Economic Growth Lecture Note 5

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Aspects of transitional dynamics. Barro-style growth regression analysis

In this lecture note we deal with three issues, all of which are related to the transitional dynamics of a growth model:

- Do poor countries necessarily tend to approach their steady state from below?
- How fast (or rather how slow) are the transitional dynamics in a growth model?
- What exactly is the theoretical foundation for a Barro-style growth regression equation?

The Solow growth model may serve as the analytical point of departure for the first two issues (and to some extent also for the third issue) and to some extent also for the third.

1 Point of departure: the Solow model

As is well-known, the fundamental differential equation for the Solow model is

$$\dot{\tilde{k}}(t) = sf(\tilde{k}(t)) - (\delta + g + n)\tilde{k}(t), \qquad \tilde{k}(0) = \tilde{k}_0 > 0, \tag{1}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $f(\tilde{k}(t)) \equiv F(\tilde{k}(t), 1)$, $A(t) = A_0 e^{gt}$, and $L(t) = L_0 e^{nt}$ (standard notation). The production function F is neoclassical with CRS and the parameters satisfy $0 < s < 1$ and $\delta + g + n > 0$. The production function on intensive form, f, therefore satisfies $f(0) \geq 0, f' > 0, f'' < 0$, and

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \frac{\delta + g + n}{s} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, that is, a unique positive solution to the equation

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{2}$$

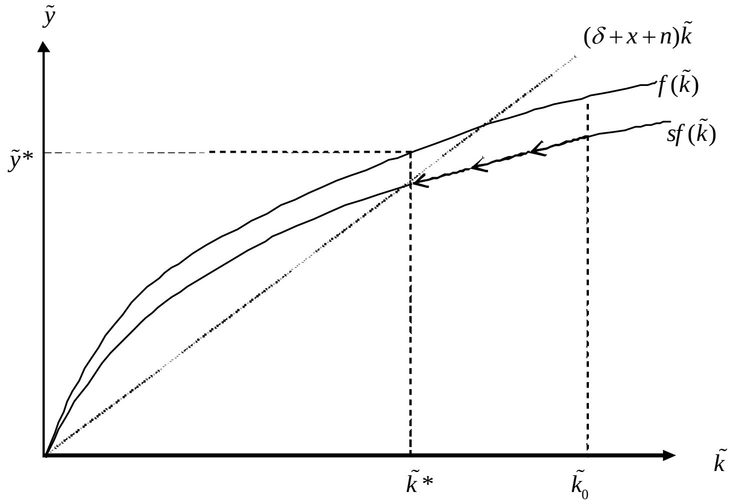

Figure 1:

Furthermore, given an arbitrary $\tilde{k}_0 > 0$, we have for all $t \geq 0$,

$$\dot{\tilde{k}}(t) \gtreqless 0 \text{ for } \tilde{k}(t) \lesseqgtr \tilde{k}^*, \tag{3}$$

respectively. The steady state, $\tilde{k}^*$, is thus *globally asymptotically stable* in the sense that for all $\tilde{k}_0 > 0$, $\lim_{t\to\infty} \tilde{k}(t) = k^*$ and this convergence is *monotonic* (in the sense that $\tilde{k}(t) - \tilde{k}^*$ does not change sign during the adjustment process).

From now on the dating of $\tilde{k}$ is suppressed unless needed for clarity. Fig. 1 illustrates the dynamics as seen from the perspective of (1) (in this and the two next figures, x should read g. Fig. 2 illustrates the dynamics emerging when we rewrite (1) this way:

$$\dot{\tilde{k}} = s\left(f(\tilde{k}) - \frac{\delta + g + n}{s}\tilde{k}\right) \gtreqless 0 \text{ for } \tilde{k} \lesseqgtr \tilde{k}^*.$$

In Fig. 3 yet another illustration is exhibited, based on rewriting (1) this way:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\frac{f(\tilde{k})}{\tilde{k}} - (\delta + g + n),$$

where $sf(\tilde{k})/\tilde{k}$ is gross saving per unit of capital, $S/K \equiv (Y - C)/K$.

An important variable in the analysis of the adjustment process towards steady state

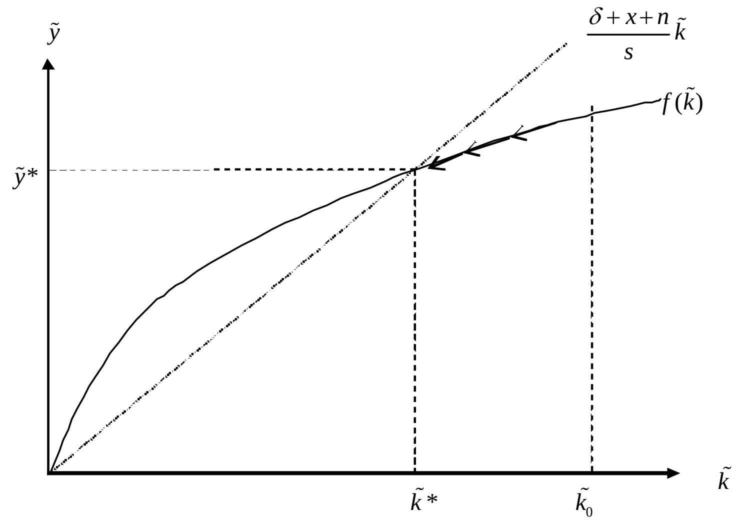

Figure 2:

