Economic Growth Lecture Note 1.

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Introduction to the Economic Growth course

1 Economic growth theory

Economic growth theory is the study of what factors and mechanisms determine the time path of *productivity* (a simple index of productivity is output per unit of labor). Thus

- productivity levels and
- productivity growth

are in focus.

Economic growth theory endogenizes productivity growth via considering human capital accumulation (formal education as well as learning-by-doing) and endogenous research and development. At a deeper level also the conditioning role of geography and juridical, political, and cultural institutions are taken into account.

The terms "New Growth Theory" and "endogenous growth models" refer to theory and models which attempt at explaining sustained growth in per capita output as an outcome of some internal mechanism in the model rather than just a reflection of exogenous technical progress as in "Old Growth Theory".

Among the themes that are considered in the course Economic Growth are:

- How is the world income distribution evolving?
- Why do income levels and growth rates differ so much across countries and regions?
- What are the roles of human capital and technology innovation in economic growth? Getting the questions right.
- Catching-up and increasing speed of information and technology diffusion.

- Economic growth and income distribution.
- Economic growth, natural resources, and the environment (including the climate). What are the limits to growth?
- Policies to ignite and sustain productivity growth.
- The prospects of growth in the future.

The focus in the course is primarily on *mechanisms* behind the evolution of productivity in the industrialized world. The emphasis is on micro-based formal models (understanding them, being able to evaluate them, from both a theoretical and empirical perspective, and to use them to analyze specific questions). The course is calculus intensive.

2 Some long-run data

Figure 1 shows the time path of real GDP (per year) and real GDP per capita 1870-2001 (a log scale is used on the vertical axis). Let Y denote real GDP (per year) and let N be population size. Then Y/N is GDP per capita. Further, let g_Y denote the average growth rate of Y per year since 1870 and let $g_{Y/N}$ denote the average (compound) growth rate of Y/N per year since 1870. Table 1 gives these growth rates for four countries using the continuous time method with continuous compounding we will encounter in the next section.

	g_Y	$g_{Y/N}$
Denmark	2,66	1,87
UK	1,90	1,41
USA	3,35	1,86
Japan	3,54	2,55

Table 1: Average annual growth rate of GDP and GDP per capita in percent, 1870–2001. Source: Maddison, A: The World Economy: Historical Statistics, 2006, Table 1b, 1c and 5c.

Figure 1 displays the Danish time path of GDP and GDP per capita 1870-2001 along with exponential regression lines estimated by OLS. The slope of these lines implies slightly higher growth rates of GDP and GDP per capita of 2,79 and 1,94 percent than

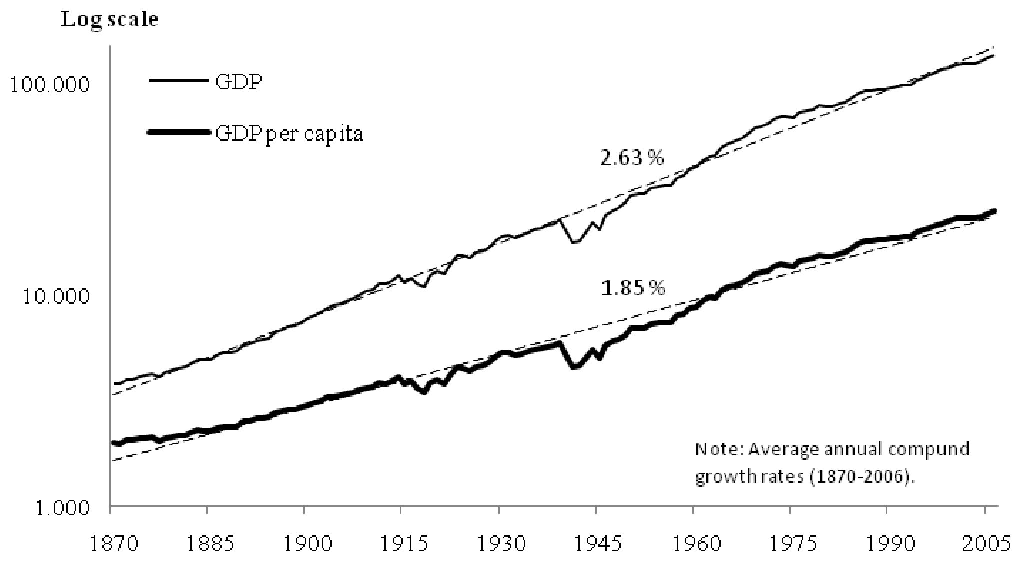

Figure 1: GDP and GDP per capita, Denmark 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison

in Table 1 (this is because of the estimated initial trend level being lower than the actual initial level). Figure 2 displays the development in UK, USA, and Japan 1870-2001.

