilde{c}_t$. From (15.26) follows $\dot{s}_t = \lambda - (\lambda + g_N) s_t$, which combined with (15.27) yields

$$\dot{s}_t = \lambda - \left(\lambda + \eta \left(B_1 + B_2 s_t - \tilde{c}_t L \right) \right) s_t. \tag{15.28}$$

Similarly, from $\tilde{c}_t \equiv c_t / N_t$ follows $\dot{\tilde{c}}_t / \tilde{c}_t = g_c - g_N = g_c^* - g_N$, by (15.24). So,

$$\dot{\tilde{c}}_t = \left(g_c^* - \eta \left(B_1 + B_2 s_t - \tilde{c}_t L \right) \right) \tilde{c}_t. \tag{15.29}$$

The differential equations (15.28) and (15.29) constitute a dynamic system with two endogenous variables, s_t and $\tilde{c}_t$, the first of which is a predetermined variable while the second is a jump variable (forward-looking variable).

15.7 Long-run growth

In a steady state ($\dot{s}_t = 0 = \dot{\tilde{c}}_t$), by definition of $\tilde{c}_t$, $g_N = g_c = g_c^*$, defined in (15.24).In view of (15.26) and $g_N = g_c^*$, the steady-state value of s_t

$$s^* = \frac{\lambda}{\lambda + g_c^*}. \tag{15.30}$$

Finally, the steady-state value of $\tilde{c}_t$ is $\tilde{c}^* = (B_1 + B_2 s^* - g_c^*/\eta)/L$.

In the steady state there is balanced growth in the sense that $Y_t, c_t, N_t, N_t^{(c)}$, and Y_t grow at the same constant rate as c_t, namely the rate g_c^* given in (15.24). This follows from the constancy of $\tilde{c}$ and s in steady state together with the expression (15.13) for the aggregate production function in sector 1. Moreover, as to the total supply, Q_t, of intermediate goods we have $Q_t = N_t^{(m)} x^{(m)} + N_t^{(c)} x^{(c)} = (N_t - N_t^{(c)}) x^{(m)} + N_t^{(c)} x^{(c)} = \left[(1 - s^*) x^{(m)} + s^* x^{(m)}\right] N_t$ in the steady state, showing that Q_t is proportional to N_t in the steady state. And by (15.2), the delivery of basic goods to sector 2 is $X_t = \psi Q_t$, which is thus also proportional to N_t in the steady state. Hence, in steady state both Q_t and X_t grow at the same rate as N_t, the rate g_c^*. The same is true for R&D investment $Z_t = \dot{N}_t/\eta = g_N^* N_t/\eta$. As shown in Appendix B, where also a phase diagram is sketched, the steady state is a saddle point; an only half-finished dynamic analysis suggests that for any given initial $N_0^{(c)}/N_0 \in (0, 1)$, there exists a unique solution to the model and it converges to the steady state for $t \to \infty$.

Like Model I, Model II thus generates fully endogenous growth with a long-run per capita growth rate equal to g_c^* in (15.24). What makes fully endogenous growth possible is, as usual, that the "growth engine" of the economy features constant returns to scale w.r.t. producible inputs. Generally, as defined in Chapter 13.5, the *growth engine* of a model is the set of input-producing sectors using their own output as an input. After having derived the aggregate production function in sector 1 as expressed in (15.13), sector 2 can be considered integrated in sector 1. On this basis, sector 1 and sector 3 constitute the growth engine in the model. Basic goods, $Y = X + C + Z$, and technical knowledge, represented by the number, N, of varieties of intermediate goods, are the two kinds of producible inputs. Sector 1 delivers the input flow X to itself and the input flow Z to sector 3. And sector 3 delivers the input flow N to sector 1. The production functions (15.13) (with $N^{(c)}/N = s^*$) and (15.5) show that in steady state there are constant returns to scale w.r.t. these two producible inputs.

The long-run per capita growth rate, g_c^*, depends on those parameters that are common with Model I in qualitatively the same way as in that model, see last section in Chapter 14. It is noteworthy that the long-run per capita growth rate is smaller than in Model I with perpetual monopolies, cf. (15.24). In turn, the latter growth rate, which we named g^*, is smaller than the social planner's growth rate, g_{SP}, cf. Exercise VII.4. That is

$$g_c^* < g^* < g_{SP}.$$

The reason that our present g_c^* ends up not only lower than the social planner's growth rate, but also lower than in a corresponding economy with perpetual monopolies, is that the erosion of monopoly power implies less protection of private ownership of the inventions. This reduces the private profitability of R&D and thereby the incentive to do R&D. Indeed, (15.23) indicates that a larger λ, i.e., a smaller expected duration, $1/\lambda$, of the status as a monopolist, implies a lower per capita growth rate, g_c^*. So, whereas erosion of monopoly power leads to a *static efficiency gain* compared with perpetual monopoly as described in Section 15.3, it aggravates the underinvestment in R&D and thereby the *dynamic distortion* in the system. In this way long-run growth is reduced even *more*, relative to the social optimum, than in the case of perpetual monopolies.

15.8 Economic policy

At the theoretical level the analysis expose the presence of static and dynamic distortions. At the empirical level for instance Jones and Williams (1998) find that R&D investment in the U.S. economy is only about a fourth of the social optimum. So government intervention seems motivated.

While in Model I, solving the static efficiency problem automatically implies solving also the dynamic efficiency problem, this is not so in the present model.

Two policy instruments are needed. To counteract the monopolist price distortion and encourage demand for monopolized intermediate goods, a subsidy to purchases of monopolized intermediate goods will work. This will also, indirectly, encourage R&D but, because of the imperfect protection of innovations, not to the extent needed for the first-best solution. A direct stimulus, a subsidy, to R&D investment, is called for. Taxation on consumption and labor income may provide the financing.

By comparing with the social planner's allocation, it is possible to find exact formulas for the subsidy rates and provide non-distortionary financing such that the social planner's allocation is replicated in a decentralized way.

Dilemmas in the design of patent systems

There are many dilemmas regarding how to design patent systems. Model II above illustrates one of them, namely the question what the period length of patents should be. The inverse of λ can be interpreted as a measure of the average duration of patents. A larger λ (shorter duration) reduces static inefficiency in the economy but it also aggravates the underinvestment in R&D and thereby increases the dynamic inefficiency in the economy. We could more generally interpret λ as reflecting strictness of antitrust policy and the conclusion would be similar.

Going outside the present specific model, there a many further aspects to take into account which we shall not attempt to do here. A survey is contained in Hall and Harhoff (2012). We end this chapter by a citation from Wikipedia (30-04-2014):

> Legal scholars, economists, scientists, engineers, activists, policymakers, industries, and trade organizations have held differing views on patents and engaged in contentious debates on the subject. Recent criticisms primarily from the scientific community focus on the core tenant of the intended utility of patents, as now some argue they are retarding innovation. Critical perspectives emerged in the nineteenth century, and recent debates have discussed the merits and faults of software patents, nanotechnology patents and biological patents. These debates are part of a larger discourse on intellectual property protection which also reflects differing perspectives on copyright.

15.9 Appendix

A. Taking the time derivative of V_t in (15.16)

We shall apply *Leibniz's' formula*[7] which says:

$$\begin{aligned} F(t) &= \int_{a(t)}^{b(t)} f(\tau,t)d\tau => \\ F'(t) &= f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(\tau,t)}{\partial t} d\tau. \end{aligned}$$

In the present case we have from (15.16), $V_t = \pi^{(m)}F(t)$, where

$$F(t) = \int_t^\infty e^{-\int_t^\tau (r_s+\lambda)ds} d\tau,$$

whereby $b(t) = \infty$ and $a(t) = t$, so that $b'(t) = 0$ and $a'(t) = 1$. We get $\dot{V}_t = \pi^{(m)}F'(t)$, that is,

$$\begin{aligned} \frac{\dot{V}_t}{\pi^{(m)}} &= F'(t) = 0 - e^{-\int_t^t (r_s+\lambda)ds} + \int_t^\infty e^{-\int_t^\tau (r_s+\lambda)ds}(r_t+\lambda)d\tau \\ &= -1 + (r_t+\lambda)F(t) = -1 + (r_t+\lambda)\frac{V_t}{\pi^{(m)}}. \end{aligned}$$

[7] For details, see any Math textbook, e.g., Sydsæter vol. II.

Reordering gives

$$\frac{\pi^{(m)} + \dot{V}_t}{V_t} = r_t + \lambda,$$

which is the no-arbitrage condition (15.17).

