Economic Growth Exercises

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Problem Set IV

IV.1 *(on persistent technology differences)* In continuation of the article by Bernard and Jones (1996)[1] we consider a set of countries, indexed by $i = 1, 2, \ldots, N$, with aggregate production functions

$$Y_i(t) = K_i(t)^{\alpha_i}(A_i(t)L_i(t))^{1-\alpha_i}, \qquad 0 < \alpha_i < 1, \tag{1}$$

where A_i is the technology level of country i. There is a country-specific capital depreciation rate, δ_i, which is assumed constant over time. Technological catching-up occurs according to

$$\frac{\dot{A}_i(t)}{A_i(t)} = \xi_i \frac{A_w(t)}{A_i(t)}, \tag{2}$$

where $A_w(t) = A_w(0)e^{gt}$ is the world frontier technology level, $g > 0$. We assume $A_i(0) \leq A_w(0)$ and $0 < \xi_i \leq g$, for all $i = 1, 2, \ldots, N$. Let "labor productivity" be measured by $y_i(t) \equiv Y_i(t)/L_i(t)$.

In their extended Solow-style setup Bernard and Jones find that even in the long run there need not be a tendency for $y_i(t)/y_w(t)$ to approach 1 and they show which potential structural differences (parameter differences) are responsible for this. They conclude that their setup:

> leads to a world in which similar steady state outcomes are the exception rather than the rule.

Bernard and Jones present data for 14 OECD countries over the period 1970-87 to substantiate this conclusion. Over this period, however, financial capital was not as mobile as it is today. This raises two questions. How, if at all, does perfect capital mobility affect the theoretical conclusion? And what does more recent data show?

Here we shall deal with the first question.[2] We replace the Solow-style setup by a setup where the countries trade in a fully integrated world market for goods and financial

[1] Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

[2] The second question may be suitable for a Master Thesis project!

capital. Assume perfect competition and that the countries face a constant real interest rate $r > 0$ in the capital market. Finally, assume that (1) and (2) still hold and that the world frontier technology is identical with the technology of one of the countries in the considered set, namely the "world leader" (say USA). We let α_w denote the output elasticity w.r.t. capital in this country.

a) Examine whether in this case there is a tendency for $y_i(t)/y_w(t)$ to approach 1 in the long run. *Hint 1:* Profit maximizing firms will under perfect competition choose a time-independent effective capital intensity, $\tilde{k}_i^*$, satisfying

$$f_i'(\tilde{k}_i^*) = r + \delta_i.$$

Hint 2. Consider the ratio $x(t) \equiv A_i(t)/A_w(t)$, a measure of country i's *lag* relative to the frontier; express the growth rate of x in terms of x, ξ_i and g; this should give you a linear first-order differential equation with constant coefficients; then apply the brief math manual in Appendix A.

b) Does the answer to a) depend on whether the countries differ w.r.t. their saving rate, s_i, and labor force growth rate, n_i? Why or why not?

Let $\text{TFP}_i(t)$ denote the total factor productivity of country i at time t.

c) Express $\text{TFP}_i(t)$ in terms of the labor-augmenting technology level $A_i(t)$.

d) Find the limit of the ratio $\frac{\text{TFP}_i(t)}{\text{TFP}_w(t)}$ for $t \to \infty$; there may be alternative cases to be considered.

e) Will there be a tendency for TFP of the different countries to differ in the long run? Why or why not?

f) On the basis of the above results, do you think the comparative analysis in terms of TFP growth adds anything of economic interest to the comparative analysis in terms of the labor-augmenting technology level A and labor productivity, y, cf. a)?

g) "Long-run growth in the ratio of two countries' TFP may misrepresent the economic meaning of technical progress when output elasticities w.r.t. capital differ and technical progress is Harrod-neutral." Do you agree? Why or why not?[3]

[3] Outside this exercise problem, Appendix B briefly comments on the concept of Total Technological Productivity, $\text{TTP}_i(t)$, introduced in Bernard and Jones (1996).