3 Calculation of the average growth rate

3.1 Discrete compounding

Let $y \equiv Y/N$. The average annual growth rate of y, with discrete compounding, is that G which satisfies

$$
\begin{aligned}
y_t &= y_0(1+G)^t, \qquad t = 0, 1, 2, ... , \qquad \text{or} && (1)\\
1+G &= (\frac{y_t}{y_0})^{1/t}, \text{ i.e.,} \\
G &= (\frac{y_t}{y_0})^{1/t} - 1. && (2)
\end{aligned}
$$

"Compounding" means adding the one-period "net return". Thus G will generally be quite different from the arithmetic average of the year-by-year growth rates. To underline this G is sometimes called the "average compound growth rate" or the "geometric average growth rate".

Note that t in the formula (2) equals the number of periods *minus 1*.

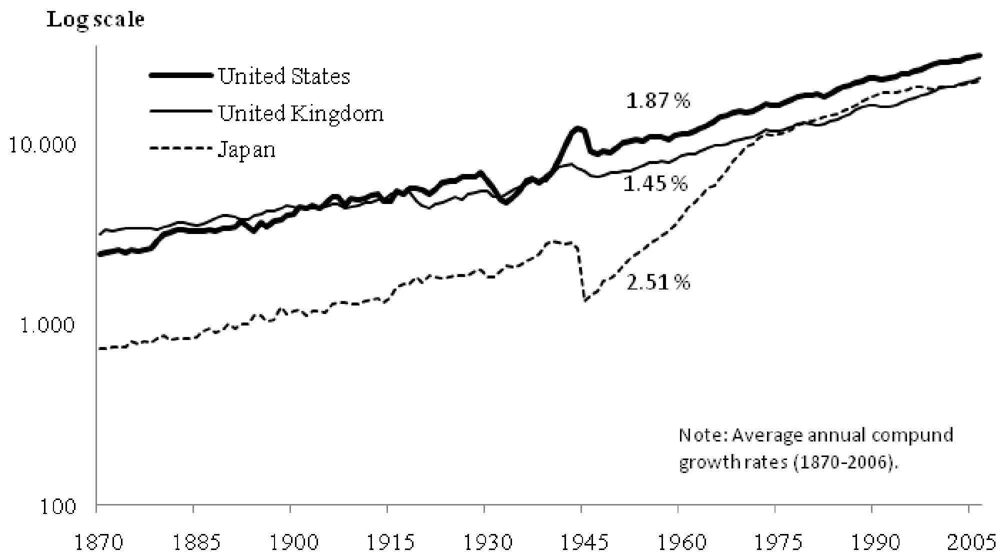

Figure 2: GDP per capita (1990 Geary-Khamis dollars) in UK, USA and Japan, 1870-2006. Kilde: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison

3.2 Continuous compounding

The average annual growth rate of y, with continuous compounding, is that g which satisfies

$$y(t) = y(0)e^{gt}, \tag{3}$$

where e denotes the Euler number, i.e., the base of the natural logarithm.[1] Solving for g gives

$$g = \frac{\ln \frac{y(t)}{y(0)}}{t} = \frac{\ln y(t) - \ln y(0)}{t}. \tag{4}$$

Here, the first formula is convenient for calculation with a pocket calculator, whereas the second formula is perhaps closer to intuition. Another name for g is the "exponential average growth rate".

Again, the t in the formula equals the number of periods minus 1.

[1] See Section 4. We let $\ln x$ denote the natural logarithm of a positive variable x, whereas B & S write it as $\log x$.