B. Stability analysis

The Jacobian matrix, evaluated in the steady state, is

$$\begin{aligned} J^* &= \begin{bmatrix} \partial\dot{s}/\partial s & \partial\dot{s}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial s & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix}_{|(s,\tilde{c})=(s^*,\tilde{c}^*)} \\ &= \begin{bmatrix} -(\lambda + g_N + \eta B_2 s^*) & \eta L u^* \\ -\eta B_2 \tilde{c}^* & \eta L \tilde{c}^* \end{bmatrix}. \end{aligned}$$

The determinant of this matrix is

$$\det J^* = -(\lambda + g_N + \eta B_2 s^*)\eta L\tilde{c}^* + \eta L u^* \eta B_2 \tilde{c}^* = -(\lambda + g_N)\eta L\tilde{c}^* < 0.$$

Hence, the eigenvalues are of opposite sign and the steady state is a saddle point. A possible configuration of the phase diagram is sketched in Fig. 15.1. In the steady state the TVC of the households is satisfied in that

$$\begin{aligned} a_t e^{-r^*t} &= \frac{N_t^{(m)} V}{L} e^{-r^*t} = \frac{N_t^{(m)}}{L\eta} e^{-r^*t} = \frac{N_t - N_t^{(c)}}{L\eta} e^{-r^*t} \\ &= \frac{(1-s_t)N_t}{L\eta} e^{-r^*t} = \frac{(1-s^*)N_0 e^{g_c^* t}}{L\eta} e^{-r^*t} \to 0 \text{ for } t \to \infty, \end{aligned}$$

since $r^* \equiv r^{(m)} - \lambda$ and (A2) combined with (15.24) implies $r^* > g_c^*$. The TVC is therefore also satisfied along the unique converging path.

15.10 References

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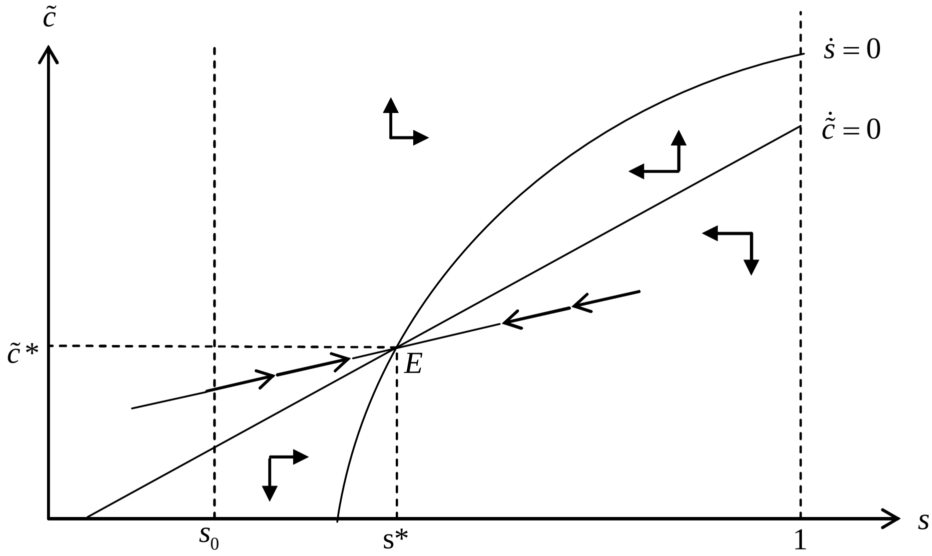

Figure 15.1: Phase diagram.

Chapter 16

Natural resources and economic growth

In this course, up to now, the relationship between economic growth and the earth's finite natural resources has been touched upon in connection with: the discussion of returns to scale (Chapter 2), the transition from a pre-industrial to an industrial economy (in Chapter 7), and the environmental problem of global warming (Chapter 8). In a more systematic way the present chapter reviews how natural resources, including the environment, relate to economic growth.

The contents are:

- Classification of means of production.
- The notion of sustainable development.
- Renewable natural resources.
- Non-renewable natural resources.
- Natural resources and the issue of limits to economic growth.

The first two sections aim at establishing a common terminology for the discussion.

16.1 Classification of means of production

We distinguish between different categories of production factors, also called means of production. First the two broad categories:

1. *Producible* means of production, also called man-made inputs.

2. *Non-producible* means of production.

The first category includes:

1.1 *Physical inputs* like processed raw materials, intermediate goods, machines, and buildings.

1.2 *Human inputs* of a produced character like technical knowledge (available in books, USB sticks etc.) and human capital.

The second category includes:

2.1 Human inputs of a non-produced character, sometimes called "raw labor".[1]

2.2 Natural resources. By definition in limited supply on this earth.

Natural resources can be sub-divided into:

2.2.1 *Renewable resources*, that is, natural resources the stock of which can be replenished by a natural self-regeneration process. Hence, if the resource is not over-exploited, it can be sustained in a more or less constant amount. Examples: ground water, fertile soil, fish in the sea, clean air, national parks.

2.2.2 *Non-renewable resources*, that is, natural resources which have no natural regeneration process (at least not within a relevant time scale). The stock of a non-renewable resource is thus depletable. Examples: fossil fuels, many non-energy minerals, virgin wilderness, endangered species, ozone layer.

The climate change problem due to greenhouse gasses can be seen as belonging to somewhere between category 2.2.1 or 2.2.2 in that the atmosphere *has* a natural self-regeneration ability, but the speed of regeneration is very low.

Given the scarcity of natural resources and the pollution problems caused by economic activity, key issues are:

a. Is sustainable development possible?

b. Is sustainable economic growth (in a per capita welfare sense) possible?

But what does "sustainable" and "sustainability" really mean"?

[1] Outside a slave society, biological reproduction is usually not considered as part of the economic sphere of society even though formation and maintainance of raw labor requires child rearing, health, food etc. and is thus conditioned by economic circumstances.

16.2 The notion of sustainable development

The basic idea in the notion of sustainable development is to emphasize intergenerational responsibility. The Brundtland Commission (1987) defined sustainable development as "development that meets the needs of present generations without compromising the ability of future generations to meet theirs".

In more standard economic terms we may define *sustainable economic development* as a time path along which per capita welfare remains non-decreasing across generations forever. An aspect of this is that current economic activities should not impose significant economic risks on future generations. The "forever" in the definition can not, of course, be taken literally, but as equivalent to "for a very long time horizon". We know that the sun will eventually (in some billion years) burn out and consequently life on earth will become extinct.

Note also that our definition emphasizes *welfare,* which should be understood in a broad sense, that is, as more or less synonymous with "quality of life", "living conditions", or "well-being" (the term used in Smulders, 1995). What may matter is thus not only the per capita amount of marketable consumption goods, but also things like health, life expectancy, enjoyment of services from the ecological system, and capability to lead a worthwhile live.

To make this more specific, consider the period utility function of a typical individual. Suppose two variables enter as arguments, namely consumption, c, of a marketable produced good and some measure, q, of the quality of services from the eco-system. Suppose further that the period utility function is of CES form:[2]

$$u(c,q) = \left[\alpha c^{\beta} + (1-\alpha)q^{\beta}\right]^{1/\beta}, \qquad 0 < \alpha < 1, \beta < 1. \tag{16.1}$$

The parameter β is called the *substitution parameter.* The elasticity of substitution between the two goods is $\sigma = 1/(1-\beta) > 0$, a constant. When $\beta \to 1$ (from below), the two goods become perfect substitutes (in that $\sigma \to \infty$). The smaller is β, the less substitutable are the two goods. When $\beta < 0$, we have $\sigma < 1$, and as $\beta \to -\infty$, the indifference curves become near to right angled.[3] According to many environmental economists, there are good rea-

[2] CES stands for Constant Elasticity of Substitution.

[3] By L'Hôpital's rule for "0/0" it follows that, for fixed c and q,

$$\lim_{\beta \to 0, \beta \neq 0} \left[\alpha c^{\beta} + (1-\alpha)q^{\beta}\right]^{1/\beta} = c^{\alpha} q^{1-\alpha}.$$

So the Cobb-Douglas utility function, which has elasticity of substitution between the goods equal to 1, is an intermediate case, corresponding to $\beta = 0$. More details in the sons to believe that $\sigma < 1$, since water, basic foodstuff, clean air, absence of catastrophic climate change, etc. are difficult to replace by produced goods and services. In this case there is a limit to the extent at which a rising c, along with a rising per capita income, can compensate for falling q.

At the same time the techniques by which the ordinary consumption good is currently produced may be "dirty" and thereby *cause* a falling q. An obvious policy response is the introduction of pollution taxes that increase the incentive for firms to replace these techniques with cleaner ones. For certain forms of pollution (e.g., sulfur dioxide, SO_2, in the air) there is evidence of an inverted U-curve relationship between the degree of pollution and the level of economic development measured by GDP per capita – the *environmental Kuznets curve*.

So an important element in *sustainable* economic development is that the economic activity of current generations does not spoil the environmental conditions for future generations. Living up to this requirement necessitates economic and environmental strategies consistent with the planet's endowments. This means recognizing the role of environmental constraints for economic development. A complicating factor is that specific abatement policies vis-a-vis particular environmental problems may face resistance from interest groups.