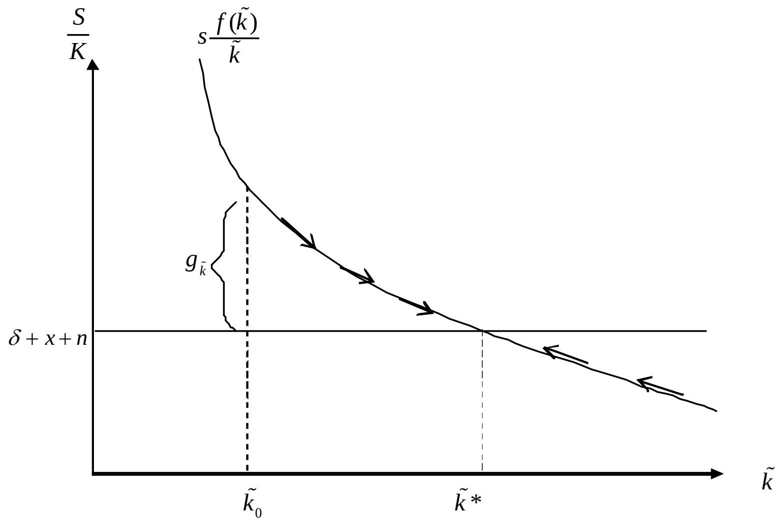

Figure 3:

is the output elasticity w.r.t. capital:

$$\frac{K}{Y}\frac{\partial Y}{\partial K} = \frac{\tilde{k}}{f(\tilde{k})}f'(\tilde{k}) \equiv \varepsilon(\tilde{k}), \tag{4}$$

where $0 < \varepsilon(\tilde{k}) < 1$ for all $\tilde{k} > 0$.

2 Do poor countries tend to approach their steady state from below?

From some textbooks (for instance Barro and Sala-i-Martin, 2004) one gets the impression that poor countries tend to approach their steady state *from below*. But this is *not* what the Penn World Table data seems to indicate. And from a theoretical point of view the size of $\tilde{k}_0$ relative to $\tilde{k}^*$ is certainly ambiguous, whether the country is rich or poor. To see this, consider a poor country with initial effective capital intensity

$$\tilde{k}_0 \equiv \frac{K_0}{A_0 L_0}.$$

Here K_0/L_0 will typically be small for a poor country (the country has not yet accumulated much capital relative to its fast-growing population). The technology level, A_0, however, *also* tends to be small for a poor country. Hence, whether we should expect $\tilde{k}_0 < \tilde{k}^*$ or $\tilde{k}_0 > \tilde{k}^*$ is not obvious *apriori*. Or equivalently: whether we should expect that a poor country's GDP at an arbitrary point in time grows at a rate higher or lower than the country's steady-state growth rate, $g + n$, is not obvious *apriori*.

While Fig. 3 illustrates the case where the inequality $\tilde{k}_0 < \tilde{k}^*$ holds, Fig. 1 and 2 illustrate the opposite case. There *exists* some empirical evidence indicating that poor countries tend to approach their steady state *from above*. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

The prejudice that poor countries *apriori* should tend to approach their steady state from below seems to come from a confusion of conditional and unconditional β convergence. The Solow model predicts - and data supports - that within a group of countries with similar structural characteristics (approximately the same f, A_0, g, s, n, and δ), the initially poorer countries will grow faster than the richer countries. This is because the poorer countries (small $y(0) = f(\tilde{k}_0)A_0$) will be the countries with relatively small initial capital-labor ratio, k_0. As all the countries in the group have approximately the same A_0, the poorer countries thus have $\tilde{k}_0 \equiv k_0/A_0$ relatively small, i.e., $\tilde{k}_0 < \tilde{k}^*$. From $y \equiv Y/L \equiv \tilde{y}A = f(\tilde{k})A$ follows that the growth rate in output per worker of these poor countries tends to exceed g. Indeed, we have generally

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + g = \frac{f'(\tilde{k})\dot{\tilde{k}}}{f(\tilde{k})} + g \gtreqqless g \text{ for } \dot{\tilde{k}} \gtreqqless 0, \text{ i.e., for } \tilde{k} \lesseqqgtr \tilde{k}^*.$$

So, *within* the group, the poor countries tend to approach the steady state, $\tilde{k}^*$, *from below*.

The countries in the world as a whole, however, differ a lot w.r.t. their structural characteristics, including their A_0. Unconditional β convergence is definitely rejected by the data. Then there is no reason to expect the poorer countries to have $\tilde{k}_0 < \tilde{k}^*$ rather than $\tilde{k}_0 > \tilde{k}^*$. Indeed, according to the mentioned study by Cho and Graham (1996), it turns out that the data for the relatively poor countries favors the latter inequality rather than the first.

3 Convergence speed and adjustment time

Our next issue is: How fast (or rather how slow) are the transitional dynamics in a growth model? To put it another way: according to a given growth model with convergence, how fast does the economy approach its steady state? The answer turns out to be: not very fast - to say the least. This is a rather general conclusion and is confirmed by the empirics: adjustment processes in a growth context are quite time consuming.

In Acemoglu's textbook we meet the concept of speed of convergence at p. 54 (under an alternative name, rate of adjustment) and p. 81 (in connection with Barro-style growth regressions). Here we shall go more into detail with the issue of speed of convergence.

Again the Solow model is our frame of reference. We search for a formula for the *speed of convergence* of $\tilde{k}(t)$ and $y(t)/y^*(t)$ in a closed economy described by the Solow model. So our analysis is concerned with *within-country convergence*: how fast do variables such as $\tilde{k}$ and y approach their steady state paths in a closed economy? The key adjustment mechanism is linked to diminishing returns to capital (falling marginal productivity of capital) in the process of capital accumulation. The problem of *cross-country convergence* (which is what "β convergence" and "σ convergence" are about) is in principle more complex because also such mechanisms as technological catching-up and cross-country factor movements are involved.

