Economic Growth, June 2011.

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A suggested solution to the problem set at the exam in Economic Growth, June 15, 2011

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (45 %)

For convenience, key equations are repeated here:

$$\dot{K}_t = Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ is given.} \tag{1.1}$$

The production function of firm i is

$$Y_{it} = K_{it}^{\alpha}(A_t L_{it})^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{1.2}$$

where A_t is the economy-wide technology level, $\sum_i K_{it} = K_t$, and $\sum_i L_{it} = L_t$, where L_t is the labor force (= employment = population). Each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables, including A_t.

a) We suppress the time index when not needed for clarity. Consider firm i. Its maximization of profits, $\Pi_i = K_i^{\alpha}(AL_i)^{1-\alpha} - (r+\delta)K_i - wL_i$, leads to the first-order conditions

$$\begin{aligned} \partial \Pi_i/\partial K_i &= \alpha K_i^{\alpha-1}(AL_i)^{1-\alpha} - (r+\delta) = 0, \\ \partial \Pi_i/\partial L_i &= (1-\alpha)K_i^{\alpha}A^{1-\alpha}L_i^{-\alpha} - w = 0. \end{aligned} \tag{1.3}$$

We can write (1.3) as

$$\alpha A^{1-\alpha}k_i^{\alpha-1} = r + \delta, \tag{1.4}$$

[1]The solution below contains *more* details and more precision than can be expected at a three hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

where $k_i \equiv K_i/L_i$. From this follows that the chosen k_i will be the same for all firms, say $\bar{k}$. In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $K = \sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital intensity, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{1.5}$$

As a consequence we can use (1.4) to *determine* the equilibrium interest rate:

$$r_t = \alpha A^{1-\alpha} k^{\alpha-1} - \delta. \tag{1.6}$$

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i k_i^\alpha A^{1-\alpha} L_i = k^\alpha A^{1-\alpha} \sum_i L_i = k^\alpha A^{1-\alpha} L \\ &= k^\alpha L^\alpha A^{1-\alpha} L^{1-\alpha} = K^\alpha (AL)^{1-\alpha} = A^{1-\alpha} K^\alpha L^{1-\alpha} \equiv T K^\alpha L^{1-\alpha}. \end{aligned} \tag{1.7}$$

b) $\text{TFP}_t = T_t = A_t^{1-\alpha}$.

c) We get

$$g_Y = \alpha g_K + (1-\alpha) g_L + g_T,$$

where g_T is the residual, often named the Solow residual (we have $g_T = (1-\alpha) g_A$).

d) The TFP growth rate is

$$g_T = g_Y - \alpha g_K - (1-\alpha) g_L.$$

The gross income share of capital is

$$\frac{(r+\delta)K}{Y} = \frac{\frac{\partial Y}{\partial K} K}{Y} = \frac{\alpha \frac{Y}{K} K}{Y} = \alpha. \tag{1.8}$$

The labor income share is

$$\frac{wL}{Y} = \frac{\frac{\partial Y}{\partial L} L}{Y} = \frac{(1-\alpha)\frac{Y}{L} L}{Y} = 1 - \alpha. \tag{1.9}$$

We now assume that A_t evolves according to

$$A_t = e^{\varepsilon t} K_t^\lambda, \qquad \varepsilon > 0, \quad 0 < \lambda < 1, \tag{*}$$

where ε and λ are given constants.

e) The assumption (*) says that the technology level has an exogenous component, $e^{\varepsilon t}$, growing at the exogenous rate ε, and an endogenous component, K_t^{λ}. The latter can be interpreted as reflecting "learning by investing". The idea is that investment – the production of capital goods – as an unintended *by-product* results in *experience* or what we may call on-the-job *learning*. This adds to the knowledge about how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive and better satisfy the needs of the users. The idea stems from (Arrow, 1962) who hypothesized that the primary basis for learning is *gross* investment. Yet, the term K_t^{λ} in (*), where λ is called the "learning parameter", indicates that the basis for learning is *net* investment, so that cumulative learning - the technology level - is an increasing function of cumulative *net* investment, $\int_{-\infty}^{t} I_s^n ds = K_t$. This latter hypothesis is more popular for the only reason that it leads to simpler dynamics. Another way in which the specification in (*) deviates from Arrow's original ideas is by assuming that technical progress is disembodied rather than embodied.

