Exercise Problems for Economic Growth

Part 1

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Contents of Part 1

Preface

This is Part 1 of a collection of exercise problems that have been used in recent years in the course Economic Growth within the Master Program in economics at Department of Economics, University of Copenhagen. The majority of the exercise problems have been tried out in previous years and exams but until now they were scattered in many different problem sets.

For constructive criticism I thank Niklas Brønager and a lot of previous students who have suffered for bad wording and obscurities in earlier versions of the problems. No doubt, it is still possible to find obscurities. Hence, I very much welcome comments and suggestions of any kind relating to these exercises.

February 10, 2012

Christian Groth

Remarks on notation

Given a production function $Y = F(K, AL)$, we use the notation $y \equiv Y/L$, $k \equiv K/L$, $\tilde{y} \equiv Y/(AL)$, and $\tilde{k} \equiv K/(AL)$. Acemoglu (2009) uses y and k in the same way. The corresponding variables defined on a "per unit of effective labor basis" he writes, however, in the asymmetrical way as $\hat{y} \equiv Y/(AL)$ and $k \equiv K/(AL)$.

To indicate the "level of technology" (assumed measurable by a single number), depending on convenience, we sometimes use A (as above), sometimes T.

Unless otherwise specified, whether we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

From Chapter 3 and onward, the time argument of a variable, x, may be written as a subscript t, that is, as x_t, rather than as $x(t)$ (depending on convenience).

I

A refresher on basic concepts. The Solow model in continuous time

I.1 In the last four decades China has had very high growth in real GDP per capita, cf. Table 1. Answer questions a), b), and c) presupposing that the growth performances of China and the U.S. continue to be like what they have been 1980-2007.

a) How many years does it take for China's GDP per capita to be doubled? You should explain your method.

b) How many years does it take for GDP per capita in the U.S. to be doubled?

c) How long time, reckoned from 2007, will it take for China to catch up with the US in terms of income per capita? You should explain your method.

d) Do you find it likely that the actual course of events will be (approximately) like that? Why or why not?

Table 1. GDP per capita in USA and China 1980 - 2007 (I$ in 2005 Constant Prices)

country	year	rgdpch
United States	1980	24537.41
United States	2007	42886.92
China	1980	1133.21
China	2007	7868.28

Source: PWT 6.3. Note: For China the Version 2 data series is used.

I.2 In a popular magazine on science the data in Table 2 was reported:

Table 2. World income per capita relative to income per capita in the US: 1952-96

Year	*Percent*
1952	13.0
1962	13.3
1972	13.0
1982	13.8
1992	15.1
1996	17.7

Source: Knowledge, Technology, & Policy 13, no. 4, 2001, p. 52.
Note. Countries' per capita income are weighted by population as a fraction of the world population.

a) Briefly, discuss this data relative to concepts of income convergence and divergence and relative to your knowledge of the importance of weighting by population size.

b) What is meant by the terms unconditional (or absolute) income convergence and conditional convergence?

c) Give a short list of mechanisms that could in principle explain the data above.

I.3 *Stocks versus flows* Two basic elements in growth models are often presented in the following way. The aggregate production function is described by

$$Y_t = F(K_t, L_t, A_t), \tag{*}$$

where Y_t is output (aggregate value added), K_t capital input, L_t labor input, and A_t the "level of technology". The time index t may refer to period t, that is the time interval $[t, t+1)$, or to a point in time, depending on the context. And accumulation of the stock of capital in the (closed) economy is described by

$$K_{t+1} - K_t = Y_t - C_t - \delta K_t, \qquad (**)$$

where δ is an (exogenous) rate of (physical) depreciation of capital, $0 \leq \delta \leq 1$. In continuous time models the corresponding equation is

$$\dot{K}(t) \equiv \frac{dK(t)}{dt} = Y(t) - C(t) - \delta K(t), \qquad \delta \geq 0.$$

a) At the theoretical level, what denominations (dimensions) should be attached to output, capital input, and labor input in a production function?

b) What is the denomination (dimension) attached to K in the accumulation equation?

c) Is there any consistency problem in the notation used in (*) vis-à-vis (**)? Explain.

d) Suggest an interpretation that ensures that there is no consistency problem.

e) Suppose there are two countries. They have the same technology, the same capital stock, the same number of employed workers, and the same number of man-hours per worker per year. Country a does not use shift work, but country b uses shift work, that is, two work teams of the same size and the same number of hours per day. Elaborate the formula (*) so that it can be applied to both countries.

f) Suppose F is a neoclassical production function with CRS w.r.t. K and L. Compare the output levels in the two countries, assuming they have the same capital depreciation rate. Comment.

g) In continuous time we write aggregate (real) gross saving as $S(t) \equiv Y(t) - C(t)$. What is the denomination of $S(t)$.

h) In continuous time, does the expression $K(t) + S(t)$ make sense? Why or why not?

i) In discrete time, how can the expression $K_t + S_t$ be meaningfully interpreted?

I.4 *Short questions* (answering requires only a few well chosen sentences)

a) Consider an economy where all firms' technology is described by the same neoclassical production function, $Y_i = F(K_i, L_i)$, $i = 1, 2, ..., N$, with decreasing returns to scale everywhere (standard notation). Suppose there is "free entry and exit" and perfect competition in all markets. Then a paradoxical situation arises in that no equilibrium with a finite number of firms (plants) would exist. Explain.

b) In the Diamond OLG model as in many other macro models, the technology is assumed to have constant returns to scale (CRS) with respect to capital and labor taken together. Often the so-called *replication argument* is put forward as a reason to expect that CRS should hold in the real world. What is the replication argument? Do you find an appeal to the replication argument to be a convincing argument for the assumption of CRS with respect to capital and labor? Why or why not?

c) Does the validity of the replication argument, considered as an argument about a property of technology, depend on the availability of the different inputs.

d) Suppose that for a certain historical period there has been something close to constant returns to scale and perfect competition, but then, after a shift to new technologies in the different industries, increasing returns to scale arise. What is likely to happen to the market form? Why?

