Exercise Problems for Economic Growth

Part 2

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V

Human capital and economic growth

V.1 *Human capital considered as just another form of capital* A famous paper by Mankiw, Romer, and Weil (1992) carries out a cross-country regression analysis (98 countries, 1960-1985) based on the aggregate production function

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}, \qquad 0 < \alpha < \alpha + \beta < 1, \tag{*}$$

where Y is GDP, K aggregate capital input, H aggregate human capital input, A the technology level, and L input of man-hours, $L_t = L_0 e^{nt}$, n constant. The gross investment rates in the two types of capital are a fraction s_K and s_H of GDP, respectively. Assuming that $A_t = A_0 e^{gt}$, $g \geq 0$, is the same for all countries in the sample (apart from a noise term affecting A_0), the authors conclude that $\alpha = \beta = 1/3$ fits the data quite well.

Let h denote average human capital, i.e., $h \equiv H/L$, and suppose all workers at any time t have the same amount of human capital, equal to h_t.

a) Show that (*) can be rewritten on the form $Y_t = F(K_t, X_t L_t)$, where F is homogeneous of degree one. Indicate what X_t must be in terms of h and A and what the implied "labor quality" function is.

b) When we study individual firms' decisions, this alternative way of writing the production function is more convenient than the form (*). Explain why.

c) Within a Ramsey-style set-up, where s_K and s_H are endogenous and time-dependent, it can be shown that the economy converges to a steady state with $\tilde{y} \equiv Y/(AL) = (\tilde{k}^*)^{\alpha}(\tilde{h}^*)^{\beta}$, where $\tilde{k}^*$ and $\tilde{h}^*$ are the constant steady state values of $\tilde{k} \equiv K/(AL)$ and $\tilde{h} \equiv h/A$. Find the long-run growth rate of $y \equiv Y/L$. Is per capita growth in the long run driven by human capital accumulation?

In Section 11.2 of the textbook by Acemoglu the author presents a Ramsey-style one-sector approach to human and physical capital accumulation. The production function is

$$Y_t = F(K_t, h_t L_t), \tag{**}$$

where F is a neoclassical production function with CRS satisfying the Inada conditions. We shall compare the implications of (*) and (**) under the assumption that A_t in (*) is time-independent and equals 1.

d) Does (*) and (**) imply the same or different answers to the last question in c)? Comment.

e) Briefly evaluate the set-up in Section 11.2 of the Acemoglu textbook from a theoretical as well as empirical perspective.

f) If we want a linear labor quality function, as implicit in (**), to be empirically realistic, there is an alternative approach that might do better. What approach is that?

V.2 *A life-cycle approach to human capital* We consider a market economy. Suppose people are alike and that if they attend school for S years, they obtain individual human capital

$$h = S^{\varphi}, \qquad \varphi > 0. \tag{*}$$

An individual "born" at time 0 chooses S to maximize

$$HW_0 = \int_S^{\infty} \hat{w}_t h e^{-(r+m)t} dt, \tag{**}$$

subject to (*). Here $\hat{w}_t$ is the market determined real wage per year *per unit of human capital* at time t, r is a constant real interest rate, and m is a parameter such that the probability of surviving at least until age τ is $e^{-m\tau}$. It is assumed that owing to technical progress,

$$\hat{w}_t = \hat{w}_0 e^{gt}, \tag{***}$$

where g is a nonnegative constant satisfying $g < r + m$.

a) Interpret the decision problem, including the parameter m. Is there a sense in which the infinite horizon in (**) can be defended as an approximation?

b) Let the optimal S for a person be denoted S^*. Given (*), (**), and (***), it can be shown that S^* satisfies the first-order condition $h'(S^*)/h(S^*) = r+m-g$. Derive this first-order condition and provide the economic intuition behind it. *Hint:* substitute (*) into (**) and maximize w.r.t. S.

c) Solve for the optimal S. The second-order condition can be shown to be that the elasticity of h' w.r.t. S is smaller than the elasticity of h w.r.t. S. Is it always satisfied?

d) With one year as the time unit, let the parameter values be $\varphi = 0.6$, $r = 0.06$, $m = 0.01$, and $g = 0.015$. What is the value of the optimal S measured in years? Comment.

e) How does an increase in life expectancy affect m and the optimal S, respectively? What is the intuition?