The learning is assumed to benefit essentially all firms in the economy due to *knowledge spillovers* across firms. Such spillovers are reasonably fast relative to the time horizon relevant for growth theory.

f) Combined with (*), (1.7) implies

$$Y_t = (e^{\varepsilon t} K_t^{\lambda})^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha} = e^{(1-\alpha)\varepsilon t} K_t^{\alpha+(1-\alpha)\lambda} L_t^{1-\alpha} \tag{1.10}$$

so that

$$g_Y = (\alpha + (1-\alpha)\lambda)\, g_K + (1-\alpha) g_L + \text{ residual}, \tag{1.11}$$

where the residual is $(1-\alpha)\varepsilon$.

g) To compare standard growth accounting with this, let the weights attached to g_K and g_L be denoted η_K and η_L, respectively. Then, in standard growth accounting we have $\eta_K = (r+\delta)K/Y$ and $\eta_L = wL/Y$, respectively. Hence, by (1.8), the "contribution" to output growth from growth in capital is set equal to αg_K. This is less than the "true contribution" to output growth from growth in capital which, by (1.11), is

$$(\alpha + (1-\alpha)\lambda)\, g_K. \tag{1.12}$$

In this sense, standard growth accounting "underestimates" the "contribution" to output growth from growth in capital. This is because the market price r does not reflect the positive externality from capital investment.

We now assume $L_t = L_0 e^{nt}$, where $n > 0$, constant.

h) In view of (1.1), under balanced growth with positive saving, $g_Y = g_K$. By (1.11) and $g_L = n$ we then have $g_Y = (\alpha + (1-\alpha)\lambda)\, g_Y + (1-\alpha)g_L + (1-\alpha)\varepsilon$, from which follows

$$g_Y = \frac{n+\varepsilon}{1-\lambda}.$$

With $y \equiv Y/L$, this implies

$$g_y = g_Y - n = \frac{\lambda n + \varepsilon}{1-\lambda} \; (= g_k = g_A). \tag{1.13}$$

i) According to the model there are, according to (1.13), two ultimate sources of per capita growth (along a BGP), learning by investing, represented by the term λn, and an exogenous source, represented by the parameter ε.

The first source, learning, is more powerful, the higher is the population growth rate, n. This role of population growth derives from the fact that at the economy-wide level there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. The more fundamental background is that technical knowledge is partly endogenous in the model and is at the same time a *non-rival good* – its use by one firm does not (in itself) limit the amount of knowledge available to other firms. In a large economic system *more* people benefit from a given increase in knowledge than in a small economic system. At the same time the per capita cost (here per capita net investment) of creating the increase in knowledge is less in the large system than in the small system.

In contrast, the role of the *exogenous* component of the technology is not expanded by the population growth rate n.

The learning *mechanism*, however, expands the role of *both* sources of per capita growth. This is manifested by the appearance of the "multiplier" $1/(1-\lambda) > 1$ in (1.13).

According to the growth accounting in c),

$$g_y = g_Y - n = \alpha(g_K - g_L) + g_T = \alpha g_k + g_T, \tag{1.14}$$

where $k \equiv K/L$ and $g_T = (1-\alpha)g_A = (1-\alpha)(\lambda n + \varepsilon)/(1-\lambda) = (1-\alpha)g_y$ under balanced growth.

