29.03.2011 Christian Groth

A note on the concepts of TFP and growth accounting: Two warnings[1]

1 Introduction

This note ends up with two warnings regarding uncritical use of the concepts of Total Factor Productivity (TFP) and TFP growth.

First, however, we should provide a precise definition of the TFP level which is in fact a tricky concept. Unfortunately, Acemoglu (p. 78) does not make a clear distinction between TFP level and TFP growth. Moreover, Acemoglu's point of departure (p. 77) *assumes* a priori that the way the production function is time-dependent can be represented by a one-dimensional index, $A(t)$. The TFP concept and the applicability of growth accounting are, however, not limited to this case.

For convenience, in this note we treat time as continuous (although the timing of the variables is indicated merely by a subscript t).

2 TFP level and TFP growth

Let the aggregate production function for a sector or the economy as a whole be

$$Y_t = \tilde{F}(K_t, N_t; t), \tag{1}$$

where Y_t is an output aggregate (value added in fixed prices), K_t is input of physical capital, and N_t is an index of quality-adjusted labor input, all at time t.[2] The "quality-adjustment" of the input of labor (man-hours) aims at taking educational level and work experience into account. In fact, all these three variables are aggregates of heterogeneous

[1] I thank Niklas Brønager for useful comments to this lecture note.

[2] Although natural resources (land, oil wells, coal mines, etc.) constitute a third primary production factor, the role of this factor is often left unmentioned.

elements. The involved measurement problems are large and there are different opinions in the growth accounting literature about how to best deal with them. Here we ignore these problems. The third argument in (1) is time, t, indicating that the production function $\tilde{F}(\cdot\ ,\cdot\ ;t)$ is time-dependent due to technical change. We assume $\tilde{F}$ is continuously differentiable w.r.t. its three arguments. When the partial derivative $\tilde{F}_t \equiv \partial \tilde{F}/\partial t > 0$, technical change is of the *progress* kind. Below we assume, as is usual in simple growth accounting, that $\tilde{F}$ is neoclassical.[3]

The reason that quality-adjusted labor input in (1) is not denoted L_t is that we usually use L_t to denote just the number of man-hours whatever the quality (productivity) involved is. On the other hand, since the fundamentals of TFP and TFP growth can be described without taking the changing quality of the labor input into account, we shall in fact from now ignore this aspect and thus simplifyingly *assume* that labor quality is constant. Then (1) is reduced to the simpler case,

$$Y_t = \tilde{F}(K_t, L_t; t), \tag{2}$$

where L_t *is* simply the number of man-hours.

Let TFP_t denote Total Factor Productivity at time t. The measurement of TFP is generally easier in growth terms than in level terms. So we start with a growth calculation. Taking logs and differentiating w.r.t. t in (2) give

$$\begin{aligned} g_{Y,t} &\equiv \frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left[\tilde{F}_K(K_t, L_t; t)\dot{K}_t + \tilde{F}_L(K_t, L_t; t)\dot{L}_t + \tilde{F}_t(K_t, L_t; t)\cdot 1\right] \\ &= \frac{K_t\tilde{F}_K(K_t, L_t; t)}{Y_t} g_{K,t} + \frac{L_t\tilde{F}_L(K_t, L_t; t)}{Y_t} g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} \\ &\equiv \varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t}, \end{aligned}$$

where $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$ are the output elasticities w.r.t. capital and labor at time t, respectively, and $\tilde{F}_t(K_t, L_t; t)$ represents the partial derivative w.r.t. the third argument of the function $\tilde{F}$ (that is, K_t and L_t are kept fixed), evaluated at the point (K_t, L_t, t).

Then the TFP *growth rate* is defined as

$$g_{\text{TFP},t} \equiv g_{Y,t} - (\varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} g_{L,t}) \equiv \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t}, \tag{3}$$

[3] Sometimes in growth accounting the left-hand side variable, Y, in (2) is gross product rather than value added. Then non-durable intermediate inputs should be taken into account as a third production factor and enter as an additional argument of $\tilde{F}$ in (2). Since production in non-market activities is difficult to measure, the government sector is usually excluded from the Y in (2). Total Factor Productivity is by some authors called *Multifactor Productivity* and abbreviated MFP.