I.5 *The Solow model; local and global asymptotic stability* The Solow growth model in continuous time can be set up in the following way. A closed economy is considered. There is an aggregate production function,

$$Y(t) = F(K(t), T(t)L(t)),$$

where F is a neoclassical production function with CRS, Y is output, K is capital input, T is the technology level, and L is labor input. There is full employment. It is assumed that

$$\begin{aligned} T(t) &= T(0)e^{gt}, \qquad T(0) = T_0, \qquad g \geq 0, \\ L(t) &= L(0)e^{nt}, \qquad L(0) = L_0, \qquad n \geq 0. \end{aligned}$$

Aggregate gross saving, S, is assumed proportional to gross aggregate income which, in a closed economy, equals real GDP, Y:

$$S(t) = sY(t), \qquad 0 < s < 1.$$

Capital accumulation is described by

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t), \qquad 0 < \delta \leq 1,$$

where δ is the rate of (physical) depreciation of capital. Finally, from national income accounting,

$$S(t) \equiv Y(t) - C(t).$$

The symbols g, n, s, and δ represent parameters (given constants) and the initial values T_0, L_0, and K_0, are given (exogenous) positive numbers.

a) What kind of technical progress is assumed in the model?

b) Let $\tilde{y} \equiv y/T \equiv Y/(TL)$. Let the production function in intensive form be denoted f. Derive f from F. Sign f' and f''. From the given information, can we be sure that $f(0) = 0$? Why or why not?

c) To get a grasp of the evolution of the economy over time, derive a first-order differential equation in the (effective) capital intensity, $\tilde{k} \equiv k/T \equiv K/(TL)$, that is, an equation of the form $\dot{\tilde{k}} = \varphi(\tilde{k})$. *Hint:* $\dot{\tilde{k}}/\tilde{k} = \dot{k}/k - \dot{T}/T = \dot{K}/K - \dot{L}/L - g$.[1]

d) If in c) you were able to write $\varphi(\tilde{k})$ on the form $\varphi(\tilde{k}) = \psi(\tilde{k}) - a\tilde{k}$, where a is a constant, you are one the right track. Draw a "Solow diagram Version 1", that is, a diagram displaying the graphs of the functions $\psi(\tilde{k})$ and $a\tilde{k}$ in the $(\tilde{k}, \tilde{y})$ plane. You may, at least initially, draw the diagram such that the two graphs cross each other for some $\tilde{k} > 0$.

e) Suppose there exists a (non-trivial) steady state, $\tilde{k}^* > 0$. Indicate $\tilde{k}^*$ in the diagram. Can there be more than one (non-trivial) steady state? Why or why not?

f) In the same or a new diagram, draw a "Solow diagram Version 2", that is, a diagram displaying the graphs of the functions $\psi(\tilde{k})/s$ and $a\tilde{k}/s$ in the $(\tilde{k}, \tilde{y})$ plane. At what value of $\tilde{k}$ will these two graphs cross?

g) Suppose capital is *essential*, that is, $F(0, TL) = 0$ for all TL. In terms of limiting values of f' for the capital intensity approaching zero and infinity, respectively, write down a necessary and sufficient condition for existence of a (non-trivial).

[1] Recall the following simple continuous time rule: Let $z = y/x$, where z, y, and x are differentiable functions of time t and positive for all t. Then $\dot{z}/z = \dot{y}/y - \dot{x}/x$, exactly. *Proof:* We have $\log z = \log y - \log x$. Now take the time derivative on both sides of the equation.

h) Suppose the steady state, $\tilde{k}^*$, is locally asymptotically stable? What is meant by this and will it always be true for the Solow model? Why or why not?

i) Suppose the steady state, $\tilde{k}^*$, is globally asymptotically stable? What is meant by this and will it always be true for the Solow model? Why or why not?

j) Find the long-run growth rate of output per unit of labor.

I.6 This problem is about the same model as Problem I.5, the standard version of the Solow model.

a) Suppose the economy is in steady state until time t_0. Then, for some extraneous reason, an upward shift in the saving rate occurs. Illustrate by the Solow diagram the evolution of the economy from t_0 onward.

b) Draw the time profile of $\ln y$ in the $(t, \ln y)$ plane.

c) How, if at all, is the level of y affected by the shift in s?

d) How, if at all, is the growth rate of y affected by the shift in s? Here you may have to distinguish between transitory and permanent effects.

e) Explain by words the economic mechanisms behind your results in c) and d).

f) As Solow once said (in a private correspondence with Amartya Sen[2]): "The idea [of the model] is to trace full employment paths, no more." What market form is theoretically capable of generating permanent full employment?

g) Even if we recognize that the Solow model only attempts to trace hypothetical time paths with full employment (or rather employment corresponding to the "natural" or "structural" rate of unemployment), the model has several important limitations. What is in your opinion the most important limitations?

I.7 Set up a Solow model where, although there is no technical progress, sustained per capita growth occurs. Comment. *Hint:* a simple approach can be based on the production

[2] *Growth Economics. Selected Readings*, edited by Amartya Sen, Penguin Books, Middlesex, 1970, p. 24.

function $Y = BK^{\alpha}L^{1-\alpha} + AK$, where $A > 0, B > 0, 0 < \alpha < 1$. "Sustained per capita growth" is said to occur if $\lim_{t\to\infty} \dot{y}/y > 0$ or $\lim_{t\to\infty} \dot{c}/c > 0$ (standard notation).

I.8 Consider a closed economy with technology described by the aggregate production function

$$Y = F(K, L),$$

where F is a neoclassical production function with CRS and satisfying the Inada conditions, Y is output, K is capital input and L is labor input = labor force = population (there is always full employment). A constant fraction, s, of *net* income is saved (and invested). Capital depreciates at the constant rate $\delta > 0$.

a) Assuming a constant population growth rate n, derive the fundamental differential equation of the model and illustrate the dynamics by a phase diagram. Comment.

b) Assume instead that the population growth rate n is a smooth function of per capita income, i.e., $n = n(y)$, where $y \equiv Y/L$. At very low levels of per capita income, n is zero, at higher per capita income, n is a hump-shaped function of y, and at very high levels of y, n tends to zero, that is, for some $\bar{y} > 0$ we have

$$n'(y) \gtreqless 0, \text{ for } y \lesseqgtr \bar{y}, \text{ respectively,}$$

whereas $n(y) \approx 0$ for y considerably above $\bar{y}$. Show that this may give rise to a dynamics quite different from that of the Solow model. Comment.

