Exercise Problems for Economic Growth

Third edition

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Contents

Preface to the third edition

This is a collection of exercise problems that have been used in recent years in the course Economic Growth within the Master's Program in Economics at the Department of Economics, University of Copenhagen. The majority of the exercise problems have been tried out in class in previous years and at exams.

Compared with the second edition (February 2013) typos and similar are corrected and a few new exercise problems are added.

For constructive criticism I thank Niklas Brønager, instructor since 2012. I also thank a lot of previous students who suffered for bad wording and obscurities in earlier versions of the problems. No doubt, it is still possible to find obscurities. Hence, I very much welcome comments and suggestions of any kind relating to these exercises.

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Remarks on notation

Given a production function $Y = F(K, AL)$, we use the notation $y \equiv Y/L$, $k \equiv K/L$, $\tilde{y} \equiv Y/(AL)$, and $\tilde{k} \equiv K/(AL)$. Acemoglu (2009) uses y and k (p. 36) in the same way; the corresponding variables defined on a "per unit of effective labor basis" he writes, however, in the asymmetrical way as $\hat{y} \equiv Y/(AL)$ and $k \equiv K/(AL)$.

To indicate the "level of technology" (assumed measurable along a single dimension), depending on convenience we sometimes use A (as above), sometimes T.

Unless otherwise specified, whether we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

In spite of the continuous time approach, the time argument of a variable, x, is in these exercises often written as a subscript t, that is, as x_t, rather than $x(t)$ (depending on convenience).

Chapter 1

A refresher on basic concepts and simple models

I.1 In the last four decades China has had very high growth in real GDP per capita, cf. Table 1. Answer questions a), b), and c) presupposing that the growth performances of China and the U.S. continue to be like what they have been 1980-2007.

a) How many years does it take for China's GDP per capita to be doubled? You should explain your method.

b) How many years does it take for GDP per capita in the U.S. to be doubled?

c) How long time, reckoned from 2007, will it take for China to catch up with the US in terms of income per capita? You should explain your method.

d) Do you find it likely that the actual course of events will be (approximately) like that? Why or why not?

Table 1. GDP per capita in USA and China 1980 - 2007 (I$ in 2005 Constant Prices)

country	year	rgdpch
United States	1980	24537.41
United States	2007	42886.92
China	1980	1133.21

China	2007	7868.28

Source: PWT 6.3. Note: For China the Version 2 data series is used.

I.2 In a popular magazine on science, the data in Table 2 was reported:

Table 2. World income per capita relative to income per capita in the US: 1952-96

Year	*Percent*
1952	13.0
1962	13.3
1972	13.0
1982	13.8
1992	15.1
1996	17.7

Source: Knowledge, Technology, & Policy 13, no. 4, 2001, p. 52.
Note. Countries' per capita income are weighted by population as a fraction of the world population.

a) Briefly, discuss this data relative to concepts of income convergence and divergence and relative to your knowledge of the importance of weighting by population size.

b) What is meant by the terms unconditional (or absolute) income convergence and conditional convergence?

c) Give a short list of mechanisms that could in principle explain the data above.

I.3 *Stocks versus flows* Two basic elements in growth models are often presented in the following way. The aggregate production function is described by

$$Y_t = F(K_t, L_t, A_t), \tag{*}$$

where Y_t is output (aggregate value added), K_t capital input, L_t labor input, and A_t the "level of technology". The time index t may refer to period t, that is the time interval $[t, t+1)$, or to a point in time, depending on the context. And accumulation of the stock of capital in the (closed) economy is described by

$$K_{t+1} - K_t = Y_t - C_t - \delta K_t, \tag{**}$$

where δ is an (exogenous) rate of (physical) depreciation of capital, $0 \leq \delta \leq 1$. In continuous time models the corresponding equation is

$$\dot{K}(t) \equiv \frac{dK(t)}{dt} = Y(t) - C(t) - \delta K(t), \qquad \delta \geq 0.$$

a) At the theoretical level, what denominations (dimensions) should be attached to output, capital input, and labor input in a production function?

b) What is the denomination (dimension) attached to K in the accumulation equation?

c) Is there any consistency problem in the notation used in (*) vis-à-vis (**)? Explain.

d) Suggest an interpretation that ensures that there is no consistency problem.

e) Suppose there are two countries. They have the same technology, the same capital stock, the same number of employed workers, and the same number of man-hours per worker per year. Country a does not use shift work, but country b uses shift work, that is, two work teams of the same size and the same number of hours per day. Elaborate the formula (*) so that it can be applied to both countries.

f) Suppose F is a neoclassical production function with CRS w.r.t. K and L. Compare the output levels in the two countries. Comment.

g) In continuous time we write aggregate (real) gross saving as $S(t) \equiv Y(t) - C(t)$. What is the denomination of $S(t)$.

h) In continuous time, does the expression $K(t) + S(t)$ make sense? Why or why not?

i) In discrete time, how can the expression $K_t + S_t$ be meaningfully interpreted?

I.4 *Short questions* (answering requires only a few well chosen sentences)

a) Consider an economy where all firms' technology is described by the same neoclassical production function, $Y_i = F(K_i, L_i)$, $i = 1, 2, ..., N$, with decreasing returns to scale everywhere (standard notation). Suppose there is "free entry and exit" and perfect competition in all markets. Then a paradoxical situation arises in that no equilibrium with a finite number of firms (plants) would exist. Explain.

b) In many macro models the technology is assumed to have constant returns to scale (CRS) with respect to capital and labor taken together. Often the so-called *replication argument* is put forward as a reason to expect that CRS should hold in the real world. What is the replication argument? Do you find an appeal to the replication argument to be a convincing argument for the assumption of CRS with respect to capital and labor? Why or why not?

c) Does the validity of the replication argument, considered as an argument about a property of technology, depend on the availability of the different inputs? Comment.

d) Suppose that for a certain historical period there has been something close to constant returns to scale and perfect competition, but then, after a shift to new technologies in the different industries, increasing returns to scale arise. What is likely to happen to the market form? Why?

I.5 *The Solow model; local and global asymptotic stability* The Solow growth model in continuous time can be set up in the following way. A closed economy is considered. There is an aggregate production function,

$$Y(t) = F(K(t), T(t)L(t)),$$

where F is a neoclassical production function with CRS, Y is output, K is capital input, T is the technology level, and L is labor input. There is full employment. It is assumed that

$$\begin{aligned} T(t) &= T(0)e^{gt}, \qquad T(0) = T_0, \qquad g \geq 0, \\ L(t) &= L(0)e^{nt}, \qquad L(0) = L_0, \qquad n \geq 0. \end{aligned}$$

Aggregate gross saving, S, is assumed proportional to gross aggregate income which, in a closed economy, equals real GDP, Y:

$$S(t) = sY(t), \qquad 0 < s < 1.$$

Capital accumulation is described by

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t), \qquad \delta > 0,$$

where δ is the rate of (physical) depreciation of capital. Finally, from national income accounting,

$$S(t) \equiv Y(t) - C(t).$$

The symbols g, n, s, and δ represent parameters (given constants) and the initial values T_0, L_0, and K_0, are given (exogenous) positive numbers.

a) What kind of technical progress is assumed in the model?

b) Let $\tilde{y} \equiv y/T \equiv Y/(TL)$. Let the production function on intensive form be denoted f. Derive f from F. Sign f' and f''. From the given information, can we be sure that $f(0) = 0$? Why or why not?

c) To get a grasp of the evolution of the economy over time, derive a first-order differential equation in the (effective) capital intensity, $\tilde{k} \equiv k/T \equiv K/(TL)$, that is, an equation of the form $\dot{\tilde{k}} = \varphi(\tilde{k})$. *Hint:* $\dot{\tilde{k}}/\tilde{k} = \dot{k}/k - \dot{T}/T = \dot{K}/K - \dot{L}/L - g$.[1]

d) If in c) you were able to write $\varphi(\tilde{k})$ on the form $\varphi(\tilde{k}) = \psi(\tilde{k}) - a\tilde{k}$, where a is a constant, you are one the right track. Draw a "Solow diagram Version 1", that is, a diagram displaying the graphs of the functions $\psi(\tilde{k})$ and $a\tilde{k}$ in the $(\tilde{k}, \tilde{y})$ plane. You may, at least initially, draw the diagram such that the two graphs cross each other for some $\tilde{k} > 0$.

e) Suppose there exists a (non-trivial) steady state, $\tilde{k}^* > 0$. Indicate $\tilde{k}^*$ in the diagram. Can there be more than one (non-trivial) steady state? Why or why not?

f) In a new diagram, draw a "Solow diagram Version 2", that is, a diagram displaying the graphs of the functions $s\psi(\tilde{k})$ and $sa\tilde{k}$ in the $(\tilde{k}, s\tilde{y})$ plane. At what value of $\tilde{k}$ will these two graphs cross?

g) Suppose capital is *essential*, that is, $F(0, TL) = 0$ for all TL. In terms of limiting values of f' for the capital intensity approaching zero and infinity, respectively, write down a necessary and sufficient condition for existence of a (non-trivial) steady state.

h) Suppose the steady state $\tilde{k}^* > 0$ is locally asymptotically stable. What is meant by this and will it always be true for the Solow model? Why or why not?

i) Suppose the steady state, $\tilde{k}^*$, is globally asymptotically stable. What is meant by this and will it always be true for the Solow model? Why or why not?

[1] Recall the following simple continuous time rule: Let $z = y/x$, where z, y, and x are differentiable functions of time t and positive for all t. Then $\dot{z}/z = \dot{y}/y - \dot{x}/x$, exactly. *Proof:* We have $\log z = \log y - \log x$. Now take the time derivative on both sides of the equation.

j) Find the long-run growth rate of output per unit of labor.

I.6 This problem is about the same model as Problem I.5, the standard version of the Solow model.

a) Suppose the economy is in steady state until time t_0. Then, for some extraneous reason, an upward shift in the saving rate occurs. Illustrate by the Solow diagram the evolution of the economy from t_0 onward.

b) Draw the time profile of $\ln y$ in the $(t, \ln y)$ plane.

c) How, if at all, is the level of y affected by the shift in s?

d) How, if at all, is the growth rate of y affected by the shift in s? Here you may have to distinguish between transitory and permanent effects.

e) Explain by words the economic mechanisms behind your results in c) and d).

f) As Solow once said (in a private correspondence with Amartya Sen[2]): "The idea [of the model] is to trace full employment paths, no more." What market form is theoretically capable of generating permanent full employment?

g) Even if we recognize that the Solow model only attempts to trace hypothetical time paths with full employment (or rather employment corresponding to the "natural" or "structural" rate of unemployment), the model has several important limitations. What is in your opinion the most important limitations?

I.7 Set up a Solow model where, although there is no technical progress, sustained per capita growth occurs. Comment. *Hint:* a simple approach can be based on the production function $Y = BK^{\alpha}L^{1-\alpha} + AK$, where $A > 0, B > 0, 0 < \alpha < 1$. "Sustained per capita growth" is said to occur if $\lim_{t\to\infty} \dot{y}/y > 0$ or $\lim_{t\to\infty} \dot{c}/c > 0$ (standard notation).

I.8 Consider a closed economy with technology described by the aggregate production function

$$Y = F(K, L),$$

[2] *Growth Economics. Selected Readings*, edited by Amartya Sen, Penguin Books, Middlesex, 1970, p. 24.

where F is a neoclassical production function with CRS and satisfying the Inada conditions, Y is output, K is capital input and L is labor input = labor force = population (there is always full employment). A constant fraction, s, of *net* income is saved (and invested). Capital depreciates at the constant rate $\delta > 0$.

a) Assuming a constant population growth rate n, derive the fundamental differential equation of the model and illustrate the dynamics by a phase diagram. Comment.

b) Assume instead that the population growth rate n is a smooth function of per capita income, i.e., $n = n(y)$, where $y \equiv Y/L$. At very low levels of per capita income, n is zero, at higher per capita income, n is a hump-shaped function of y, and at very high levels of y, n tends to zero, that is, for some $\bar{y} > 0$ we have

$$n'(y) \gtreqless 0, \text{ for } y \lesseqgtr \bar{y}, \text{ respectively,}$$

whereas $n(y) \approx 0$ for y considerably above $\bar{y}$. Show that this may give rise to a dynamics quite different from that of the Solow model. Comment.

I.9 *Short questions*

a) "The Cobb-Douglas production function has the property that, under technical progress, it satisfies all three neutrality criteria if it satisfies one of them." True or false? Explain why.

b) Write down a Cobb-Douglas production function that displays non-neutral technical change.

c) "If the production function is Cobb-Douglas with CRS and time-independent output elasticity w.r.t. capital, the standard Solow model with competitive markets and existence of a steady state predicts that the labor income share of national income is constant in the long run." True or false? Give a reason for your answer.

I.10 *Short question* "Relatively homogeneous groups of countries, such as for example the 12 old EU member countries, tend to experience income convergence in the sense that the standard deviation of income per capita across the countries diminishes over time." True or not true as an empirical statement? Comment.

I.11 Consider the production function $Y = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$.

a) Does the function imply constant returns to scale?

b) Is the production function neoclassical? *Hint:* after checking criterion (a) of the definition of a neoclassical production function, you may use claim (iii) of Section 3 in Lecture Note 2 together with your answer to a).

c) Given this production function, is capital an essential production factor? Is labor?

d) If we want to extend the domain of definition of the production function to include $(K, L) = (0, 0)$, how can this be done while maintaining continuity of the function?

I.12 *Never trust authorities - form your own opinion (I)* Sometimes misleading graphs and/or figure texts about across-country income convergence are published. For example, Figure 1 shows a copy of a figure from a publication by the Danish Ministry of Finance, 1996. In English, the headline reads "Standard deviation in GDP per capita in EU-12".

a) What is the problem with this data material as presented, including the headline? *Hint:* if you need help, you may consult Section 4.2 and 4.5 in Lecture Note 1.

Similarly, Figure 2 shows a copy of a figure from the Danish Economic Council, 1997. In English, the headline reads: Standard deviation in GDP per capita and per worker across the EU countries.[3]

b) What is the problem with this data material as presented, including the headline?

c) There is a certain property of the standard deviation of income per capita that is important for its relevance as a criterion for whether σ convergence is present or not. Briefly discuss.

[3] The note to this figure says that GDP per capita and per worker are weighted with size of population and size of labor force, respectively.

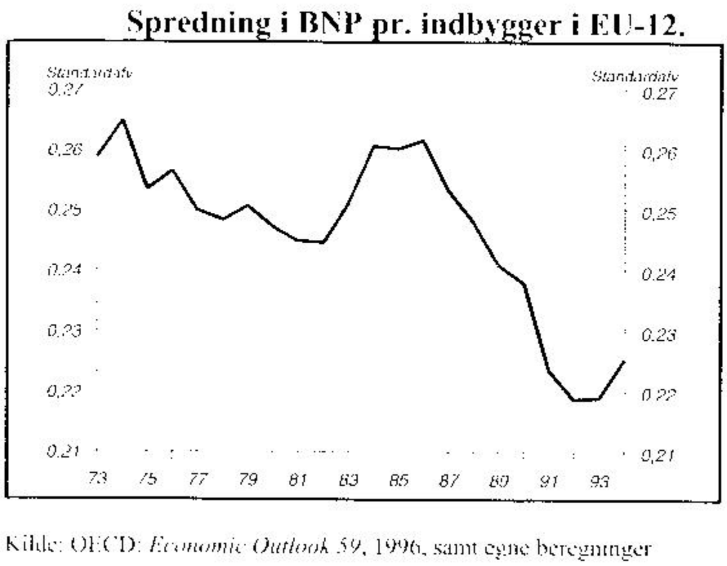


Figure 1.1: Source: Finansredegørelse 96. Bilag p. 53. Finansministeriet, Dec. 1996.