3.1 Convergence speed for $\tilde{k}(t)$

The ratio of $\dot{\tilde{k}}(t)$ to $(\tilde{k}(t) - \tilde{k}^*) \neq 0$ can be written

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t) - \tilde{k}^*} = \frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*}, \tag{5}$$

since $d\tilde{k}^*/dt = 0$. We define the *instantaneous speed of convergence* at time t as the (proportionate) rate of *decline* of the distance $\left|\tilde{k}(t) - \tilde{k}^*\right|$ at time t and we denote it $\mathrm{SOC}_t(\tilde{k})$.[1] Thus,

$$\mathrm{SOC}_t(\tilde{k}) \equiv -\frac{d\left(\left|\tilde{k}(t) - \tilde{k}^*\right|\right)/dt}{\left|\tilde{k}(t) - \tilde{k}^*\right|} = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*}, \tag{6}$$

where the equality sign is valid for monotonic convergence.

Generally, $\mathrm{SOC}_t(\tilde{k})$ depends on both the absolute size of the difference $\tilde{k} - \tilde{k}^*$ at time t and its sign. But if the difference is already "small", $\mathrm{SOC}_t(\tilde{k})$ will be "almost" constant for increasing t and we can find an approximate measure for it. Let the function $\varphi(\tilde{k})$ be defined by $\varphi(\tilde{k}) \equiv sf(\tilde{k}) - m\tilde{k}$, where $m \equiv \delta + g + n$. A first-order Taylor approximation of $\varphi(\tilde{k})$ around $\tilde{k} = \tilde{k}^*$ gives

$$\varphi(\tilde{k}) \approx \varphi(\tilde{k}^*) + \varphi'(\tilde{k}^*)(\tilde{k} - \tilde{k}^*) = 0 + (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*).$$

For $\tilde{k}$ in a small neighborhood of the steady state, $\tilde{k}^*$, we thus have

$$\begin{aligned}
\dot{\tilde{k}} &= \varphi(\tilde{k}) \approx (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*) \\
&= (\frac{sf'(\tilde{k}^*)}{m} - 1)m(\tilde{k} - \tilde{k}^*) \\
&= (\frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (2))} \\
&\equiv (\varepsilon(\tilde{k}^*) - 1)m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (4))}.
\end{aligned}$$

Applying the definition (6) and the identity $m \equiv \delta + g + n$, we now get

$$\mathrm{SOC}_t(\tilde{k}) = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \approx (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*) > 0. \tag{7}$$

This result tells us how fast, approximately, the economy approaches its steady state if it starts "close" to it. If, for example, $\beta(\tilde{k}^*) = 0.02$ per year, then 2 percent of the gap

[1] Synonyms for speed of convergence are *rate of convergence, rate of adjustment* or *adjustment speed.*

between $\tilde{k}(t)$ and $\tilde{k}^*$ vanishes per year. We also see that everything else equal, a higher output elasticity w.r.t. capital implies a lower speed of convergence.

In the limit, for $\left|\tilde{k} - \tilde{k}^*\right| \to 0$, the instantaneous speed of convergence coincides with what is called the *asymptotic speed of convergence*, defined as

$$\text{SOC}^*(\tilde{k}) \equiv \lim_{\left|\tilde{k}-\tilde{k}^*\right| \to 0} \text{SOC}_t(\tilde{k}) = \beta(\tilde{k}^*). \tag{8}$$

Multiplying through by $-(\tilde{k}(t) - \tilde{k}^*)$, the equation (7) takes the form of a homogeneous linear differential equation (with constant coefficient), $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. With $x(t) = \tilde{k}(t) - \tilde{k}^*$ and "$=$" replaced by "$\approx$", we get in the present case

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t}. \tag{9}$$

This is the approximative time path for the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ and shows how the gap becomes smaller and smaller at the rate $\beta(\tilde{k}^*)$.

One of the reasons that the speed of convergence is important is that it indicates what weight should be placed on transitional dynamics of a growth model relative to the steady-state behavior. The speed of convergence matters for instance for the evaluation of growth-promoting policies. In growth models with diminishing marginal productivity of production factors, successful growth-promoting policies have transitory growth effects and permanent level effects. Slower convergence implies that the full benefits are slower to arrive.

3.2 Convergence speed for $\log \tilde{k}(t)$

We have found an approximate expression for the convergence speed of $\tilde{k}$. Since models in empirical analysis and applied theory are often based on log-linearization, we might ask what the speed of convergence of $\log \tilde{k}$ is. The answer is: approximately the same and asymptotically exactly the same as that of $\tilde{k}$ itself! Let us see why.

A first-order Taylor approximation of $\log \tilde{k}(t)$ around $\tilde{k} = \tilde{k}^*$ gives

$$\log \tilde{k}(t) \approx \log \tilde{k}^* + \frac{1}{\tilde{k}^*}(\tilde{k}(t) - \tilde{k}^*). \tag{10}$$

By definition

$$\begin{aligned}
\mathrm{SOC}_t(\log \tilde{k}) &= -\frac{d(\log \tilde{k}(t) - \log \tilde{k}^*)/dt}{\log \tilde{k}(t) - \log \tilde{k}^*} = -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)(\log \tilde{k}(t) - \log \tilde{k}^*)} \\
&\approx -\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)\frac{\tilde{k}(t)-\tilde{k}^*}{\tilde{k}^*}} = \frac{\tilde{k}^*}{\tilde{k}(t)}\mathrm{SOC}_t(\tilde{k}) \to \mathrm{SOC}^*(\tilde{k}) = \beta(\tilde{k}^*) \text{ for } \tilde{k}(t) \to \tilde{k}^*, \quad (11)
\end{aligned}$$

where in the second line we have used, first, the approximation (10), second, the definition in (7), and third, the definition in (8).