I.9 *Short questions*

a) "The Cobb-Douglas production function has the property that, under technical progress, it satisfies all three neutrality criteria if it satisfies one of them." True or false? Explain why.

b) Write down a Cobb-Douglas production function that displays non-neutral technical change.

c) "If and only if the production function is Cobb-Douglas with CRS and time-independent output elasticity w.r.t. capital, does the Solow model with competitive markets predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

I.10 *Short question* "Relatively homogeneous groups of countries, such as for example the 12 old EU member countries, tend to experience income convergence in the sense that the standard deviation of income per capita across the countries diminishes over time." True or not true as an empirical statement? Comment.

I.11 Consider the production function $Y = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$.

a) Does the function imply constant returns to scale?

b) Is the production function neoclassical? *Hint:* after checking criterion (a) of the definition of a neoclassical production function, you may use claim (iii) of Section 2.1.3 together with your answer to a).

c) Given this production function, is capital an essential production factor? Is labor?

d) If we want to extend the domain of definition of the production function to include $(K, L) = (0, 0)$, how can this be done while maintaining continuity of the function?

I.12 *Never trust authorities I* Sometimes misleading graphs and/or figure texts about across-country income convergence are published. For example, Fig. 1 shows a copy of a figure from a publication by the Danish Ministry of Finance, 1996. In English, the headline reads "Standard deviation in GDP per capita in EU-12".

a) What is the problem with this data material as presented, including the headline? *Hint:* if you need help, you may consult Section 4.2 and 4.5 in Lecture Note 1.

Similarly, Fig. 2 shows a copy of a figure from the Danish Economic Council, 1997. In English, the headline reads: Standard deviation in GDP per capita and per worker across the EU countries.[3]

b) What is the problem with this data material as presented, including the headline?

c) There is a certain property of the standard deviation that is important for its relevance as a criterion for whether σ convergence is present or not. Briefly discuss.

[3] The note to this figure says that GDP per capita and per worker are weighted with size of population and size of labor force, respectively.

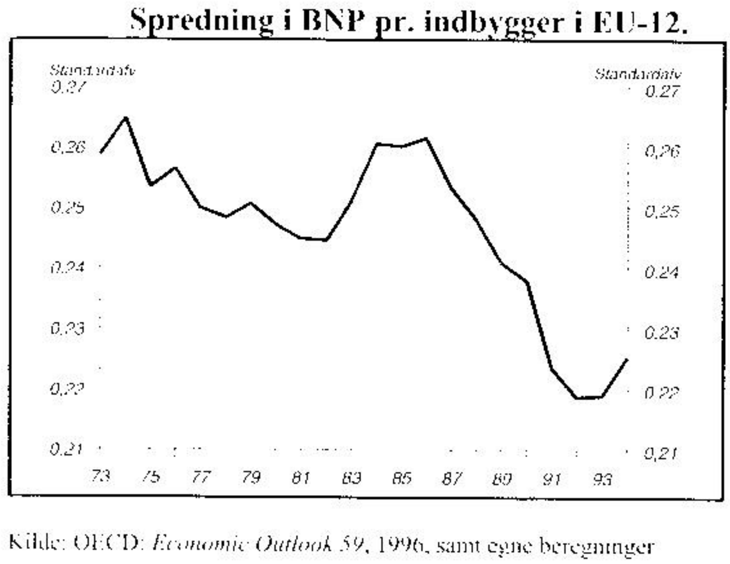

Figure 1: Source: Finansredegørelse 96. Bilag p. 53. Finansministeriet, Dec. 1996.

I.13 *Never trust authorities II* Also prominent economists sometimes make elementary mistakes. For example, in the context of the Solow growth model we read in Acemoglu's textbook (Acemoglu 2009, p. 53) the statement: "In addition, k^* is increasing in α", where

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

What is the problem with the cited statement?

I.14 An important aspect of growth analysis is to pose good questions in the sense of questions that are brief, interesting, and manageable. If we set aside an hour or so in one of the later lectures, what question would you suggest should be discussed?

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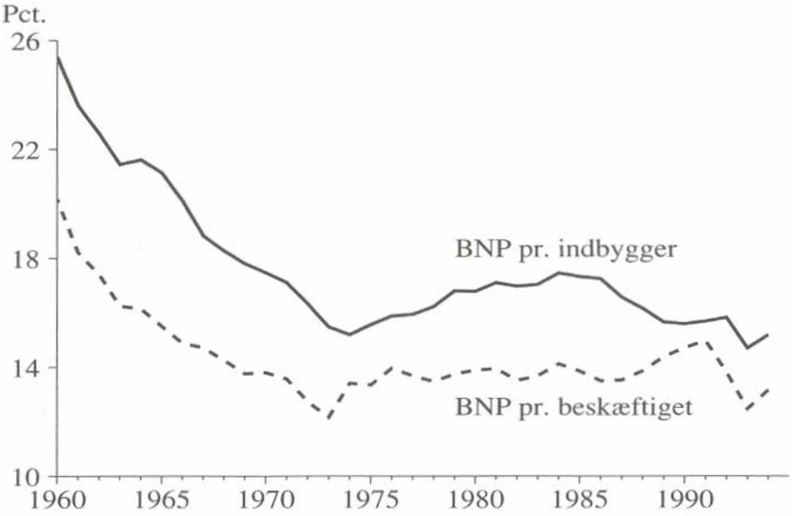
Figur IV.4 Spredningen i EU-landenes indkomst pr. indbygger og pr. beskæftiget



Anm.: Spredningen er beregnet som den vægtede spredning i EU-landenes købekraftskorrigerede BNP pr. indbygger og pr. beskæftiget. Som landevægte er anvendt hhv. befolkningstal og antal beskæftigede.

Kilde: OECD, *Economic Outlook* og egne beregninger.

Figure 2: Source: Det økonomiske Råd, Dansk økonomi Forår 1997, p. 147.