I.13 *Never trust authorities - form your own opinion (II)* Also prominent economists sometimes make elementary mistakes. For example, in the context of the Solow growth model we read in Acemoglu's textbook (Acemoglu 2009, p. 53) the statement: "In addition, k^* is increasing in α", where

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

What is wrong with the cited statement?

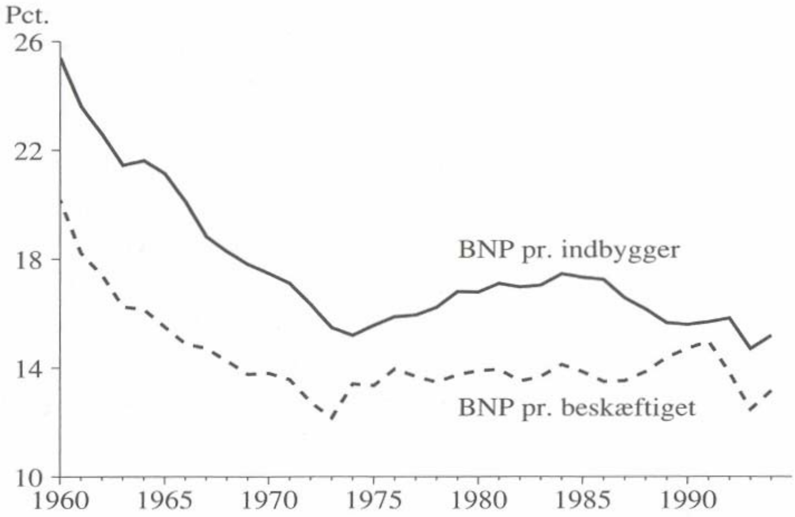
I.14 *Short questions*

"A relatively homogeneous group of countries such as for example the EU countries tend to experience income convergence in the sense that the standard deviation of income per capita across the countries diminishes over time." True or not true as an empirical statement? Explain.

I.15 Several spending items which in national income accounting are classified as "public consumption" are from an economic point of view better described as public investment. List some examples.

I.16 An important aspect of growth analysis is to pose good questions in the sense of questions that are brief, interesting, and manageable. If we

Figur IV.4 Spredningen i EU-landenes indkomst pr. indbygger og pr. beskæftiget



Anm.: Spredningen er beregnet som den vægtede spredning i EU-landenes købekraftskorrigerede BNP pr. indbygger og pr. beskæftiget. Som landevægte er anvendt hhv. befolkningstal og antal beskæftigede.

Kilde: OECD, *Economic Outlook* og egne beregninger.

Figure 1.2: Source: Det økonomiske Råd, Dansk økonomi Forår 1997, p. 147.

set aside an hour or so in one of the last lectures at the end of the semester, what question would you suggest should be discussed?

Chapter 2

Transitional dynamics. Speed of convergence

II.1 *Within-country speed of convergence according to the Solow model*
We know that the Solow growth model can be reduced to the following differential equation ("law of motion")

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + g + n)\tilde{k}, \qquad (*)$$

where $\tilde{k} \equiv k/A \equiv K/(AL)$ (standard notation). Assume

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \frac{\delta + g + n}{s} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}).$$

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, which is globally asymptotically stable.

a) Illustrate this result graphically in the $(\tilde{k}, \tilde{y})$ plane, where $\tilde{y} \equiv y/A \equiv Y/(AL)$.

b) Suppose the economy has been in steady state until time 0. Then an upward shift in the saving rate occurs. Illustrate graphically the evolution of the economy from time 0 onward.

c) We are interested not only in the ultimate effect of this parameter shift, but also in how fast the adjustment process occurs. For a general differentiable adjustment process, $(x(t))$, an answer to this can be based on the (proportionate) rate of decline at time t of the distance to the steady state:

$$\mathrm{SOC}_t(x) \equiv -\frac{d(|x(t) - x^*|)/dt}{|x(t) - x^*|}, \qquad (**)$$

where $|a|$ is our notation for the absolute value of a real number, a (SOC stands for Speed of Convergence). In the context of the Solow model the corresponding expression simplifies to

$$\mathrm{SOC}_t(x) = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} = -\frac{\dot{\tilde{k}}}{\tilde{k}(t) - \tilde{k}^*}. \qquad (***)$$

How?

The right-hand side of (***) is called the *instantaneous speed of convergence* of the technology-corrected capital intensity at time t. For $t \to \infty$ both the denominator and the numerator approach zero. Yet the ratio of the two has a well-defined limit for $t \to \infty$, the *asymptotic speed of convergence*, often simply denoted SOC. By applying a first-order Taylor approximation of $\dot{\tilde{k}}$ about the steady state, one can find SOC.

d) Applying a first-order Taylor approximation of $\dot{\tilde{k}}$, derive a formula for the asymptotic speed of convergence and calculate its value, given an assumption of perfect competition together with the following data (time unit one year): $\delta = 0.07$, $g = 0.02$, $n = 0.01$, and gross capital income share $= 1/3$. *Hint:* for a differentiable non-linear function $\varphi(x)$, a first-order Taylor approximation about x^* is $\varphi(x) \approx \varphi(x^*) + \varphi'(x^*)(x - x^*)$.

II.2 *Example where an explicit solution for the time-path can be found* Sometimes an explicit solution for the whole time path during the adjustment process can be found. In the Solow model this is so when the production function is Cobb-Douglas: $Y = K^\alpha (AL)^{1-\alpha}$, $\quad 0 < \alpha < 1$, $A(t) = A(0)e^{gt}$. In standard notation the fundamental differential equation is

$$\dot{\tilde{k}} = s\tilde{k}^\alpha - (\delta + g + n)\tilde{k}, \qquad 0 < s < 1,\ \delta + g + n > 0. \qquad (*)$$

Let $\tilde{k}(0) > 0$ be arbitrary. Since (*) is a so-called Bernoulli equation (the right-hand side of the ordered differential equation is the sum of a power function term and a linear term), we can find an explicit solution of (*):

a) Let $x \equiv \tilde{k}^{1-\alpha} = \tilde{k}/\tilde{y} = k/y$. So x is simply the capital-output ratio. Derive a differential equation for x.

b) Write down the solution of this differential equation. *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$, has the solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}.$$

c) Determine the instantaneous speed of convergence of the capital-output ratio at time t. *Hint:* apply (**) of Problem II.1.

d) Generally, for variables belonging to a nonlinear model the instantaneous speed of convergence is time-dependent and at best an acceptable *approximation* to the asymptotic speed of convergence. Compare to your result from c). *Hint:* Compare the asymptotic speed of convergence of $\tilde{k}$ in the present model to your result from c).

II.3 Consider a Solow model for a closed economy with perfect competition. The rate of Harrod-neutral technical progress is 1.8 percent per year, the rate of population growth is 0.5 percent per year, capital depreciates at the rate 0.6 per year, and in steady state the share of labor income in steady state is 2/3.

a) Find the asymptotic speed of convergence of $\tilde{k}$. *Hint:* given the production function on intensive form, $f(\tilde{k})$, the asymptotic speed of convergence is $(1 - \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)})(\delta + g + n)$ (standard notation).

b) Find the approximate *half-life* of the initial distance of $\tilde{k}$ to its steady-state value, $\tilde{k}^*$. *Hint:* Let the time path of a variable x be $x(t) = x(0)e^{-\beta t}$ where β is a positive constant. Then the half-life of the initial distance to the steady state (i.e., the time it takes for half of the initial gap to be eliminated) is given as $-\ln \frac{1}{2}/\beta = \ln 2/\beta$.

c) Empirical cross-country analyses of conditional convergence point at speeds of convergence between 2% per year and 9% per year, depending on the selection of countries and the econometric method. Although this throws empirical light on "cross-country convergence" while the Solow model throws theoretical light on "within-country convergence", compare your result from a) to this empirical knowledge.

d) What is the *doubling-time* of income per capita implied by the model?

e) What is the long-run per capita growth rate implied by the model?

II.4 *Barro regressions and the formulas in Acemoglu (2009, pp. 80-81)* Consider a closed economy with aggregate production function $Y = F(K, AL)$, where Y is GDP, K capital input, A the technology level, and L labor input (the dating of the variables is implicit). It is assumed that F is neoclassical with CRS and that A and L grow exogenously at the constant rates $g > 0$ and $n \geq 0$, respectively. Capital moves according to

$$\dot{K} = Y - C - \delta K, \qquad \delta \geq 0,$$

where C is aggregate consumption.

Suppose it is known that in the absence of shocks, $\tilde{k} \equiv K/(AL)$ converges towards a unique steady state value, $\tilde{k}^* > 0$, for $t \to \infty$. Suppose further that in a small neighborhood of the steady state, the instantaneous speed of convergence of $\tilde{k}$ is

$$\mathrm{SOC}_t(k) \equiv -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \approx (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*), \qquad (*)$$

where $\varepsilon(\tilde{k})$ is the output elasticity w.r.t. capital, evaluated at the effective capital intensity $\tilde{k}$.

a) Show that (*) implies

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t}.$$

Hint: when a variable $x > 0$ has a constant growth rate, γ, its time path (in continuous time) is $x(t) = x(0)e^{\gamma t}$ (this follows from the hint to Problem II.2b above) with $a = -\gamma$ and $b = 0$).

Since time series of economic data are often (for good reasons) given in logarithmic form and many approximative models are based on log-linearization, it is of interest what the asymptotic speed of convergence of $\log \tilde{k}(t) - \log \tilde{k}^*$ is.

b) To pursue this, explain why a first-order Taylor approximation of $\log \tilde{k}$ about $\log \tilde{k}^*$ gives $\log \tilde{k} \approx \log \tilde{k}^* + (\tilde{k} - \tilde{k}^*)/\tilde{k}^*$.

c) Find the asymptotic speed of convergence of $\log \tilde{k}$, $\mathrm{SOC}_t(\log \tilde{k})$. Comment.

For a variable $x > 0$, let $g_x \equiv \dot{x}/x$.

d) Show that
$$g_{\tilde{k}} \approx -\beta(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*).$$
Hint: use your result in c).

Define $y \equiv Y/L \equiv \tilde{y}A$.

e) Show that $g_y = g + \varepsilon(\tilde{k})g_{\tilde{k}}$.

f) Show that
$$g_y \approx g - \beta(\tilde{k}^*)\varepsilon(\tilde{k})(\log \tilde{k} - \log \tilde{k}^*) \approx g - \beta(\tilde{k}^*)\varepsilon(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*). \quad (**)$$

Define $y^*(t) \equiv f(\tilde{k}^*)A(t)$, where $f(\tilde{k}) \equiv F(\tilde{k}, 1)$.

g) Interpret $y^*(t)$. Draw the time profile of $\log y^*(t)$. In the same diagram, draw illustrating time profiles of $\log y(t)$ for the cases $\tilde{k}(0) < \tilde{k}^*$ and $\tilde{k}(0) > \tilde{k}^*$, respectively.

h) Show that
$$\log y - \log y^* \approx \varepsilon(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*). \quad (***)$$
Hint: you may start from the left-hand side of (***), apply the principle in b), use that $\tilde{y}$ is a function of $\tilde{k}$, apply a first-order Taylor approximation on this function, and apply again the principle in b).

i) Based on (**) combined with (***), write down a Barro-style growth regression equation. *Hint:* a discrete-time approximation of g_y can be based on the principle in b).

II.5 This problem presupposes that you have already solved Problem II.4. The setup is the same as in the introduction to Problem II.4.

a) "The SOC of $\log \tilde{k}$ must equal the SOC of $(\log \tilde{k} - \log \tilde{k}^*)$". True or false? Why? *Hint:* recall the general definition of SOC of a converging variable x.

b) Find the approximate speed of convergence of the vertical distance $\log y - \log y^*$ in your graph from g) of Problem II.4. *Hint:* use your result in a) together with (***) and the conclusion in c) of Problem II.4.

II.6 A key variable in the adjustment process of a growth model is the saving-capital ratio. As an example we take the Solow model where this ratio is given as $S/K = sf(\tilde{k})/\tilde{k}$. Suppose f satisfies the Inada conditions.

a) On the basis of (*) in Problem II.1, illustrate the adjustment over time of S/K in the $(\tilde{k}, S/K)$ plane, assuming $\delta + g + n > 0$ (standard notation). For an arbitrary $\tilde{k}_0 > 0$, indicate $\dot{\tilde{k}}/\tilde{k}$ in the diagram. Comment.

Let $g = 0$. Suppose the production function is Cobb-Douglas such that $f(\tilde{k}) = Bk^{\alpha}$, $B > 0$, $\alpha \in (0, 1)$, and $s > (\delta + n)/B$.

b) Let $\alpha \to 1$. What happens to S/K and $y \equiv Y/L$ in the limit where $\alpha = 1$? Comment.

Chapter 3

Growth accounting vs. explanation of growth

III.1 Consider an economy with aggregate production function

$$Y_t = \tilde{F}(K_t, L_t, t),$$

where $\tilde{F}$ is a neoclassical production function w.r.t. K and L, Y is GDP, K capital input, A the technology level, and L labor input. We apply the convenient notation: $g_z \equiv \dot{z}/z$.

a) By the standard growth accounting method, decompose the output growth rate into its three basic components.

b) How is the TFP growth rate, x, defined? Interpret the concept TFP growth rate.

From now on assume $\tilde{F}$ has CRS w.r.t. K and L and can be written

$$\tilde{F}(K_t, L_t, t) = F(K_t, A_t L_t), \tag{*}$$

where A_t grows at a given constant rate $g > 0$ and employment grows at a given constant rate $n > 0$. Moreover, the increase in capital per time unit is given by

$$\dot{K}_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \tag{**}$$

where C is aggregate consumption and not all of Y is consumed.

c) Determine g_Y and g_K under balanced growth. *Hint:* in view of CRS, we know something about the sum of the output elasticities w.r.t. the two production factors and in view of the given additional information we know something about the relationship between g_Y and g_K under balanced growth.

d) Let $y_t \equiv Y_t/L_t$. Determine g_y and x under balanced growth.

e) Is there a sense in which technical progress explains more than what the growth accounting under a) and b) suggested? Explain.

III.2 This problem presupposes that you have already solved Problem III.1. In that problem, technological change was taken as exogenous. There are many ways to endogenize g_A. One is the "learning by investing" hypothesis according to which the evolution of A in (*) of Problem III.1 is a by-product of capital accumulation, for example in the following simple form:

$$A_t = K_t^{\lambda}, \qquad 0 < \lambda < 1.$$

a) Let $y_t \equiv Y_t/L_t$. Maintaining the production function F in (*) as well as (**) and $g_L = n > 0$ from Problem III.1, determine g_y under balanced growth.

b) We may say that now the mechanism that drives long-run productivity growth is the dynamic interaction of capital accumulation and learning (they stimulate each other). At a deeper level we may emphasize two aspects as being of key importance for this mechanism to be able to *sustain* productivity growth: one aspect pertains to the technology and the other to demography. What are these two aspects?