So, at least in a neighborhood of the steady state, the instantaneous rate of decline of the logarithmic distance of $\tilde{k}$ to the steady-state value of $\tilde{k}$ approximates the instantaneous rate of decline of the distance of $\tilde{k}$ itself to its steady-state value. The asymptotic speed of convergence of $\log \tilde{k}$ coincides with that of $\tilde{k}$ itself and is exactly $\beta(\tilde{k}^*)$.

In the Cobb-Douglas case (where $\varepsilon(\tilde{k}^*)$ is a constant, say α) it is possible to find an explicit solution to the Solow model, see Acemoglu p. 53 and Exercise II.2. It turns out that the instantaneous speed of convergence in a finite distance from the steady state is a constant and equals the asymptotic speed of convergence, $(1-\alpha)(\delta+g+n)$.

3.3 Convergence speed for $y(t)/y^*(t)$

The variable which we are interested in is usually not so much $\tilde{k}$ in itself, but rather labor productivity, $y(t) \equiv \tilde{y}(t)A(t)$. In the interesting case where $g > 0$, labor productivity does not converge towards a constant. We therefore focus on the ratio $y(t)/y^*(t)$, where $y^*(t)$ denotes the hypothetical value of labor productivity at time t, conditional on the economy being on its steady-state path, i.e.,

$$y^*(t) \equiv \tilde{y}^* A(t). \quad (12)$$

We have

$$\frac{y(t)}{y^*(t)} \equiv \frac{\tilde{y}(t)A(t)}{\tilde{y}^*A(t)} = \frac{\tilde{y}(t)}{\tilde{y}^*}. \quad (13)$$

As $\tilde{y}(t) \to \tilde{y}^*$ for $t \to \infty$, the ratio $y(t)/y^*(t)$ converges towards 1 for $t \to \infty$.

Taking logs on both sides of (13), we get

$$
\begin{aligned}
\log \frac{y(t)}{y^*(t)} &= \log \frac{\tilde{y}(t)}{\tilde{y}^*} = \log \tilde{y}(t) - \log \tilde{y}^* \\
&\approx \log \tilde{y}^* + \frac{1}{\tilde{y}^*}(\tilde{y}(t) - y^*) - \log \tilde{y}^* \quad \text{(first-order Taylor approx. of } \log \tilde{y}\ ) \\
&= \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}(t)) - f(\tilde{k}^*)) \\
&\approx \frac{1}{f(\tilde{k}^*)}(f(\tilde{k}^*) + f'(\tilde{k}^*)(\tilde{k}(t) - \tilde{k}^*) - f(\tilde{k}^*)) \quad \text{(first-order approx. of } f(\tilde{k})) \\
&= \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)} \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \equiv \varepsilon(\tilde{k}^*) \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} \\
&\approx \varepsilon(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \qquad \text{(by (10))}. \qquad (14)
\end{aligned}
$$

Multiplying through by $-(\log \tilde{k}(t) - \log \tilde{k}^*)$ in (11) and carrying out the differentiation w.r.t. time, we find an approximate expression for the growth rate of $\tilde{k}$,

$$
\begin{aligned}
\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)} &\equiv g_{\tilde{k}}(t) \approx -\frac{\tilde{k}^*}{\tilde{k}(t)} \mathrm{SOC}_t(\tilde{k})(\log \tilde{k}(t) - \log \tilde{k}^*) \\
&\to -\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) \quad \text{for } \tilde{k}(t) \to \tilde{k}^*, \qquad (15)
\end{aligned}
$$

where the convergence follows from the last part of (11). We now calculate the time derivative on both sides of (14) to get

$$
\begin{aligned}
d(\log \frac{y(t)}{y^*(t)})/dt &= d(\log \frac{\tilde{y}(t)}{\tilde{y}^*})/dt = \frac{d\tilde{y}(t)/dt}{\tilde{y}(t)} \equiv g_{\tilde{y}}(t) \\
&\approx \varepsilon(\tilde{k}^*) g_{\tilde{k}}(t) \approx -\varepsilon(\tilde{k}^*)\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*). \qquad (16)
\end{aligned}
$$

from (15). Dividing through by $-\log(y(t)/y^*(t))$ in this expression, taking (14) into account, gives

$$
-\frac{d(\log \frac{y(t)}{y^*(t)})/dt}{\log \frac{y(t)}{y^*(t)}} = -\frac{d(\log \frac{y(t)}{y^*(t)} - \log 1)/dt}{\log \frac{y(t)}{y^*(t)} - \log 1} \equiv \mathrm{SOC}_t(\log \frac{y}{y^*}) \approx \beta(\tilde{k}^*), \qquad (17)
$$

in view of $\log 1 = 0$. So the logarithmic distance of y from its value on the steady-state path at time t has approximately the same rate of decline as the logarithmic distance of $\tilde{k}$ from $\tilde{k}$'s value on the steady-state path at time t. The asymptotic speed of convergence for $\log y(t)/y^*(t)$ is exactly the same as that for $\tilde{k}$, namely $\beta(\tilde{k}^*)$.

What about the speed of convergence of $y(t)/y^*(t)$ itself? Here the same principle as in (11) applies. The asymptotic speed of convergence for $\log(y(t)/y^*(t))$ is the same as that for $y(t)/y^*(t)$ (and vice versa), namely $\beta(\tilde{k}^*)$.

With one year as our time unit, standard parameter values are: $g = 0.02$, $n = 0.01$, $\delta = 0.05$, and $\varepsilon(\tilde{k}^*) = 1/3$. We then get $\beta(\tilde{k}^*) = (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) = 0.053$ per year. In the empirical Chapter 11 of Barro and Sala-i-Martin (2004), it is argued that a lower value of $\beta(\tilde{k}^*)$, say 0.02 per year, fits the data better. This requires $\varepsilon(\tilde{k}^*) = 0.75$. Such a high value of $\varepsilon(\tilde{k}^*)$ ($\approx$ the income share of capital) may seem difficult to defend. But if we reinterpret K in the Solow model so as to include *human* capital (skills embodied in human beings and acquired through education and learning by doing), a value of $\varepsilon(\tilde{k}^*)$ at that level may not be far out.