There is perfect competition and the representative firm chooses capital input, K_t, and labor input (measured in man-years), L_t, in order to maximize profit, given the production function

$$Y_t = F(K_t, A_t h L_t),$$

where Y_t is output, A_t is the technology level, and F is a neoclassical production function with constant returns to scale. There is a constant capital depreciation rate $\delta > 0$. Suppose the country considered is fully integrated in the world market for goods and financial capital and that the real interest rate in this market is constant and equal to r for a long time.

f) Let the equilibrium real wage *per year* at time t for a typical member of the labor force be denoted w_t. Find w_t.

g) Find $\hat{w}_t$. What is the growth rate over time of w_t according to the information given in the introductory paragraph above? And what is the implied growth rate of A?

V.3 *Human capital and catching up* Consider a country which is fully integrated in the world market for goods and financial capital. Suppose that the real interest rate in the world market is a constant, $r > 0$. Let the aggregate production function be $Y_t = F(K_t, A_t h L_t)$ (standard notation). The technology level A_t evolves according to the catching-up hypothesis

$$\frac{\dot{A}_t}{A_t} = \xi \frac{\tilde{A}_t}{A_t},$$

where $\xi > 0$, and $\tilde{A}_t = \tilde{A}_0 e^{gt}$ is the world frontier technology level, $g > 0$.[1] We assume $A_0 < \tilde{A}_0$ and $0 < \xi < g$.

[1] Cf. Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

a) Will the country's technology level be able to catch up in the long run? *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$, has the solution $x(t) = (x_0 - x^*)e^{-at} + x^*$, where $x^* = b/a$; let $x(t) \equiv A_t/\tilde{A}_t$ and express the growth rate of x in terms of x, ξ, and g.

Let m be a measure of the country's mortality rate and suppose the country is a developing country with average human capital $h = \varphi/(r+m-g)$, where $r+m > g$, and φ is a positive parameter.

b) Let the catching-up ability be an increasing function of average human capital, h, i.e., $\xi = \xi(h)$, $\xi' > 0$. Can a general health improvement in the country help in catching up? Why or why not?

V.4 *AK model with human and physical capital* We consider a closed market economy with education in private schools that charge a fee from students. Under perfect competition the representative firm chooses capital input, K^d, and labor input, L^d, in order to maximize profit, given the production function

$$Y = F(K^d, \pi L^d), \tag{1}$$

where Y is output, π is "quality" (or "productivity") of labor, and F is a neoclassical production function with constant returns to scale.

a) Given π and the aggregate supplies of capital, K, and labor, L, respectively, determine the real rental rate, $\hat{r}$, for capital and the real wage, $\hat{w}$, per unit of *effective* labor input in equilibrium.

We shall in this exercise assume that $\pi = h \equiv H/L$, where H is aggregate human capital in the labor force formed in the following way. Aggregate output (= aggregate gross income) is used for consumption, C, investment, I_K, in physical capital and investment, I_H, in human capital, i.e.,

$$Y = C + I_K + I_H.$$

The dating of the variables is suppressed where not needed for clarity. The increase per time unit in the two kinds of capital is given by

$$\begin{aligned} \dot{K} &= I_K - \delta_K K, \quad \text{and} \\ \dot{H} &= I_H - \delta_H H, \end{aligned} \tag{2}$$

respectively. The depreciation rates, δ_K and δ_H, are positive constants.

The representative household (dynasty) has infinite horizon and consists of L members, where $L = L_0 e^{nt}$, $n \geq 0$, $L_0 > 0$. Each family member supplies inelastically one unit of labor per time unit. From (2) and the definition $H \equiv hL$ follows the per capita human capital accumulation equation:

$$\dot{h} = i - (\delta_H + n)h, \tag{3}$$

where $i \equiv I_H/L$ is the per capita educational cost (in real terms) per time unit.

b) Present a derivation of (3).

Let θ and ρ be positive constants, where $\rho > n$. Let a be per capita financial wealth, r the real interest rate, and $c_t \equiv C_t/L_t$. The representative household chooses a path $(c_t, i_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned}
U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} & (4)\\
c_t &\geq 0,\ i_t \geq 0, & (5)\\
\dot{a}_t &= (r_t - n)a_t + \hat{w}_t h_t - i_t - c_t, \qquad a_0 \text{ given}, & (6)\\
\dot{h}_t &= i_t - (\delta_H + n)h_t, \qquad h_0 > 0 \text{ given}, & (7)\\
\lim_{t\to\infty} a_t e^{-\int_t^\tau (r_s - n)ds} &\geq 0, & (8)\\
h_t &\geq 0 \text{ for all } t. & (9)
\end{aligned}$$

c) Briefly interpret the six elements in this decision problem. Why is there a non-negativity constraint on i_t?