III.3 *Population breeds ideas*[1] Consider a pre-industrial economy described by:

$$\begin{aligned}
Y_t &= A_t^{\sigma} L_t^{\alpha} Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, &(1)\\
\dot{A}_t &= \lambda A_t^{\varepsilon} L_t, \qquad \lambda > 0, \varepsilon \leq 1, \quad A_0 > 0 \text{ given}, &(2)\\
L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, &(3)
\end{aligned}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), and Z the amount of land (fixed). Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene; that is, (3) should be seen as a short-cut.

a) Interpret the model, including the parameter $\bar{y}$.

[1] This builds on Kremer, QJE 1993, §I-II. Acemoglu, 2009, pp. 113-114, discusses two special cases of the model.

From now, let $Z = 1$.

b) Show that the dynamics of the model reduce to the "law of motion":

$$\dot{A} = \hat{\lambda} A^{\varepsilon + \frac{\sigma}{1-\alpha}}, \qquad \text{where } \hat{\lambda} \equiv \lambda \varphi^{\frac{1}{1-\alpha}}.$$

c) Define $\mu \equiv \varepsilon + \frac{\sigma}{1-\alpha}$ and solve the differential equation $\dot{A} = \hat{\lambda} A^{\mu}$. *Hint:* consider the cases $\mu = 1$ and $\mu \neq 1$ separately; for $\mu \neq 1$, consider the implied differential equation for $x \equiv A^{1-\mu}$.

d) Show that "growth acceleration" ($\dot{A}/A$ rising over time) arises if and only if $\varepsilon > 1 - \frac{\sigma}{1-\alpha}$.

e) Show that the "growth acceleration" in this model takes a very dramatic form.

f) For fixed $\alpha = \bar{\alpha}$, illustrate in the (σ, ε) plane the region leading to "growth acceleration". Need ε be positive for "growth acceleration" to happen?

g) If the parameters are such that the economy belongs to the region mentioned in f), we can conclude something about the possibility of persistence of the Malthusian regime as described by the model. What can we conclude?

III.4 It is preferable to solve Problem III.3 before this problem. Consider a Solow-Malthus model of a subsistence economy:

$$\begin{aligned} Y_t &= K_t^{\varepsilon} L_t^{\alpha} Z^{1-\varepsilon-\alpha}, \qquad && 0 < \varepsilon < \varepsilon + \alpha < 1, \\ \dot{K}_t &= sY_t, \quad 0 < s < 1, && K_0 > 0, \\ L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \quad && \bar{y} > 0, \end{aligned}$$

where Y is aggregate output, K input of physical capital, L the labor force (= population), and Z the amount of land (fixed). For simplicity, we ignore capital depreciation. Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene.

a) Interpret the equations, including $\bar{y}$. The exponents to K, L, and Z sum to one. What could be the argument for this?

From now, let $Z = 1$.

b) Derive the law of motion of the economy.

c) Does this model lead us to predict that the economy must sooner or later transcend the Malthusian subsistence regime? Why or why not? *Hint:* consider the behavior of $\dot{K}/K$ for $t \to \infty$.

d) Comment in relation to Problem III.3.

III.5 *Short questions.*

a) "If there are constant returns to scale with respect to physical capital and labor taken together, then, considering technical knowledge as a third production factor, there will be increasing returns w.r.t. to all three production factors taken together." True or false? Explain why.

b) Consider a set of countries, $j = 1, 2, ..., N$. Country j has the aggregate production function
$$Y_{jt} = F(K_{jt}, A_{jt}L_{jt}),$$
where F is neoclassical and has CRS (standard notation). The technology level A_{jt} evolves according to $A_{jt} = A_{j0}e^{gt}$, where A_{j0} differs widely across the countries. The positive constant g as well as the function F and the capital depreciation rate are, however, the same across the countries. Assume that (i) the countries trade in a fully integrated world market for goods and financial capital; (ii) they face a constant real interest rate $r > 0$ in this market; and (iii) there is perfect competition in all markets. "In this setup there will be a strong economic incentive for workers to migrate." True or false? Explain why.

III.6 *On persistent technology differences* Inspired by the article by Bernard and Jones (1996)[2], we consider a group of countries, indexed by $i = 1, 2, \ldots, N$, with aggregate production functions

$$Y_i(t) = K_i(t)^{\alpha_i}(A_i(t)L_i(t))^{1-\alpha_i}, \qquad 0 < \alpha_i < 1, \tag{1}$$

where A_i is the technology level of country i. There is a country-specific capital depreciation rate, δ_i, which is assumed constant over time. Technological catching-up occurs according to

$$\frac{\dot{A}_i(t)}{A_i(t)} = \xi_i \frac{A_w(t)}{A_i(t)}, \tag{2}$$

[2] Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

where $A_w(t) = A_w(0)e^{gt}$ is the world frontier technology level, $g > 0$. We assume $A_i(0) \leq A_w(0)$ and $0 < \xi_i \leq g$, for all $i = 1, 2, \ldots, N$. Let "labor productivity" be measured by $y_i(t) \equiv Y_i(t)/L_i(t)$.

In their Solow-style setup,

$$\dot{K}_i = s_i K_i^{\alpha_i}(A_i L_i)^{1-\alpha_i} - \delta_i K_i, \qquad \text{(standard notation)}$$

Bernard and Jones find that even in the long run there need not be a tendency for $y_i(t)/y_w(t)$ to approach 1 and they show which potential structural differences (parameter differences) are responsible for this. They conclude that their setup "leads to a world in which *similar* steady state outcomes are the exception rather than the rule".

Bernard and Jones present data for 14 OECD countries over the period 1970-87 to substantiate this conclusion. Over this period, however, financial capital was not as mobile as it is today. This raises two questions. How, if at all, does perfect capital mobility affect the theoretical conclusion that *similar* steady state outcomes are the exception rather than the rule? And what does more recent data show?

Here we shall deal with the first question.[3] We replace the Solow-style setup by a setup where the countries trade in a fully integrated world market for goods and financial capital. Assume perfect competition and that the countries face a constant real interest rate $r > 0$ in the market for financial capital while, however, labor is entirely immobile. Finally, assume that (1) and (2) still hold and that the world frontier technology is identical with the technology of one of the countries in the considered set, namely the "world leader" (say USA). We let α_w denote the output elasticity w.r.t. capital in this country.

a) Examine whether in this case there is a tendency for $y_i(t)/y_w(t)$ to approach 1 in the long run. *Hint 1:* Profit maximizing firms will under perfect competition choose a time-independent effective capital intensity, $\tilde{k}_i^*$, satisfying
$$f_i'(\tilde{k}_i^*) = r + \delta_i.$$
Hint 2. Consider the ratio $x(t) \equiv A_i(t)/A_w(t)$, a measure of country i's *lag* relative to the frontier; express the growth rate of x in terms of x, ξ_i, and g; this should give you a linear first-order differential equation with constant coefficients; then apply the brief math manual in the appendix.

b) Assess the hypothesis that *similar* steady state outcomes are the exception rather than the rule.

[3] The second question may be suitable for a Master Thesis project!

c) Interpret the parameter ξ_i.

d) Does the answer to a) depend on whether the countries differ w.r.t. their saving rate, s_i, and labor force growth rate, n_i? Why or why not?

III.7 This problem presupposes that you have already solved Problem III.6. Let $\mathrm{TFP}_i(t)$ denote the total factor productivity of country i at time t.

a) Express $\mathrm{TFP}_i(t)$ in terms of the labor-augmenting technology level $A_i(t)$.

b) Find the limit of the ratio $\frac{\mathrm{TFP}_i(t)}{\mathrm{TFP}_w(t)}$ for $t \to \infty$; there may be alternative cases to be considered.

c) Will there be a tendency for TFP of the different countries to differ in the long run? Why or why not?

d) On the basis of the above results, do you think the comparative analysis in terms of TFP growth adds anything of economic interest to the comparative analysis in terms of the labor-augmenting technology level A and labor productivity, y, cf. a) of Problem III.6? Discuss.

e) "Long-run growth in the ratio of two countries' TFP may misrepresent the economic meaning of technical progress when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral." Do you agree? Why or why not?

III.8 *Short questions*

a) "Growth accounting pinpoints the determinants of growth." True or false? Explain.

b) "If there are constant returns to scale with respect to physical capital, labor, and land taken together, then, considering technical knowledge as a fourth production factor, there will be increasing returns w.r.t. to all four production factors taken together." True or false? Explain why.

III.9 In poor countries capital per worker measured as K/L, tends to be much lower than in rich countries. Can we, while accepting the neoclassical assumption of diminishing marginal productivity of capital, explain why the capital flows from rich to poor countries are not much larger than they are? Why or why not?

III.10 Briefly describe Acemoglu's distinction between proximate and fundamental determinants of differences in economic performance.

III.11 Consider a simple closed economy where two goods get produced: corn and new ideas. Time is continuous. The labor force (in this problem the same as population), L_t, grows at an exogenous constant rate $n \geq 0$, i.e., $L_t = L_0 e^{nt}$. People can work as farmers, L_{Yt}, or researchers, L_{At}, where $L_{Yt} + L_{At} = L_t$ and $L_{At} = s_A L_t$, where, for simplicity, s_A is assumed constant, $0 < s_A < 1$. The production functions are:

$$\begin{aligned} Y_t &= A_t^{\sigma} J^{\alpha} L_{Yt}^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, && (*) \\ \dot{A}_t &= \mu A_t^{\varphi} L_{At}, \qquad \mu > 0, \varphi < 1, && (**) \end{aligned}$$

where Y_t is corn output, J is the amount of land (fixed), and A_t the level of technical knowledge. Let the growth rate of a variable $x > 0$ at a given point in time be denoted g_x (not necessarily a constant). Let $y \equiv Y/L$.

a) Express g_y in terms of g_A and n.

b) Sign g_A.

c) Show that $g_A \to n/(1-\varphi)$ for $t \to \infty$. *Hint:* The easiest approach is to calculate the growth rate of g_A on the basis of (**); by reordering you get an expression of the form $\dot{x} = (a - bx)x$ where $a \geq 0$ and $b > 0$; now use the principle that when $x > 0$, we have $\dot{x} \gtreqless 0$ for $x \lesseqgtr a/b$, respectively.

d) Let the long-run value of g_y be denoted g_y^*. Find g_y^*.

e) Let $n > 0$. Under three alternative conditions involving the values of α, σ, and φ, three alternative cases, $g_y^* > 0, g_y^* = 0$, and $g_y^* < 0$, are possible. Show this.

f) In the case $g_y^* > 0$ show that $\partial g_y^*/\partial n > 0$. What is the economic intuition behind this result?

g) What is the economic intuition behind that the alternative case, $g_y^* < 0$, is also possible in the model?

h) Would this alternative case be possible if the marginal productivity of labor in corn production were a positive constant? Why or why not?

Chapter 4

Applying the Ramsey model

IV.1 *Agents' behavior, equilibrium factor prices, and dynamic system of the Ramsey model* Consider the Ramsey model for a market economy with perfect competition, CRRA utility function, and exogenous Harrod-neutral technical progress at a constant rate $g > 0$.

a) Write down the dynamic budget identity and the NPG condition for the representative household expressed in absolute terms (not per capita terms).

b) Derive the corresponding dynamic budget constraint and NPG condition expressed in per capita terms.

c) Set up the consumption-saving problem of the representative household and derive the first-order conditions and the transversality condition.

d) Derive the Keynes-Ramsey rule.

e) Under the assumption of perfect competition, characterize the representative firm's behavior and determine the equilibrium (real) factor prices and the equilibrium real interest rate.

f) The model can be reduced to two coupled differential equations in the technology-corrected capital per head and the technology-corrected consumption per head. Derive these two differential equations.

IV.2 *A positive technology shock* Consider a Ramsey model for a closed economy. The model can be reduced to two differential equations

$$\begin{aligned} \dot{\tilde{k}}_t &= f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, && (*) \\ \dot{\tilde{c}}_t &= \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g)\tilde{c}_t, && (**) \end{aligned}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s)-\delta-g-n)ds} = 0. \qquad (***)$$

Notation is: $\tilde{k}_t = K_t/(T_t L_t)$ and $\tilde{c}_t = C_t/(T_t L_t) = c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, and L_t is population = labor supply, all at time t. Further, T_t is a measure of the technology level and f is a production function on intensive form, satisfying $f' > 0$, $f'' < 0$, and the Inada conditions. The remaining symbols stand for parameters and all these are positive. Moreover, $\rho - n > (1-\theta)g$.

a) Briefly interpret the equations (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive $\tilde{k}_0$. Can the divergent paths be ruled out? Why or why not?

c) Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then for some external reason an unanticipated technology shock occurs so that T_0 is replaced by $T_0' > T_0$. After this shock everybody rightly expects T to grow forever at the same rate as before. We now study short- and long-run effects of this shock.

d) Illustrate by means of the phase diagram what happens to $\tilde{k}$ and $\tilde{c}$ on impact, i.e., immediately after the shock, and in the long run.

e) What happens to the real interest rate on impact and in the long run?

f) Why is the sign of the impact effect on the real wage ambiguous (at the theoretical level) as long as f is not specified further?[1]

g) Compare the real wage in the long run to what it would have been without the shock.

h) Suppose $\theta = 1$. Why is the sign of the impact effect on per capita consumption ambiguous? *Hint:* $c = (\rho - n)(k + h)$.

[1]Remark: for "empirically realistic" production functions (having elasticity of factor substitution larger than elasticity of production w.r.t. capital), the impact effect *is* positive, however.

i) Compare per capita consumption in the long run to what it would have been without the shock.

IV.3 *Short questions* (can be answered by a few well chosen sentences)

a) Can a path *below* the saddle path in the $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

b) Can a path *above* the saddle path in the $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

c) Answer questions b) and c) now presuming that we are dealing with the solution of the problem from the point of view of a social planner in the Ramsey model.

d) In what sense does the Ramsey model imply a more concise theory of the long-run rate of return than do, e.g., the Solow model or the Diamond OLG model?

e) Briefly, assess the theory of the long-run rate of return implied by the Ramsey model. That is, mention what you regard as strengths and weaknesses of the theory.