3.4 Adjustment time

Let τ_ω be the time that it takes for the fraction $\omega \in (0,1)$ of the initial gap between $\tilde{k}$ and $\tilde{k}^*$ to be eliminated, i.e., τ_ω satisfies the equation

$$\frac{\left|\tilde{k}(\tau_\omega) - \tilde{k}^*\right|}{\left|\tilde{k}(0) - \tilde{k}^*\right|} = \frac{\tilde{k}(\tau_\omega) - \tilde{k}^*}{\tilde{k}(0) - \tilde{k}^*} = 1 - \omega, \tag{18}$$

where $1 - \omega$ is the fraction of the initial gap still remaining at time τ_ω. In (18) we have applied that $sign(\tilde{k}(t) - \tilde{k}^*) = sign(\tilde{k}(0) - \tilde{k}^*)$ in view of monotonic convergence.

By (9), we have

$$\tilde{k}(\tau_\omega) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

In view of (18), this implies

$$1 - \omega \approx e^{-\beta(\tilde{k}^*)\tau_\omega}.$$

Taking logs on both sides and solving for τ_ω gives

$$\tau_\omega \approx -\frac{\log(1-\omega)}{\beta(\tilde{k}^*)}. \tag{19}$$

This is the approximate *adjustment time* required for $\tilde{k}$ to eliminate the fraction ω of the initial distance of $\tilde{k}$ to its steady-state value, $\tilde{k}^*$, when the adjustment speed (speed of convergence) is $\beta(\tilde{k}^*)$.

Often we consider the *half-life* of the adjustment, that is, the time it takes for half of the initial gap to be eliminated. To find the half-life of the adjustment of $\tilde{k}$, we put $\omega = \frac{1}{2}$ in (19). Again we use one year as our time unit. With the previous parameter values, we have $\beta(\tilde{k}^*) = 0.053$ per year and thus

$$\tau_{\frac{1}{2}} \approx -\frac{\log\frac{1}{2}}{0.053} \approx \frac{0.69}{0.053} = 13,1 \text{ years.}$$

As noted above, Barro and Sala-i-Martin (2004) estimate the asymptotic speed of convergence to be $\beta(\tilde{k}^*) = 0.02$ per year. With this value, the half-life is approximately

$$\tau_{\frac{1}{2}} \approx -\frac{\log \frac{1}{2}}{0.02} \approx \frac{0.69}{0.02} = 34.7 \text{ years.}$$

And the time needed to eliminate three quarters of the initial distance to steady state, $\tau_{3/4}$, will then be about 70 years ($= 2 \cdot 35$ years, since $1 - 3/4 = \frac{1}{2} \cdot \frac{1}{2}$).

Among empirical analysts there is not general agreement about the size of $\beta(\tilde{k}^*)$. Some authors, for example Islam (1995), using a panel data approach, find speeds of convergence considerably larger, between 0.05 and 0.09. McQuinne and Whelan (2007) get similar results. There is a growing realization that the speed of convergence differs across periods and groups of countries. Perhaps an empirically reasonable range is $0.02 < \beta(\tilde{k}^*) < 0.09$. Correspondingly, a reasonable range for the half-life of the adjustment will be 7.6 years $< \tau_{\frac{1}{2}} <$ 34.7 years.

Most of the empirical studies of convergence use a variety of cross-country regression analysis of the kind described in the next section. Yet the theoretical frame of reference is often the Solow model - or its extension with human capital (Mankiw et al., 1992). These models are closed economy models with exogenous technical progress and deal with "within-country" convergence. It is not obvious that they constitute an appropriate framework for studying cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and where some countries are pushing the technological frontier further out, while others try to imitate and catch up. At least one should be aware that the empirical estimates obtained may reflect mechanisms in addition to the falling marginal productivity of capital in the process of capital accumulation.

4 Barro-style growth regression equations

Barro-style growth regression analysis, which became very popular in the 1990s, draws upon transitional dynamics aspects (including the speed of convergence) as well as steady state aspects of neoclassical growth theory (for instance the Solow model or the Ramsey model).

In his Section 3.2 of Chapter 3 Acemoglu presents Barro's growth regression equations in an unconventional form, see Acemoglu's equations (3.12), (3.13), and (3.14). The left-hand side appears as if it is just the growth rate of y (output per unit of labor) from one year to the next. But the true left-hand side of a Barro equation is the average compound annual growth rate of y over many years. Moreover, since Acemoglu's text is very brief about the formal links to the underlying neoclassical theory of transitional dynamics, we will spell the details out here.

Most of the preparatory work has already been done above. The point of departure is a neoclassical one-sector growth model for a closed economy:

$$\dot{\tilde{k}}_t = s(\tilde{k}_t)f(\tilde{k}_t) - (\delta + g + n)k_t, \tag{20}$$

where $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, $A(t) = A_0 e^{gt}$, and $L(t) = L_0 e^{nt}$ as above. The Solow model is the special case where the saving-income ratio, $s(\tilde{k}_t)$, is a constant $s \in (0, 1)$.