IV.2 *(human capital and catching up)* We consider a market economy. Suppose people are alike and that if they attend school for s years, they obtain individual human capital

$$h = s^{\varphi}, \qquad \varphi > 0. \tag{*}$$

An individual "born" at time 0 chooses s to maximize

$$HW_0 = \int_s^{\infty} \hat{w}_t h e^{-(r+m)t} dt, \tag{**}$$

subject to (*). Here $\hat{w}_t$ is the market determined real wage per year *per unit of human capital* at time t, r is a constant real interest rate, and m is a parameter such that the probability of surviving at least until age τ is $e^{-m\tau}$. For simplicity m is assumed independent of age and calendar time. It is assumed that owing to technical progress,

$$\hat{w}_t = \hat{w}_0 e^{gt}, \tag{***}$$

where g is a nonnegative constant.

a) Interpret the decision problem.

b) Let the optimal s for a person be denoted s^*. Given (*), (**), and (***), it can be shown that s^* satisfies the first-order condition $h'(s^*)/h(s^*) = r+m-g$. Sketch how this first-order condition can be formally derived and provide the economic intuition behind it.

c) Solve for the optimal s. *Hint:* the second-order condition is that the elasticity of h' w.r.t. s is smaller than the elasticity of h w.r.t. s.

d) With one year as the time unit, let the parameter values be $\varphi = 0.6$, $r = 0.06$, $m = 0.01$, and $g = 0.015$. What is the value of the optimal s measured in years? Comment.

e) How does an increase in life expectancy affect m and the optimal s, respectively? What is the intuition?

There is perfect competition and the representative firm chooses capital input, K_t, and labor input (measured in man-years), L_t, in order to maximize profit, given the production function

$$Y_t = F(K_t, A_t h L_t),$$

where Y_t is output, A_t is the technology level, and F is a neoclassical production function with constant returns to scale. There is a constant capital depreciation rate $\delta > 0$. Suppose the country considered is fully integrated in the world market for goods and financial capital and that the real interest rate in this market is constant and equal to r for a long time.

f) Let the equilibrium real wage *per year* at time t for a typical member of the labor force be denoted w_t. Find w_t.

g) How is this real wage related to $\hat{w}_t$? What is its growth rate over time according to the information given in the introductory paragraph above? And what is the implied growth rate of A?

h) We now change this assumption about how A moves. We introduce the Bernard and Jones technological catching-up hypothesis, thus assuming that

$$\frac{\dot{A}_t}{A_t} = \xi \frac{\tilde{A}_t}{A_t},$$

where $\xi > 0$ and $\tilde{A}_t = \tilde{A}_0 e^{gt}$ is the world frontier technology level, $g > 0$.[4] We assume $A_0 < \tilde{A}_0$ and $0 < \xi < g$. Will the country's technology level be able to catch up in the long run? *Hint:* consider Hint 2 in Problem IV.1a).

i) Suppose the country considered is a developing country and that its catching-up ability is an increasing function of average human capital, i.e., $\xi = \xi(h)$, $\xi' > 0$. Can a general health improvement in the country help in catching up? Why or why not?

Appendix A: Solution formulas for linear differential equations of first order

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}.$$

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} \int_0^t b(s)e^{as} ds.$$

[4] Cf. Bernard and Jones, Technology and convergence, *Economic Journal*, vol. 106, 1996.

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} c \int_0^t e^{(a+h)s} ds = (x_0 - \frac{c}{a+h}) e^{-at} + \frac{c}{a+h} e^{ht}.$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-\int_0^t a(\tau)d\tau} + e^{-\int_0^t a(\tau)d\tau} \int_0^t b(s) e^{\int_0^s a(\tau)d\tau} ds.$$

Appendix B: A remark on the TTP concept in Bernard and Jones (1996)

In their paper Bernard and Jones introduce a concept they call Total Technological Productivity, $\text{TTP}_i(t)$, of a country i at time t. Applying the concept to the Cobb-Douglas case in Problem IV.1, the definition is $\text{TTP}_i(t) = A_i(t)^{1-\alpha_i} k^{\alpha_i}$, where k is the median (or average) capital-labor ratio in the initial year across the selection of countries considered. It is not obvious why this concept should be more appropriate than the simpler concept, TFP. On top of this comes that, as Exercise IV.1 shows, both comparative TFP growth and comparative TTP growth may misrepresent the economic meaning of technical progress when this is Harrod-neutral.

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