IV.4 *Productivity slowdown* Consider a Ramsey model of a market economy with perfect competition in all markets. The model can be reduced to two coupled differential equations (using standard notation):

$$\begin{aligned} \dot{\tilde{k}} &= f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, && (*) \\ \dot{\tilde{c}} &= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}, && (**) \end{aligned}$$

together with the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = 0. \qquad (***)$$

a) Equation (*) generally holds for a closed economy with a per capita production function f, a capital depreciation rate δ, a growth rate n of the labour force and a growth rate g of labour efficiency. Explain this in more detail.

b) Equation (**) emerges from assumptions specific to the Ramsey model. Give a brief account of this.

c) From now on it is assumed that $f(0) > 0$. State in words the economic interpretation of this assumption.

d) Construct a phase diagram to illustrate the dynamics of the model.

e) Assume that $\tilde{k}_0 > k^*$, where $\tilde{k}_0$ is the initial capital intensity and $\tilde{k}^*$ is the capital intensity of the economically interesting steady state (which is thus assumed to exist). Show in the phase diagram the evolution over time brought about by the model. Next, show in a graph having time on the horizontal axis (i.e. a "time diagram") the evolution of $\tilde{k}$, $\tilde{c}$, r and w/A (standard notation).

f) Assume instead that the economy has been in steady state until time t_0. Then g unexpectedly shifts down to a lower constant level g'. The economic agents will immediately after time t_0 form expectations about the future that include the new lower growth rate in labour efficiency. Using a phase diagram, show how $\tilde{k}$ and $\tilde{c}$ evolve in the economy for $t \geq t_0$. As for $\tilde{c}$ the sign of the immediate change cannot be determined without more information (why not?); but the direction of movement in the future can be determined unambiguously.

g) Show in a diagram the qualitative features of the time profiles of $\tilde{k}$, $\tilde{c}$, r and w/A for $t \geq t_0$. *Hint:* it is important to realize how the shift in g may affect the $\dot{\tilde{k}} = 0$ locus and the $\dot{\tilde{c}} = 0$ locus.

h) What is the growth rate of output per worker and the real wage, respectively, in the long run? Are these growth rates diminishing or increasing over time in the adjustment process towards the new steady state? Give a reason for your answer.

IV.5 *Short questions*

a) Germany and Japan had a very high per-capita growth rate after the second world war (and up to the mid 1970s). "As predicted by neoclassical growth theory (Solow or Ramsey style), sooner or later the very fast growth came to an end." Do you think this statement makes sense? Briefly explain.

b) Consider a Ramsey model with exogenous Harrod-neutral technical progress and a neoclassical CRS production function which is not Cobb-Douglas. Can the predictions of the model be consistent with Kaldor's "stylized facts"? Give a reason for your answer.

IV.6 *Short questions* Consider the Ramsey model for a market economy with perfect competition.

a) "If and only if the production function is Cobb-Douglas with CRS and time-independent output elasticity w.r.t. capital, is the Ramsey model consistent with Kaldor's stylized facts." True or false? Why?

b) "The Ramsey model predicts that for countries with similar structural characteristics, the further away from its steady state a country is, the higher is its per capita growth rate." True or false? Why?

IV.7 *Aggregate saving and the return to saving* Consider a Ramsey model for a closed competitive market economy with public consumption, transfers, and capital income taxation. The government budget is always balanced. The model leads to the following differential equations (standard notation)

$$\begin{aligned}
\dot{\tilde{k}} &= f(\tilde{k}) - \tilde{c} - \tilde{\gamma} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \text{ given}, && (*) \\
\dot{\tilde{c}} &= \frac{1}{\theta}\left[(1-\tau_r)(f'(\tilde{k}) - \delta) - \rho - \theta g\right]\tilde{c}, && (**)
\end{aligned}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left[(1-\tau_r)(f'(\tilde{k}_s)-\delta)-g-n\right]ds} = 0. \qquad (***)$$

All parameters are positive and it is assumed that $\rho > n$ and

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \frac{\rho + \theta g}{1-\tau_r} > n + g > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta.$$

The government controls $\tilde{\gamma}$, $\tau_r \in (0,1)$, and the transfers. Until further notice $\tilde{\gamma}$ and τ_r are kept constant over time and the transfers are continuously adjusted so that the government budget remains balanced.

a) Briefly interpret (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path that the economy follows, for a given $\tilde{k}_0 > 0$. Comment.

c) Is it possible for a steady state to exist without assuming f satisfies the Inada conditions? Why or why not?

d) Suppose the economy has been in steady state until time t_0. Then, suddenly τ_r is increased to a higher constant level. Illustrate by a phase diagram what happens in the short and long run. Give an economic interpretation of your result.

e) Does the direction of movement of $\tilde{k}$ depend on θ? Comment.

f) Suppose $\theta = 1$. It is well-known that in this case the substitution effect and the income effect on current consumption of an increase in the (after-tax) rate of return offset each other. Can we from this conclude that aggregate saving does not change in response to the change in fiscal policy? Why or why not? Add some economic intuition. *Hint* regarding the latter*:* when $\theta = 1$, $c_t = (\rho - n)(a_t + h_t)$, where

$$h_t \equiv \int_t^{\infty} (w_s + x_s) e^{-\int_t^s [(1-\tau_r)r_\tau - n]d\tau} ds;$$

here, x_s is per capita transfers at time s. Four "effects" are in play, not only the substitution and income effects.

IV.8 *Short questions* We assume that a given selection of countries (considered as closed economies) can be described by the Ramsey model for a closed economy with Harrod-neutral technical progress at a constant positive rate. For each country parameters and initial conditions are such that an equilibrium path and a steady state exist (standard notation).

a) "The model predicts that for countries with the same technology (same F, A_0, g, and δ), differences in per capita growth rates are only temporary and due to the transitional dynamics." True or false? Comment.

b) "The model predicts that for countries with the same technology, differences in per capita income are only temporary and due to the transitional dynamics." True or false? Comment.

IV.9 *Command optimum* Consider a Ramsey setup with CRRA utility and exogenous technical progress at the constant rate $g \geq 0$. Suppose resource allocation is not governed by market mechanisms, but by a "social planner" – by which is meant an "all-knowing and all-powerful" central authority. The social planner is not constrained by other limitations than those from technology and initial resources and can thus ultimately decide on the resource allocation within these confines.

The decision problem of the social planner is (standard notation):

$$\begin{aligned} \max_{(c_t)_{t=0}^{\infty}} U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} &(1)\\ c_t &\geq 0, &(2)\\ \dot{\tilde{k}}_t &= f(\tilde{k}_t) - \frac{c_t}{A_t} - (\delta + g + n)\tilde{k}_t, &(3)\\ \tilde{k}_t &\geq 0 \qquad \text{for all } t \geq 0, &(4) \end{aligned}$$

where $\delta + g > 0$ and $\theta > 0$ (in case $\theta = 1$, the expression $(c^{1-\theta} - 1)/(1-\theta)$ should be interpreted as $\ln c$). Assume $\rho - n > (1-\theta)g$ and that the production function satisfies the Inada conditions.

a) Briefly interpret the problem, including the parameters. Comment on the inequality $\rho - n > (1-\theta)g$.

b) Derive a characterization of the solution to the problem.

c) Compare the solution with the equilibrium path generated by a competitive market economy described by a Ramsey model with the same preferences and technology as above. Comment.

IV.10 *Some quotations.*

a) Two economists – one from MIT and one from Chicago – are walking down the street. The MIT economist sees a 100 dollar note lying on the sidewalk and says: "Oh, look, what a fluke!". "Don't be silly, obviously it is false", laughs the Chicago economist, "if it wasn't, someone would have picked it up". Discuss in relation to the theoretical concepts of arbitrage and equilibrium.

b) A riddle asked by Paul Samuelson (Nobel Prize winner 1970): A physicist, a chemist, and an economist are stranded on an island, with nothing to eat. A can of soup washes ashore. But how to open it? The physicist says “let us smash the can open with a rock”. The chemist says “let us build a fire and heat the can first”. Guess what the economist says?

IV.11 “When the steady state of a dynamic system is a saddle point, then the system is saddle-point stable.” True or false? Why?

Chapter 5

Human capital and economic growth

V.1 *Human capital considered as just another form of capital* A famous paper by Mankiw, Romer, and Weil (1992) carries out a cross-country regression analysis (98 countries, 1960-1985) based on the aggregate production function

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}, \qquad 0 < \alpha < \alpha + \beta < 1, \tag{*}$$

where Y is GDP, K aggregate capital input, H aggregate human capital input, A the technology level, and L input of man-hours, $L_t = L_0 e^{nt}$, n constant. The gross investment rates in the two types of capital are a fraction s_K and s_H of GDP, respectively. Assuming that $A_t = A_0 e^{gt}$, $g \geq 0$, is the same for all countries in the sample (apart from a noise term affecting A_0), the authors conclude that $\alpha = \beta = 1/3$ fits the data quite well.

Let h denote average human capital, i.e., $h \equiv H/L$, and suppose all workers at any time t have the same amount of human capital, equal to h_t.

a) Show that (*) can be rewritten on the form $Y_t = F(K_t, X_t L_t)$, where F is homogeneous of degree one. Indicate what X_t must be in terms of h and A and what the implied "labor quality" function is.

b) When we study individual firms' decisions, this alternative way of writing the production function is more convenient than the form (*). Explain why.

c) Within a Ramsey-style set-up, where s_K and s_H are endogenous and time-dependent, it can be shown that the economy converges to a steady state with $\tilde{y} \equiv Y/(AL) = (\tilde{k}^*)^{\alpha}(\tilde{h}^*)^{\beta}$, where $\tilde{k}^*$ and $\tilde{h}^*$ are the constant steady state values of $\tilde{k} \equiv K/(AL)$ and $\tilde{h} \equiv h/A$. Find the long-run growth rate of $y \equiv Y/L$. Is per capita growth in the long run driven by human capital accumulation?

In Section 11.2 of the textbook by Acemoglu the author presents a Ramsey-style one-sector approach to human and physical capital accumulation. The production function is

$$Y_t = F(K_t, h_t L_t), \tag{**}$$

where F is a neoclassical production function with CRS and satisfying the Inada conditions. We shall compare the implications of (*) and (**) under the assumption that A_t in (*) is time-independent and equals 1.

d) Does (*) and (**) imply the same or different answers to the last question in c)? Comment.

e) Briefly evaluate the set-up in Section 11.2 of the Acemoglu textbook, that is (**), from a theoretical as well as empirical perspective.

f) If we want a linear labor quality function, as implicit in (**), to be empirically realistic, there is an alternative approach that might do better. What approach is that?

V.2 *A life-cycle approach to human capital* We consider a market economy where people first attend school full-time and then work full-time until death. People are born with the same abilities. If a person attends school for S years, he or she obtains human capital $h = h(S)$, $h' > 0$. A person "born" at time v (v arbitrary) chooses S to maximize

$$HW_v = \int_{v+S}^{\infty} \hat{w}_t h(S) e^{-(r+m)(t-v)} dt, \tag{*}$$

where $\hat{w}_t$ is the market-determined real wage per year per unit of human capital at time t, r is a constant real interest rate, and m is a parameter such that the probability of surviving at least until age $\tau > 0$ is $e^{-m\tau}$. It is assumed that owing to technical progress,

$$\hat{w}_t = \hat{w}_0 e^{gt}, \tag{**}$$

where g is an exogenous constant satisfying $0 < g < r + m$.

a) Interpret the decision problem, including the parameter m. Is there a sense in which the infinite horizon in (*) can to some extent be defended as an approximation?

b) Let the optimal S for a person be denoted S^*. Given (*) and (**), show that S^* satisfies the first-order condition $h'(S^*)/h(S^*) = r + m - g \equiv \tilde{r}$. *Hint:* Substitute (**) into (*) and move $h(S)$, and possibly other constants, outside the integral; then perform the integration by applying that for given constants a, b, and c, where $a < 0$, $\int_b^\infty e^{a(t-c)}dt = \left.\frac{e^{a(t-c)}}{a}\right|_b^\infty = \frac{e^{a(b-c)}}{-a}$; finally, maximize w.r.t. S.

c) Provide the economic intuition behind this first-order condition.

Let

$$h(S) = S^\eta, \qquad \eta > 0. \tag{***}$$

d) Solve for S^*. It can be shown that for the second-order condition to ensure that the first-order condition gives an optimum, the elasticity of h' w.r.t. S must be smaller than the elasticity of h w.r.t. S at least at $S = S^*$. Check whether this condition is satisfied in the case (***).

e) With one year as the time unit, let the parameter values be $\eta = 0.6$, $r = 0.06$, $m = 0.008$, and $g = 0.018$. What is the value of the optimal S measured in years? Comment.

f) How will an increase in life expectancy affect m and the optimal S, respectively? What is the intuition?

Suppose there is perfect competition in all markets and that the representative firm chooses capital input, K_t, and labor input (measured in man-years), L_t, in order to maximize profit, given the production function

$$Y_t = F(K_t, A_t h L_t),$$

where Y_t is output, A_t is the technology level (exogenous), and F is a neoclassical production function with constant returns to scale and satisfying the Inada conditions. There is a constant capital depreciation rate $\delta > 0$. Suppose further that the country considered is a small open economy which is fully integrated in the world market for goods and financial capital. Let the real interest rate in this market be a constant equal to r.

g) Let the equilibrium real wage *per year* at time t for a typical member of the labor force be denoted w_t. Find w_t. *Hint:* Determine $\tilde{k} \equiv K_t/(A_t h L_t)$ from one of the firm's first-order conditions.

h) Express w_t in terms of $\hat{w}_t$. What is the growth rate of w_t according to the information given in the introductory paragraph above? And what is the implied growth rate of A_t?

i) Let $y_t \equiv Y_t/L_t$. Does the level of education affect the level of y_t in this economy? Find the growth rate of y_t. Does education affect the growth rate of y_t? Comment.

V.3 *Human capital and catching up* Consider a country which is fully integrated in the world market for goods and financial capital. Suppose that the real interest rate in the world market is a constant, $r > 0$. Let the aggregate production function be $Y_t = F(K_t, A_t hL_t)$ (standard notation). The technology level A_t evolves according to the catching-up hypothesis

$$\frac{\dot{A}_t}{A_t} = \xi \frac{\tilde{A}_t}{A_t},$$

where $\xi > 0$, and $\tilde{A}_t = \tilde{A}_0 e^{gt}$ is the world frontier technology level, $g > 0$.[1] We assume $A_0 < \tilde{A}_0$ and $0 < \xi < g$.

a) Will the country's technology level be able to catch up in the long run? *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$, has the solution $x(t) = (x_0 - x^*)e^{-at} + x^*$, where $x^* = b/a$; let $x(t) \equiv A_t/\tilde{A}_t$ and express the growth rate of x in terms of x, ξ, and g.

Let m be a measure of the country's mortality rate and suppose the country is a developing country with average human capital $h = \eta/(r+m-g)$, where $r + m > g$, and η is a positive parameter.

b) Let the catching-up ability be an increasing function of aggregate human capital, H, i.e., $\xi = \xi(H)$, $\xi' > 0$. Can a general health improvement in the country help in catching up? Why or why not?

V.4 *AK model with human and physical capital* We consider a closed market economy with education in private schools that charge a fee from students. Under perfect competition the representative firm chooses capital input, K^d, and labor input, L^d, in order to maximize profit, given the production function

$$Y = F(K^d, qL^d), \tag{5.1}$$

where Y is output, q is "quality" (or "productivity") of labor, and F is a neoclassical production function with constant returns to scale.

[1] Cf. Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

a) Given q and the aggregate supplies of capital, K, and labor, L, respectively, determine the real rental rate, $\hat{r}$, for capital and the real wage, $\hat{w}$, per unit of *effective* labor input in equilibrium.

We shall in this exercise assume that $q = h \equiv H/L$, where H is aggregate human capital in the labor force formed in the following way. Aggregate output (= aggregate gross income) is used for consumption, C, investment, I_K, in physical capital and investment, I_H, in human capital, i.e.,

$$Y = C + I_K + I_H.$$

The dating of the variables is suppressed where not needed for clarity. The increase per time unit in the two kinds of capital is given by

$$\begin{aligned} \dot{K} &= I_K - \delta_K K, \qquad \text{and} \\ \dot{H} &= I_H - \delta_H H, \end{aligned} \tag{5.2}$$

respectively. The depreciation rates, δ_K and δ_H, are positive constants.