As defined, a criterion for sustainable economic *development* to be present is that per capita welfare remains *non-decreasing* across generations. A subcategory of this is *sustainable* economic *growth* which is present if per capita welfare is *growing* across generations. Here we speak of growth in a *welfare* sense, not in a physical sense. Permanent exponential per capita output growth in a *physical* sense is of course not possible with limited natural resources (matter or energy). The issue about sustainable *growth* is whether, by combining the natural resources with man-made inputs (knowledge, human capital, and physical capital), an output stream of *increasing quality*, and therefore increasing *economic value,* can be maintained. In modern times capabilities of many digital electronic devices provide conspicuous examples of exponential growth in *quality* (or *efficiency*). Think of processing speed, memory capacity, and efficiency of electronic sensors. What is known as *Moore's Law* is the rule of thumb that there is a doubling of the efficiency of microprocessors within every 18 months. The evolution of the internet has provided faster and widened dissemination of information and fine arts.

Of course there are intrinsic difficulties associated with measuring sustainability in terms of well-being. There now exists a large theoretical and applied literature dealing with these issues. A variety of extensions and

appendix.

modifications of the standard national income accounting GDP has been developed under the heading *Green NNP* (green net national product). An essential feature in the measurement of Green NNP is that from the conventional GDP (which essentially just measures the level of economic activity) is subtracted the depreciation of not only the physical capital but also the environmental assets. The latter depreciate due to pollution, overburdening of renewable natural resources, and depletion of reserves of non-renewable natural resources.[4] In some approaches the focus is on whether a comprehensive measure of *wealth* is maintained over time. Along with reproducible assets and natural assets (including the damage to the atmosphere from greenhouse gasses), Arrow et al. (2012) take in health, human capital, and "knowledge capital" in their measure of "wealth". They apply this measure in a study of the United States, China, Brazil, India, and Venezuela over the period 1995-2000 and find that all five countries satisfy the sustainability criterion of non-decreasing wealth in this broad sense. Indeed their wealth measure is found to be growing in all five countries.[5] Note that it is sustainability that is claimed, not optimality.

In the next two sections we will go more into detail with the challenge to sustainability and growth coming from renewable and non-renewable resources, respectively. We shall primarily deal with the issues from the point of view of technical feasibility of non-decreasing, and possibly rising, per-capita consumption. Concerning the big questions about appropriate institutions the reader is referred to the specialized literature.

We begin with renewable resources.

16.3 Renewable resources

A useful analytical tool is the following simple model of the stock dynamics associated with a renewable resource.

Let $S_t \geq 0$ denote the *stock* of the renewable resource at time t. Then we may write

$$\dot{S}_t \equiv \frac{dS_t}{dt} = G_t - R_t = G(S_t) - R_t, \qquad (16.2)$$

[4]The depreciation of these environmental and natural assets is evaluated in terms of the social planner's shadow prices. See, e.g., Heal (1998), Weitzman (2001, 2003), and Stiglitz et al. (2010).

[5]Of course, many measurement uncertainties and disputable issues of weighting are involved; brief discussions, and questioning, of the study are contained in Solow (2012), Hamilton (2012), and Smulders (2012). Regarding Denmark 1990-2009, a study by Lind and Schou (2013), along lines somewhat similar to those of Arrow et al. (2012), also suggests sustainability to hold.

where G_t is the self-regeneration of the resource per time unit and $R_t \geq 0$ is the extraction (and use) of the resource per time unit at time t. If for instance the stock refers to the number of fish in the sea, the flow R_t represents the number of fish caught per time unit. And if, in a pollution context, the stock refers to "cleanness" of the air in cities, R_t measures, say, the emission of sulfur dioxide, SO_2, per time unit. The self-regenerated amount per time unit depends on the available stock through the function $G(S_t)$, known as a *self-regeneration function*.[6]

Until further notice, we stick to the first interpretation, that of S indicating the size of a fish population. The self-regeneration function will often have a bell-shape as illustrated in the upper panel of Figure 16.1. Essentially, the self-regeneration ability is due to the flow of solar energy continuously entering the the eco-system of the earth. This flow of solar energy is constant and beyond human control.

There is a lower threshold, $\underline{S}(0) \geq 0$, below which even with $R = 0$ there are too few female fish to generate offspring, and the population necessarily shrinks and eventually reaches zero. We may call $\underline{S}(0)$ the minimum sustainable stock.

At the other intersection with the horizontal axis, $\bar{S}(0)$ represents the maximum sustainable stock. The eco-system cannot support further growth in the fish population. The reason may be food scarcity or spreading of diseases because of high population density.[7] The value MSY, indicated on the vertical axis, in the upper panel equals $= \max_S g(S)$. This value is thus the *maximum sustainable yield* per time unit. It is sustainable, presupposing the size of the fish population is initially at least of size $S_{MSY} = \arg\max_S g(S)$ which is that value of S where $G(S) = MSY$. The size, S_{MSY}, of the fish population is consistent with maintaining the harvest MSY per time unit forever in a steady state.

The lower panel in Figure 16.1 illustrates the dynamics in the $(S, \dot{S})$ plane, given a fixed level of $R = \bar{R} \in (0, MSY]$. The arrows indicate the direction of movement. In the long run, if $R = \bar{R}$ for all t, the stock will settle down at the size $\bar{S}(\bar{R})$. The stippled curve in the upper panel indicates $G(S) - \bar{R}$, which is the same as $\dot{S}$ in the lower panel which presumes $R = \bar{R}$. The stippled curve in the lower panel indicates the dynamics in case $R = MSY$. In this case the steady-state stock, $\bar{S}(MSY) = S_{MSY}$, is unstable. In this

[6]Even if S represents the stock of a *non-renewable* resource, the equation (16.2) will still be valid if we impose that there is no self-regeneration, i.e., $G(S) \equiv 0$.

[7]Popular mathematical specifications of $G(\cdot)$ include the logistic function $G(S) = \alpha S(1-S/\beta)$, where $\alpha > 0$, $\beta > 0$, and the quasi-logistic function $G(S) = \alpha S(1-S/\beta)(S/\gamma-1)$, where also $\gamma > 0$. In both cases $\bar{S}(0) = \beta$, but $\underline{S}(0)$ equals 0 in the first case and γ in the second.

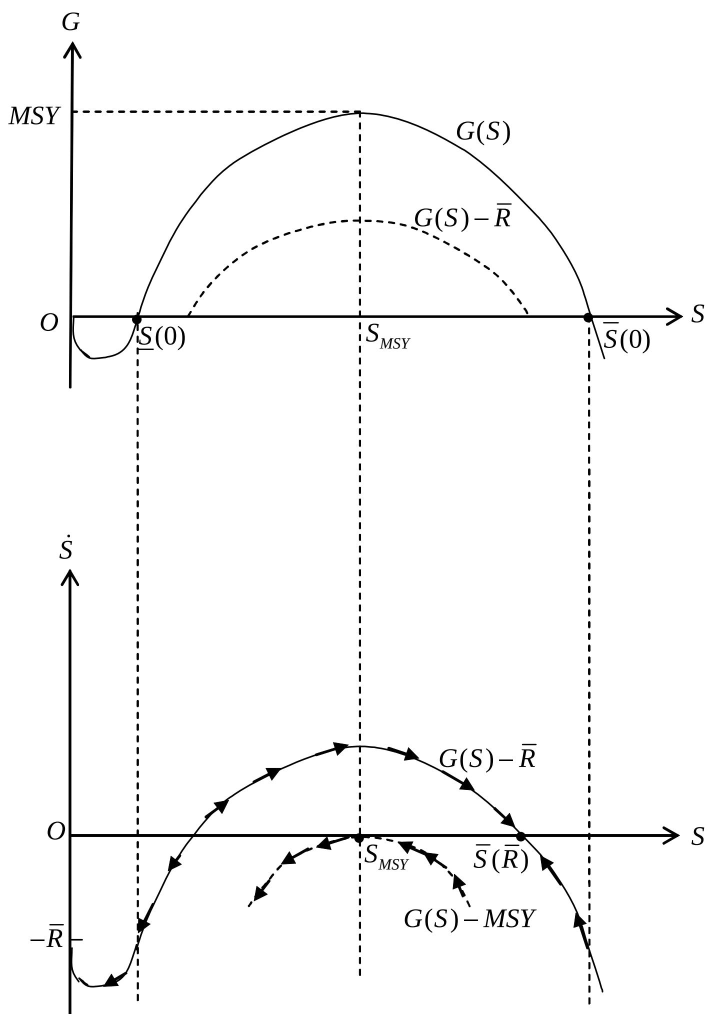

Figure 16.1: The self-generation function (upper panel) and stock dynamics for $R = \bar{R} \in (0, MSY]$ (lower panel).

state a small negative shock to the stock will not lead to a gradual return but to a self-reinforcing reduction of the stock as long as the extraction $R = MSY$ is maintained.

MSY is an ecological maximum and not necessarily in any sense an economic optimum. Indeed, since the search and extraction costs may be a decreasing function of the fish density, hence of the stock, it may be worthwhile to increase the stock beyond S_{MSY}, thus settling for a smaller harvest per time unit. Moreover, a microeconomic calculation will maximize the sum of discounted expected profits per time unit, taking into account the expected evolution of the market price of fish, the cost function, and the dynamic relationship (16.2).