d) Apply the Maximum Principle (for the case with two control variables and two state variables) to find the first-order conditions for an interior solution.

e) Derive from the first-order conditions the Keynes-Ramsey rule.

f) Set up a no-arbitrage equation showing a relationship between $\hat{w}$ and r. You may either apply your intuition or derive the relationship from the first-order conditions. In case you apply your intuition, check whether it is consistent with the first-order conditions. *Hint:* along an interior optimal path the household should be indifferent between placing the marginal unit of saving in a financial asset yielding the rate of return r or in education to obtain one more unit of human capital.

Assume now for simplicity that the aggregate production function is:

$$Y = K^{\alpha}(\pi L)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

g) Determine the real interest rate in equilibrium in this case.

Suppose parameters are such that $\dot{c}/c > 0$ and U_0 is bounded.

h) The no-arbitrage equation from f) (which is needed for an *interior* solution to the household's decision problem) requires a specific value of $\hat{k} \equiv K/H$ to be present. Assuming, for simplicity, that $\delta_K = \delta_H = \delta$, determine the required value of $\hat{k}$. Let this value be denoted $\hat{k}^*$. Explain what happens to begin with if the historically given initial $\hat{k}$ in the economy differs from $\hat{k}^*$.

i) Suppose the historically given initial $\hat{k} \neq \hat{k}^*$. Illustrate graphically the time profile of $\hat{k}$. Will $\hat{k}$ reach $\hat{k}^*$ in finite time?

j) After some time the economy behaves like an AK model with an endogenous per capita growth rate that depends negatively on the rate of impatience. Explain. Find "A" (the factor of proportionality between Y and aggregate capital, $\tilde{K} \equiv K + H$).

k) Briefly evaluate the model from a theoretical and empirical perspective. Doing this, you may compare the model with the extended Solow model by Mankiw, Romer, and Weil (1992).

V.5 *Subsidizing education* The point of departure is the same model as in Problem V.4 (so it is an advantage if you have already solved that problem). For simplicity, assume $\delta_K = \delta_H = 0$.

a) Consider a constant subsidy, $\sigma \in (0, 1)$, to education such that per unit of investment in education the private cost is only $1 - \sigma$. That is, i_t in (7) is replaced by $(1 - \sigma)i_t$. Suppose the subsidy is financed by lump-sum taxes. Will such a subsidy affect long-run growth in this model? Explain. *Hint:* In answering, you may use your intuition or make a formal derivation. A quick approach can be based on the no-arbitrage condition in the new situation.

b) Assuming the social welfare function is the same as the objective function of the representative household, will the subsidy (combined with lump-sum taxation) increase or decrease welfare? Explain.

c) Since there is a representative household and no externalities in the model as it stands, it could be argued that there is no need for a subsidy. Going outside the model, what kinds of motivations for subsidizing education in the real world might be put forward?

V.6 *Short questions* These questions relate to the model in Problem V.4.

a) Comment on the model in relation to the concepts of fully endogenous growth and semi-endogenous growth.

b) Comment on the model in relation to the issue of scale effects.

c) What do you guess will be the consequence w.r.t. long-run per capita growth of assuming $\pi = h^{\varphi}$, $0 < \varphi < 1$? Comment.

V.7 *A model with human capital and R&D* Consider a closed economy with two production sectors, manufacturing and R&D. For simplicity we imagine that the R&D sector is fully managed by the government. Time is continuous. At the aggregate level we have:

$$\begin{aligned} Y_t &= A_t^{\gamma} K_t^{\alpha} (\bar{h}_t L_{Y,t})^{1-\alpha}, \qquad \gamma > 0, 0 < \alpha < 1, & (10) \\ \dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \delta \geq 0, & (11) \\ \dot{A}_t &= \eta A_t^{\varphi} (\bar{h}_t L_{A,t})^{1-\varepsilon}, \qquad \eta > 0, \varphi < 1, 0 \leq \varepsilon < 1, & (12) \\ L_{Y,t} + L_{A,t} &= L_t = \text{ labor force.} & (13) \end{aligned}$$

Here A_t measures the stock of technical knowledge and $\bar{h}_t$ is average human capital in the labor force at time t (otherwise notation is standard).