If we take logs on both sides of (1), we get

$$\begin{aligned} \ln \frac{y_t}{y_0} &= t \ln(1+G) \ \Rightarrow \\ \ln(1+G) &= \frac{\ln \frac{y_t}{y_0}}{t}. \end{aligned} \tag{5}$$

Thus, $g = \ln(1+G) < G$ for $G > 0$. Yet, by a first-order Taylor approximation around $G = 0$ we have

$$\ln(1+G) \approx G \ \text{ for } G \text{ “small”}. \tag{6}$$

Thus,

$$G \approx \frac{\ln \frac{y_t}{y_0}}{t} = g. \tag{7}$$

For a given data set the G calculated from (2) will be slightly above the g calculated from (4). The reason is that a given growth force is more powerful when compounding is continuous rather than discrete. Anyway, the difference between G and g is usually immaterial. If for example G refers to the annual GDP growth rate, it will be a small number, and the difference between G and g immaterial. For example, to $G = 0.040$ corresponds $g \approx 0.039$. Even if $G = 0.10$ (think of China in recent decades), the corresponding g is 0.0953. But if G stands for the inflation rate and there is high inflation, the difference is substantial. During hyperinflation the monthly inflation rate may be $G = 100\%$, but the corresponding g is only 69%.

For calculation with a pocket calculator the continuous compounding formula, (4), is slightly easier to use than the discrete compounding formula, (2).

4 Doubling time

How long time does it take for y to double if the growth rate with discrete compounding is G? Knowing G, we rewrite the formula (5):

$$t = \frac{\ln \frac{y(t)}{y(0)}}{\ln(1+G)} = \frac{\ln 2}{\ln(1+G)} \approx \frac{0.6931}{\ln(1+G)}$$

$$G = (\frac{y_t}{y_0})^{1/t} - 1.$$

How long time does it take for y to double if the growth rate with continuous compounding is g? The answer is based on rewriting the formula (4):

$$t = \frac{\ln \frac{y(t)}{y(0)}}{g} = \frac{\ln 2}{g} \approx \frac{0.6931}{g}.$$

With $g = 0.0186$, cf. Table 1, we find

$$t \approx \frac{0.6931}{0.0186} = 37.3 \text{ years.}$$

Again, with a pocket calculator the continuous compounding formula is slightly easier to use.

5 Kaldor's stylized facts

Kaldor's "stylized facts" for the more developed industrialized countries are:[2]

1. Real output per worker (in principle, per man-hour) grows at a more or less constant rate over fairly long periods of time. (Of course, there are short-run fluctuations superposed around this trend.)

2. The stock of physical capital (crudely measured) grows at a more or less constant rate exceeding the growth rate of the labor input.

3. The ratio of output to capital (properly measured) shows no systematic trend.

4. The rate of return to capital shows no systematic trend.

5. The income shares of labor and capital (broadly defined, including land and other natural resources), respectively, are nearly constant.

6. The growth rate of output per worker differs substantially across countries.

Although Solow's growth model (Solow, 1956) can be seen as the first relatively successful attempt at building a model consistent with Kaldor's "stylized facts", Solow once remarked about them: "There is no doubt that they are stylized, though it is possible to question whether they are facts." (Solow, 1970). At least they seem to fit the US and UK surprisingly well, see, e.g., Attfield and Temple (2006). The sixth Kaldor fact is of course well documented empirically (a nice summary is contained in Pritchett, L., 1997).

6 On continuous time analysis

Let us start from a discrete time framework: the run of time is divided into successive periods of constant length, taken as the time-unit. Let financial wealth at the beginning

[2] Kaldor (1961)..

of period i be denoted a_i, $i = 0, 1, 2, \ldots$. Then wealth accumulation in discrete time can be written

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

6.1 Transition to continuous time

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt be equal to one time unit. Then $a(i\Delta t) = a_i$. Consider the forward first difference in a, $\Delta a(t) \equiv a(t+\Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t + \Delta t)$ approach zero, i.e., let $\Delta t \to 0$. Assuming $a(\cdot)$ is a differentiable function, then $\lim_{\Delta t \to 0} \Delta a(t)/\Delta t$ exists and is denoted the *derivative of* $a(\cdot)$ at t, usually written $da(t)/dt$ or just $\dot{a}(t)$. That is,

$$\dot{a}(t) = \frac{da(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t + \Delta t) - a(t)}{\Delta t}.$$

By implication, wealth accumulation in continuous time is written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{8}$$

where $s(t)$ is the saving at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t + 1) - a(t) \approx s(t)$.