In addition to its importance for regeneration, the stock, S, may have amenity value and thus enter the instantaneous utility function. Then again some conservation of the stock over and above S_{MSY} will be motivated.

A dynamic model with a renewable resource Consider a simple model consisting of (16.2) together with

$$\begin{aligned} Y_t &= F(K_t, L_t, R_t, t), \qquad \partial F/\partial t \geq 0, \\ \dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ given}, \\ L_t &= L_0 e^{nt}, \qquad n \geq 0, \end{aligned} \tag{16.3}$$

where Y_t is aggregate output and K_t, L_t, and R_t are inputs of capital, labor, and a renewable resource, respectively, per time unit at time t. Let the aggregate production function, F, be neoclassical[8] with constant returns to scale w.r.t. the rival inputs K, L, and R. The assumption $\partial F/\partial t \geq 0$ represents exogenous technical progress. Further, C_t is aggregate consumption ($\equiv c_t L_t$, where c_t is per capita consumption) and δ denotes a constant rate of capital depreciation. There is no distinction between employment and population, L_t. The population growth rate, n, is assumed constant.

Is sustainable economic development in this setting technically feasible? The answer will be yes if non-decreasing per capita consumption can be sustained forever. As the issue is about technical feasibility, we disregard problems of "tragedy of the commons". Or rather, we assume this problem is avoided by appropriate institutions.

Suppose the use of the renewable resource is kept constant at a sustainable level $\bar{R} \in (0, MSY)$. To begin with, suppose $n = 0$ so that $L_t = L$ for all $t \geq 0$. Assume that at $R = \bar{R}$, the system is "productive" in the sense that

$$\lim_{K \to 0} F_K(K, L, \bar{R}, 0) > \delta > \lim_{K \to \infty} F_K(K, L, \bar{R}, 0). \tag{A1}$$

[8] That is, marginal productivities of the production factors are positive, but diminishing; and the upper contour sets are strictly convex.

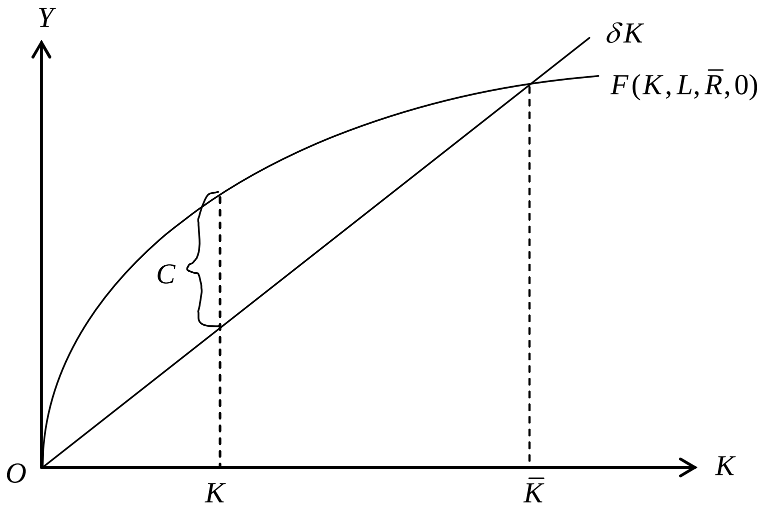

Figure 16.2: Sustainable consumption in the case of $n = 0$ and no technical progress (L and $\bar{R}$ fixed).

This condition is satisfied in Figure 16.2 where the value $\bar{K}$ has the property $F(\bar{K}, L, \bar{R}, 0) = \bar{K}$. Given the circumstances, this value is the least upper bound for a sustainable capital stock in the sense that

$$\begin{aligned} \text{if } K &\geq \bar{K}, \text{ we have } \dot{K} < 0 \text{ for any } C > 0, \text{ while} \\ \text{if } 0 &< K < \bar{K}, \text{ we have } \dot{K} = 0 \text{ for } C = F(K, L, \bar{R}, 0) - \delta K > 0. \end{aligned}$$

For such a C, illustrated in Figure 16.2, a constant $Y = F(K, L, \bar{R}, 0)$ is maintained forever which implies non-decreasing per-capita income, $y \equiv Y/L$, forever. So, in spite of the limited availability of the natural resource, a *non-decreasing* level of consumption is technically feasible even without technical progress. A forever *growing* level of consumption will, of course, require sufficient technical progress capable of substituting for the natural resource.

Now consider the case $n > 0$. Along a balanced growth path (if it exists) we have

$$1 = F(\frac{K_t}{Y_t}, \frac{L_t}{Y_t}, \frac{\bar{R}}{Y_t}, t), \tag{16.4}$$

where K_t/Y_t and C_t/Y_t must be constant, cf. Chapter 4. Maintaining C_t/L_t $(= (C_t/Y_t)/(L_t/Y_t))$ constant along this path when $n > 0$, requires that L_t/Y_t is constant and thereby that Y_t grows at the rate n. But then $\bar{R}/Y_t$ will be declining over time. To compensate for this in (16.4), sufficient technical progress is necessary. This of course holds, a fortiori, for sustained *growth* in per-capita consumption to occur.

As technical progress in the far future is by its very nature uncertain and unpredictable, there can be no guarantee for sustained per capita growth.

Pollution As hinted at above, the concern that certain production methods involve pollution is commonly incorporated into economic analysis by subsuming environmental quality into the general notion of renewable resources. In that context S in (16.2) and Figure 16.1 will represent the "level of environmental quality" and R_t will be the amount of dirty emissions per time unit. Since the level of the environmental quality is likely to be an argument in both the utility function and the production function, again some limitation of the "extraction" (the pollution flow) is motivated. Pollution taxes may help to encourage abatement activities and make technical innovations towards cleaner production methods more profitable.

16.4 Non-renewable resources

Whereas extraction and use of renewable resources can be sustained at a more or less constant level (if not too high), the situation is different with non-renewable resources. They have no natural regeneration process (at least not within a relevant time scale) and so continued extraction per time unit of these resources will inevitably have to decline and approach zero in the long run.

To get an idea of the implications, we will consider the Dasgupta-Heal-Solow-Stiglitz model (DHSS model) from the 1970s.[9]

16.4.1 The DHSS model

The production side of the model is described by:

$$\begin{aligned}
Y_t &= F(K_t, L_t, R_t, t), \qquad \partial F/\partial t \geq 0, && (16.5)\\
\dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \quad K_0 > 0 \text{ given}, && (16.6)\\
\dot{S}_t &= -R_t \equiv -u_t S_t, \qquad S_0 > 0 \text{ given}, && (16.7)\\
L_t &= L_0 e^{nt}, \qquad n \geq 0. && (16.8)
\end{aligned}$$

The new element is the replacement of (16.2) with (16.7), where S_t is the *stock of the non-renewable* resource (e.g., oil reserves), and u_t is the depletion

[9] See, e.g., Stiglitz, 1974.

rate. Since we must have $S_t \geq 0$ for all t, there is a finite upper bound on cumulative resource extraction:

$$\int_0^\infty R_t dt \leq S_0. \qquad (16.9)$$

Since the resource is non-renewable, no re-generation function appears in (16.7). Uncertainty is ignored and the extraction activity involve no costs.[10] As before, there is no distinction between employment and population, L_t.

The model was formulated as a response to the pessimistic Malthusian views of the Club of Rome (Meadows et al., 1972). Stiglitz (and fellow economists) asked the question: what are the technological conditions needed to avoid falling per capita consumption in the long run in spite of the inevitable decline in resource use? The answer is that there are three ways in which this decline in resource use may be counterbalanced:

1. input substitution;

2. resource-saving technical progress;

3. increasing returns to scale.

Let us consider each of them in turn (although in practice the three mechanisms tend to be intertwined).

Input substitution

By input substitution is meant the gradual replacement of the input of the exhaustible natural resource by man-made input, capital. Substitution of fossil fuel energy by solar, wind, tidal and wave energy resources is an example. Similarly, more abundant lower-grade non-renewable resources can substitute for scarce higher-grade non-renewable resources - and this *will* happen when the scarcity price of these has become sufficiently high. A rise in the price of a mineral makes a synthetic substitute cost-efficient or lead to increased recycling of the mineral. Finally, the composition of final output can change toward goods with less material content. Overall, capital accumulation can be seen as the key background factor for such substitution processes (though also the arrival of new technical knowledge may be involved - we come back to this).

[10]This simplified description of resource extraction is the reason that it is common to classify the model as a *one-sector* model, notwithstanding there are two productive activities in the economy, manufacturing and resource extraction.