From now we ignore the explicit dating of the variables unless needed for clarity. Let the growth rate of a variable $x > 0$ be denoted g_x (not necessarily positive and not necessarily constant over time). Assume that all variables in the model are positive and remain so.

a) Interpret the case $0 < \varphi < 1$ versus the case $\varphi < 0$. Interpret the case $0 < \varepsilon < 1$ versus the case $\varepsilon = 0$.

b) Write down a growth accounting relation expressing g_Y in terms of g_A, g_K, $g_{\bar{h}}$, and g_{L_Y}. In addition, express g_A in terms of A, $\bar{h}$, L_A.

c) Presupposing $g_A > 0$, express the growth rate of g_A in terms of g_A, $g_{\bar{h}}$, and g_{L_A}.

Let the time unit be one year. Suppose an individual "born" at time v (v for "vintage") spends the first S years of life in school and then enters the labor market with a human capital level which at time $t \geq v + S$ is $h(S)$, where $h' > 0$. After leaving school, the individual works full-time until death (for simplicity). We ignore the role of teachers and schooling equipment. At least to begin with, we assume that S is a constant and thus independent of v. Then with a stationary age distribution in society,

$$L_t = (1 - \beta)N_t, \tag{14}$$

where N_t is the size of the (adult) population at time t and β is the constant fraction of this population under education (β will be an increasing function of S). We assume that life expectancy is constant and that the population grows at a constant rate $n > 0$:

$$N_t = N_0 e^{nt}. \tag{15}$$

Let a balanced growth path (BGP) in this economy be defined as a path along which g_Y, g_C, g_K, g_A, $g_{\bar{h}}$, g_{L_A}, and g_{L_Y} are constant.

d) Can we be sure that $g_{L_Y} = n$ under balanced growth? Why or why not?

e) Show that

$$g_A = \frac{(1-\varepsilon)(g_{\bar{h}} + n)}{1 - \varphi}.$$

f) From a certain general proposition we can be sure that along a BGP, $g_Y = g_K$. What proposition and why?

g) Defining $y \equiv Y/L$, it follows that under balanced growth,

$$g_y = \frac{\gamma g_A}{1 - \alpha} + g_{\bar{h}}.$$

How?

h) It is possible to express g_y under balanced growth in terms of only one endogenous variable, $g_{\bar{h}}$. Show this.

i) Comment on the role of n in the resulting formula for g_y.

V.8 This problem presupposes that you have solved Problem V.7, in particular question h).

a) Consider two connected statements: "The model in Problem V.7 assumes diminishing marginal productivity of knowledge in knowledge creation" and "hence, sustained exponential per capita growth requires $n > 0$ or $g_{\bar{h}} > 0$". Evaluate these statements.

b) Given the prospect of non-increasing population in the world economy in the long run, what is the prospect of sustained exponential per capita growth in the world economy according to the model?

Suppose $h(S) = S^{\mu}$, $\mu > 0$.

c) Demographic data exists saying that life expectancy in industrialized countries tends to grow arithmetically, in fact, almost by a quarter of a year per year. Assuming this to continue, and going a little outside the model, what is the prospect of sustained exponential per capita growth in the world economy? Discuss.

d) Although hardly realistic, suppose $h(S)$ is exponential as in the Mincer equation. Then again answer c).

V.9 *Short questions*

a) In the theory of human capital and economic growth we encounter different assumptions about the schooling technology. List some examples. Briefly comment.

b) "Arrow's learning-by-investing model predicts that the share of capital income in national income is constant in the long run if and only if the aggregate production function is Cobb-Douglas." True or false? Why?

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VI

Simple reduced-form AK models

VI.1 *The learning-by-investing model: two alternative cases* Consider a closed market economy with N profit maximizing firms, operating under perfect competition (N "large"). The size of the labor force (= employment = population) is $L_t = L_0 e^{nt}$, where n is constant, $n \geq 0$. Aggregate output at time t is Y_t per time unit. Output is used for consumption, $C_t \equiv c_t L_t$, and investment in physical capital K_t so that $\dot{K}_t = Y_t - C_t - \delta K_t$, where $K_0 > 0$ is given and $\delta \geq 0$ is the rate of physical decay of capital. The initial value $K_0 > 0$ is given. There is a perfect market for loans with a short-term real interest rate r_t. There is no uncertainty (perfect foresight).