The representative household (dynasty) has infinite horizon and consists of L members, where $L = L_0 e^{nt}$, $n \geq 0$, $L_0 > 0$. Each family member supplies inelastically one unit of labor per time unit. From (5.2) and the definition $H \equiv hL$ follows the per capita human capital accumulation equation:

$$\dot{h} = i - (\delta_H + n)h, \tag{5.3}$$

where $i \equiv I_H/L$ is the per capita educational cost (in real terms) per time unit.

b) Present a derivation of (5.3).

Let θ and ρ be positive constants, where $\rho > n$. Let a be per capita financial wealth, r the real interest rate, and $c_t \equiv C_t/L_t$. The representative household chooses a path $(c_t, i_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} && (5.4) \\ c_t &\geq 0, \; i_t \geq 0, && (5.5) \\ \dot{a}_t &= (r_t - n)a_t + \hat{w}_t h_t - i_t - c_t, \qquad a_0 \text{ given}, && (5.6) \\ \dot{h}_t &= i_t - (\delta_H + n)h_t, \qquad h_0 > 0 \text{ given}, && (5.7) \\ \lim_{t\to\infty} a_t e^{-\int_t^{\tau}(r_s - n)ds} &\geq 0, && (5.8) \\ h_t &\geq 0 \text{ for all } t. && (5.9) \end{aligned}$$

c) Briefly interpret the six elements in this decision problem. Why is there a non-negativity constraint on i_t?

d) Apply the Maximum Principle (for the case with two control variables and two state variables) to find the first-order conditions for an interior solution.

e) Derive from the first-order conditions the Keynes-Ramsey rule.

f) Set up a no-arbitrage equation showing a relationship between $\hat{w}$ and r. You may either apply your intuition or derive the relationship from the first-order conditions. In case you apply your intuition, check whether it is consistent with the first-order conditions. *Hint:* along an interior optimal path the household should be indifferent between placing the marginal unit of saving in a financial asset yielding the rate of return r or in education to obtain one more unit of human capital.

Assume now for simplicity that the aggregate production function is:

$$Y = K^{\alpha}(qL)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

g) Determine the real interest rate in equilibrium in this case.

Suppose parameters are such that $\dot{c}/c > 0$ and U_0 is bounded.

h) The no-arbitrage equation from f) (which is needed for an *interior* solution to the household's decision problem) requires a specific value of $\hat{k} \equiv K/H$ to be present. Assuming, for simplicity, that $\delta_K = \delta_H = \delta$, determine the required value of $\hat{k}$. Let this value be denoted $\hat{k}^*$. Explain what happens to begin with if the historically given initial $\hat{k}$ in the economy differs from $\hat{k}^*$.

i) Suppose the historically given initial $\hat{k} \neq \hat{k}^*$. Illustrate graphically the time profile of $\hat{k}$. Will $\hat{k}$ reach $\hat{k}^*$ in finite time?

V.5 "*Broad capital*". *Subsidizing education* The point of departure is the same model as in Problem V.4 (so it is an advantage if you have already solved that problem).

a) After some time the economy behaves like an AK model with an endogenous per capita growth rate that depends negatively on the rate of impatience. Explain. Find "A" (the factor of proportionality between Y and "aggregate capital" $\tilde{K} \equiv K + H$).

b) Briefly evaluate the model from a theoretical and empirical perspective. Doing this, you may compare the model with the extended Solow model of Mankiw, Romer, and Weil (1992).

We shall now discuss subsidizing education. For simplicity, we assume $\delta_K = \delta_H = 0$.

c) Consider a constant subsidy, $\sigma \in (0,1)$, to education such that per unit of investment in education the private cost is only $1-\sigma$. That is, i_t in (5.7) is replaced by $(1-\sigma)i_t$. Suppose the subsidy is financed by lump-sum taxes. Will such a subsidy affect long-run growth in this model? Explain. *Hint:* In answering, you may use your intuition or make a formal derivation. A quick approach can be based on the no-arbitrage condition in the new situation.

d) Assuming the social welfare function is the same as the objective function of the representative household, will the subsidy (combined with lump-sum taxation) increase or decrease welfare? Explain.

e) Since there is a representative household and no externalities in the model as it stands, it could be argued that there is no need for a subsidy. Going outside the model, what kinds of motivations for subsidizing education in the real world might be put forward?

V.6 *Short questions* These questions relate to the model in Problem V.4.

a) Comment on the model in relation to the concepts of fully endogenous growth and semi-endogenous growth.

b) Comment on the model in relation to the issue of scale effects.

c) What do you guess will be the consequence w.r.t. long-run per capita growth of assuming $q = h^{\varphi}$, $0 < \varphi < 1$? Comment.

V.7 *A model with human capital and R&D* Consider a closed economy with two production sectors, manufacturing and R&D. For simplicity we imagine that the R&D sector is fully managed by the government. Time is continuous. At the aggregate level we have:

$$\begin{aligned}
Y_t &= A_t^{\gamma} K_t^{\alpha} (\bar{h}_t L_{Y,t})^{1-\alpha}, \qquad \gamma > 0, 0 < \alpha < 1, && (1)\\
\dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, && (2)\\
\dot{A}_t &= \eta A_t^{\varphi} (\bar{h}_t L_{A,t})^{1-\varepsilon}, \qquad \eta > 0, \varphi < 1, 0 \leq \varepsilon < 1, && (3)\\
L_{Y,t} + L_{A,t} &= L_t = \text{ labor force}. && (4)
\end{aligned}$$

Here A_t measures the stock of technical knowledge and $\bar{h}_t$ is average human capital in the labor force at time t (otherwise notation is standard).

From now we ignore the explicit dating of the variables unless needed for clarity. Let the growth rate of a variable $x > 0$ be denoted g_x (not necessarily positive and not necessarily constant over time). Assume that all variables in the model are positive and remain so.

a) Interpret the case $0 < \varphi < 1$ versus the case $\varphi < 0$. Interpret the case $0 < \varepsilon < 1$ versus the case $\varepsilon = 0$.

b) Write down a growth accounting relation expressing g_Y in terms of g_A, g_K, $g_{\bar{h}}$, and g_{L_Y}. In addition, express g_A in terms of A, $\bar{h}$, L_A.

c) Presupposing $g_A > 0$, express the growth rate of g_A in terms of g_A, $g_{\bar{h}}$, and g_{L_A}.

Let the time unit be one year. Suppose an individual "born" at time v (v for "vintage") spends the first S years of life in school and then enters the labor market with a human capital level which at time $t \geq v + S$ is $h(S)$, where $h' > 0$. We ignore the role of teachers and schooling equipment. We assume that life expectancy is constant over time and that S is the same for all individuals independently of time of birth. After leaving school, individuals works full-time until death. We assume that the population grows at a constant rate $n > 0$:

$$N_t = N_0 e^{nt}. \tag{*}$$

Then, with a stationary age distribution in society,

$$L_t = (1 - \beta) N_t, \tag{**}$$

where β is the constant fraction of this population under education (β will be an increasing function of S).

From now on assume that (i) the economy is in balanced growth, defined as a path along which g_Y, g_C, g_K, g_A, $g_{\bar{h}}$, g_{L_A}, and g_{L_Y} are constant; (ii) $Y - cN > 0$ for all t.

d) Can we be sure that $g_{L_Y} = n$ along a balanced growth path? Why or why not?

e) Show that

$$g_A = \frac{(1 - \varepsilon)(g_{\bar{h}} + n)}{1 - \varphi}.$$

f) From a certain general proposition we can be sure that along a BGP, $g_Y = g_K$. What proposition and why?

g) Defining $y \equiv Y/L$, it follows that under balanced growth,

$$g_y = \frac{\gamma g_A}{1-\alpha} + g_{\bar{h}}.$$

How?

h) It is possible to express g_y under balanced growth in terms of only one endogenous variable, $g_{\bar{h}}$. Show this.

i) Comment on the role of n in the resulting formula for g_y.

V.8 This problem presupposes that you have solved Problem V.7, in particular question h).

a) Consider two connected statements: "The model in Problem V.7 assumes diminishing marginal productivity of knowledge in knowledge creation" and "hence, sustained exponential per capita growth requires $n > 0$ or $g_{\bar{h}} > 0$". Evaluate these two statements.

b) It seems plausible that the world population will be non-increasing in the long run. In this perspective, what is the prospect of sustained exponential per capita growth in the world economy according to the model?

Suppose $h(S) = S^{\mu}$, $\mu > 0$.

c) Demographic data exists saying that life expectancy in industrialized countries tends to grow arithmetically, in fact, almost by a quarter of a year per year. How is S likely to be affected by this?

d) Assuming this to continue, and going a little outside the model, what is the prospect of sustained exponential per capita growth in the world economy in the long run? Discuss.

e) Although hardly realistic, suppose $h(S)$ is exponential as sometimes assumed in the literature. Then again answer c).

V.9 *Short questions*

a) In the theory of human capital and economic growth we encounter different assumptions about the schooling technology. List some possible specifications. Briefly comment.

b) In the Acemoglu textbook a model where long-run productivity growth is driven by a combination of physical and human capital accumulation is presented. Set up the aggregate production function of the model. Briefly comment on what you think are strengths and/or weaknesses of the model.

c) It is sometimes argued that results like the Arrow result, $g_y = \lambda n/(1-\lambda)$ (standard notation), are from an empirical point of view falsified by the fact that cross-country growth regressions do not tend to indicate a positive correlation between per capita economic growth and population growth. Evaluate this argument.

Chapter 6

Simple reduced-form AK models

VI.1 *The learning-by-investing model: two alternative cases* Consider a closed market economy with N profit maximizing firms, operating under perfect competition (N "large"). The size of the labor force (= employment = population) is $L_t = L_0 e^{nt}$, where n is constant, $n \geq 0$. Aggregate output at time t is Y_t per time unit. Output is used for consumption, $C_t \equiv c_t L_t$, and investment in physical capital K_t so that $\dot{K}_t = Y_t - C_t - \delta K_t$, where $K_0 > 0$ is given and $\delta \geq 0$ is the rate of physical decay of capital. The initial value $K_0 > 0$ is given. There is a perfect market for loans with a short-term real interest rate r_t. There is no uncertainty (perfect foresight).

The production function for firm i ($i = 1, 2, ..., N$) is

$$Y_{it} = F(K_{it}, A_t L_{it}), \tag{1}$$

where F is neoclassical and has CRS. The variable A_t evolves according to

$$A_t = e^{\varepsilon t} K_t^{\lambda}, \quad \varepsilon \geq 0, 0 < \lambda \leq 1, \tag{2}$$

where ε and λ are given constants and $K_t = \sum_i K_{it}$.

a) Briefly interpret (2), including the parameters and the variable A.

Suppose each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables.

b) In general equilibrium, determine r and the aggregate production function at time t.

c) Assume $\varepsilon > 0$, $\lambda < 1$, and $n > 0$. Show that without knowing anything in detail about household saving behavior we can determine the growth rate of Y and $y \equiv Y/L$ under balanced growth, assuming gross saving is positive. *Hint:* If a production function $Y = F(K, XL)$ is homogeneous of degree one, then
$$\frac{Y}{K} = F(1, \frac{XL}{K});$$
combine this with a certain general balanced growth property.

d) What type of endogenous growth is this model (as given in c)) capable of generating?

From now, let $\lambda = 1$ and $\varepsilon = n = 0$. Moreover, assume the household sector is Ramsey-style with inelastic labor supply, instantaneous CRRA utility of per capita consumption with parameter $\theta > 0$, and a constant utility discount rate $\rho > 0$. Finally, assume that the inequalities $F_1(1, L) > \delta + \rho$ and $\rho > (1 - \theta)\gamma$ hold, where γ is the equilibrium growth rate of c.

e) Determine γ. Comment.

f) Determine the equilibrium growth rate for k $(\equiv K/L)$ and y $(\equiv Y/L)$, respectively. *Hint:* If you have shown that the real interest rate is a constant and that the aggregate production function is AK-style, then it is enough to refer to general knowledge about reduced-form AK models.

g) What would happen if $n > 0$? Why?

VI.2 *A production subsidy in the learning-by-investing model: Paul Romer's case* This problem presupposes that you have already solved Problem VI.1. In continuation of the last part of that problem, we consider the case $\lambda = 1$ and $\varepsilon = n = 0$ and the same Ramsey-style household sector.

a) There is a certain feature of the economy which "invites" government intervention in the market mechanism. What is this feature?

We introduce a government which contemplates to implement a production subsidy and finance it by a consumption tax. The idea is to subsidize production at a constant rate $\sigma > 0$ so that if firm i produces and sells Y_i, its revenue is $(1 + \sigma)Y_i$. Assume you, as an economic advisor, are asked by the government to suggest an optimal size of σ, given that the social welfare function coincides with the criterion function of the representative household.

b) Derive a formula for the recommendable size of σ. *Hint:* Set up the social planner's problem,[1] derive the first-order conditions and the TVC. Determine the implied growth rate of c_t. Next, use your general knowledge about reduced-form AK models to determine the growth rates of k_t and y_t (a brief verbal account is enough). Finally, use that for σ to be optimal, σ should ensure that the net rate of return to capital investment implied by the aggregate production technology equals the net rate of return on saving faced by the consumer.

c) Assume that the government always balances the budget and has no other expenditures than the production subsidy. Find the consumption tax rate, τ, needed to finance the subsidy. *Hint:* At a certain stage in the argument you will need knowledge about what value is taken by c_t/k_t in the social planner's solution. You do not have to derive this value; it is given here: $c/k = F(1, L) - \delta - g_{SP}$.

d) Let F in (1) be $Y_{it} = BK_{it}^{\alpha}(A_t L_{it})^{1-\alpha}$, $0 < \alpha < 1$, and assume $\alpha = 1/3$. What is the implied value of σ according to your formula in b)?

e) With one year as the time unit, let $B = 0.003$, $L = 1000$, $\delta = 0.05$, $\theta = 2$, and $\rho = 0.02$. Check whether the implied value of r under laissez-faire makes sense empirically. Next find the implied value of τ according to your formula in c). (Do not expect a "modest" result for τ, given that neither the model nor the found value for σ are "modest".)

f) Given the model, *is* the suggested policy, (σ, τ), optimal or might there for example be distortionary effects associated with the financing? Briefly discuss.

g) Whatever the answer to f), briefly suggest other subsidy policies which could do the job.

h) Briefly evaluate the present model.

VI.3 *A subsidy to saving in Paul Romer's learning-by-investing model* Consider a closed market economy with perfect competition where firm no. i has the production function

$$Y_{it} = F(K_{it}, T_t L_{it}),$$

[1] Recall that a "social planner" is a hypothetical "all-knowing and all-powerful" central authority who can fully decide on the resource allocation within the constraints given by technology and initial resources. That is, by definition, a social planner need not care about market mechanisms and market prices.

where F is a neoclassical production function with CRS and satisfying the Inada conditions (standard notation). It is assumed that the technology level T_t satisfies

$$T_t = K_t^{\lambda}, \qquad 0 < \lambda \leq 1.$$

Time, t, is continuous. There is no uncertainty. At the aggregate level,

$$\dot{K}_t \equiv \frac{dK_t}{dt} = Y_t - C_t - \delta K_t, \qquad \delta > 0, \quad K_0 > 0 \text{ given.}$$

a) Determine the equilibrium real interest rate, r, and the aggregate production function. Comment.