II

Transitional dynamics. Speed of convergence

II.1 *Within-country speed of convergence according to the Solow model* We know that the Solow growth model can be reduced to the following differential equation ("law of motion")

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + g + n)\tilde{k}, \qquad (*)$$

where $\tilde{k} \equiv k/A \equiv K/(AL)$ (standard notation). Assume

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \frac{\delta + g + n}{s} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}).$$

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, which is globally asymptotically stable.

a) Illustrate this result graphically in the $(\tilde{k}, \tilde{y})$ plane, where $\tilde{y} \equiv y/A \equiv Y/(AL)$.

b) Suppose the economy has been in steady state until time 0. Then an upward shift in the saving rate occurs. Illustrate graphically the evolution of the economy from time 0 onward.

c) We are interested not only in the ultimate effect of this parameter shift, but also in how fast the adjustment process occurs. For a general differentiable adjustment process, $(x(t))$, an answer to this can be based on the (proportionate) rate of decline at time t of the distance to the steady state:

$$\text{SOC}(t) \equiv -\frac{d(|x(t) - x^*|)/dt}{|x(t) - x^*|}, \qquad (**)$$

where $|a|$ is our notation for the absolute value of a real number, a (SOC stands for Speed of Convergence). In the context of the Solow model the corresponding expression simplifies to

$$\text{SOC}(t) = -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} = -\frac{\dot{\tilde{k}}}{\tilde{k}(t) - \tilde{k}^*}. \qquad (***)$$

How?

The right-hand side of (***) is called the *instantaneous speed of convergence* of the technology-corrected capital intensity at time t. For $t \to \infty$ both the denominator and the numerator approach zero. Yet the ratio of the two has a well-defined limit for $t \to \infty$, the so-called *asymptotic speed of convergence*, often simply denoted SOC. By applying a first-order Taylor approximation of $\dot{\tilde{k}}$ about the steady state, one can find SOC.

d) Applying a first-order Taylor approximation of $\dot{\tilde{k}}$, derive a formula for the asymptotic speed of convergence and calculate its value, given an assumption of perfect competition together with the following data (time unit one year): $\delta = 0.07$, $g = 0.02$, $n = 0.01$, and gross capital income share $= 1/3$. *Hint:* for a differentiable non-linear function $\varphi(x)$, a first-order Taylor approximation about x^* is $\varphi(x) \approx \varphi(x^*) + \varphi'(x^*)(x - x^*)$.

II.2 *Example where an explicit solution for the time-path can be found* Sometimes an explicit solution for the whole time path during the adjustment process can be found. In the Solow model this is so when the production function is Cobb-Douglas: $Y = K^\alpha (AL)^{1-\alpha}$, $\quad 0 < \alpha < 1$, $A(t) = A(0)e^{gt}$. In standard notation the fundamental differential equation is

$$\dot{\tilde{k}} = s\tilde{k}^\alpha - (\delta + g + n)\tilde{k}, \qquad 0 < s < 1,\ \delta + g + n > 0. \qquad (*)$$

Let $\tilde{k}(0) > 0$ be arbitrary. Since (*) is a so-called Bernoulli equation (the right-hand side of the ordered differential equation is the sum of a power function term and a linear term), we can find an explicit solution of (*):

a) Let $x \equiv \tilde{k}^{1-\alpha} = \tilde{k}/\tilde{y} = k/y$. So x is simply the capital-output ratio. Derive a differential equation for x.

b) Write down the solution of this differential equation. *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$, has the solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}.$$

c) Determine the instantaneous speed of convergence of the capital-output ratio at time t. *Hint:* apply (**) of Problem II.1.

d) Generally, for variables belonging to a nonlinear model the instantaneous speed of convergence is time-dependent and at best an acceptable *approximation* to the asymptotic speed of convergence. Compare to your result from c).

e) Compare the asymptotic speed of convergence of $\tilde{k}$ in the present model to your result from c).

II.3 Consider a Solow model for a closed economy with perfect competition. The rate of Harrod-neutral technical progress is 1.8 percent per year, the rate of population growth is 0.5 percent per year, capital depreciates at the rate 0.6 per year, and in steady state the share of labor income in steady state is 2/3.

a) Find the asymptotic speed of convergence of $\tilde{k}$. *Hint:* given the production function on intensive form, $f(\tilde{k})$, the asymptotic speed of convergence is $(1-\frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)})(\delta+g+n)$ (standard notation).

b) Find the approximate *half-life* of the initial distance of $\tilde{k}$ to its steady-state value, $\tilde{k}^*$. *Hint:* Let the time path of a variable x be $x(t) = x(0)e^{-\beta t}$ where β is a positive constant. Then the half-life of the initial distance to the steady state (i.e., the time it takes for half of the initial gap to be eliminated) is given as $-\ln \frac{1}{2}/\beta = \ln 2/\beta$.

c) Empirical cross-country analyses of conditional convergence point at speeds of convergence between 2% per year and 9% per year, depending on the selection of countries and the econometric method. Although this throws empirical light on "across-country convergence" while the Solow model throws theoretical light on "within-country convergence", compare your result from a) to this empirical knowledge.

d) What is the *doubling-time* of income per capita implied by the model?

e) What is the long-run per capita growth rate implied by the model?

II.4 *Accounting for the formulas in Acemoglu (2009, pp. 80-81)* Consider a closed economy with aggregate production function $Y = F(K, AL)$, where Y is GDP, K capital input, A the technology level, and L labor input (the dating of the variables is implicit).

It is assumed that F is neoclassical with CRS and that A and L grow exogenously at the constant rates $g > 0$ and $n \geq 0$, respectively. Capital moves according to

$$\dot{K} = Y - C - \delta K, \qquad \delta \geq 0,$$

where C is aggregate consumption.

Suppose it is known that in the absence of shocks, $\tilde{k} \equiv K/(AL)$ converges towards a unique steady state value, $\tilde{k}^* > 0$, for $t \to \infty$. Suppose further that in a small neighborhood of the steady state, the instantaneous speed of convergence of $\tilde{k}$ is

$$\text{SOC}_t \equiv -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \approx (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*), \qquad (*)$$

where $\varepsilon(\tilde{k})$ is the output elasticity w.r.t. capital, evaluated at the effective capital intensity $\tilde{k}$.

a) Show that (*) implies

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t}.$$

Hint: when a variable $x > 0$ has a constant growth rate, γ, its time path (in continuous time) is $x(t) = x(0)e^{\gamma t}$ (this follows from the hint to Problem II.2b above) with $a = -\gamma$ and $b = 0$).