It is assumed that the model, (20), generates monotonic convergence, i.e., $\tilde{k}_t \to \tilde{k}^* > 0$ for $t \to \infty$. Hence, in a neighborhood of the steady state, all the above formulas, based on the Solow model, are still valid. The asymptotic speed of convergence for $y(t)/y^*(t)$ is thus still $\beta(\tilde{k}^*)$, as defined in (7). For notational convenience, we will just denote it $\beta,$ interpreted as a derived parameter, i.e.,

$$\beta = (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*). \tag{21}$$

In view of $y(t) \equiv \tilde{y}(t)A(t)$, we have $g_y(t) = g_{\tilde{y}}(t) + g$. By (16) and the definition of β,

$$g_y(t) \approx g - \varepsilon(\tilde{k}^*)\beta(\log \tilde{k}(t) - \log \tilde{k}^*) \approx g - \beta(\log y(t) - \log y^*(t)), \tag{22}$$

where the last approximation comes from (14). This is Acemoglu's Equation (3.10) (recall that Acemoglu's k^* is the same as our $\tilde{k}^*$).

With the horizontal axis representing time, Fig. 4 gives an illustration of these transitional dynamics. As $g_y(t) = d \log y(t)/dt$ and $g = d \log y^*(t)/dt$, (22) is equivalent with

$$\frac{d(\log y_t - \log y_t^*)}{dt} \approx -\beta(\log y_t - \log y_t^*). \tag{23}$$

So again we have a simple differential equation of the form $\dot{x}(t) = \beta x(t)$, the solution of which is $x(t) = x(0)e^{\beta t}$. The solution of (23) is thus

$$\log y(t) - \log y^*(t) \approx (\log y(0) - \log y^*(0))e^{-\beta t}.$$

As $y^*(t) = y^*(0)e^{gt}$, this can written

$$\log y(t) \approx \log y^*(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t}. \tag{24}$$

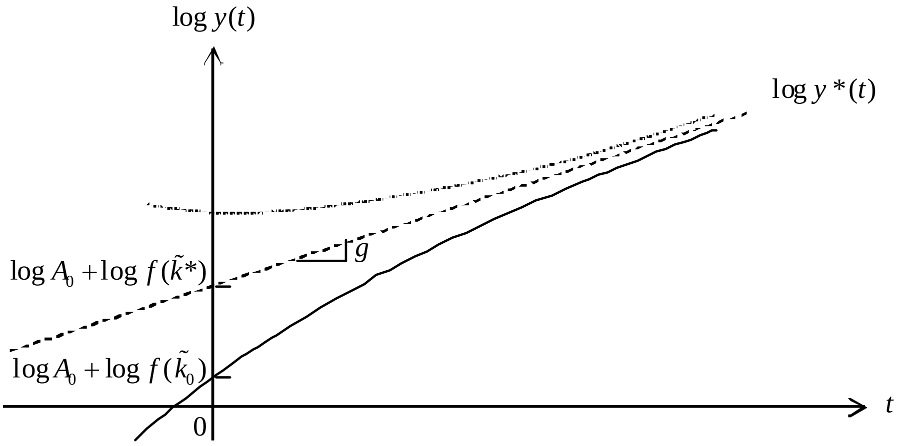

Figure 4:

The solid curve in Fig. 4 depicts the evolution of $\log y(t)$ in the case where $\tilde{k}_0 < \tilde{k}^*$ (note that $\log y^*(0) = \log f(\tilde{k}^*) + \log A_0$). The dotted curve exemplifies the case where $\tilde{k}_0 > \tilde{k}^*$. The figure illustrates per capita income convergence: low initial income is associated with a high subsequent growth rate which, however, diminishes along with the diminishing logarithmic distance of per capita income to its level on the steady state path.

For convenience, we will from now on treat (24) as an equality. Subtracting $\log y(0)$ on both sides, we get

$$\begin{aligned}\log y(t) - \log y(0) &= \log y^*(0) - \log y(0) + gt + (\log y(0) - \log y^*(0))e^{-\beta t} \\ &= gt - (1 - e^{-\beta t})(\log y(0) - \log y^*(0)).\end{aligned}$$

Dividing through by $t > 0$ gives

$$\frac{\log y(t) - \log y(0)}{t} = g - \frac{1 - e^{-\beta t}}{t}(\log y(0) - \log y^*(0)). \tag{25}$$

On the left-hand side appears the average compound annual growth rate of y from period 0 to period t, which we will denote $\bar{g}_y(0, t)$. On the right-hand side appears the initial distance of $\log y$ to its hypothetical level along the steady state path. The coefficient, $-(1 - e^{-\beta t})/t$, to this distance is negative and approaches zero for $t \to \infty$. Thus (25) is a translation into growth form of the convergence of $\log y_t$ towards the steady-state path, $\log y_t^*$, in the theoretical model without shocks. Rearranging the right-hand side, we get

$$\bar{g}_y(0, t) = g + \frac{1 - e^{-\beta t}}{t} \log y^*(0) - \frac{1 - e^{-\beta t}}{t} \log y(0) \equiv b^0 + b^1 \log y(0),$$

where both the constant $b^0 \equiv g + \left[(1 - e^{-\beta t})/t\right] \log y^*(0)$ and the coefficient $b^1 \equiv -(1 - e^{-\beta t})/t$ are determined by "structural characteristics". Indeed, β is determined by δ, g, n, and $\varepsilon(\tilde{k}^*)$ through (21), and $y^*(0)$ is determined by A_0 and $f(\tilde{k}^*)$ through (12), where, in turn, $\tilde{k}^*$ is determined by the steady state condition $s(\tilde{k}^*) f(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*$, s^* being the saving-income ratio in the steady state.