Comparison: The natural interpretation of (1.13) is that (along a BGP) all per capita growth "comes from" growth in the labor-augmenting technology level A, that is, from "technical progress". In contrast, (1.14) says that only a fraction of per capita growth is accounted for by "technical progress", the remainder being accounted for by increases in the capital intensity k. This way of characterizing the growth process is, however, superficial for two reasons. First, even in a mere accounting perspective, the "direct contribution" to g_Y from growth in K (or "direct contribution" to g_y from growth in k) is underestimated, as noted under g). Second, the growth in k is itself endogenous and would be absent if there were no learning, no population growth, and no exogenous technology growth. To see this, in (1.13) let $\lambda = n = \varepsilon = 0$. On the other hand, capital accumulation is certainly a key factor in the learning process.

2. Solution to Problem 2 (45 %)

a) Yes, the "lab-equipment" version of the expanding input variety model is consistent with this evidence. The "lab-equipment" version (in contrast to the two "knowledge-spillover" versions, see below) features no positive externality from knowledge creation, yet there is underinvestment in R&D. This is because the monopoly position (obtained through the patent system) of innovators implies that the invented specialized intermediate goods are priced above marginal costs. Consequently, "too little" of these goods is demanded, that is, the market for each variety is "too small". This results in too little remuneration of the R&D activity, which invents new types of intermediate goods, new varieties. Consequently, there is too little incentive to do R&D, and the growth rate ends up smaller than the social optimum as defined from the perspective of a social planner respecting the preferences of an assumed representative infinitely-lived household.

b) Let the government pay a subsidy at constant rate, σ, to purchases of intermediate goods such that the price of intermediate good i is $(1 - \sigma)p_i$, where p_i is the price set by the monopolist supplier of intermediate good i. Let the government finance this subsidy by taxing consumption at the constant rate τ.

(Although certainly not needed for answering the question, as phrased, we may mention that $p_i = \psi/(1 - \beta)$, where β is the inverse of the elasticity of substitution between intermediate goods in the production of basic goods, and ψ is the marginal cost of supplying intermediate good i, when its technical design has already been invented. The optimal σ then equals β. There exists a unique value of the *constant* consumption tax, τ, such that the government can finance the subsidy while still maintaining a balanced budget. This is because the model ends up as a reduced-form AK model featuring balanced growth from the beginning.)

c) The distinctive feature of two other versions of the expanding input variety model is an aggregate invention production functions of the form

$$\dot{N}_t = \eta N_t^{\varphi} L_{Rt}, \qquad \eta > 0, \varphi \leq 1, \tag{2.1}$$

where N_t is the number of existing different varieties of intermediate goods (indivisibilities are ignored) and L_{Rt} the input of research labor at time t. The number, N_t, of existing different varieties of intermediate goods can be interpreted as reflecting the stock of technical knowledge. The time derivative, $\dot{N}_t$, reflects the number of new technical designs for intermediate goods, invented per time unit at time t. Thus, $\dot{N}_t$ can be seen as the increase per time unit in technical knowledge. This increase is determined by the input of research labor, L_{Rt}, and the economy-wide research productivity, ηN_t^{φ}, exogenous to the "small" individual R&D firms. For $\varphi \neq 0$, the research productivity depends on the stock of technical knowledge. This dependency is positive if $\varphi > 0$ ("the standing on the shoulders of giants case") and negative if $\varphi < 0$ ("the fishing out case", "the standing on the toes case").

The knife-edge case $\varphi = 1$ (together with $n = 0$, where n is the growth rate of the labor force L) gives the Romer version. And the case $\varphi < 1$ (together with $n \geq 0$) gives the Jones version (the calibration by Jones, 1995, suggests $0 < \varphi < 1$).

d) Yes, both the Romer version and the Jones version (with $\varphi > 0$) are consistent with the mentioned evidence. Both the monopoly pricing mentioned under a) and the positive intertemporal externality of R&D via the economy-wide productivity term ηN_t^{φ} contribute to the tendency to underinvestment in R&D.

e) Yes, the positive intertemporal externality of R&D in these two versions calls for a research subsidy, s, in addition to the subsidy to purchases of intermediate goods mentioned under b). The cost to the R&D firm will then be $(1 - s)w$ per unit of research labor.

f) Let the patent-R&D ratio at time t be named u_t. Then then two model versions mentioned under c) imply

$$u_t = \frac{\dot{N}_t}{w_t L_{Rt}} = \frac{\dot{N}_t}{N_t} \frac{N_t}{w_t} \frac{1}{L_{Rt}}. \tag{2.2}$$

g) Under balanced growth the Romer version has $\dot{N}_t/N_t$ constant, w_t growing at the same rate as N_t, and L_{Rt} $(= L - L_{Yt})$ constant. Thus, the Romer version predicts a constant u_t over time.