Since time series of economic data are often given in logarithmic form and many approximative models are based on log-linearization, it is of interest what the asymptotic speed of convergence of $\log \tilde{k}(t) - \log \tilde{k}^*$ is.

b) To pursue this, show by a first-order Taylor approximation of $\log \tilde{k}$ about $\log \tilde{k}^*$ that $\log \tilde{k} \approx \log \tilde{k}^* + (\tilde{k} - \tilde{k}^*)/\tilde{k}^*$.

c) Find the asymptotic speed of convergence of $\log \tilde{k}$, $\text{SOC}_{\log \tilde{k}}$. Comment.

For a variable $x > 0$, let $g_x \equiv \dot{x}/x$.

d) Show that

$$g_{\tilde{k}} \approx -\beta(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*).$$

Hint: use your result in c).

Define $y \equiv Y/L \equiv \tilde{y}A$.

e) Show that $g_y = g + \varepsilon(\tilde{k})g_{\tilde{k}}$.

f) Show that

$$g_y \approx g - \beta(\tilde{k}^*)\varepsilon(\tilde{k})(\log \tilde{k} - \log \tilde{k}^*) \approx g - \beta(\tilde{k}^*)\varepsilon(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*). \qquad (**)$$

Define $y^*(t) \equiv f(\tilde{k}^*)A(t)$, where $f(\tilde{k}) \equiv F(\tilde{k}, 1)$.

g) Interpret $y^*(t)$. Draw the time profile of $\log y^*(t)$. In the same diagram, draw illustrating time profiles of $\log y(t)$ for the cases $\tilde{k}(0) < \tilde{k}^*$ and $\tilde{k}(0) > \tilde{k}^*$, respectively.

h) Show that

$$\log y - \log y^* \approx \varepsilon(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*). \qquad (***)$$

Hint: start from the left-hand side of (***), apply the principle in b), use that $\tilde{y}$ is a function of $\tilde{k}$, apply a first-order Taylor approximation on this function, and apply again the principle in b).

i) Based on (**) combined with (***), write down a Barro-style growth regression equation. *Hint:* a discrete-time approximation of g_y can be based on the principle in b).

II.5 This problem presupposes that you have already solved Problem II.4. The setup is the same as in the introduction to Problem II.4.

a) "The SOC of $\log \tilde{k}$ must equal the SOC of $(\log \tilde{k} - \log \tilde{k}^*)$". True or false? Why? *Hint:* recall the general definition of SOC of a converging variable x.

b) Find the approximate speed of convergence of the vertical distance $\log y - \log y^*$ in your graph from g) of Problem II.4. *Hint:* use your result in a) together with (***) and the conclusion in c) of Problem II.4.

II.6 A key variable in the adjustment process of a growth model is the saving-capital ratio. As an example we take the Solow model where this ratio is given as $S/K = sf(\tilde{k})/\tilde{k}$. Suppose f satisfies the Inada conditions.

a) On the basis of (*) in Problem II.1, illustrate the adjustment over time of S/K in the $(\tilde{k}, S/K)$ plane, assuming $\delta + g + n > 0$ (standard notation). For an arbitrary $\tilde{k}_0 > 0$, indicate $\dot{\tilde{k}}/\tilde{k}$ in the diagram. Comment.

Suppose the production function is Cobb-Douglas such that $f(\tilde{k}) = B\tilde{k}^{\alpha}$, $B > 0$, $\alpha \in (0, 1)$, and $s > (\delta + g + n)/B$.

b) Let $\alpha \to 1$ at the same time as $g \to 0$. What happens to the asymptotic speed of convergence in the limit when $\alpha = 1$ and $g = 0$? Comment.

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III

Growth accounting versus explanation of growth

III.1 Consider an economy with aggregate production function

$$Y_t = \tilde{F}(K_t, L_t, A_t),$$

where $\tilde{F}$ is a neoclassical production function w.r.t. K and L, Y is GDP, K capital input, A the technology level, and L labor input. We apply the convenient notation: $g_z \equiv \dot{z}/z$.

a) By the standard growth accounting method, decompose the output growth rate into its three basic components.

b) How is the TFP growth rate, x, defined? Interpret the concept TFP growth rate.

From now on assume $\tilde{F}$ has CRS w.r.t. K and L and can be written

$$\tilde{F}(K_t, L_t, A_t) = F(K_t, A_t L_t), \tag{*}$$

where A_t grows at a given constant rate $g > 0$ and employment grows at a given constant rate $n > 0$. Moreover, the increase in capital per time unit is given by

$$\dot{K}_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad \delta \geq 0, \tag{**}$$

where C is aggregate consumption.

c) Determine g_Y and g_K under balanced growth. *Hint:* in view of CRS, we know something about the sum of the output elasticities w.r.t. the two production factors and in view of the given additional information we know something about the relationship between g_Y and g_K under balanced growth.

d) Let $y_t \equiv Y_t/L_t$. Determine g_y and x under balanced growth.

e) Is there a sense in which technical progress explains more than what the growth accounting under a) and b) suggested? Explain.

III.2 This problem presupposes that you have already solved Problem III.1. In that problem, technological change was taken as exogenous. There are many ways to endogenize g_A. One is the "learning by investing" hypothesis according to which the evolution of A is a by-product of capital accumulation, for example in the following simple form:

$$A_t = K_t^{\lambda}, \qquad 0 < \lambda < 1.$$

a) Let $y_t \equiv Y_t/L_t$. Maintaining the production function F in (*) as well as (**) and $g_L = n > 0$ from Problem III.1, determine g_y under balanced growth.

b) We may say that now the mechanism that drives long-run productivity growth is the dynamic interaction of capital accumulation and learning (they stimulate each other). At a deeper level we may emphasize two aspects as being of key importance for this mechanism to be able to *sustain* productivity growth: one aspect pertains to the technology and the other to demography. What are these two aspects?

III.3 *The "general" version of Michael Kremer's subsistence economy model*[4] Consider a pre-industrial economy described by:

$$\begin{aligned} Y_t &= A_t^{\sigma} L_t^{\alpha} Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, && (1)\\ \dot{A}_t &= \lambda A_t^{\varepsilon} L_t, \qquad \lambda > 0, \varepsilon \leq 1, && (2)\\ L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, && (3) \end{aligned}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), and Z the amount of land (fixed). Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene; that is, (3) should be seen as a short-cut.

a) Interpret the equations, including $\bar{y}$.