With data for n countries, $i = 1, 2, \ldots, n$, a test of the *unconditional convergence hypothesis* may be based on the regression equation

$$\bar{g}_{y_i}(0,t) = b^0 + b^1 \log y_i(0) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2), \tag{26}$$

where ϵ_i is the error term. This can be seen as a Barro growth regression equation in its simplest form. For countries in the entire world, the theoretical hypothesis $b^1 < 0$ is clearly not supported (or, to use the language of statistics, the null hypothesis, $b^1 = 0$, is not rejected).[2]

Allowing for the considered countries having different structural characteristics, the Barro growth regression equation takes the form

$$\bar{g}_{y_i}(0,t) = b_i^0 + b^1 \log y_i(0) + \epsilon_i, \qquad b^1 < 0, \qquad \epsilon_i \sim N(0, \sigma_\epsilon^2). \tag{27}$$

In this "fixed effects" form, the equation has often been applied for a test of the *conditional convergence hypothesis,* $b^1 < 0$, often supporting this hypothesis.

From the estimate of b^1 the implied estimate of the asymptotic speed of convergence, β, is readily obtained through the formula $b^1 \equiv (1 - e^{-\beta t})/t$. Even β, and therefore also the slope, b^1, does depend, theoretically, on country-specific structural characteristics. But the sensitivity on these do not generally seem large enough to blur the analysis based on (27) which abstracts from this dependency.

With the aim of testing hypotheses about growth determinants, Barro (1991) and Barro and Sala-i-Martin (1992, 2004) decompose b_i^0 so as to reflect the role of a set of measurable potentially causal variables,

$$b_i^0 = \alpha_0 + \alpha_1 x_{it} + \alpha_2 x_{i2} + \ldots + \alpha_m x_{im},$$

where the α's are the coefficients and the x's are the potentially causal variables.[3] These variables could be measurable Solow-type parameters among those appearing in (20)

[2] Cf. Acemoglu, p. 16. For the OECD countries, however, b^1 is definitely estimated to be negative (cf. Acemoglu, p. 17).

[3] Note that our α vector is called β in Acemoglu, pp. 83-84. So Acemoglu's β is to be distinquished from our β which denotes the asymptotic speed of convergence.

or a broader set of determinants, including for instance the investment-income ratio, the educational level in the labor force, and institutional variables like rule of law and democracy. Some studies include the initial within-country inequality in income or wealth among the x's and extend the theoretical framework correspondingly.[4]

From an econometric point of view there are several problematic features in regressions of Barro's form (also called the β convergence approach). These problems are discussed in Acemoglu pp. 82-85.

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[4] See, e.g., Alesina and Rodrik (1994) and Perotti (1996), who argue for a negative relationship between inequality and growth. Forbes (2000), however, rejects that there should be a robust negative correlation between the two.

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04.02.2013 Christian Groth

On the Simon-Kremer version of the population-breeds-ideas model

This lecture note relates to Section 2 of Acemoglu's Chapter 4 and explains the details of what Acemoglu (p. 114) calls the *Simon-Kremer version* of the population-breeds-ideas model.

1 The model

Suppose a pre-industrial economy can be described by:

$$
\begin{aligned}
Y_t &= A_t^\sigma L_t^\alpha Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, & (1)\\
\dot{A}_t &= \lambda A_t^\varepsilon L_t, \qquad \lambda > 0, 0 < \varepsilon \leq 1, \qquad A_0 > 0 \text{ given}, & (2)\\
L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, & (3)
\end{aligned}
$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), Z the amount of land (fixed), and $\bar{y}$ subsistence minimum (so the φ in Acemoglu's equation (4.2) is simply the inverse of the subsistence minimum). Both Z and $\bar{y}$ are considered as constant parameters. Time is continuous and it is understood that a kind of Malthusian population mechanism (see below) is operative behind the scene.

The exclusion of capital from the aggregate production function, (1), reflects the presumption that capital (tools etc.) is quantitatively of minor importance in a pre-industrial economy. In accordance with the replication argument, the production function has CRS w.r.t. the rival inputs, labor and land. The factor A_t^σ measures total factor productivity. In view of (2), the technology level, A_t, is rising over time. The increase in A_t per time unit is seen to be an increasing function of the size of the population. This reflects the hypothesis that population breeds ideas; these are non-rival and enter the pool of technical knowledge available for society as a whole. The rate per capita, λA^ε, by which population breeds ideas is an increasing function of the already existing level of technical knowledge.

This reflects the hypothesis that the larger is the stock of ideas, the easier do new ideas arise (perhaps by combination of existing ideas).

Equation (3) is a shortcut description of a Malthusian population mechanism. Suppose the true mechanism is

$$\dot{L}_t = \beta(y_t - \bar{y})L_t, \qquad \beta > 0, \bar{y} > 0, \tag{4}$$

where $y_t \equiv Y_t/L_t$ is per capita income and $\bar{y}$ is subsistence minimum. A rise in y_t above $\bar{y}$ will lead to increases in L_t, thereby generating downward pressure on Y_t/L_t and perhaps end up pushing y_t below $\bar{y}$. When this happens, population will be decreasing for a while and so return towards its sustainable level, $Y_t/\bar{y}$. Equation (3) treats this mechanism as if the population instantaneously adjusts to its sustainable level (as if $\beta \to \infty$). The model hereby gives a long-run picture, ignoring the Malthusian ups and downs in population and per capita income about the subsistence minimum. The important feature is that the technology level and thereby Y_t as well as the sustainable population will be rising over time. This speeds up the arrival of new ideas and so raises Y_t even faster.

For simplicity, we now normalize the constant Z to be 1.