This is the *basic growth accounting relation.* The output elasticities w.r.t. capital and labor, $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$, will, under CRS, perfect competition and absence of externalities, equal the income shares of capital and labor, respectively. Time series for these income shares, together with Y, K, and L, hence also $g_{Y,t}$, $g_{K,t}$, and $g_{L,t}$, can be obtained (directly or indirectly) from national income accounts. This allows straightforward calculation of the residual, $g_{\text{TFP},t}$. This type of calculation was introduced by Solow (1957), hence the TFP growth rate is sometimes called the *Solow residual.*[4]

Now let us consider the *level* of TFP. Since we can easily calculate the Solow residual, in principle we know the growth rate of TFP for all $t \geq 0$, that is, we have a differential equation in TFP:

$$d(\text{TFP})/dt = g_{\text{TFP},t} \cdot \text{TFP}.$$

The solution of this linear differential equation is

$$\text{TFP}_t = \text{TFP}_0 e^{\int_0^t g_{\text{TFP},\tau} d\tau}. \tag{4}$$

For a given initial value $\text{TFP}_0 > 0$, the time path of TFP is given by the right-hand side of (4).

Unfortunately, the TFP level and its growth may not be very informative concepts unless technical progress is Hicks neutral (which it does not generally seem to be).

The first problem is that the TFP level does not have a clear *intuitive* meaning unless technical progress is Hicks neutral. To see this, let us assume that technical change is neutral (either in the Hicks, Solow, or Harrod sense), that is, technical change can be represented by the evolution of a one-dimensional variable, B_t, with a given initial value $B_0 > 0$. Then in the case of Hicks neutrality, Y_t in (2) can be specified as

$$Y_t = \tilde{F}(K_t, L_t; t) = B_t F(K_t, L_t), \tag{5}$$

and the TFP level at any t is simply identical to the level of B_t if we normalize the initial value of both B and TFP to be one, i.e., $\text{TFP}_0 = B_0 = 1$. Indeed, under Hicks neutrality the TFP growth rate, calculated from the formula (3), is

$$g_{\text{TFP},t} = \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} = \frac{\dot{B}_t F(K_t, L_t)}{B_t F(K_t, L_t)} = \frac{\dot{B}_t}{B_t} \equiv g_{B,t}, \tag{6}$$

[4] Of course, data are in discrete time. So to make actual calculations we have to translate (3) into discrete time. This is done in Acemoglu, p. 79.

where the second equality comes from the fact that K_t and L_t are kept fixed when the *partial* derivative of $\tilde{F}$ w.r.t. t is calculated. Hence the formula (4) gives

$$\text{TFP}_t = B_0 \cdot e^{\int_0^t g_{B,\tau} d\tau} = B_t.$$

The nice feature of Hicks neutrality is that we can write

$$\text{TFP}_t = \frac{B_t F(K_t, L_t)}{B_0 F(K_t, L_t)} = B_t,$$

using the normalization $B_0 = 1$. That is, under Hicks neutrality, current TFP appears as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.

This clear and intuitive interpretation of TFP is, however, only valid under Hicks-neutral technical change. Neither under general technical change nor even under Solow- or Harrod-neutral technical change (unless the production function is Cobb-Douglas), will current TFP appear as the ratio between the current output level and the hypothetical output level that would have resulted from the current inputs of capital and labor in case of no technical change since time 0.

To see this, let us return to the general time-dependent production function in (2). Let X denote the ratio between the current output level at time t and the hypothetical output level that would have obtained with the current inputs of capital and labor in case of no change in the technology since time 0. That is,

$$X \equiv \frac{\tilde{F}(K_t, L_t; t)}{\tilde{F}(K_t, L_t; 0)} \equiv \frac{\tilde{F}(K_t, L_t; t)}{G(K_t, L_t)}, \tag{7}$$

where the new function $G(K_t, L_t)$ is defined by $G(K_t, L_t) \equiv \tilde{F}(K_t, L_t; 0)$ and represents the hypothetical output level that would have obtained with the current inputs of capital and labor in case of no change in the technology since time 0.

If this X should indicate the level of TFP, then, according to the general definition of the TFP growth rate in (3), the growth rate of X should equal $\tilde{F}_t(K_t, L_t; t)/Y_t$. Generally, it does not, however. Indeed,

$$\begin{aligned}
g_{X,t} &\equiv \frac{d\tilde{F}(K_t, L_t; t)/dt}{\tilde{F}(K_t, L_t; t)} - \frac{dG(K_t, L_t)/dt}{G(K_t, L_t)} \\
&= \frac{1}{Y_t}\left[\tilde{F}_K(K_t, L_t; t)\dot{K}_t + \tilde{F}_L(K_t, L_t; t)\dot{L}_t + \tilde{F}_t(K_t, L_t; t)\cdot 1\right] \\
&\quad -\frac{1}{G(K_t, L_t)}\left[G_K(K_t, L_t)\dot{K}_t + G_L(K_t, L_t)\dot{L}_t\right] \\
&= \varepsilon_{K,t}g_{K,t} + \varepsilon_{L,t}g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} - (\varepsilon_{K,0}g_{K,t} + \varepsilon_{L,0}g_{L,t}) \\
&= (\varepsilon_{K,t} - \varepsilon_{K,0})g_{K,t} + (\varepsilon_{L,t} - \varepsilon_{L,0})g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} \\
&\neq \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t}
\end{aligned}$$

generally. So, the X defined in (7) will generally *not* equal TFP.