From now we assume $\lambda = 1$.

b) Determine the equilibrium real interest rate, r, and the aggregate production function in this case. Comment.

There is a representative Ramsey household with instantaneous utility function of CRRA type,

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0,$$

where c is per capita consumption ($c \equiv C/L$). The rate of time preference is a constant $\rho > 0$. There is no population growth ($n = 0$). Until further notice there are no taxes and subsidies.

c) Determine the equilibrium growth rate of c and name it g_c^*.

From now, assume (A1) $F_1(1, L) - \delta > \rho$ and (A2) $\rho > (1 - \theta)g_c^*$.

d) What could be the motivation for these two assumptions?

e) Determine the growth rate of $k \equiv K/L$ and $y \equiv Y/L$. A detailed derivation involving the transversality condition need not be given; instead you may refer to a general property of AK and reduced-form AK models in a Ramsey framework where (A2) holds.

f) Set up and solve the social planner's problem, assuming the same criterion function as that of the representative household. *Hint*: the linear differential equation $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$ has the solution:

$$x(t) = (x(0) - \frac{c}{a+h})e^{-at} + \frac{c}{a+h}e^{ht}.$$

g) Now consider again the decentralized market economy, but suppose there is a government that wants to establish the social planner's allocation by use of a subsidy, σ, to private saving such that the after-subsidy-rate of return on private saving is $(1+\sigma)r$. Let the subsidy be financed by a lump-sum tax on all households. Determine σ such that the social planner's allocation is established, if this is possible. Comment.

VI.4 *Productive government services* Consider a closed market economy with constant population, L utility maximizing households, and M profit maximizing firms, operating under perfect competition (L and M are constant, but "large"). There is also a government that free of charge supplies a service G per time unit. Each household has an infinite horizon and supplies inelastically one unit of labor per time unit. Aggregate output is Y per time unit and output is used for private consumption, $C \equiv cL$, the public service, G, and investment, I, in (physical) capital, i.e., $Y = C + G + I$. The stock of capital, K, changes according to $\dot{K} = I - \delta K$, where $\delta \geq 0$ is the rate of physical decay of capital. Variables are dated implicitly. The initial value $K_0 > 0$ is given. The capital stock in society is owned, directly or indirectly (through bonds and shares), by the households. There is perfect competition in the labor market. The equilibrium real wage is called w. There is a perfect market for loans with a real interest rate, r, and there is no uncertainty.

The government chooses G so that

$$G = \gamma Y,$$

where the constant $\gamma \in (0, 1-\alpha]$ is an exogenous policy parameter, and α is defined below. The government budget is always balanced and the service G is the only public expenditure. Only households are taxed. The tax revenue is

$$\left[\tau(ra + w) + \tau_\ell\right] L = G, \tag{GBC}$$

where a is per capita financial wealth, and τ and τ_ℓ denote the income tax rate and a lump-sum tax, respectively. The tax rate τ is a given constant, $0 \leq \tau < 1$, whereas τ_ℓ is adjusted when needed for (GBC) to be satisfied.

The production function for firm i is

$$Y_i = AK_i^\alpha (GL_i)^{1-\alpha}, \quad 0 < \alpha < 1, A > 0, \quad i = 1, 2, ..., M. \tag{*}$$

a) Comment on the nature of G.

b) Show that in equilibrium

$$\begin{aligned} r &= \alpha\bar{A} - \delta, \quad \text{where} \quad k \equiv K/L \quad \text{and} \quad \bar{A} \equiv A^{\frac{1}{\alpha}}(\gamma L)^{\frac{1-\alpha}{\alpha}}, \\ Y &= \sum_i Y_i = \sum_i y_i L_i = y \sum_i L_i = yL = Ak^\alpha G^{1-\alpha} L = A^{1/\alpha}(\gamma L)^{(1-\alpha)/\alpha} kL \equiv \bar{A}K. \end{aligned}$$

Suppose the households, all alike, have a constant rate of time preference $\rho > 0$ and an instantaneous utility function with (absolute) elasticity of marginal utility equal to a constant $\theta > 0$.

c) Set up the optimization problem of a household and derive the Keynes-Ramsey rule, given the described taxation system.

d) Write down the transversality condition in a form comparable to the No-Ponzi-Game condition of the household. Comment.

e) Find the growth rate of $k \equiv K/L$ and $y \equiv Y/L$ in this economy (an informal argument, based on your general knowledge about reduced-form AK models, is enough). In case, you need to introduce a restriction on some parameters to ensure existence of equilibrium with growth, do it.

f) Sign $\partial g_c^*/\partial\gamma$ and $\partial g_c^*/\partial L$. Comment in relation to the scale effect issue.

VI.5 This problem relates to Problem VI.4. The model of that problem is essentially the model in Barro (JPE, 1990). Several aspects of the model have been questioned in the literature. One critical aspect is that G enters (*) in a very powerful but arbitrary way. Let q be an index of labor-augmenting productivity considered as a function of the public productive service G, i.e., $q = q(G)$. We assume that taxes are lump-sum. Then it is reasonable to assume that $q' > 0$. Still, $q(G)$ could be strictly concave, for example in the form $q = G^\lambda, 0 < \lambda < 1$. Barro assumes apriori that $\lambda = 1$ and $n = 0$, where n is the population growth rate (= growth rate of the labor force).

a) Suppose $0 < \lambda \leq 1$. Given K and L, what level of G and γ, respectively, maximizes $Y - G$ (i.e., the amount of output which is left for private consumption and capital investment)? Briefly provide the intuition behind your result. *Hint:* by a procedure analogue to that in question b) of Problem VI.4 it can be shown that in equilibrium the aggregate production now is $Y = AK^\alpha(G^\lambda L)^{1-\alpha}$.

From now, let $0 < \lambda < 1$ and $n \geq 0$.

b) Find first g_Y, then g_y, along a BGP. "The public productive service has no effect on the growth rate of y along a balanced growth path." True or false? Why? *Hint:* use that if a production function $Y = F(K, XL)$ is homogeneous of degree one, then

$$1 = F(\frac{K}{Y}, \frac{XL}{Y});$$

combine with the balanced growth equivalence theorem.

c) Compare with the results from f) of Problem VI.4. Comment.

VI.6 This problem presupposes that you have solved Problem VI.4. Indeed, we consider essentially the same economy as that described above with the firm production function (*). There is one difference, however, namely that lump-sum taxation is not feasible. Hence, let $\tau_\ell = 0$ for all $t \geq 0$.

a) Examine whether it is possible to fix τ at a level (constant over time and < 1) such that the government budget is still balanced in equilibrium for all $t \geq 0$? *Hint:* find the solution for w; if you need a new restriction on parameters to ensure $\tau < 1$, introduce it.

b) If the welfare of the representative household is the criterion, what proposal to the government do you have w.r.t. the size of γ?

c) With respect to the *form* of taxation (given that a direct lump-sum tax is not feasible), let us see if we can suggest an appropriate tax scheme:

1. is an income tax non-distortionary? Why or why not?
2. is a pure labor income tax likely to work "in practice"? *Hint:* perhaps the needed labor income tax rate is too large in some sense.
3. will a consumption tax work?

VI.7 *Learning by investing as an illustration of the difference between growth accounting and a model of sources of growth* Consider a closed economy with profit maximizing firms, operating under perfect competition. The size of the labor force (= employment = population) is L_t. Aggregate output (GDP) at time t is Y_t per time unit. Output is used for consumption and investment in physical capital, K_t, so that $\dot{K}_t = Y_t - C_t - \delta K_t$, where C_t is consumption and δ is the rate of physical decay of capital, $\delta \geq 0$. The initial value $K_0 > 0$ is given. There is a perfect market for loans with a short-term real interest rate r_t. Time is continuous and there is no uncertainty.

The production function of firm i is

$$Y_{it} = K_{it}^{\alpha}(A_t L_{it})^{1-\alpha}, \qquad 0 < \alpha < 1, \qquad i = 1, 2, \ldots, N,$$

where A_t is the economy-wide technology level, $\sum_i K_{it} = K_t$, $\sum_i L_{it} = L_t$, and N is "large". Suppose each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables, including A_t.

a) In general equilibrium, determine r and the aggregate production function at time t.

b) For a given A_t, find the TFP level (total factor productivity) at time t.

For any variable $x > 0$, let g_x denote its growth rate, $\dot{x}/x$.

c) Following the basic idea in growth accounting, express g_Y analytically in terms of the "contributions" from growth in K, L, and a residual, respectively.

d) Find expressions for the TFP growth rate, the gross income share of capital (aggregate gross income to capital owners divided by GDP = GNP), and the labor income share, respectively.

From now, suppose A_t evolves according to

$$A_t = e^{\varepsilon t} K_t^{\lambda}, \quad \varepsilon > 0, \quad 0 < \lambda < 1, \tag{*}$$

where ε and λ are given constants.

e) Briefly interpret (*).

f) Given (*), express g_Y analytically in terms of the "contributions" from growth in K, L, and a residual, respectively.

g) As a thought experiment, suppose we have empirical data for this economy. Will applying standard growth accounting on the basis of these data lead to over- or underestimation of the "contribution" to output growth from growth in capital? Why?

Let $L_t = L_0 e^{nt}$, where n is a positive constant.

h) Determine the growth rate of $y \equiv Y/L$ under balanced growth, assuming saving is positive. *Hint:* use a certain general balanced growth property.

i) Briefly explain what constitute the ultimate sources of per capita growth according to the model. Compare with what the growth accounting in c) suggested.

VI.8 It is sometimes argued that results like the Arrow result, $g_y = \lambda n/(1-\lambda)$ (standard notation), are from an empirical point of view falsified by the fact that cross-country growth regressions do not tend to indicate a positive correlation between per capita economic growth and population growth. Evaluate this argument.

VI.9 Some researchers emphasize that sharp class differences in a society may hamper economic growth through creating social and political instability and lack of "social capital" (social trust). Comment on this hypothesis in relation to Table X which contains comparative data for South Korea and Philippines (column 1 shows the annual GDP per capita growth rate 1960-90, columns 2-9 provide different descriptive statistics 1960, and columns 10 and 11 give the Gini coefficient for household income before tax).

VI.10 List a few theoretical reasons that may be put forward in support of Acemoglu's hypothesis that differences in the economic institutions (rules of the game) across countries constitute the key for an understanding of the cross-country differences in income per capita.

VI.11 In endogenous growth theory two alternative kinds of scale effects may be present.

a) Give a brief account.

b) Link two alternative learning-by-investing models to these two kinds of scale effects.

c) "Models with scale effects are always problematic." Discuss.

VI.12 *Growth accounting versus sources of growth* We consider a closed competitive economy. Time is continuous. The aggregate capital stock, measured in constant efficiency units, grows according to

$$\dot{K}_t = T_t I_t - \delta K_t, \qquad \delta > 0, \tag{1}$$

where I_t is aggregate gross investment at time t and T_t measures the "quality" (productivity) of newly produced investment goods. We assume T_t is determined by

$$T_t = \tilde{\xi}\left(\int_{-\infty}^{t} I_\tau d\tau\right)^{\lambda}, \qquad \tilde{\xi} > 0,\ 0 < \lambda < \frac{1-\alpha}{\alpha}. \tag{2}$$

The aggregate production function is

$$Y_t = A_t {K_t}^{\alpha} {L_t}^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{3}$$

where Y_t is output, L_t is labor input, growing at a given constant rate $n \geq 0$, and A_t is TFP, growing at a given constant rate $\gamma \geq 0$. Finally, by national income accounting,

$$Y_t = I_t + C_t, \tag{4}$$

where C_t is aggregate consumption.

a) Briefly interpret (1) and (2).

b) What might the empirical motivation for a model like this be?

c) Derive from (2) a differential equation for T_t.

The growth rate of a variable x is denoted g_x.

d) Use the result in c) to find a formula for g_T and on the basis of this express $\dot{g}_T/g_T$ in terms of g_T and g_I.

We now consider a balanced growth path (BGP) with $s \equiv I/Y$ constant and $0 < s < 1$. With the aim of finding g_I along BGP we go through a series of steps.

e) Why must g_I and TI/K be constant along BGP?

f) Why must g_T then be constant along BGP?

g) Express g_T in terms of g_I and then g_K in terms of g_I. *Hint:* first, combine f) and d), then implicate e).

h) Determine g_I and g_Y. *Hint:* use constancy of s; take growth rates on both sides of (**??**).

i) Determine g_y, where $y \equiv Y/L$. Write down the contributions to g_y from disembodied and embodied technical progress, respectively.

j) Imagine we have data for the economy described by this model and that the capital stock is measured in constant efficiency units. Applying the standard growth accounting method, what value for the TFP growth rate (Solow residual) would we find?

k) Suppose $\gamma = 0$. What value for the TFP growth rate would we then find? Compare with the true contribution from technical progress in this case. Comment.

VI.13 *An investment subsidy in Paul Romer's learning-by-investing model* Consider a closed market economy with N profit maximizing firms, operating under perfect competition (N "large"). The size of the labor force (= employment = population) is a constant, L. Aggregate output at time t is Y_t per time unit. Output is used for consumption, $C_t \equiv c_t L$, and investment in physical capital K_t so that $\dot{K}_t = Y_t - C_t - \delta K_t$, where $K_0 > 0$ is given and $\delta \geq 0$ is the rate of physical decay of capital. The initial value $K_0 > 0$ is given. There is a perfect market for loans with a short-term real interest rate r_t. There is no uncertainty (perfect foresight).

The production function for firm i ($i = 1, 2, ..., N$) is

$$Y_{it} = F(K_{it}, A_t L_{it}),$$

where F is neoclassical and has CRS. The variable A_t evolves according to

$$A_t = K_t,$$

where $K_t = \sum_i K_{it}$.

a) Briefly interpret the assumption that $A_t = K_t$ as a special case of some general hypothesis.

Suppose each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables.

b) In general equilibrium, determine r and the aggregate production function at time t.

Let the household sector be Ramsey-style with inelastic labor supply, instantaneous CRRA utility of per capita consumption with parameter $\theta > 0$, and a constant utility discount rate $\rho > 0$. Finally, we assume that $\rho > (1-\theta)g$, where g is the equilibrium growth rate of c.

c) Determine g. What parameter restriction is needed to ensure $g > 0$? From now, assume this restriction satisfied.

d) Determine the equilibrium growth rate for k $(\equiv K/L)$ and $y(\equiv Y/L))$, respectively. *Hint:* In answering you may refer to general knowledge about a certain class of models provided you have shown that the present model belongs to this class.

e) What would happen to the growth rate of c if the population were growing? Why?

We introduce a government which contemplates (i) to pay an investment subsidy $\sigma \in (0,1)$ to the firms so that their capital costs are reduced to $(1-\sigma)(r+\delta)$ per unit of capital per time unit; (ii) to finance this subsidy by a consumption tax rate, τ.