The production function for firm i ($i = 1, 2, ..., N$) is

$$Y_{it} = F(K_{it}, A_t L_{it}), \tag{1}$$

where F is neoclassical and has CRS. The variable A_t evolves according to

$$A_t = e^{\varepsilon t} K_t^{\lambda}, \qquad \varepsilon \geq 0, 0 < \lambda \leq 1, \tag{2}$$

where ε and λ are given constants and $K_t = \sum_i K_{it}$.

a) Briefly interpret (2), including the parameters and the variable A.

Suppose each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables.

b) In general equilibrium, determine r and the aggregate production function at time t.

c) Assume $\varepsilon > 0$ and $\lambda < 1$. Show that without knowing anything about household saving behavior we can determine the growth rate of Y and $y \equiv Y/L$ under balanced growth, assuming saving is positive. *Hint:* If a production function $Y = F(K, XL)$ is homogeneous of degree one, then

$$\frac{Y}{K} = F(1, \frac{XL}{K});$$

combine this with a certain general balanced growth property.

d) What type of endogenous growth is this model (as given in c)) capable of generating?

From now, let $\lambda = 1$ and $\varepsilon = n = 0$. Moreover, assume the household sector is Ramsey-style with inelastic labor supply, instantaneous CRRA utility of per capita consumption with parameter $\theta > 0$, and a constant utility discount rate $\rho > 0$. Finally, assume that the inequalities $F_1(1, L) > \delta + \rho$ and $\rho > (1 - \theta)\gamma$ hold, where γ is the equilibrium growth rate of c.

e) Determine γ. Comment.

f) Determine the equilibrium growth rate for k ($\equiv K/L$) and y, respectively. *Hint:* If you have shown that the real interest rate is a constant and that the aggregate production function is AK-style, then it is enough to refer to general knowledge about reduced-form AK models.

g) What would happen if $n > 0$? Why?

VI.2 *The learning-by-investing model: Paul Romer's case* This problem presupposes that you have already solved Problem VI.1. In continuation of the last part of that problem, we consider the case $\lambda = 1$ and $\varepsilon = n = 0$ and the same Ramsey-style household sector.

a) There is a certain feature of the economy which "invites" government intervention in the market mechanism. What is this feature?

We introduce a government which contemplates to implement a production subsidy and finance it by a consumption tax. The idea is to subsidize production at a constant rate $\sigma > 0$ so that if firm i produces and sells Y_i, its revenue is $(1 + \sigma)Y_i$. Assume you, as an economic advisor, are asked by the government to suggest an optimal size of σ, given that the social welfare function coincides with the criterion function of the representative household.

b) Derive a formula for the recommendable size of σ. *Hint:* Set up the social planner's problem,[2] derive the first-order conditions, and determine the implied time path of

[2] Recall that a "social planner" is a hypothetical "all-knowing and all-powerful" central authority who can fully decide on the resource allocation within the constraints by technology and initial resources. That is, by definition, a social planner need not care about market mechanisms and market prices.

c_t. Next, use your general knowledge about reduced-form AK models to determine the time paths of k_t and y_t (a brief verbal account is enough). Finally, use that for σ to be optimal, σ should ensure that the net rate of return to capital investment implied by the aggregate production technology equals the net rate of return on saving faced by the consumer.

c) Assume that the government always balances the budget and has no other expenditures than the production subsidy. Find the consumption tax rate, τ, needed to finance the subsidy. *Hint:* At a certain stage in the argument you will need knowledge about what value is taken by c_t/k_t in the social planner's solution. You do not have to derive this value; it is given here: $c/k = F(1, L) - \delta - g_{SP}$.

d) Let F in (1) be $Y_{it} = BK_{it}^{\alpha}(A_t L_{it})^{1-\alpha}$, $0 < \alpha < 1$, and assume $\alpha = 1/3$. What is the implied value of σ according to your formula in b)?

e) With one year as the time unit, let $B = 0.003$, $L = 1000$, $\delta = 0.05$, $\theta = 2$, and $\rho = 0.02$. Check whether the implied value of r under laissez-faire makes sense empirically. Next find the implied value of τ according to your formula in c). (Do not expect a "modest" result for τ, given that neither the model nor the found value for σ are "modest".)

f) Given the model, *is* the suggested policy, (σ, τ), optimal or might there for example be distortionary effects associated with the financing? Why or why not?

g) Whatever the answer to f), briefly suggest other subsidy policies which could do the job.

h) Briefly evaluate the present model.