Whether capital accumulation can do the job depends crucially on the degree of substitutability between K and R. To see this, let the production function F be a three-factor CES production function. Suppressing the explicit dating of the variables when not needed for clarity, we have.

$$Y = \left(\alpha_1 K^\beta + \alpha_2 L^\beta + \alpha_3 R^\beta\right)^{1/\beta},\ \alpha_1, \alpha_2, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 1, \beta < 1, \beta \neq 0. \tag{16.10}$$

The important parameter is β, the *substitution parameter*. Let p_R denote the cost to the firm per unit of the resource flow and let $\hat{r}$ be the cost per unit of capital (generally, $\hat{r} = r + \delta$, where r is the real rate of interest). Then $p_R/\hat{r}$ is the relative factor price, which may be expected to increase as the resource becomes more scarce. The *elasticity of substitution* between K and R is $[d(K/R)/d(p_R/\hat{r})]\,(p_R/\hat{r}\ )/(K/R)$ evaluated along an isoquant curve, i.e., the percentage rise in the K-R ratio that a cost-minimizing firm will choose in response to a one-percent rise in the relative factor price, $p_R/\hat{r}$. Since we consider a CES production function, this elasticity is a constant $\sigma = 1/(1-\beta) > 0$. Indeed, the three-factor CES production function has the property that the elasticity of substitution between any pair of the three production factors is the same.

First, suppose $\sigma > 1$, i.e., $0 < \beta < 1$. Then, for fixed K and L, $Y \to \left(\alpha_1 K^\beta + \alpha_2 L^\beta\right)^{1/\beta} > 0$ when $R \to 0$. In this case of high substitutability the resource is seen to be *inessential* in the sense that it is not necessary for a positive output. That is, from a production perspective, conservation of the resource is not vital.

Suppose instead $\sigma < 1$, i.e., $\beta < 0$. Although increasing when R decreases, output per unit of the resource flow is then bounded from above. Consequently, the finiteness of the resource inevitably implies doomsday sooner or later if input substitution is the only salvage mechanism. To see this, keeping K and L fixed, we get

$$\frac{Y}{R} = Y(R^{-\beta})^{1/\beta} = \left[\alpha_1(\frac{K}{R})^\beta + \alpha_2(\frac{L}{R})^\beta + \alpha_3\right]^{1/\beta} \to \alpha_3{}^{1/\beta} \text{ for } R \to 0, \tag{16.11}$$

since $\beta < 0$. Even if K and L are increasing, $\lim_{R\to 0} Y = \lim_{R\to 0}(Y/R)R = \alpha_3^{1/\beta} \cdot 0 = 0$. Thus, when substitutability is low, the resource is *essential* in the sense that output is nil in the absence of the resource.

What about the intermediate case $\sigma = 1$? Although (16.10) is not defined for $\beta = 0$, using L'Hôpital's rule (as for the two-factor function, cf. Appendix), it can be shown that $\left(\alpha_1 K^\beta + \alpha_2 L^\beta + \alpha_3 R^\beta\right)^{1/\beta} \to K^{\alpha_1} L^{\alpha_2} R^{\alpha_3}$ for $\beta \to 0$. In the limit a three-factor Cobb-Douglas function thus appears. This function has $\sigma = 1$, corresponding to $\beta = 0$ in the formula $\sigma = 1/(1-\beta)$.

The interesting aspect of the Cobb-Douglas case is that it is the only case where the resource is essential while at the same time output per unit of the resource is unbounded from above (since $Y/R = K^{\alpha_1}L^{\alpha_2}R^{\alpha_3-1} \to \infty$ for $R \to 0$).[11] Under these circumstances it was considered an open question whether non-decreasing per capita consumption could be sustained. Therefore the Cobb-Douglas case was studied intensively. For example, Solow (1974) showed that if $n = \delta = 0$, then a necessary and sufficient condition that a constant positive level of consumption can be sustained is that $\alpha_1 > \alpha_3$. This condition in itself seems fairly realistic, since, empirically, α_1 is many times the size of α_3 (Nordhaus and Tobin, 1972, Neumayer 2000). Solow added the observation that under competitive conditions, the *highest* sustainable level of consumption is obtained when investment in capital exactly equals the resource rent, $R \cdot \partial Y/\partial R$. This result was generalized in Hartwick (1977) and became known as *Hartwick's rule*. If there is population growth ($n > 0$), however, not even the Cobb-Douglas case allows sustainable per capita consumption unless there is sufficient technical progress, as equation (16.15) below will tell us.

Neumayer (2000) reports that the empirical evidence on the elasticity of substitution between capital and energy is inconclusive. Ecological economists tend to claim the poor substitution case to be much more realistic than the optimistic Cobb-Douglas case, not to speak of the case $\sigma > 1$. This invites considering the role of technical progress.

Technical progress

Solow (1974) and Stiglitz (1974) analyzed the theoretical possibility that resource-saving technological change can overcome the declining use of non-renewable resources that must be expected in the future. In this context the focus is not only on whether a non-decreasing consumption level can be maintained, but also on the possibility of sustained per capita *growth* in consumption.

New production techniques may raise the efficiency of resource use. For example, Dasgupta (1993) reports that during the period 1900 to the 1960s, the quantity of coal required to generate a kilowatt-hour of electricity fell from nearly seven pounds to less than one pound.[12] Further, technological developments make extraction of lower quality ores cost-effective and make more durable forms of energy economical. On this background we incorporate resource-saving technical progress at the rate γ_3 along with labor-saving

[11] To avoid misunderstanding: by "Cobb-Douglas case" we refer to any function where R enters in a "Cobb-Douglas fashion", i.e., any function like $Y = \tilde{F}(K, L)^{1-\alpha_3}R^{\alpha_3}$.

[12] For a historical account of energy technology, see Smil (1994).

technical progress at the rate γ_2. So the CES production function reads

$$Y = \left(\alpha_1 K^\beta + \alpha_2 (A_2 L)^\beta + \alpha_3 (A_3 R)^\beta\right)^{1/\beta}, \tag{16.12}$$

where $A_2 = e^{\gamma_2 t}$ and $A_3 = e^{\gamma_3 t}$, assuming γ_2 and γ_3 to be exogenous positive constants. If the (proportionate) rate of decline of R is kept smaller than γ_3, then the "effective" resource input is no longer decreasing over time. As a consequence, even if $\sigma < 1$ (the poor substitution case), the finiteness of nature need not be an insurmountable obstacle to non-decreasing per capita consumption.

Actually, a technology with $\sigma < 1$ *needs* a considerable amount of resource-saving technical progress to obtain compliance with the empirical fact that the income share of natural resources has not been rising (Jones, 2002). When $\sigma < 1$, market forces tend to increase the income share of the factor that is becoming relatively more scarce. Empirically, K/R and Y/R have increased systematically. However, with a sufficiently increasing A_3, the income share $p_R R/Y$ need not increase in spite of $\sigma < 1$. Similarly, for the model to comply with Kaldor's "stylized facts" (more or less constant growth rates of K/L and Y/L and stationarity of the output-capital ratio, the income share of labor, and the rate of return on capital), we need labor-saving technical change (A_2 growing over time).[13] The motivation for not introducing a rising A_1 and replacing K in (16.12) by $A_1 K$, is that this would be at odds with Kaldor's "stylized facts", in particular the absence of a trend in the rate of return to capital.

With $\gamma_3 > \gamma_2 + n$, we end up with conditions allowing a *balanced growth path* (BGP for short), defined as a path along which the quantities Y, C, and K change at constant proportionate rates (some or all of which may be negative). It is well-known that compliance with Kaldor's "stylized facts" is close to equivalent to existence of a balanced growth path. It can be shown that along the BGP, $Y/(A_2 L)$ is constant and so $g_y = \gamma_2$ (hence also $g_c = \gamma_2$).[14] Of course, one thing is that such a combination of assumptions allows for constant growth in per capita consumption - which is more or less what we have seen since the industrial revolution. Another thing is: will the needed assumptions be satisfied for a long time in the future? Since we have considered *exogenous* technical change, there is so far no hint from theory. But, even taking endogenous technical change into account, there will be many uncertainties about what technological changes will come through in the future and how fast.

[13] Although the two forms of technical change are by many authors called "resource-augmenting" and "labor-augmenting", respectively, we prefer the more intuitive names, "resource-saving" and "labor-saving".

[14] For any positive variable z, g_z denotes the growth rate, $\dot{z}/z$.

Balanced growth in the Cobb-Douglas case Let us end this discussion by some remarks about the Cobb-Douglas case. By making capital-saving, labor-saving, and resource-saving technical progress indistinguishable, the Cobb-Douglas case again constitutes a convenient intermediate case. Technical progress can simply be represented by

$$Y = BK^{\alpha_1}L^{\alpha_2}R^{\alpha_3}, \qquad \alpha_1, \alpha_2, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 1, \tag{16.13}$$

where total factor productivity, B, is growing over time. This, together with (16.6) - (16.8), is now the model under examination.

Let us assume B grows at some constant rate $\tau > 0$. Log-differentiating w.r.t. time in (16.13) yields the growth-accounting relation

$$g_Y = \tau + \alpha_1 g_K + \alpha_2 n + \alpha_3 g_R. \tag{16.14}$$

By a simple extension of the method in Chapter 4, it is easily shown that along a BGP, $g_K = g_Y = g_C \equiv g_c + n$ and, if nothing of the resource is left unutilized forever, $g_R = g_S \equiv \dot{S}/S = -R/S =$ constant < 0.[15] With the depletion rate, R/S, denoted u, (16.14) thus implies

$$g_c = g_y = \frac{1}{1-\alpha_1}(\tau - \alpha_3 n - \alpha_3 u), \tag{16.15}$$

since $\alpha_1 + \alpha_2 - 1 = -\alpha_3$.