Suppose you, as an economic advisor, are asked by the government to suggest an optimal size of σ, given that the social welfare function coincides with the criterion function of the representative household.

f) Derive a formula for the recommendable size of σ. *Hint:* Set up the social planner's problem, derive the first-order conditions and the transversality condition. Determine the implied growth rate of c_t. Next, use your general knowledge about a certain class of models to determine the growth rates of k_t and y_t (a brief verbal account is enough). Finally, use that for σ to be optimal, σ should ensure that the net rate of return on saving faced by the consumer equals the net rate of return to capital investment implied by the aggregate production technology.

g) Assume that the government always balances the budget and has no other expenditures than the investment subsidy. Find the consumption tax rate, τ, needed to balance the budget. *Hint:* At a certain stage in the argument you will need knowledge about what value is taken by c/k in the social planner's solution. You do not have to derive this value; it is given here: $c/k = F(1, L) - \delta - g_{SP}$.

h) Given the model, *is* the suggested policy (σ, τ) optimal or might there for example be distortionary effects associated with the financing? Discuss.

i) Whatever the answer to h), briefly suggest other subsidy policies which might do the job within the desired economy.

j) Briefly evaluate the model.

VI.14 *Short questions*

a) Define the concepts of fully endogenous growth and semi-endogenous growth. Many fully endogenous growth models rely on knife-edge conditions; give at least three examples. Why may knife-edge conditions be considered problematic?

b) "Economic policy will have no effect on the long-term economic performance of an economy described by a semi-endogenous growth model". True or false? Why?

c) "Arrow's learning-by-investing model predicts that the share of capital income in national income is constant in the long run if and only if the aggregate production function is Cobb-Douglas." True or false? Why?

VI.15 "In models where technical knowledge is endogenous there is a built-in tendency for either weak or strong scale effects (i.e., scale effects on either levels or growth, respectively) to arise." True or false? Explain why.

Chapter 7

R&D and horizontal innovations

VII.1 *The production side of the Lab-Equipment Model (avoiding arbitrary parameter links)* Consider a closed economy with a given aggregate labor supply L, constant over time. There are three production sectors:

Firms in Sector 1 produce basic goods, in the amount Y_t per time unit, under perfect competition.

Firms in Sector 2 produce specialized intermediate goods, in the amount Q_t per time unit, under monopolistic competition and barriers to entry.

Firms in Sector 3 perform R&D to develop technical designs ("blueprints") for new specialized intermediate goods under conditions of perfect competition and free entry.

Basic goods and intermediate goods are nondurable goods. There is no physical capital in the economy. There is a labor market and a market for loans, both with perfect competition. All firms are profit maximizers. Time is continuous.

The representative firm in Sector 1 has the production function

$$Y_t = A \left(\sum_{i=1}^{N_t} x_{it}^{1-\beta} \right) L_t^{\beta}, \qquad A > 0, \quad 0 < \beta < 1, \tag{1}$$

where Y_t is output per time unit, x_{it} is input of intermediate good i ($i = 1, 2, ..., N$), N_t is the number of different types of intermediate goods available at time t, and L_t is labor input. At any point in time, the firms in Sector 1 take this number of "varieties" as given. Aggregate labor supply equals a constant, L, and Sector 1 is the only sector that directly uses labor. In view of clearing in the labor market (where perfect competition is assumed to rule), $L_t = L$, which can be substituted into (1).

The output of basic goods is used partly for consumption, $C_t \equiv c_t L$, partly as input in sector 2, X_t, and partly for R&D investment in Sector 3, Z_t :

$$Y_t = X_t + C_t + Z_t, \tag{*}$$

where in equilibrium $X_t = \psi Q_t$, $\psi > 0$.

Unless needed for clarity, the dating of the time-dependent variables is from now on implicit. Let the basic good be the numeraire and let p_i denote the price of intermediate good i.

a) Find the demand for intermediate good i conditional on full employment. What is the price elasticity of this demand?

b) Suppose $p_i = p$, $\forall i$. Show that the assumed production function, (1), in this case is in conformity with the classical idea from Adam Smith that "there are gains by specialization and division of labor" or, with another formulation, "variety is productive". *Hint:* check how a rise in N affects Y for given L and given total input of intermediates, Q_t.

After having invented the technical design i, the inventor in Sector 3 has taken out (free of charge) a perpetual patent on the commercial use of this design. The inventor then entered Sector 2, starting to supply the new intermediate good corresponding to this design, that is, the intermediate good i. Performing this role, the inventor is called *firm* i. Given the technical design i, firm i can effortlessly transform basic goods into intermediate goods of type i simply by pressing a button on a computer, thereby activating a computer code. The following linear transformation rule applies to all $i = 1, \ldots, N$:

it takes $\psi x_i > 0$ units of the basic good to supply x_i units of intermediate good i,

that is, ψ is the marginal = average cost of supplying intermediate goods.

The market value of firm i in Sector 2 can be written

$$V_{it} = \int_t^\infty \pi_{is} e^{-\int_t^s r_\tau d\tau} ds,$$

where π_{is} is the profit at time s and r_τ is the discount rate at time τ. Since there is a time lag between R&D outlay and a successful R&D outcome and this time lag is stochastic, research is risky. It is assumed, however, that all risk is ideosyncratic and that the economy is "large" with "many" firms in all sectors. By holding their financial wealth in the form of balanced portfolios consisting of diversified equity shares in innovative firms in Sector 2 and 3, investors (the households) can thus essentially avoid risk. This allows the research labs to act in a risk-neutral manner.

c) Interpret the expression for V_{it}. What is the relevant discount rate, r_τ?

Being a monopolist, firm i is a price setter and thus chooses a time path $(p_{is})_{s=t}^{\infty}$ so as to maximize the market value of the firm.

d) This problem can be reduced to a series of static profit maximization problems. Why? Solve the problem. Comment.

e) Show that in general equilibrium,

$$\begin{aligned} x_{it} &= \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv \varphi(A,\beta,\psi)L \equiv x, \text{ for all } i,t, \\ \pi_{it} &= (p_{it}-\psi)x_{it} = (\frac{\psi}{1-\beta}-\psi)x = \frac{\beta}{1-\beta}\psi\varphi(A,\beta,\psi)L \equiv \pi \text{ for all } i,t, \\ V_{it} &= \pi\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds \equiv V_t \text{ for all } i, \end{aligned}$$

To simplify the formulas Acemoglu (pp. 434, 436) introduces two (not entirely innocent) parameter links:

$$A = \frac{1}{1-\beta} \quad \text{and} \quad \psi = 1-\beta. \tag{**}$$

f) Find x_{it} and π_{it} in this special case. In what sense may introducing parameter links in a model be "risky"?

From now we ignore (**) and return to the general case where A and ψ are independent of β.

All the R&D firms in Sector 3 face the same simple "research technology". The rate at which successful research outcomes arrive is proportional to the flow investment of basic goods into research. Consider R&D firm j. Let z_{jt} be the amount of basic goods per time unit the firm devotes at time t in its endeavor to make an invention. With η_{jt} denoting the instantaneous success arrival rate, we have

$$\eta_{jt} = \eta z_{jt}, \qquad \eta > 0,$$

where η is a given parameter reflecting "research productivity".

g) Give a verbal intuitive argument for the claim that the expected payoff per unit of basic goods devoted to R&D per time unit is $V_t\eta$, where V_t is the market value of an arbitrary firm in Sector 2.

h) At time t, let there be J_t R&D firms, indexed by $j = 1, 2, \ldots, J_t$. So aggregate research input is $Z_t \equiv \sum_{j=1}^{J_t} z_{jt}$. "In equilibrium with $Z_t > 0$, we must have $V_t\eta = 1$." True or false? Why?

i) Show that the risk-free real interest rate in equilibrium is a constant and equals $\eta\pi$. *Hint:* consider the no-arbitrage condition for the asset markets.

Under the simplifying assumption of independence, no memory, and no overlap in research, the expected aggregate number of inventions per time unit at time t is ηZ_t.

j) Ignoring indivisibilities and appealing to the law of large numbers, relate $\dot{N}_t$ ($\equiv dN_t/dt$) to Z_t.

Problem VII.2 below considers this economy from a national income accounting perspective. Problem VII.3 introduces a household sector into the model and considers the growth rate of c_t and N_t in general equilibrium.

VII.2 *National income accounting in the lab-equipment model* Here we consider the same model and use the same notation as in Problem VII.1 (it is an advantage if you have already solved at least e) and h) of that problem). We assume that general equilibrium obtains in the economy.

a) A correct answer to e) of Problem VII.1 implies that, the total quantity, Q_t, of intermediate goods produced per time unit at time t can be written $Q_t = xN_t$. Why?

b) Referring to (*), we have $X_t = \psi x N_t$. Why?

c) Show that

$$GDP_t = Y_t - \psi x N_t.$$

Hint: add up the value added in the three sectors and apply the conclusion to h) of Problem VII.1.

d) We also have

$$GDP_t = C_t + S_t,$$

and

$$GDP_t = w_t L + \pi N_t,$$

where S_t is aggregate net saving, w_t is the real wage, and π is profit per firm in Sector 2. Explain these two equations.

VII.3 *R&D-driven fully endogenous growth* We consider the same model and use the same notation as in Problem VII.1 (it is an advantage if you have already solved that problem). We "close" the model by specifying the household sector.

Suppose there are L infinitely-lived households (L "large"), all alike. Each household supplies inelastically one unit of labor per time unit. Given $\theta > 0$ and $\rho > 0$, each household chooses a plan $(c_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{c_t{}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= r a_t + w_t - c_t, \qquad a_0 \text{ given,} \\ \lim_{t\to\infty} a_t e^{-rt} &\geq 0, \end{aligned}$$

where a_t is financial wealth.

a) Express a_t in terms of V and N_t as defined in Problem VII.1; comment on the absence of a time subscript on r.

b) Find the growth rate of c_t in general equilibrium; comment on your result. *Hint:* results in i) and e) of Problem VII.1 are useful here.

We assume that the parameter values are such that there is positive consumption growth.

c) Write down the required parameter restriction.

d) Write down the parameter restriction needed to ensure that the utility integral U_0 is bounded.

e) By defining $\hat{A}$ appropriately, in an equilibrium with $Z_t > 0$ we have the following relationship (which is useful in many contexts):

$$Y_t = \hat{A} L N_t = \psi x N_t + C_t + \eta^{-1} \dot{N}_t. \qquad (\Delta)$$

Show that

$$\hat{A} = A \left(\frac{A(1-\beta)^2}{\psi} \right)^{(1-\beta)/\beta}$$

and derive the second equality in (Δ). *Hint:* as to the second equality in (Δ), the conclusion to j) of Problem VII.1 may be of help.

f) Find the growth rate of N_t and Y_t; comment on your result. *Hint:* there are two features of the model that indicates it is a kind of reduced-form AK model; this allows you to give a quick answer.

g) How does the growth rate of c depend on η and L, respectively? Comment on the intuition.

h) "The resource allocation in the economy is not Pareto optimal'". True or false? Why?

VII.4 An economy described by the Lab-Equipment Model will under laissez-faire suffer from a certain kind of inefficiency.

a) Briefly describe the kind of inefficiency hinted at.

We now imagine that there is a "social planner" with the same criterion function as that of the representative household. The social planner's decision problem can be split into a static problem and a dynamic problem. The static problem is to ensure that Sector 1 (the basic-goods sector) uses the "right" quantity of intermediate goods. Output in the sector is

$$Y_t = A\left(\sum_{i=1}^{N_t} x_{it}^{1-\beta}\right)L^{\beta}, \qquad A > 0, \quad 0 < \beta < 1,$$

where Y_t is output per time unit, x_{it} is input of intermediate good i, N_t is the number of different types of intermediate goods available at time t, and L is labor input = the exogenous and constant labor supply. The output of basic goods is used partly as input, X_t, in Sector 2, partly for consumption, $C_t \equiv c_t L$, and partly for R&D investment, Z_t, in Sector 3:

$$Y_t = X_t + C_t + Z_t = X_t + C_t + \frac{\dot{N}_t}{\eta}, \qquad \eta > 0,$$

and

$$X_t = \sum_{i=1}^{N_t} \psi x_{it}, \qquad \psi > 0.$$

b) Interpret the parameters η and ψ.

Although the textbook by Acemoglu focuses on the special case $A = (1-\beta)^{-1}$ and $\psi = 1-\beta$, we shall here study the general case which does not rely on such arbitrary parameter links.

The social planner first solves the following static problem (the dating of the variables is suppressed for convenience):

$$\max_{x,\dots,x_N} Y - X = A\left(\sum_{i=1}^{N} x_i^{1-\beta}\right) L^{\beta} - \sum_{i=1}^{N} \psi x_i.$$

c) Why is this problem of relevance from an efficiency point of view? Let x_{SP} denote the solution for x_i (the solution will be the same for all i). Find x_{SP}.

d) The market outcome under laissez-faire is $x_i = \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv x$ for all i. Compare x_{SP} with this market outcome. What is the economic explanation of the quantitative difference?

e) Show that net output of basic goods can be written $Y - X = \tilde{A}N$, where $\tilde{A}$ is a positive constant.

The dynamic problem faced by the social planner is to choose $(c_t)_{t=0}^{\infty}$ so as to:

$$\max U_0 = \int_0^{\infty} \frac{c_t{}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.}$$

$$0 \le c_t \le \frac{\tilde{A}N_t}{L}, \tag{*}$$

$$\dot{N}_t = \eta(\tilde{A}N_t - Lc_t), \qquad N_0 > 0 \text{ given}, \tag{**}$$

$$N_t \ge 0 \text{ for all } t \ge 0, \tag{***}$$

where θ and ρ are given parameters, $\theta > 0$. In (**) indivisibilities are ignored and N is regarded as a continuous and differentiable function of time t. In view of (*) and (**), condition (***) will automatically hold and can therefore be deleted.

f) Explain why in (*) the control variable is bounded above; and explain how the dynamic constraint (**) arises.

g) Derive the first-order conditions for an interior solution of the social planner's dynamic problem. Determine the implied growth rate of c_t. Next, apply your general knowledge about reduced-form AK models to determine the time paths of N_t and Y_t (a brief verbal account is enough).

h) Write down the parameter restrictions required for positive growth and boundedness of the utility integral, respectively.

i) Recalling that the equilibrium interest rate in the laissez-faire market economy is $r^* = \eta\pi = \eta(\frac{\psi}{1-\beta} - \psi)x$, with x given as in d), compare the social planner's consumption growth rate with that of the laissez-faire market economy. *Hint:* an answer can be reached by showing that $\tilde{A}$ $= (\frac{\psi}{1-\beta} - \psi)x_{SP}$ and bearing in mind the result from d).

j) Now consider a government that attempts to implement the social planner's allocation in a decentralized way. The government pays a subsidy at constant rate, σ, to purchases of intermediate goods such that the net price of intermediate good i is $(1-\sigma)p_i$, where $p_i = \psi/(1-\beta)$ is the price set by the monopolist supplier of good i. The government finances this subsidy by taxing consumption at a constant rate τ. It can be shown that a proper choice of σ and τ is sufficient to obtain the social planner's allocation in a decentralized way. Derive the required value of the subsidy rate σ. Comment.