VI.3 *A subsidy to saving in Paul Romer's learning-by-investing model* Consider a closed market economy with perfect competition where firm no. i has the production function

$$Y_{it} = F(K_{it}, T_t L_{it}),$$

where F is a neoclassical production function with CRS and satisfying the Inada conditions (standard notation). It is assumed that the technology level T_t satisfies

$$T_t = K_t^{\lambda}, \qquad 0 < \lambda \leq 1.$$

Time, t, is continuous. There is no uncertainty. At the aggregate level,

$$\dot{K}_t \equiv \frac{dK_t}{dt} = Y_t - C_t - \delta K_t, \qquad \delta > 0, \qquad K_0 > 0 \text{ given.}$$

a) Determine the equilibrium real interest rate, r, and the aggregate production function. Comment.

From now we assume $\lambda = 1$.

b) Determine the equilibrium real interest rate, r, and the aggregate production function in this case. Comment.

There is a representative Ramsey household with instantaneous utility function of CRRA type,

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0,$$

where c is per capita consumption ($c \equiv C/L$). The rate of time preference is a constant $\rho > 0$. There is no population growth ($n = 0$).

c) Determine the equilibrium growth rate of c and name it g_c^*.

From now, assume (A1) $F_1(1, L) - \delta > \rho$ and (A2) $\rho > (1-\theta)g_c^*$.

d) What could be the motivation for these two assumptions?

e) Determine the growth rate of $k \equiv K/L$ and $y \equiv Y/L$. A detailed derivation involving the transversality condition need not be given; instead you may refer to a general property of AK and reduced-form AK models in a Ramsey framework where (A2) holds.

f) Set up and solve the social planner's problem, assuming the same criterion function as that of the representative household. *Hint*: the linear differential equation $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$ has the solution:

$$x(t) = (x(0) - \frac{c}{a+h})e^{-at} + \frac{c}{a+h}e^{ht}.$$

g) Now consider again the decentralized market economy, but suppose there is a government that wants to establish the social planner's allocation by use of a subsidy, σ, to private saving such that the after-subsidy-rate of return on private saving is $(1+\sigma)r$. Let the subsidy be financed by a lump-sum tax. Determine σ such that the social planner's allocation is established, if this is possible. Comment.

VI.4 *Productive government services* Consider a closed market economy with constant population, L utility maximizing households, and M profit maximizing firms, operating under perfect competition (L and M are constant, but "large"). There is also a government that free of charge supplies a non-rival productive service G per time unit. Each household has an infinite horizon and supplies inelastically one unit of labor per time unit. Aggregate output is Y per time unit and output is used for private consumption, $C \equiv cL$, the public productive service, G, and investment, I, in (physical) capital, i.e., $Y = C + G + I$. The stock of capital, K, changes according to $\dot{K} = I - \delta K$, where $\delta \geq 0$ is the rate of physical decay of capital. Variables are dated implicitly. The initial value $K_0 > 0$ is given. The capital stock in society is owned, directly or indirectly (through bonds or shares), by the households. There is perfect competition in the labor market. The equilibrium real wage is called w. There is a perfect market for loans with a real interest rate, r, and there is no uncertainty. A dot over a variable denotes the time derivative.

The government chooses G so that

$$G = \gamma Y,$$

where the constant $\gamma \in (0, 1-\alpha]$ is an exogenous policy parameter. The government budget is always balanced and the service G is the only public expenditure. Only households are taxed. The tax revenue is

$$\left[\tau(ra + w) + \tau_\ell\right] L = G, \tag{GBC}$$

where a is per capita financial wealth, and τ and τ_ℓ denote the income tax rate and a lump-sum tax, respectively. The tax rate τ is a given constant, $0 \leq \tau < 1$, whereas τ_ℓ is adjusted when needed for (GBC) to be satisfied.

The production function for firm i is

$$Y_i = AK_i^\alpha (GL_i)^{1-\alpha}, \quad 0 < \alpha < 1, A > 0, \quad i = 1, 2, ..., M. \tag{*}$$

a) Comment on the nature of G.

b) Show that in equilibrium

$$\begin{aligned} r &= \alpha \bar{A} - \delta, \quad \text{where} \quad k \equiv K/L \quad \text{and} \quad \bar{A} \equiv A^{\frac{1}{\alpha}}(\gamma L)^{\frac{1-\alpha}{\alpha}}, \\ Y &= \sum_i Y_i = \sum_i y_i L_i = y \sum_i L_i = yL = Ak^\alpha G^{1-\alpha} L = A^{1/\alpha}(\gamma L)^{(1-\alpha)/\alpha} kL \equiv \bar{A}K. \end{aligned}$$