From now, let $Z = 1$.

b) Show that the dynamics of the model reduces to the "law of motion":

$$\dot{A} = \hat{\lambda} A^{\varepsilon + \frac{\sigma}{1-\alpha}}, \qquad \text{where } \hat{\lambda} \equiv \lambda \varphi^{\frac{1}{1-\alpha}}.$$

[4] This builds on Kremer, QJE 1993, §I-II. Acemoglu, 2009, pp. 113-114, discusses two special cases of the model.

c) Define $\mu \equiv \varepsilon + \frac{\sigma}{1-\alpha}$ and solve the differential equation $\dot{A} = \hat{\lambda} A^{\mu}$. *Hint:* for $\mu = 1$, apply the hint in a) of Problem II.4; for $\mu \neq 1$, consider the implied differential equation for $x \equiv A^{1-\mu}$.

d) Show that "growth acceleration" ($\dot{A}/A$ rising over time) arises if and only if $\varepsilon > 1 - \frac{\sigma}{1-\alpha}$.

e) Show that the "growth acceleration" in this model takes a very dramatic form.

f) For fixed $\alpha = \bar{\alpha}$, illustrate in the (σ, ε) plane the region leading to "growth acceleration". Need ε be positive for "growth acceleration" to happen?

g) If the parameters are such that the economy belongs to the region mentioned in f), we can conclude something about the sustainability of the Malthusian regime as described by the model. What can we conclude?

III.4 It is preferable to solve Problem III.3 before this problem. Consider a Solow-Malthus model of a subsistence economy:

$$\begin{aligned} Y_t &= K_t^{\varepsilon} L_t^{\alpha} Z^{1-\varepsilon-\alpha}, \qquad 0 < \varepsilon < \varepsilon + \alpha < 1, \\ \dot{K}_t &= sY_t, \qquad 0 < s < 1, \\ L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, \end{aligned}$$

where Y is aggregate output, K input of physical capital, L the labor force (= population), and Z the amount of land (fixed). For simplicity, we ignore capital depreciation. Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene.

a) Interpret the equations, including $\bar{y}$. The exponents to K, L, and Z sum to one. What could be the argument for this?

From now, let $Z = 1$.

b) Derive the law of motion of the economy.

c) Does this model lead us to predict that the economy must sooner or later transcend the Malthusian subsistence regime? Why or why not? *Hint:* consider the behavior of $\dot{K}/K$ for $t \to \infty$.

d) Comment in relation to Problem III.3.

III.5 *Short questions.*

a) "If there are constant returns to scale with respect to physical capital and labor taken together, then, considering technical knowledge as a third production factor, there will be increasing returns w.r.t. to all three production factors taken together." True or false? Explain why.

b) Consider a set of countries, $j = 1, 2, ..., N$. Country j has the aggregate production function

$$Y_{jt} = F(K_{jt}, A_{jt}L_{jt}),$$

where F is neoclassical and has CRS (standard notation). The technology level A_{jt} evolves according to $A_{jt} = A_{j0}e^{gt}$, where A_{j0} differs widely across the countries. The positive constant g as well as F and the capital depreciation rate are, however, shared by the countries. Assume that (i) the countries trade in a fully integrated world market for goods and financial capital; (ii) they face a constant real interest rate $r > 0$ in this market; and (iii) there is perfect competition in all markets. "In this setup there will be a strong economic incentive for workers to migrate." True or false? Explain why.

c) In poor countries the capital intensity measured as K/L tends to be much lower than in rich countries. Given the diminishing marginal productivity of capital, why are capital flows from rich to poor countries not much higher than they are?

III.6 *On persistent technology differences* In continuation of the article by Bernard and Jones (1996)[5] we consider a set of countries, indexed by $i = 1, 2, \ldots, N$, with aggregate production functions

$$Y_i(t) = K_i(t)^{\alpha_i}(A_i(t)L_i(t))^{1-\alpha_i}, \qquad 0 < \alpha_i < 1, \tag{1}$$

where A_i is the technology level of country i. There is a country-specific capital depreciation rate, δ_i, which is assumed constant over time. Technological catching-up occurs according to

$$\frac{\dot{A}_i(t)}{A_i(t)} = \xi_i \frac{A_w(t)}{A_i(t)}, \tag{2}$$

[5] Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

where $A_w(t) = A_w(0)e^{gt}$ is the world frontier technology level, $g > 0$. We assume $A_i(0) \leq A_w(0)$ and $0 < \xi_i \leq g$, for all $i = 1, 2, \ldots, N$. Let "labor productivity" be measured by $y_i(t) \equiv Y_i(t)/L_i(t)$.

In their extended Solow-style setup Bernard and Jones find that even in the long run there need not be a tendency for $y_i(t)/y_w(t)$ to approach 1 and they show which potential structural differences (parameter differences) are responsible for this. They conclude that their setup "leads to a world in which *similar* steady state outcomes are the exception rather than the rule".

Bernard and Jones present data for 14 OECD countries over the period 1970-87 to substantiate this conclusion. Over this period, however, financial capital was not as mobile as it is today. This raises two questions. How, if at all, does perfect capital mobility affect the theoretical conclusion? And what does more recent data show?

Here we shall deal with the first question.[6] We replace the Solow-style setup by a setup where the countries trade in a fully integrated world market for goods and financial capital. Assume perfect competition and that the countries face a constant real interest rate $r > 0$ in the market for financial capital while, however, labor is entirely immobile. Finally, assume that (1) and (2) still hold and that the world frontier technology is identical with the technology of one of the countries in the considered set, namely the "world leader" (say USA). We let α_w denote the output elasticity w.r.t. capital in this country.

a) Examine whether in this case there is a tendency for $y_i(t)/y_w(t)$ to approach 1 in the long run. *Hint 1:* Profit maximizing firms will under perfect competition choose a time-independent effective capital intensity, $\tilde{k}_i^*$, satisfying

$$f_i'(\tilde{k}_i^*) = r + \delta_i.$$

Hint 2. Consider the ratio $x(t) \equiv A_i(t)/A_w(t)$, a measure of country i's *lag* relative to the frontier; express the growth rate of x in terms of x, ξ_i and g; this should give you a linear first-order differential equation with constant coefficients; then apply the brief math manual in Appendix A.

b) Does the answer to a) depend on whether the countries differ w.r.t. their saving rate, s_i, and labor force growth rate, n_i? Why or why not?