As an implication, we can generally only say that the TFP level is given by (4) where $g_{\mathrm{TFP},t}$ is given from (3). And this is not a very intuitive meaning. Moreover, it is not obvious that measuring TFP and TFP growth is at all useful when technical progress is not Hicks neutral. This leads us to the first of the two warnings.

3 Two warnings

Warning 1 We claim that TFP and TFP growth can be misleading when technical progress is not Hicks-neutral (which it does not generally seem to be). Indeed, if technical progress is for example Harrod-neutral, relative TFP growth rates across sectors or countries can be quite deceptive.

Suppose there are n countries and that country i has the aggregate production function

$$Y_{it} = F^{(i)}(K_{it}, A_t L_{it}) \qquad i = 1, 2, ..., n,$$

where $F^{(i)}$ is a neoclassical production function with CRS and A_t is the level of labor augmenting technology which, for simplicity, we assume shared by all the countries (open economies). So technical progress is Harrod-neutral. Let the growth rate of A be a constant $g > 0$. Many models imply that $\tilde{k}_i \equiv K_{it}/(A_t L_{it})$ tends to a constant, $\tilde{k}_i^*$, in the long run, which we assume is also the case here. Then, for $t \to \infty$, $k_{it} \equiv K_{it}/L_{it} \equiv \tilde{k}_{it}A_t$ where $\tilde{k}_{it} \to \tilde{k}_i^*$ and $y_{it} \equiv Y_{it}/L_{it} \equiv \tilde{y}_{it}A_t$ where $\tilde{y}_{it} \to \tilde{y}_i^* = f^{(i)}(\tilde{k}_i^*)$; here $f^{(i)}$ is the production function on intensive form. So in the long run g_{k_i} and g_{yi} tend to $g_A = g$.

Formula (3) then gives the TFP growth rate of country i in the long run as

$$\begin{aligned} g_{\text{TFP}_i} &\equiv g_{Y_i} - (\alpha_i^* g_{K_i} + (1-\alpha_i^*) g_{L_i}) = g_{Y_i} - g_{L_i} - \alpha_i^*(g_{K_i} - g_{L_i}) \\ &= g_{y_i} - \alpha_i^* g_{k_i} = (1-\alpha_i^*) g, \end{aligned} \tag{8}$$

where α_i^* is the output elasticity w.r.t. capital, $f^{(i)\prime}(\tilde{k}_i)\tilde{k}_i/f^{(i)}(\tilde{k}_i)$, evaluated at $\tilde{k}_i = \tilde{k}_i^*$. Owing to differences in product and industry composition, the countries may have different α_i^*'s. For two different countries, i and j, we get

$$\frac{TFP_i}{TFP_j} \rightarrow \begin{cases} \infty \text{ if } \alpha_i^* < \alpha_j^*, \\ 1 \text{ if } \alpha_i^* = \alpha_j^*, \\ 0 \text{ if } \alpha_i^* > \alpha_j^*, \end{cases}$$

for $t \rightarrow \infty$.[5] Thus, in spite of long-run growth in the essential variable, y, being the same across the countries, their TFP growth is very different. Countries with low α_i^*'s appear to be technologically very dynamic and countries with high α_i^*'s appear to be lagging behind. It is all due to the difference in α across countries. And in itself α has nothing to do with technical progress.

We conclude that comparison of TFP levels across countries, sectors, or time may misrepresent the economic meaning of productivity and technical progress when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral.

Warning 2 Do not confuse growth *accounting* with *causality* in growth analysis. To talk about causality we need a theoretical model supported by the data. On the basis of such a model we can say that this or that set of exogenous factors through the propagation mechanisms of the model cause this or that phenomenon, including economic growth. In contrast, considering the growth accounting identity (3) in itself, non of the terms have priority over the others w.r.t. a causal role.

—

[5] If F is Cobb-Douglas with output elasticity w.r.t. capital equal to α_i, the result in (8) can be derived more directly by first defining $B_t = A_t^{1-\alpha_i}$, then writing the production function in the Hicks neutrality form (5), and finally use (6).