Suppose the households, all alike, have a constant rate of time preference $\rho > 0$ and an instantaneous utility function with (absolute) elasticity of marginal utility equal to a constant $\theta > 0$.

c) Set up the optimization problem of a household and derive the Keynes-Ramsey rule, given the described taxation system.

d) Write down the transversality condition in a form comparable to the No-Ponzi-Game condition of the household. Comment.

e) Find the growth rate of $k \equiv K/L$ and $y \equiv Y/L$ in this economy (an informal argument, based on your general knowledge about reduced-form AK models, is enough). In case, you need to introduce a restriction on some parameters to ensure existence of equilibrium with growth, do it.

f) Sign $\partial g_c^*/\partial \gamma$ and $\partial g_c^*/\partial L$. Comment in relation to the scale effect issue.

VI.5 This problem relates to Problem VI.4. The model of that problem is essentially the model in Barro (JPE, 1990). Several aspects of the model have been questioned in the literature. One critical aspect is that G enters (*) in a very powerful but arbitrary way. Let q be an index of labor-augmenting productivity considered as a function of the public productive service G, i.e., $q = q(G)$. We assume that taxes are lump-sum. Then it is reasonable to assume that $q' > 0$. Still $q(G)$ could be strictly concave, for example in the form $q = G^\lambda, 0 < \lambda < 1$. Barro assumes apriori that $\lambda = 1$ and $n = 0$, where n is the population growth rate (= growth rate of the labor force).

a) Suppose $0 < \lambda \leq 1$. Given K and L, what level of G and γ, respectively, maximizes $Y - G$ (i.e., the amount of output which is left for private consumption and capital investment)? Briefly provide the intuition behind your result. *Hint:* by a procedure analogue to that in question b) of Problem VI.4 it can be shown that in equilibrium the aggregate production now is $Y = AK^\alpha(G^\lambda L)^{1-\alpha}$.

From now, let $0 < \lambda < 1$ and $n \geq 0$.

b) Find first g_Y, then g_y, along a BGP. "The public productive service has no effect on the growth rate of y along a balanced growth path." True or false? Why? *Hint:* use that if a production function $Y = F(K, XL)$ is homogeneous of degree one, then

$$1 = F(\frac{K}{Y}, \frac{XL}{Y});$$

combine with the balanced growth equivalence theorem.

c) Compare with the results from f) of Problem VI.4. Comment.

VI.6 This problem presupposes that you have solved Problem VI.4. Indeed, we consider essentially the same economy as that described above with the firm production function (*). There is one difference, however, namely that lump-sum taxation is not feasible. Hence, let $\tau_\ell = 0$ for all $t \geq 0$.

a) Examine whether it is possible to fix τ at a level (constant over time and < 1) such that the government budget is still balanced in equilibrium for all $t \geq 0$? *Hint:* find the solution for w; if you need a new restriction on parameters to ensure $\tau < 1$, introduce it.

b) If the welfare of the representative household is the criterion, what proposal to the government do you have w.r.t. the size of γ?

c) With respect to the *form* of taxation (given that a direct lump-sum tax is not feasible), let us see if we can suggest an appropriate tax scheme:

1. is an income tax non-distortionary? Why or why not?
2. is a pure labor income tax likely to work "in practice"? *Hint:* perhaps the needed labor income tax rate is too large in some sense.
3. will a consumption tax work?

VI.7 Consider a closed economy with profit maximizing firms, operating under perfect competition. The size of the labor force (= employment = population) is L_t. Aggregate output (GDP) at time t is Y_t per time unit. Output is used for consumption and investment in physical capital, K_t, so that $\dot{K}_t = Y_t - C_t - \delta K_t$, where C_t is consumption and δ is the rate of physical decay of capital, $\delta \geq 0$. The initial value $K_0 > 0$ is given. There is a perfect market for loans with a short-term real interest rate r_t. Time is continuous and there is no uncertainty.

The production function of firm i is

$$Y_{it} = K_{it}^{\alpha}(A_t L_{it})^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where A_t is the economy-wide technology level, $\sum_i K_{it} = K_t$, and $\sum_i L_{it} = L_t$. Suppose each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables, including A_t.

a) In general equilibrium, determine r and the aggregate production function at time t.

b) For a given A_t, find the TFP level (total factor productivity) at time t.