Let $\text{TFP}_i(t)$ denote the total factor productivity of country i at time t.

[6] The second question may be suitable for a Master Thesis project!

c) Express $\text{TFP}_i(t)$ in terms of the labor-augmenting technology level $A_i(t)$.

d) Find the limit of the ratio $\frac{\text{TFP}_i(t)}{\text{TFP}_w(t)}$ for $t \to \infty$; there may be alternative cases to be considered.

e) Will there be a tendency for TFP of the different countries to differ in the long run? Why or why not?

f) On the basis of the above results, do you think the comparative analysis in terms of TFP growth adds anything of economic interest to the comparative analysis in terms of the labor-augmenting technology level A and labor productivity, y, cf. a)?

g) "Long-run growth in the ratio of two countries' TFP may misrepresent the economic meaning of technical progress when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral." Do you agree? Why or why not?

Appendix A: Solution formulas for linear differential equations of first order

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}.$$

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} \int_0^t b(s)e^{as} ds.$$

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} c \int_0^t e^{(a+h)s} ds = (x_0 - \frac{c}{a+h})e^{-at} + \frac{c}{a+h} e^{ht}.$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-\int_0^t a(\tau)d\tau} + e^{-\int_0^t a(\tau)d\tau} \int_0^t b(s) e^{\int_0^s a(\tau)d\tau} ds.$$

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IV

Applying the Ramsey model

IV.1 *Agents' behavior, equilibrium factor prices, and dynamic system of the Ramsey model* Consider the Ramsey model for a market economy with perfect competition, CRRA utility function, and Harrod-neutral technical progress at a constant rate $g > 0$.

a) Write down the dynamic budget identity and the NPG condition for the representative household expressed in absolute terms (not per capita terms).

b) Derive the corresponding dynamic budget constraint and NPG condition expressed in per capita terms.

c) Set up the consumption-saving problem of the representative household and derive the first-order conditions and the transversality condition.

d) Derive the Keynes-Ramsey rule.

e) Under the assumption of perfect competition, characterize the representative firm's behavior and determine the equilibrium (real) factor prices and the equilibrium real interest rate.

f) The model can be reduced to two coupled differential equations in the technology-corrected capital per head and the technology-corrected consumption per head. Derive these two differential equations.

IV.2 *A positive technology shock* Consider a Ramsey model for a closed economy. The model can be reduced to two differential equations

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given,} \tag{*}$$

$$\dot{\tilde{c}}_t = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g)\tilde{c}_t, \tag{**}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = 0. \tag{***}$$

Notation is: $\tilde{k}_t = K_t/(T_tL_t)$ and $\tilde{c}_t = C_t/(T_tL_t) = c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, and L_t is population = labor supply, all at time t. Further, T_t is a measure of the technology level and f is a production function on intensive form, satisfying $f' > 0$, $f'' < 0$, and the Inada conditions. The remaining symbols stand for parameters and all these are positive. Moreover, $\rho - n > (1-\theta)g$.

a) Briefly interpret the equations (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive $\tilde{k}_0$. Can the divergent paths be ruled out? Why or why not?

c) Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then for some external reason an unanticipated technology shock occurs so that T_0 is replaced by $T_0' > T_0$. After this shock everybody rightly expects T to grow forever at the same rate as before. We now study short- and long-run effects of this shock.

d) Illustrate by means of the phase diagram what happens to $\tilde{k}$ and $\tilde{c}$ on impact, i.e., immediately after the shock, and in the long run.

e) What happens to the real interest rate on impact and in the long run?

f) Why is the sign of the impact effect on the real wage ambiguous (at the theoretical level) as long as f is not specified further?[7]

g) Compare the real wage in the long run to what it would have been without the shock.

h) Suppose $\theta = 1$. Why is the sign of the impact effect on per capita consumption ambiguous? *Hint:* $c = (\rho - n)(k + h)$.

i) Compare per capita consumption in the long run to what it would have been without the shock.

IV.3 *Short questions* (can be answered by a few well chosen sentences)

a) Can a path *below* the saddle path in the $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

[7] Remark: for "empirically realistic" production functions (having elasticity of factor substitution larger than elasticity of production w.r.t. capital), the impact effect *is* positive, however.

b) Can a path *above* the saddle path in the $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

c) Answer questions b) and c) now presuming that we are dealing with the solution of the problem from the point of view of a social planner in the Ramsey model.

d) In what sense does the Ramsey model imply a more concise theory of the long-run rate of return than do, e.g., the Solow model or the Diamond OLG model?

e) Briefly, assess the theory of the long-run rate of return implied by the Ramsey model. That is, mention what you regard as strengths and weaknesses of the theory.

IV.4 *Productivity slowdown* Consider a Ramsey model of a market economy with perfect competition in all markets. The model can be reduced to two coupled differential equations (using standard notation):

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, \qquad (*)$$

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}, \qquad (**)$$

together with the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = 0. \qquad (***)$$

a) Equation (*) generally holds for a closed economy with a per capita production function f, a capital depreciation rate δ, a growth rate n of the labour force and a growth rate g of labour efficiency. Explain this in more detail.

b) Equation (**) emerges from assumptions specific to the Ramsey model. Give a brief account of this.

c) From now on it is assumed that $f(0) > 0$. State in words the economic interpretation of this assumption.

d) Construct a phase diagram to illustrate the dynamics of the model.

e) Assume that $\tilde{k}_0 > k^*$, where $\tilde{k}_0$ is the initial capital intensity and $\tilde{k}^*$ is the capital intensity of the economically interesting steady state (which is thus assumed to exist). Show in the phase diagram the evolution over time brought about by the model. Next, show in a graph having time on the horizontal axis (i.e. a "time diagram") the evolution of $\tilde{k}$, $\tilde{c}$, r and w/A (standard notation).

f) Assume instead that the economy has been in steady state until time t_0. Then g unexpectedly shifts down to a lower constant level g'. The economic agents will immediately after time t_0 form expectations about the future that include the new lower growth rate in labour efficiency. Using a phase diagram, show how $\tilde{k}$ and $\tilde{c}$ evolve in the economy for $t \geq t_0$. As for $\tilde{c}$ the sign of the immediate change cannot be determined without more information (why not?); but the direction of movement in the future can be determined unambiguously.

g) Show in a diagram the qualitative features of the time profiles of $\tilde{k}$, $\tilde{c}$, r and w/A for $t \geq t_0$. *Hint:* it is important to realize how the shift in g may affect the $\dot{\tilde{k}} = 0$ locus and the $\dot{\tilde{c}} = 0$ locus.

h) What is the growth rate of output per worker and the real wage, respectively, in the long run? Are these growth rates diminishing or increasing over time in the adjustment process towards the new steady state? Give a reason for your answer.