Also the Jones version has, under balanced growth, $\dot{N}_t/N_t$ constant $(= n/(1 - \varphi))$ and w_t growing at the same rate as N_t. But the factor $1/L_{Rt}$ will be falling under balanced growth in the Jones model when $n > 0$ (which the model allows). Indeed, under balanced growth, $L_{Rt} = s_R L_t$, where the researchers' fraction, s_R, of the labor force is constant so that L_{Rt} grows at the same rate as the labor force. It follows that the Jones version predicts the patent-R&D ratio, u_t, to be falling over time.

Remark: The somewhat demanding step in this reasoning is the constancy of N_t/w_t under balanced growth. This constancy is only implicit in Acemoglu's §13.2-3. Yet, intuition should be enough to reach the conclusion. Indeed, in all the R&D-based models we have considered in the course, under balanced growth labor productivity in manufacturing, Y/L_Y, grows at the same rate as knowledge, N. In turn, the labor income share in manufacturing, wL_Y/Y, is constant. So also w grows at the same rate as N, implying that N_t/w_t is constant under balanced growth.

A more formal approach is of course to derive the conclusion about constancy of N_t/w_t in detail from profit maximization in the manufacturing sector. Suppressing the time index, profit of the representative firm in the sector is

$$\Pi = Y - \sum_{i=1}^{N} p_i x_i - wL_Y, \qquad \text{where} \qquad Y = A\left(\sum_{i=1}^{N} x_i^{1-\beta}\right) L_Y^{\beta}, \qquad A > 0 \text{ constant.}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \Pi}{\partial x_i} &= (1-\beta)Ax_i^{-\beta}L_Y^{\beta} - p_i = 0, \qquad i = 1, 2, ..., N, & (2.3) \\ \frac{\partial \Pi}{\partial L_Y} &= \beta A\left(\sum_{i=1}^{N} x_i^{1-\beta}\right) L_Y^{\beta-1} - w = 0. & (2.4) \end{aligned}$$

From (2.3)

$$x_i = (A(1-\beta))^{1/\beta} L_Y p_i^{-1/\beta}.$$

Since monopoly pricing in the intermediate goods sector leads to $p_i = \psi/(1-\beta) \equiv p$, $i = 1, 2, ..., N$, the chosen $x_i = \mathit{const} \cdot L_Y \equiv x$, the same for all i. So (2.4) implies

$$w = \beta A N x^{1-\beta} L_Y^{\beta-1} = \beta A N \cdot \mathit{const}^{1-\beta},$$

saying that w/N is constant.

h) The observed systematic decline in the empirical patent-R&D ratio in the US, at the same time as the labor force has been growing ($n > 0$), fits well with the Jones version, but not with the Romer version.

3. Solution to Problem 3 (10 %)

We define *human capital* as the stock of productive skills embodied in an individual. Increases in this stock occurs through formal education and on-the-job-training.

Consider an individual who at time 0 begins going to school (full-time) and then leaves school at time S. Then this individual has spent S time units (say years) under formal education. Suppose the individual thereafter never returns to school. Then the simplest possible version of a *schooling technology* gives the human capital, h_t, of this individual at time $t \geq S$, i.e., after leaving school, as

$$h_t = h(S), \qquad \text{where } h(0) \geq 0, \ h' > 0, \tag{3.1}$$

ignoring the effect of on-the-job training. Indeed, empirically, the primary input in education is the time spent by the students studying. This time is not used in work and gives thereby rise to an opportunity cost of studying. (In a macroeconomic context, we might perceive the costs associated with teachers' time and educational buildings and equipment as being either negligible or implicit in the function symbol $h(\cdot)$.)