For any variable $x > 0$, let g_x denote its growth rate, $\dot{x}/x$.

c) Following the basic idea in growth accounting, express g_Y analytically in terms of the "contributions" from growth in K, L, and a residual, respectively.

d) Find expressions for the TFP growth rate, the gross income share of capital (aggregate gross income to capital owners divided by GDP = GNP), and the labor income share, respectively.

From now, suppose A_t evolves according to

$$A_t = e^{\varepsilon t} K_t^{\lambda}, \quad \varepsilon > 0, \quad 0 < \lambda < 1, \tag{*}$$

where ε and λ are given constants.

e) Briefly interpret (*).

f) Given (*), express g_Y analytically in terms of the "contributions" from growth in K, L, and a residual, respectively.

g) As a thought experiment, suppose we have empirical data for this economy. Will applying standard growth accounting on the basis of these data lead to over- or underestimation of the "contribution" to output growth from growth in capital? Why?

Let $L_t = L_0 e^{nt}$, where n is a positive constant.

h) Determine the growth rate of $y \equiv Y/L$ under balanced growth, assuming saving is positive. *Hint:* use a certain general balanced growth property.

i) Briefly explain what constitute the ultimate sources of per capita growth according to the model. Compare with what the growth accounting in c) suggested.

VI.8 *Short questions.*

a) Consider a set of countries, $j = 1, 2, ..., N$. Country j has the aggregate production function

$$Y_{jt} = F(K_{jt}, A_{jt}L_{jt}),$$

where F is neoclassical and has CRS (standard notation). The technology level A_{jt} evolves according to $A_{jt} = A_{j0}e^{gt}$, where A_{j0} differs widely across the countries. The positive constant g as well as F and the capital depreciation rate are, however, shared by the countries. Assume that (i) the countries trade in a fully integrated world market for goods and financial capital; (ii) they face a constant real interest rate $r > 0$ in this market; and (iii) there is perfect competition in all markets. "In this setup there will be a strong economic incentive for workers to migrate." True or false? Explain why.

b) "In models where technical knowledge is endogenous there is a built-in tendency for either weak or strong scale effects (i.e., scale effects on either levels or growth, respectively) to arise." True or false? Explain why.

c) It is sometimes argued that results like the Arrow result, $g_Y = (\varepsilon + \lambda n)/(1 - \lambda)$, cf. Problem VI.1c), are from an empirical point of view falsified by the fact that cross-country growth regressions do not tend to indicate a positive correlation between per capita economic growth and population growth. Do you agree in the argument? Why or why not?

VI.9 Some researchers emphasize that sharp class differences in a society may hamper economic growth through creating social and political instability and lack of "social capital" (social trust). Comment on this hypothesis in relation to Table X which contains comparative data for South Korea and Philippines (column 1 shows the annual GDP per capita growth rate 1960-90, columns 2-9 provide different descriptive statistics 1960, and columns 10 and 11 give the Gini coefficient for household income before tax).

Tabel X. Vækst og initialbetingelser for Sydkorea og Filippinerne.

	Vækst p.a.	BNP pr. indbygger	Indbyggertal	Andel af befolkning bosat i hovedstad	Landbrugets andel af BNP	Industriens andel af BNP	Landbrugsvarer m.m., andel af samlet eksport	Primær skolegang, drenge	Primær skolegang, piger	Ginikoefficient[c]	
	(pct.)	(US$)[b]	(mio.)	(pct.)	(pct.)	(pct.)	(pct.)	(pct.)	(pct.)	(pct.)	
	1960-90[a]				1960					1965	1988
Sydkorea	6,7	904	25	28	37	20	86	100	99	34,3	33,6
Filippinerne	1,5	1133	28	27	26	28	96	97	89	51,3	45,7

Kilder: Søjle 1-2: Penn World Table 5.6. Søjle 3-7: Lucas (1993), s. 251. Søjle 8-9: Barro og Lee (1993). Søjle 10-11: Benabou (1996).
Anm.: a) Gennemsnitlig vækst i BNP pr. indbygger. b) 1985 PPP korrigerede. c) Målt på husstandsindkomst før skat.

VI.10 List a few theoretical reasons that may be put forward in support of Acemoglu's hypothesis that differences in the economic institutions (rules of the game) across countries constitute the key for an understanding of the cross-country differences in income per capita.

VI.11 In endogenous growth theory two alternative kinds of scale effects may be present. Give a brief account. Link two alternative learning-by-investing models to these two kinds of scale effects.

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