IV.5 *Short questions*

a) Germany and Japan had a very high per-capita growth rate after the second world war (and up to the mid 1970s). "As predicted by neoclassical growth theory (Solow or Ramsey style), sooner or later the very fast growth came to an end." Do you think this statement makes sense? Briefly explain.

b) Consider a Ramsey model with exogenous Harrod-neutral technical progress and a neoclassical CRS production function which is not Cobb-Douglas. Can the predictions of the model be consistent with Kaldor's "stylized facts"? Give a reason for your answer.

IV.6 *Short questions* Consider the Ramsey model for a market economy with perfect competition.

a) "Only if the production function is Cobb-Douglas with CRS and time-independent output elasticity w.r.t. capital, does the Ramsey model predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

b) "The Ramsey model predicts that for countries with similar structural characteristics, the further away from its steady state a country is, the higher is its per capita growth rate." True or false? Comment.

IV.7 *Aggregate saving and the return to saving* Consider a Ramsey model for a closed competitive market economy with public consumption, transfers, and capital income taxation. The government budget is always balanced. The model leads to the following differential equations (standard notation)

$$\begin{aligned} \dot{\tilde{k}} &= f(\tilde{k}) - \tilde{c} - \tilde{\gamma} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \text{ given}, && (*) \\ \dot{\tilde{c}} &= \frac{1}{\theta}\left[(1-\tau_r)(f'(\tilde{k}) - \delta) - \rho - \theta g\right]\tilde{c}, && (**) \end{aligned}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left[(1-\tau_r)(f'(\tilde{k}_s)-\delta)-g-n\right]ds} = 0. \qquad (***)$$

All parameters are positive and it is assumed that $\rho > n$ and

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \frac{\rho + \theta g}{1-\tau_r} > n + g > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta.$$

The government controls $\tilde{\gamma}$, $\tau_r \in (0,1)$, and the transfers. Until further notice $\tilde{\gamma}$ and τ_r are kept constant over time and the transfers are continuously adjusted so that the government budget remains balanced.

a) Briefly interpret (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path that the economy follows, for a given $\tilde{k}_0 > 0$. Comment.

c) Is it possible for a steady state to exist without assuming f satisfies the Inada conditions? Why or why not?

d) Suppose the economy has been in steady state until time t_0. Then, suddenly τ_r is increased to a higher constant level. Illustrate by a phase diagram what happens in the short and long run. Give an economic interpretation of your result.

e) Does the direction of movement of $\tilde{k}$ depend on θ? Comment.

f) Suppose $\theta = 1$. It is well-known that in this case the substitution effect and the income effect on current consumption of an increase in the (after-tax) rate of return offset each other. Can we from this conclude that aggregate saving does not change in response to the change in fiscal policy? Why or why not? Add some economic intuition. *Hint* regarding the latter: when $\theta = 1$, $c_t = (\rho - n)(a_t + h_t)$, where

$$h_t \equiv \int_t^\infty (w_s + x_s)e^{-\int_t^s[(1-\tau_r)r_\tau - n]d\tau}ds;$$

here, x_s is per capita transfers at time s. Four "effects" are in play, not only the substitution and income effects.

IV.8 *Short questions* We assume that a given selection of countries (considered, for simplicity, as closed economies) can be described by the Ramsey model for a closed economy with Harrod-neutral technical progress at a constant positive rate. For each country parameters and initial conditions are such that an equilibrium exists (standard notation).

a) "The model predicts that for countries with the same technology (same F, A_0, g, and δ), differences in per capita growth rates are only temporary and due to the transitional dynamics." True or false? Comment.

b) "The model predicts that for countries with the same technology, differences in per capita income are only temporary and due to the transitional dynamics." True or false? Comment.

IV.9 *Command optimum* Consider a Ramsey setup with CRRA utility and exogenous technical progress at the constant rate $g \geq 0$. Suppose resource allocation is not governed by market mechanisms, but by a "social planner" – by which is meant an "all-knowing and all-powerful" central authority. The social planner is not constrained by other limitations than those from technology and initial resources and can thus ultimately decide on the resource allocation within these confines.

The decision problem of the social planner is (standard notation):

$$\max_{(c_t)_{t=0}^{\infty}} U_0 = \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho - n)t} dt \qquad \text{s.t.} \tag{1}$$

$$c_t \geq 0, \tag{2}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{c_t}{A_t} - (\delta + g + n)\tilde{k}_t, \tag{3}$$

$$\tilde{k}_t \geq 0 \qquad \text{for all } t \geq 0, \tag{4}$$

where $\delta + g > 0$ and $\theta > 0$ (in case $\theta = 1$, the expression $(c^{1-\theta} - 1)/(1-\theta)$ should be interpreted as $\ln c$). Assume $\rho - n > (1-\theta)g$ and that the production function satisfies the Inada conditions.

a) Briefly interpret the problem, including the parameters. Comment on the inequality $\rho - n > (1-\theta)g$.

b) Derive a characterization of the solution to the problem.

c) Compare the solution with the equilibrium path generated by a market economy described by a Ramsey model with perfect competition and with the same preferences and technology as above. Comment.

IV.10 *Some quotations.*

a) Two economists – one from MIT and one from Chicago – are walking down the street. The MIT economist sees a 100 dollar note lying on the sidewalk and says: "Oh, look, what a fluke!". "Don't be silly, obviously it is false", laughs the Chicago economist, "if it wasn't, someone would have picked it up". Discuss in relation to the theoretical concepts of arbitrage and equilibrium.

b) A riddle asked by Paul Samuelson (Nobel Prize winner 1970): A physicist, a chemist, and an economist are stranded on an island, with nothing to eat. A can of soup washes ashore. The physicist says "let us smash the can open with a rock". The chemist says "let us build a fire and heat the can first". Guess what the economist says?

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