As time unit let us choose one year. Going back to discrete time, if wealth grows at the constant rate $g > 0$ per year, then after i periods of length one year (with annual compounding)

$$a_i = a_0(1 + g)^i, \quad i = 0, 1, 2, \ldots . \tag{9}$$

When compounding is n times a year, corresponding to a period length of $1/n$ year, then after i *such* periods:

$$a_i = a_0(1 + \frac{g}{n})^i. \tag{10}$$

With t still denoting time (measured in years) that has passed since the initial date (here date 0), we have $i = nt$ periods. Substituting into (10) gives

$$a(t) = a_{nt} = a_0(1 + \frac{g}{n})^{nt} = a_0 \left[(1 + \frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let n (and so m) $\to \infty$. Then, in the limit there is continuous compounding and

$$a(t) = a_0 e^{gt}, \tag{11}$$

where e is the "exponential" defined as $e \equiv \lim_{m\to\infty}(1 + 1/m)^m \simeq 2.718281828....$

The formula (11) is the analogue in continuous time (with continuous compounding) to the discrete time formula (9) with annual compounding. Thus, a geometric growth factor is replaced by an exponential growth factor.

We can also view the formulas (9) and (11) as the solutions to a difference equation and a differential equation, respectively. Thus, (9) is the solution to the simple linear difference equation $a_{i+1} = (1 + g)a_i$, given the initial value a_0. And (11) is the solution to the simple linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. With a time-dependent growth rate, $g(t)$, the corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ with solution

$$a(t) = a_0 e^{\int_0^t g(\tau)d\tau}, \tag{12}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (12) is called the *basic growth formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.

Notice that the allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{13}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, in (13) $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (13) gives $K(t) = K_0 e^{-\delta t}$ (exponential decay). This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style overlapping generations model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. Then with 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

6.2 Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption and saving). A *stock* variable is a variable measured as just a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ above are flow variables.

One cannot add a stock and a flow, because they have *different denomination*. What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so and so many machines of a certain kind or so and so many liters of oil or so and so many units of payment) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units*. Thus, the capital stock K has the denomination "quantity of machines". In contrast, investment I has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". If we change our time unit, say from quarters to years, the value of a flow variable is quadrupled (presupposing annual compounding). A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}".

Thus, in continuous time analysis expressions like $K(t) + I(t)$ or $K(t) + \dot{K}(t)$ are illegitimate. But one can write $K(t + \Delta t) \approx K(t) + (I(t) - \delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub contains 50 liters of water and the tap pours $\frac{1}{2}$ liter per second into the tub. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec.})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec.})\times\text{sec.}) = 90\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t)\Delta t$. Similarly, $s(t)$, that is, the flow of individual saving, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0) \cdot \Delta t$, cf. the hatched area in Fig. 3.

The notation commonly used in discrete time analysis blurs the distinction between

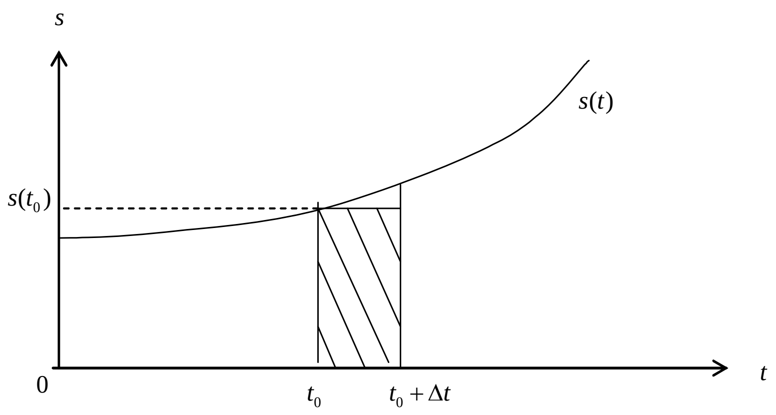

Figure 3: With Δt "small" the integral of $s(t)$ from t_0 to $t_0 + \Delta t$ is $\approx$ the hatched area.

stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. But, it is really wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

In empirical economics data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may be a good reason for putting at least microeconomic analysis in period terms. Yet, it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions and then a formulation in continuous time may be preferable. This also allows variation *within* the usually artificial periods in which the data are chopped up.[3] For example, stock markets (markets for bonds and shares) are more naturally modelled in continuous time because such markets equilibrate almost instantaneously; they respond immediately to new information.

In his discussion of this modelling issue, Allen (1967) concluded that from the point of view of the economic contents, the choice between discrete time or continuous time analysis may be a matter of taste. But from the point of view of mathematical convenience,

[3]Allowing for such variations may be necessary to avoid the *artificial* oscillations which sometimes arise in a discrete time model due to a too large period length.

the continuous time formulation, which has worked so well in the natural sciences, is preferable.[4]

7 References

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[4] At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier if one is not familiar with stochastic calculus.