Absent the need for input of limited natural resources, we would have $\alpha_3 = 0$ and so $g_c = \tau/(1-\alpha_1)$. But with $\alpha_3 > 0$, the non-renewable resource is essential and implies a *drag on per capita growth* equal to $\alpha_3(n+u)/(1-\alpha_1)$. We get $g_c > 0$ if and only if $\tau > \alpha_3(n+u)$ (where, the depletion rate, u, can in principle be chosen very small if we want a strict conservation policy).

It is noteworthy that in spite of per-capita growth being due to exogenous technical progress, (16.15) shows that there is scope for policy affecting the long-run per-capita growth rate to the extent that policy can affect the rate of depletion u in the opposite direction.[16]

When speaking of "sustained growth" in K and c, it should not be understood in a narrow physical sense. As alluded to earlier, we have to understand K broadly as "produced means of production" of rising quality and falling material intensity; similarly, c must be seen as a composite of consumer "goods" with declining material intensity over time.[17] This accords with the empirical fact that as income rises, the share of consumption expenditures

[15] Otherwise, g_Y could not be constant.

[16] Cf. Section 13.5.1 of Chapter 13.

[17] See Fagnart and Germain (2011).

devoted to agricultural and industrial products declines and the share devoted to services, hobbies, and amusement increases. Although "economic development" is perhaps a more appropriate term (suggesting qualitative and structural change), we retain standard terminology and speak of "economic growth".

In any event, simple aggregate models like the present one should be seen as no more than a frame of reference, a tool for thought experiments. At best such models might have some validity as an approximate summary description of a certain period of time. One should be aware that an economy in which the ratio of capital to resource input grows without limit might well enter a phase where technological relations (including the elasticity of factor substitution) will be very different from now. For example, along *any* economic development path, the aggregate input of non-renewable resources must in the long run asymptotically approach zero. From a physical point of view, however, there must be some minimum amount of the resource below which it can not fulfil its role as a productive input. Thus, strictly speaking, sustainability requires that in the "very long run", non-renewable resources become inessential.

A backstop technology We end this sub-section by a remark on a rather different way of modeling resource-saving technical change. Dasgupta and Heal (1974) present a model of resource-saving technical change, considering it not as a smooth gradual process, but as something arriving in a discrete once-for-all manner with economy-wide consequences. The authors envision a future major discovery of, say, how to harness a lasting energy source such that a hitherto essential resource like fossil fuel becomes inessential. The contour of such a *backstop technology* might be currently known, but its practical applicability still awaits a technological breakthrough. The time until the arrival of this breakthrough is uncertain and may well be long. In Dasgupta, Heal and Majumdar (1977) and Dasgupta, Heal and Pand (1980) the idea is pursued further, by incorporating costly R&D. The likelihood of the technological breakthrough to appear in a given time interval depends positively on the accumulated R&D as well as the current R&D. It is shown that under certain conditions an index reflecting the probability that the resource becomes unimportant acts like an addition to the utility discount rate and that R&D expenditure begins to decline after some time. This is an interesting example of an early study of *endogenous* technological change.[18]

[18] A similar problem has been investigated by Kamien and Schwartz (1978) and Just et al. (2005), using somewhat different approaches.

Increasing returns to scale

The third circumstance that might help overcoming the finiteness of nature is increasing returns to scale. For the CES function with poor substitution ($\sigma < 1$), however, increasing returns to scale, though helping, are not by themselves sufficient to avoid doomsday. For details, see, e.g., Groth (2007).

Summary on the DHSS model

Apart from a few remarks by Stiglitz, the focus of the fathers of the DHSS model is on constant returns to scale; and, as in the simple Solow and Ramsey growth models, only *exogenous* technical progress is considered. For our purposes we may summarize the DHSS results in the following way. Non-renewable resources do not really matter seriously if the elasticity of substitution between them and man-made inputs is above one. If not, however, then:

(a) absent technical progress, if $\sigma = 1$, sustainable per capita consumption requires $\alpha_1 > \alpha_3$ and $n = 0 = \delta$; otherwise, declining per capita consumption is inevitable and this is definitely the prospect, if $\sigma < 1$;

(b) on the other hand, if there is enough resource-saving and labor-saving technical progress, non-decreasing per capita consumption and even growing per capita consumption may be sustained;

(c) population growth, implying more mouths to feed from limited natural resources, exacerbates the drag on growth implied by a declining resource input; indeed, as seen from (16.15), the drag on growth is $\alpha_3(n + u)/(1 - \alpha_1)$ along a BGP.

16.4.2 Endogenous technical progress

An obvious next step is to examine how *endogenizing* technical change may throw new light on the issues, in particular the visions (b) and (c). Without going into detail here, we may mention that because of the non-rival character of technical knowledge, endogenizing knowledge creation may have profound implications, in particular concerning point (c). Indeed, the relationship between population growth and economic growth may be circumvented when endogenous creation of ideas (implying a form of increasing returns to scale) is considered. In Groth (2007) a series of innovation-based endogenous growth models with non-renewable resources dealing with this is surveyed. The article also touches on aspects of environmental policy aiming at enhancing the prospects of sustainable development or even sustainable economic growth. Among other things, it is shown that the utilitarian principle of discounted utility maximization *may* clash with a requirement of sustainability.

16.5 Natural resources and the issue of limits to economic growth

Two distinguished professors were asked by a journalist: Are there limits to economic growth?
The answers received were:

Clearly YES:

- A finite planet!
- The amount of cement, oil, steel, and water that we can use is limited!

Clearly NO:

- Human creativity has no bounds!
- The quality of wine, TV transmission of concerts, computer games, and medical treatment knows no limits!

An aim of this chapter has been to bring to mind that it would be strange if there were no limits to growth. So a better question is:

What determines the limits to economic growth?

The answer suggested is that these limits are determined by the capability of the economic system to substitute limited natural resources by man-made goods the variety and quality of which are expanded by creation of new ideas. In this endeavour frontier countries, first the U.K. and Western Europe, next the United States, have succeeded at a high rate for two and a half century. To what extent this will continue in the future nobody knows. Some economists, e.g. Gordon (2012), argue there is an enduring tendency to slowing down of innovation and economic growth (the low-hanging fruits have been taken). Others, e.g. Brynjolfsson and McAfee (2014), disagree. They reason that the potentials of information technology and digital communication are on the verge of the point of ubiquity and flexible application. For these authors the prospect is "The Second Machine Age" (the title of their book), by which they mean a new innovative epoch where smart machines and new ideas are combined and recombined - with pervasive influence on society.

16.6 Appendix: The CES function

The CES (Constant Elasticity of Substitution) function is used in consumer theory as a specification of preferences and in production theory as a specification of a production function. Here we consider it as a production function.

It can be shown[19] that if a neoclassical production function with CRS has a constant elasticity of (factor) substitution different from one, it must be of the form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}}, \tag{16.16}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function.* For a given choice of measurement units, the parameter A reflects efficiency and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (16.16) is not defined when either K or L (or both) equal 0. We can circumvent this problem by extending the domain of the CES function and assign the function value 0 to these points when $\beta < 0$. Continuity is maintained in the extended domain.

By taking partial derivatives in (16.16) and substituting back we get

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta}\left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}\left(\frac{Y}{L}\right)^{1-\beta}, \tag{16.17}$$

where $Y/K = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and $Y/L = A\left[\alpha k^{\beta} + 1 - \alpha\right]^{\frac{1}{\beta}}$. The marginal rate of substitution of K for L therefore is

$$MRS = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha}k^{1-\beta} > 0.$$

Consequently,

$$\frac{dMRS}{dk} = \frac{1-\alpha}{\alpha}(1-\beta)k^{-\beta},$$

where the inverse of the right-hand side is the value of $dk/dMRS$. Substituting these expressions into the general definition of the *elasticity of substitution* between capital and labor, evaluated at the point (K, L),

$$\tilde{\sigma}(K,L) = \frac{MRS}{K/L}\frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} = \frac{\frac{d(K/L)}{K/L}}{\frac{dMRS}{MRS}}_{|Y=\bar{Y}}. \tag{16.18}$$

[19] See, e.g., Arrow et al. (1961).

gives

$$\tilde{\sigma}(K, L) = \frac{1}{1-\beta} \equiv \sigma, \tag{16.19}$$

confirming the constancy of the elasticity of substitution, given (16.17). Since $\beta < 1$, $\sigma > 0$ always. A higher substitution parameter, β, results in a higher elasticity of substitution, σ. And $\sigma \lesseqgtr 1$ for $\beta \lesseqgtr 0$, respectively.