VII.5 *Hidden parameter links in the simple Lab-Equipment Model* Here we consider an extension of what we here call the *simple Lab-Equipment Model* (i.e., the model of Problem VII.1, notation the same). In the simple Lab-Equipment Model, even without Acemoglu's simplification (**) in Problem VII.1, the aggregate production function (**??**) contains three simplifying, but arbitrary parameter links.

In what we shall call the *extended Lab-Equipment Model*, (**??**) is replaced by

$$Y_t = AN_t^{\gamma}(CES_t)^{1-\beta}L^{\beta}, \qquad A > 0,\ \gamma > 0, 0 < \beta < 1, \tag{Y}$$

where the parameter γ reflects "gains to specialization" (see d) below), and CES_t is a CES aggregate of the quantities $x_{1t}, ..., x_{N_t t}$:

$$CES_t \equiv N_t\left(N_t^{-1}\sum_{i=1}^{N_t} {x_{it}}^{\varepsilon}\right)^{\frac{1}{\varepsilon}}, \qquad 0 < \varepsilon < 1, \tag{CES}$$

This is the standard definition of a CES aggregate. The parameter ε is called the *substitution parameter* in that the elasticity of substitution between the different specialized input goods is $1/(1-\varepsilon) > 1$ and thereby an increasing function of ε.

In (Y) it is understood that employment in Sector 1 equals the constant labor supply, L. The institutional setting is a laissez-faire market economy.

a) Show that the right-hand side of (CES) has CRS with respect to the inputs $x_{1t}, ..., x_{N_t t}$.

This CRS property is convenient because it opens up for "gains to specialization" to be represented by an independent parameter, γ, appearing explicitly *outside* the CES index as in (Y).

b) "The specification (**??**) in Problem VII.1 is a special case of (Y)-(CES), namely the case $\gamma = \beta$ together with $\varepsilon = 1 - \beta$". True or false? Comment.

In equilibrium, because of symmetry and the fact that the prices of intermediate goods will all be set at the same level, the representative firm in Sector 1 chooses $x_{it} = x_t$, for all i.

c) With $x_{it} = x_t$ for all i, (CES) reduces to $CES_t = N_t x_t$. Show this.

Ignoring for a moment the issue whether the specialized input goods are durable or non-durable, we may think of $N_t x_t$ as the total input of physical capital, K_t, in the representative firm of Sector 1.

d) Write (CES) as a Cobb-Douglas production function with CRS to the rival inputs, labor and capital. In this interpretation, if N_t grows at the constant rate g_N, what is then the growth rate of total factor productivity?

Let us for a while keep time fixed and suppress the explicit timing of the variables. The representative firm in Sector 1 faces given input prices, $p_1, \ldots, p_N$, and w. The demand for the specialized input good i can be shown[1] to have price elasticity equal to $-1/(1-\varepsilon) < -1$.

Given the technical design corresponding to intermediate good i, the marginal cost of supplying this good is $\psi > 0$ for all i.

e) Show that the monopoly price is $p_i = \psi/\varepsilon$ for all i. *Hint:* MR $= p_i + x_i dp_i/dx_i = (1 + (x_i/p_i)dp_i/dx_i)p_i =$ MC.

f) Does the monopoly power (defined by the markup on marginal cost) depend on the output elasticity w.r.t. labor input? Compare with the simple Lab-Equipment Model.

g) In equilibrium, $x_{it} = x_t$ and $\pi_{it} = \pi_t$ for all i. Why? Express x_t and π_t, respectively, in terms of N_t. *Hint:* It can be shown (see appendix) that in general equilibrium with $p_i = \psi/\varepsilon$ for all i,

$$K = Nx = ((1-\beta)AN^{\gamma})^{1/\beta}(\frac{\psi}{\varepsilon})^{-1/\beta}L.$$

[1] See the note at the end of this problem.

h) Show that in general equilibrium

$$Y_t = A\left(\frac{(1-\beta)\varepsilon}{\psi}\right)^{(1-\beta)/\beta} LN_t^{\gamma+\gamma(1-\beta)/\beta}.$$

i) What is the necessary and sufficient condition for π_t being independent of N_t as in Problem VII.1? And what is the necessary and sufficient condition for Y_t being proportional to N_t? Relate your answers to your answer to b).

j) Suppose gains to specialization is less than the elasticity of Y w.r.t. L. In this case, would you think the economy is capable of generating fully endogenous growth? Why or why not?

k) Suppose gains to specialization is larger than the elasticity of Y w.r.t. L. This case has an implication that makes it implausible. What implication could that be?

Note to d) of Problem VII.5

We claimed that the demand for the specialized input good i has price elasticity equal to $-1/(1-\varepsilon) < -1$. This follows from microeconomic duality theory. Here we give a brief account of how the demand for intermediate good i is determined. There are two steps:

Step 1. For a given size $K > 0$ of CES, choose $(x_1, \ldots, x_N)$ so as to minimize the cost of obtaining K. That is, solve the problem:

$$\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} p_i x_i \quad \text{s.t.} \quad N\left(N^{-1}\sum_{i=1}^{N} x_i{}^{\varepsilon}\right)^{\frac{1}{\varepsilon}} = K.$$

This problem can be shown to have the solution

$$x_i = \frac{K}{N}\left(\frac{p_i}{\lambda}\right)^{-\frac{1}{1-\varepsilon}} \equiv x_i^*, \tag{i}$$

where

$$\lambda = \left(N^{-1}\sum_{i=1}^{N} p_i^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \equiv \bar{p}$$

is the Lagrange multiplier, $i = 1, \ldots, N$. We see that the solution, $(x_1^*, \ldots, x_N^*)$, is proportional to K (as expected in view of CES being homogeneous of degree one in $x_1, \ldots, x_N$. It can be shown that $\sum_{i=1}^{N} p_i x_i^* = \bar{p}K$ and so $\bar{p}$ can be interpreted as the minimum cost per unit of K. (Note that $\bar{p}$ is a kind of average of the p_i's in the sense that (a) if $p_i = p$ for all i, then $\bar{p} = p$; and (b) if for any $\mu > 0$, p_i is replaced by $p_i' = \mu p_i$, then $\bar{p}' = \mu\bar{p}$.)

Step 2. Choose K and L so as to maximize $\Pi = AN^{\gamma}K^{1-\beta}L^{\beta} - \bar{p}K - wL$. The first-order condition w.r.t. K is

$$\frac{\partial \Pi}{\partial K} = (1-\beta)AN^{\gamma}K^{-\beta}L^{\beta} - \bar{p} = 0,$$

so that, given $\bar{p}$ and N, the profit maximizing $k \equiv K/L$ is $k = ((1-\beta)AN^{\gamma})^{1/\beta}\bar{p}^{-1/\beta}$ (in view of CRS, $\bar{p}$ determines only the profit maximizing factor *ratio*). If moreover L is considered given, we have

$$K = ((1-\beta)AN^{\gamma})^{1/\beta}\bar{p}^{-1/\beta}L. \tag{ii}$$

The supplier of intermediate good i is "small" relative to the economy as a whole and takes $\bar{p}$, and thereby K, as given. Hence the perceived price elasticity of the demand for intermediate good i is given by (i) as $-1/(1-\varepsilon) < -1$.

VII.6 *Knowledge-spillover models* The bulk of empirical evidence suggests that market economies do too little R&D investment compared to the optimal level as defined from the perspective of a social planner respecting the preferences of an assumed representative infinitely-lived household.

a) Is the "lab-equipment" version of the expanding input variety model consistent with this evidence? Briefly discuss.

b) What kind of subsidy and taxation scheme is capable of implementing the social planner's allocation in the "lab-equipment" model?

c) Our syllabus describes two other versions of the expanding input variety model. The aggregate invention production functions in these two versions are two alternative cases within the common form

$$\dot{N}_t = \eta N_t^{\varphi} L_{Rt}, \qquad \eta > 0, \varphi \leq 1,$$

where N_t is the number of existing different varieties of intermediate goods (indivisibilities are ignored) and L_{Rt} the input of research labor at time t (time is continuous). Briefly interpret.

d) Are these versions consistent with the mentioned evidence? Why or why not?

e) Are there features in these versions that may call for additional policy measures compared with b)? Briefly discuss.

f) The patent-R&D ratio is defined as the number of new patents per year divided by aggregate R&D expenditures (in real terms). With w_t denoting the real wage, write down an expression for the patent-R&D ratio according to the model versions mentioned under c).

g) What predictions concerning the time path of the patent-R&D ratio can we derive from the two alternative model versions mentioned under c), assuming balanced growth? Why?

h) Since the nineteen fifties, in the US a systematic decline in the empirical patent-R&D ratio has taken place. Briefly relate to your result in g).

VII.7 Consider the original Jones (1995) R&D-based growth model for a closed economy. Like the original Romer (1990) knowledge-spillover model, the model includes durable physical capital goods. Indeed, the specialized (non-durable) intermediate goods in Acemoglu's §13.2-3 are replaced by specialized capital goods both in the original Jones and the original Romer model. The number of different varieties of capital goods is called A_t and is treated as an index of the general level of technical knowledge.

There is a Romer-style microeconomic story about the behavior of competitive basic-goods firms demanding specialized capital goods supplied under conditions of monopolistic competition; in addition, the individual R&D firms ignore the knowledge-spillovers. We skip the details and go directly to the implied aggregate level. With Y denoting output of basic goods (not GDP) and K denoting the cumulative non-consumed output of basic goods (otherwise the notation is standard), the aggregate model is:

$$\begin{aligned}
Y_t &= K_t^{\alpha}(A_t L_{Yt})^{1-\alpha}, \qquad 0<\alpha<1, && (1)\\
\dot{K}_t &= Y_t - c_t L_t - \delta K_t, \qquad \delta \geq 0, && (2)\\
\dot{A}_t &= \eta A_t^{\varphi} L_{At}, \qquad \eta>0, \varphi<1, && (3)\\
L_{Yt} + L_{At} &= L_t, && (4)\\
L_t &= L_0 e^{nt}, \; n>0. && (5)
\end{aligned}$$

For simplicity we have ignored the R&D duplication externality in Jones (1995).

There is a representative Ramsey household with pure rate of time preference, ρ, and a CRRA instantaneous utility function with parameter $\theta > 0$. To ensure boundedness of the utility integral we assume $\rho - n > (1-\theta)n/(1-\varphi)$.

a) Find an expression for the growth rate of "knowledge", A, under the assumption that this growth rate is positive and constant. *Hint:* start from an expression for g_A derived from (3) and consider the growth rate of g_A.

As the model has *two* state variables, K_t and A_t, it will necessarily exhibit transitional dynamics. The dynamic system will consist of *four* coupled differential equations and is thus relatively complicated.[2] Hence, we shall here concentrate on the balanced growth path (BGP) defined as a path along which Y, K, $C \equiv cL$, and A are positive and g_Y, g_K, g_C and g_A are constant.

b) Along a BGP with $g_A > 0$ and $g_K > -\delta$, find g_Y and g_y, where $y \equiv Y/L$. *Hint:* a possible approach is to divide by $Y = Y^\alpha Y^{1-\alpha}$ on both sides of (1) and then use one of the basic balanced growth theorems.

c) Find the growth rate of c under balanced growth.

It can be shown that the equilibrium real interest rate at time t equals $\alpha^2 Y_t/K_t - \delta$. This information is useful for some of the next questions.

d) Suppose $s_A \equiv L_A/L$ in balanced growth is increased by an R&D subsidy.

  d1) Will this increase affect the long-run per capita growth rate? Comment.

  d2) Will the increase in s_A affect *levels* under balanced growth? Comment. *Hint:* Find an expression for y in terms of $\tilde{k} \equiv K/(AL_Y)$, s_A, and A under balanced growth. Then find an expression for A in terms of L_A under balanced growth. Check that $\tilde{k}$ is independent of s_A; use here that the output-capital ratio in balanced growth can be found from the Keynes-Ramsey rule of the representative household.

e) Is the level of the y path a monotonic function of s_A? Why or why not?

f) The laissez-faire market economy can be shown to generate too little R&D compared to the social planner's solution? What factors might explain this feature?

[2] Yet the presence of saddle point stability can be established, cf. Arnold (Rev. Econ. Dynamics, 2006, 143-152).

g) Check whether there is a scale effect on levels in the model. Comment. *Hint:* From Jones (1995, p. 769) we have that s_A under balanced growth is independent of L. Show by use of the Keynes-Ramsey rule that also $\tilde{k}$ under balanced growth is independent of L. Then the stated question can be answered on the basis of the result in d2).

h) Do you view the presence of scale effects on levels in an endogenous growth model as a strength or weakness of the model? Why?

Appendix A

Solutions to linear differential equations

For a general differential equation of first order, $\dot{x}(t) = \varphi(x(t), t)$, with $x(t_0) = x_{t_0}$ and where φ is a continuous function, we have, at least for t in an interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$,

$$x(t) = x_{t_0} + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau. \tag{*}$$

To get a confirmation, calculate $\dot{x}(t)$ from (*).

For the special case of a linear differential equation of first order, $\dot{x}(t) + a(t)x(t) = b(t)$, we can specify the solution. Three sub-cases of rising complexity are:

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = (x_{t_0} - x^*)e^{-a(t-t_0)} + x^*, \text{ where } x^* = \frac{b}{a}.$$

If $a = 0$, we get, directly from (*), the solution $x(t) = x_{t_0} + bt$.[1]

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0} e^{-a(t-t_0)} + e^{-a(t-t_0)} \int_{t_0}^{t} b(s) e^{a(s-t_0)} ds.$$

[1]Some non-linear differential equations can be transformed into this simple case. For simplicity let $t_0 = 0$. Consider the equation $\dot{y}(t) = \alpha y(t)^{\beta}$, $y_0 > 0$ given, $\alpha \neq 0, \beta \neq 1$ (a Bernoulli equation). To find the solution for $y(t)$, let $x(t) \equiv y(t)^{1-\beta}$. Then, $\dot{x}(t) = (1-\beta)y(t)^{-\beta}\dot{y}(t) = (1-\beta)y(t)^{-\beta}\alpha y(t)^{\beta} = (1-\beta)\alpha$. The solution for this is $x(t) = x_0 + (1-\beta)\alpha t$, where $x_0 = y_0^{1-\beta}$. Thereby the solution for $y(t)$ is $y(t) = x(t)^{1/(1-\beta)} = \left(y_0^{1-\beta} + (1-\beta)\alpha t\right)^{1/(1-\beta)}$, which is defined for $t > -y_0^{1-\beta}/((1-\beta)\alpha$.

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)}c\int_{t_0}^{t} e^{(a+h)(s-t_0)}ds = (x_{t_0} - \frac{c}{a+h})e^{-a(t-t_0)} + \frac{c}{a+h}e^{h(t-t_0)}.$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau} + e^{-\int_{t_0}^{t} a(\tau)d\tau}\int_{t_0}^{t} b(s)e^{\int_{t_0}^{s} a(\tau)d\tau}ds.$$

Special case: $b(t) = 0$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau}.$$

Even more special case: $b(t) = 0$ and $a(t) = a$, a constant. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)}.$$

Remark 1 For $t_0 = 0$, most of the formulas will look simpler.

Remark 2 To check whether a suggested solution *is* a solution, calculate the time derivative of the suggested solution and add an arbitrary constant. By appropriate adjustment of the constant, the final result should be a replication of the original differential equation together with its initial condition.