A desirable property of a schooling technology is that the "stuff" it generates, which we call h_t, is approximately proportional to labor productivity and is thereby, at least under perfect competition, approximately proportional to the obtained real wage.

One popular specification is $h(S) = S^{\varphi}$, $\varphi > 0$.

Another specification is the sometimes used "Mincerian equation", $h(S) = h(0)e^{\psi S}$, $\psi > 0$. As noted in one of the lecture notes, such an exponential form comes from a false analogy between Jacob Mincer's cross-sectional evidence on relative wages at a given point time and a production function for human capital. Moreover, the strong convexity implied by an exponential specification has several awkward properties, including that the second-order condition in a standard human wealth maximization problem is violated.

To take effects of on-the-job training into account, we may extend (3.1) to

$$h_t = h(S, t - S), \qquad \text{where } h(0, t) \geq 0, \ h'_S > 0, \ h'_{t-S} > 0.$$

assuming that the person in question is a "full-time worker".

To take cohort-effects into account, we may assume that a person who is "born" at time v (v for "vintage") and spends the first S years of life full-time in school, at $t \geq v + S$ has human capital

$$h_t = h(v, S), \qquad \text{where } h(v, 0) \geq 0, \ h'_S > 0.$$

If efficiency of schooling is improving over time, $h'_v > 0$.

In more complicated settings, a schooling technology is specified as a differential equation. Let $s_t \in [0, 1]$ denote the fraction of time spent in school at time t. This allows the individual to go to school only part-time and for example spend the remainder of time working, in which case $1 - s_t =$ fraction of time spent working at time t. The time dependency of s_t takes into account that an individual may spend a varying fraction of available time in school and the remainder in work.

Here a schooling technology may, for $t \geq 0$, be given in the differential form

$$\dot{h}_t \equiv \frac{dh_t}{dt} = G(t, h_t, s_t), \qquad h_0 \ \geq 0 \text{ given.}$$

The "static" specification, $h(S) = S^\varphi$, $\varphi > 0$, mentioned above, is in fact a solution to this differential equation in the special case where

$$\dot{h}_t = \varphi t^{\varphi - 1} s_t, \qquad h_0 \ = 0 \text{ given,} \tag{3.2}$$

and

$$s_t = \begin{cases} 1 \text{ for } 0 \leq t < S, \\ 0 \text{ for } t \geq S. \end{cases} \tag{3.3}$$

Indeed, integrating (3.2) we have for $t \geq S$

$$h_t = h_0 + \int_0^t \dot{h}_\tau d\tau = 0 + \int_0^S \varphi \tau^{\varphi - 1} d\tau + 0 = [\tau^\varphi]_0^S = S^\varphi.$$

Also the Mankiw, Romer, and Weil (1992) approach (MRW) to aggregate human capital formation may be seen as representing a "schooling technology". Let aggregate human capital be $H \equiv hL$, where h is average human capital in the labor force. Then, MRW assume

$$\dot{H} \equiv \frac{dH}{dt} = I_H - \delta_H H, \qquad \delta_H \geq 0,$$

where δ_H is the depreciation rate for human capital, I_H is educational gross investment, defined (in principle) as

$$I_H = Y - I_K - C,$$

where Y is GDP, I_K is gross investment in physical capital, and C is consumption. Based on their cross-country regression analysis, MRW find that the following production function for a country's GDP is an acceptable approximation:

$$Y = K^{1/3}(hL)^{1/3}(AL)^{1/3} = K^{1/3}(A^{1/2}h^{1/2}L)^{2/3},$$

where A represents the level of technology and is growing over time.

These three equations together can be seen as representing a schooling technology in a broad sense. It may be claimed, however, that a weakness of this approach is that labor productivity is not proportional to the "stuff", h, generated this way and by the authors called "human capital". Indeed, the "quality function", $q(h)$, is not linear, but a strictly concave function, $h^{1/2}$.

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