Since $\beta = 0$ is not allowed in (16.16), at first sight we cannot get $\sigma = 1$ from this formula. Yet, $\sigma = 1$ can be introduced as the *limiting* case of (16.16) when $\beta \to 0$, which turns out to be the Cobb-Douglas function. Indeed, one can show[20] that, for fixed K and L,

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to AK^{\alpha}L^{1-\alpha}, \text{ for } \beta \to 0.$$

By a similar procedure as above we find that a Cobb-Douglas function always has elasticity of substitution equal to 1; this is exactly the value taken by σ in (16.19) when $\beta = 0$. In addition, the Cobb-Douglas function is the *only* production function that has unit elasticity of substitution everywhere.

Another interesting limiting case of the CES function appears when, for fixed K and L, we let $\beta \to -\infty$ so that $\sigma \to 0$. We get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\min(K, L), \text{ for } \beta \to -\infty. \tag{16.20}$$

So in this case the CES function approaches a Leontief production function, the isoquants of which form a right angle, cf. Figure 16.3. In the limit there is *no* possibility of substitution between capital and labor. In accordance with this the elasticity of substitution calculated from (16.19) approaches zero when β goes to $-\infty$.

Finally, let us consider the "opposite" transition. For fixed K and L we let the substitution parameter rise towards 1 and get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\left[\alpha K + (1-\alpha)L\right], \text{ for } \beta \to 1.$$

Here the elasticity of substitution calculated from (16.19) tends to ∞ and the isoquants tend to straight lines with slope $-(1-\alpha)/\alpha$. In the limit, the production function thus becomes linear and capital and labor become *perfect substitutes.*

Figure 16.3 depicts isoquants for alternative CES production functions and their limiting cases. In the Cobb-Douglas case, $\sigma = 1$, the horizontal and vertical asymptotes of the isoquant coincide with the coordinate axes.

[20] For proofs of this and the further claims below, see Appendix E of Chapter 4 in Groth (2013).

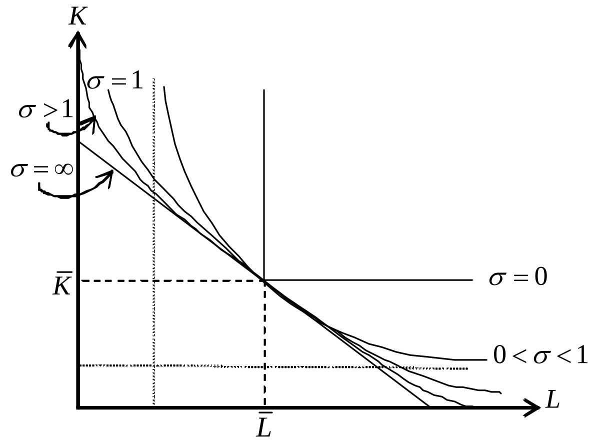

Figure 16.3: Isoquants for the CES production function for alternative values of $\sigma = 1/(1-\beta)$.

When $\sigma < 1$, the horizontal and vertical asymptotes of the isoquant belong to the interior of the positive quadrant. This implies that both capital and labor are essential inputs. When $\sigma > 1$, the isoquant terminates in points *on* the coordinate axes. Then neither capital, nor labor are essential inputs. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for industrialized economies. The elasticity also differs across the production sectors. A recent thorough econometric study (Antràs, 2004) of U.S. data indicate the aggregate elasticity of substitution to be in the interval $(0.5, 1.0)$.

The CES production function in intensive form

Dividing through by L on both sides of (16.16), we obtain the CES production function in intensive form,

$$y \equiv \frac{Y}{L} = A(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}, \tag{16.21}$$

where $k \equiv K/L$. The marginal productivity of capital can be written

$$MPK = \frac{dy}{dk} = \alpha A \left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}} = \alpha A^{\beta} \left(\frac{y}{k}\right)^{1-\beta},$$

which of course equals $\partial Y / \partial K$ in (16.17). We see that the CES function violates either the lower or the upper Inada condition for MPK, depending on the sign of β. Indeed, when $\beta < 0$ (i.e., $\sigma < 1$), then for $k \to 0$ both y/k

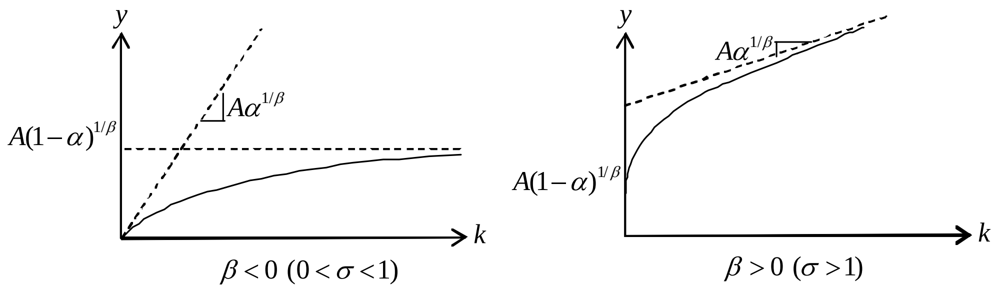

Figure 16.4: The CES production function for $\sigma < 1$ (left panel) and $\sigma > 1$ (right panel).

and dy/dk approach an upper bound equal to $A\alpha^{1/\beta} < \infty$, thus violating the lower Inada condition for MPK (see the right-hand panel of Figure 2.3 in Chapter 2). It is also noteworthy that in this case, for $k \to \infty$, y approaches an upper bound equal to $A(1-\alpha)^{1/\beta} < \infty$. These features reflect the low degree of substitutability when $\beta < 0$.

When instead $\beta > 0$, there is a high degree of substitutability ($\sigma > 1$). Then, for $k \to \infty$ both y/k and $dy/dk \to A\alpha^{1/\beta} > 0$, thus violating the upper Inada condition for MPK (see the right-hand panel of Figure 16.4). It is also noteworthy that for $k \to 0$, y approaches a positive lower bound equal to $A(1-\alpha)^{1/\beta} > 0$. Thus, in this case capital is not essential. At the same time $dy/dk \to \infty$ for $k \to 0$ (so the lower Inada condition for the marginal productivity of capital holds).

The marginal productivity of labor is

$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha)A(\alpha k^{\beta} + 1 - \alpha)^{(1-\beta)/\beta} \equiv w(k),$$

from (16.17).

Since (16.16) is symmetric in K and L, we get a series of symmetric results by considering output per unit of capital as $x \equiv Y/K = A\left[\alpha + (1-\alpha)(L/K)^{\beta}\right]^{1/\beta}$. In total, therefore, when there is low substitutability ($\beta < 0$), for fixed input of either of the production factors, there is an upper bound for how much an unlimited input of the other production factor can increase output. And when there is high substitutability ($\beta > 0$), there is no such bound and an unlimited input of either production factor take output to infinity.

The Cobb-Douglas case, i.e., the limiting case for $\beta \to 0$, constitutes in several respects an intermediate case in that *all* four Inada conditions are satisfied and we have $y \to 0$ for $k \to 0$, and $y \to \infty$ for $k \to \infty$.

Generalizations

The CES production function considered above has CRS. By adding an elasticity of scale parameter, γ, we get the generalized form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \qquad \gamma > 0. \tag{16.22}$$

In this form the CES function is homogeneous of degree γ. For $0 < \gamma < 1$, there are DRS, for $\gamma = 1$ CRS, and for $\gamma > 1$ IRS. If $\gamma \neq 1$, it may be convenient to consider $Q \equiv Y^{1/\gamma} = A^{1/\gamma}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}$ and $q \equiv Q/L$ $= A^{1/\gamma}(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$.

The elasticity of substitution between K and L is $\sigma = 1/(1-\beta)$ whatever the value of γ. So including the limiting cases as well as non-constant returns to scale in the "family" of production functions with constant elasticity of substitution, we have the simple classification displayed in Table 16.1.

Table 16.1 The family of production functions with constant elasticity of substitution.

$\sigma = 0$	$0 < \sigma < 1$	$\sigma = 1$	$\sigma > 1$
Leontief	CES	Cobb-Douglas	CES

Note that only for $\gamma \leq 1$ is (16.22) a *neoclassical* production function. This is because, when $\gamma > 1$, the conditions $F_{KK} < 0$ and $F_{NN} < 0$ do not hold everywhere.

We may generalize further by assuming there are n inputs, in the amounts $X_1, X_2, ..., X_n$. Then the CES production function takes the form

$$Y = A\left[\alpha_1 {X_1}^{\beta} + \alpha_2 {X_2}^{\beta} + ...\alpha_n {X_n}^{\beta}\right]^{\frac{\gamma}{\beta}}, \ \alpha_i > 0 \text{ for all } i, \sum_i \alpha_i = 1, \gamma > 0. \tag{16.23}$$

In analogy with (16.18), for an n-factor production function the *partial elasticity of substitution* between factor i and factor j is defined as

$$\sigma_{ij} = \frac{MRS_{ij}}{X_i/X_j}\frac{d(X_i/X_j)}{dMRS_{ij}}_{|Y=\bar{Y}},$$

where it is understood that not only the output level but also all X_k, $k \neq i, j$, are kept constant. Note that $\sigma_{ji} = \sigma_{ij}$. In the CES case considered in (16.23), all the partial elasticities of substitution take the same value, $1/(1-\beta)$.