2 Law of motion

The dynamics of the model can be reduced to one differential equation, the law of motion of technical knowledge. By (3), $L_t = \varphi Y_t = \varphi A_t^{\sigma} L_t^{\alpha}$. Consequently $L_t^{1-\alpha} = \varphi A_t^{\sigma}$ so that

$$L_t = \varphi^{\frac{1}{1-\alpha}} A_t^{\frac{\sigma}{1-\alpha}}.$$

Substituting this into (2) gives the law of motion of technical knowledge:

$$\dot{A}_t = \lambda \varphi^{\frac{1}{1-\alpha}} A_t^{\varepsilon + \frac{\sigma}{1-\alpha}} \equiv \hat{\lambda} A_t^{\varepsilon + \frac{\sigma}{1-\alpha}}. \tag{5}$$

Define $\mu \equiv \varepsilon + \frac{\sigma}{1-\alpha}$ and assume $\mu > 1$. Then (5) can be written

$$\dot{A}_t = \hat{\lambda} A_t^{\mu}, \tag{6}$$

which is a nonlinear differential equation in A.[1] Let $x \equiv A^{1-\mu}$. Then

$$\dot{x}_t = (1-\mu) A_t^{-\mu} \hat{\lambda} A_t^{\mu} = (1-\mu)\hat{\lambda}, \tag{7}$$

[1] The differential equation, (6), is a special case of what is known as the *Bernoulli equation*.

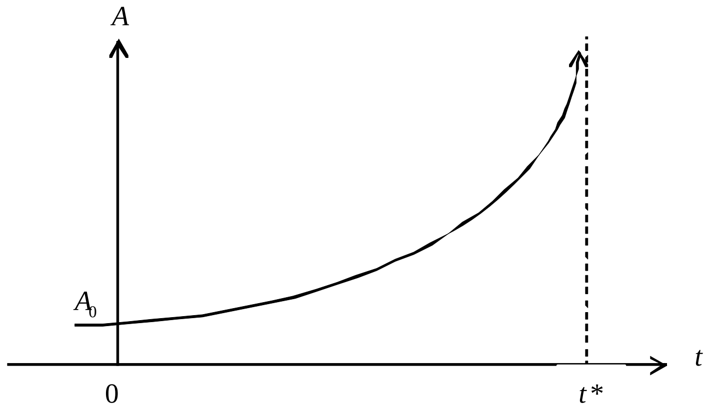

Figure 1:

a constant. To find x_t from this, we only need simple integration:

$$x_t = x_0 + \int_0^t \dot{x}_\tau d\tau = x_0 + (1-\mu)\hat{\lambda}t.$$

As $A = x^{\frac{1}{1-\mu}}$ and $x_0 = A_0^{1-\mu}$, this implies

$$A_t = x_t^{\frac{1}{1-\mu}} = \left[A_0^{1-\mu} + (1-\mu)\hat{\lambda}t\right]^{\frac{1}{1-\mu}} = \frac{1}{\left[A_0^{1-\mu} - (\mu-1)\hat{\lambda}t\right]^{\frac{1}{\mu-1}}}. \tag{8}$$

3 The inevitable ending of the Malthusian regime

The result (8) helps us in understanding why the Malthusian regime must come to an end (at least if the model is an acceptable description of the Malthusian regime).

Although to begin with, A_t may grow extremely slowly, the growth in A_t will be *accelerating* because of the *positive feedback* (visible in (2)) from both rising population and rising A_t. Indeed, since $\mu > 1$, the denominator in (8) will be decreasing over time and approach zero in finite time, namely as t approaches the finite value $t^* = A_0^{1-\mu}/((\mu-1)\hat{\lambda})$. Fig. 1 illustrates. The evolution of technical knowledge becomes explosive as t approaches t^*.

It follows from (1) that explosive growth in A implies explosive growth in Y. The acceleration in the evolution of Y will sooner or later make Y move fast enough so that the Malthusian population mechanism (which for biological reasons has to be slow) can not catch up. Then, what was in the Malthusian population mechanism, equation (4), earlier only a transitory excess of y_t over $\bar{y}$, will sooner or later become a permanent excess and take the form of sustained growth in y_t.

According to equation (4), this should lead to a permanently rising population growth rate. As economic history has testified, however, the rising standard of living *changed* the demographics and resulted in the "demographic transition" with fertility declining faster than mortality. This results in completely different dynamics about which the present model has nothing to say.[2] As to the demographic transition as such, explanations suggested by economists include: higher opportunity costs of raising children, the trade-off between "quality" (educational level) of the offspring and their "quantity", skill-biased technical change, and improved contraception technology.

The present model is about dynamics in the Malthusian regime of the pre-industrial epoch. The story told by the model is the following. When the feedback parameter, μ, is above one, the Malthusian regime has to come to an end because the battle between scarcity of land (or natural resources more generally) and technological progress will inevitably be won by the latter.[3]

4 Closing remarks

The cases $\mu < 1$ and $\mu = 1$ are considered in Exercise III.3. The case $\mu = 1$ corresponds to Acemoglu's first version (p. 113) of the population-breeds-ideas model. In that version, σ has the value $1 - \alpha$ and $\varepsilon = 0$ (two arbitrary knife-edge conditions). Then a constant growth rate in A, L, and Y results and y remains at $\bar{y}$ forever.

On the basis of demographers' estimates of the growth in global population over most of human history, Kremer (1993) finds empirical support for $\mu > 1$.

5 Appendix

Mathematically, the background for the explosion result is that the solution to a first-order differential equation of the form $\dot{x}(t) = \alpha + bx(t)^c$, $c > 1$, $b \neq 0$, $x(0) = x_0$ given, is always explosive. Indeed, the solution, $x = x(t)$, will have the property that $x(t) \rightarrow \pm\infty$

[2] Kremer (1993), however, also includes an extended model taking some of these changed dynamics into account.

[3] The mathematical background for the explosion result is explained in the appendix.

for $t \to t^*$ for some fixed $t^* > 0$; and thereby the solution is defined only on a bounded time interval.

Take the differential equation $\dot{x}(t) = 1 + x(t)^2$ as an example. As is well-known, the solution is $x(t) = \tan t = \sin t / \cos t$, defined on the interval $(-\pi/2, \pi/2)$.

6 References

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