16.7 Literature

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Chapter 17

Addendum to Chapter 2

The contents of this chapter are identical to the contents of Short Note 1 as of 21.02.2014.

17.1 Skill-biased technical change in the sense of Hicks: An example

Let output be produced through a differentiable three-factor production function $\tilde{F}$:

$$Y = \tilde{F}(K, L_1, L_2, t), \qquad \partial\tilde{F}/\partial t > 0,$$

where K is capital input, L_1 is input of unskilled labor, and L_2 is input of skilled labor. Suppose technological change is such that the production function can be rewritten

$$\tilde{F}(K, L_1, L_2, t) = F(K, H(L_1, L_2, t)), \tag{17.1}$$

where the function $H(L_1, L_2, t)$ represents a "human capital" aggregate. Let the function H have CRS-neoclassical properties w.r.t. (L_1, L_2) and let $\partial H/\partial t > 0$.

In equilibrium under perfect competition in the labor markets the relative wage, the "skill premium", will be

$$\frac{w_2}{w_1} = \frac{\partial Y/\partial L_2}{\partial Y/\partial L_1} = \frac{F_H \partial H/\partial L_2}{F_H \partial H/\partial L_1} = \frac{H_2(L_1, L_2, t)}{H_1(L_1, L_2, t)} = \frac{H_2(1, L_2/L_1, t)}{H_1(1, L_2/L_1, t)}, \tag{17.2}$$

where we have used Euler's theorem[1] (saying that if H is homogeneous of degree one in its first two arguments, then the partial derivatives of H are homogeneous of degree zero w.r.t. these arguments).

[1] Acemoglu, p. 29.

Hicks' definitions are now: If for all $L_2/L_1 > 0$,

$$\frac{d\left(\frac{H_2(1,L_2/L_1,t)}{H_1(1,L_2/L_1,t)}\right)}{dt}\Bigg|_{\frac{L_2}{L_1}\text{ constant}} \gtreqless 0, \text{ then technical change is}$$

$$\begin{cases} \text{skill-biased in the sense of Hicks,} \\ \text{skill-neutral in the sense of Hicks,} \\ \text{blue collar-biased in the sense of Hicks,} \end{cases} \tag{17.3}$$

respectively. Combining with (17.2), we see that if the skill-premium has an upward trend for fixed relative supplies of skilled and unskilled labor, a possible explanation is that technological change is skill-biased in the sense of Hicks.

In the US the skill premium (measured by the wage ratio for college grads vis-a-vis high school grads) has had an upward trend since 1950 (see Jones and Romer, 2010).[2] If in the same period the relative supply of skilled labor had been roughly constant, a suggested explanation could be skill-biased technical change. In practice the relative supply of skilled labor has also been rising over the same period (in fact even faster than the skill premium). This suggests that the extend of "skill-biasedness" has been even stronger.[3]

An additional aspect of the story is that skill-biasedness helps *explain* the observed increase in the relative *supply* of skilled labor. If for a constant relative supply of skilled labor the skill premium is increasing, this increase strengthens the incentive to go to college. Thereby the fraction of skilled labor in the labor force tends to increase.

17.2 Capital-skill complementarity

Another potential source of a rising skill premium is *capital-skill complementarity*. Consider the production function

$$Y = \tilde{F}(K, L_1, L_2, t) = F(K, A_{1t}L_1, A_{2t}L_2) = (K + A_{1t}L_1)^{\alpha}(A_{2t}L_2)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where A_{1t} and A_{2t} are technical coefficients that may be rising over time. In this production function capital and unskilled labor are perfectly substitutable (the partial elasticity of factor substitution is $+\infty$). On the other

[2] On the other hand, over the years 1915 - 1950 the skill premium had a downward trend (Jones and Romer, 2010).

[3] As the H function has CRS-neoclassical properties w.r.t. L_1 and L_2, $H_{22} < 0$ and $H_{12} > 0$, cf. LN 2. Hence, with skill-neutral technical change we should have observed a *declining* skill premium (even more so with blue collar-biased technical change).

hand there is *direct complementarity* between capital and skilled labor ($\partial^2 Y/(\partial L_2 \partial K) > 0$).

In equilibrium under perfect competition the skill premium is

$$\frac{w_2}{w_1} = \frac{\partial Y/\partial L_2}{\partial Y/\partial L_2} = \frac{(K + A_{1t}L_1)^{\alpha}(1-\alpha)(A_{2t}L_2)^{-\alpha}A_{2t}}{\alpha(K + A_{1t}L_1)^{\alpha-1}A_{1t}(A_{2t}L_2)^{1-\alpha}} = \frac{1-\alpha}{\alpha}\left(\frac{K + A_{1t}L_1}{A_{2t}L_2}\right)\frac{A_{2t}}{A_{1t}}. \tag{17.4}$$

Here, even without technical change (A_{1t} and A_{2t} constant), a rising capital stock will, for fixed L_1 and L_2, raise the skill premium.

Equilibrium under perfect competition also implies

$$\frac{\partial Y}{\partial K} = \alpha(K + A_{1t}L_1)^{\alpha-1}(A_{2t}L_2)^{1-\alpha} = \alpha\left(\frac{K + A_{1t}L_1}{A_{2t}L_2}\right)^{\alpha-1} = r_t + \delta, \tag{17.5}$$

where r_t is the real interest rate at time t and δ is the (constant) capital depreciation rate. If in the long run r_t tends to be constant (cf. Kaldor's stylized facts), then also $(K+A_{1t}L_1)/(A_{2t}L_2)$ will tend to be constant. In this case, (17.4) shows that capital-skill complementarity is *not sufficient* for a rising skill premium. For the skill premium to remain increasing in this case, we need that technical change brings about a rising A_{2t}/A_{1t}. This amounts to skill-biasedness in a strong form.

The above observations are consistent with a story where capital equipment gradually replaces unskilled labor and a rising skill premium induces more and more people to go to college. The rising level of education in the labor force contributes to productivity. This together with continued technical change constitutes the basis for further capital accumulation and productivity increases.

In particular since the early 1980s the skill premium has been sharply increasing in the US (see Acemoglu, p. 498). This is also the period where ICT technologies took off.

17.3 Literature

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Appendix

Errata to lecture notes in Economic Growth, Spring 2014
(as updated 7/5 2014)

4/3 Ch. 2, p. 38, line 1 from below: replace *QY* by *qY*.
1/4 More to Ch.2: p. 37, line 3 from below: "higher *At, Bt,* or both" should be "higher *at*, *bt*, or both".

Ch. 4, p. 61, middle: "CRS production function (5.2)" should read "CRS production function (4.6)".
p. 62, line 1: replace (5.2) by (4.6).
p. 63, line 8: replace (5.2) by (4.6).

Ch. 5, p. 74, line 7: "absence of externalities" should read "absence of externalities and increasing returns to scale".

24/3 Ch. 8, p. 127, headline of the appendix: "certainty loss" should read "certainty-equivalent loss".
Ch. 9, p. 136, line 7: in the formula starting with *w* the factor *h* before the third equality sign should be *Ah*.
p. 138: line 9-17 should be deleted.
p. 144, eq. (9.19): the lower limit of integration should not be *S* but *v + S*.
p. 152, line 9: "and thus in better harmony" should read "and is thus in better harmony".
line 11: "*h = a*e^eta*S*" should read "*h = a*e^eta*lnS*".

1/4 Ch. 9, p. 145, line 9 from below: "no utility from leisure" should read "no utility from leisure and no bequest motive".
- line 1 from below: "= *HW(S)*" should read "= *HW(v, S)*".

p. 150, line 6: "spirit of assumption 7" should read "spirit of assumption 5".

7/4 Ch. 10, p. 168, line 12 from below: "in chapters 12 and 16" should read "in chapters 13 and 16".

28/4 Ch. 13, p. 205, line 14: "Even within" should read "Within".
p. 217, line 10 from below: "(??)" should be "(13.14)".

29/4 Ch. 14, p. 245-46: in (14.12), (14.13), and (14.14) the variables *Pi, pi, xi, w, Y,* and *N* should be indexed by a *t*.
p. 246, line 4: delete "14.17n".
p. 246, line 5: "in (14.12) should read "in (14.12) is".
p. 246, in (14.16) as well as four lines lower: "1" should be "psi".
p. 247, line 2: delete the factor *xi/pi* before El*xipi*.
p. 253, line 7: "which (without" should read "(that have no".

2/5 Ch. 15, p. 265: in (A1) in line 2 from below delete theta.

6/5 Ch. 14, p. 251, n. 6: "drives share prices down and the rate of return, *rt*, up" should read "drives the interest rate, *rt*, up".

p. 255, line 5: "Hence, the so far unknown" should read "Hence, the until now unknown".

Ch. 15, p. 263-264: every time you see a *N_dot_t* on these two pages, add a "+" as a top index on the *N_dot_t.* This is meant to indicate that what is meant is the conditional capital gain, that is, the increase per time unit in the market value of the monopoly firm at time t, *conditional* on its monopoly position remaining in place also in the next moment.

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