Economic Growth Exercises

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Problem Set III

III.1 Consider a closed economy with aggregate production function $Y = F(K, AL)$, where Y is GDP, K capital input, A the technology level, and L labor input (the dating of the variables is implicit). It is assumed that F is neoclassical with CRS and that A and L grow exogenously at the constant rates $g > 0$ and $n \geq 0$, respectively. Capital moves according to

$$\dot{K} = Y - C - \delta K, \qquad \delta \geq 0,$$

where C is aggregate consumption.

Suppose it is known that in the absence of shocks, $\tilde{k} \equiv K/(AL)$ converges towards a unique steady state value, $\tilde{k}^* > 0$, for $t \to \infty$. Further, it is known that in a small neighborhood of the steady state, the speed of convergence for $\tilde{k}$ is

$$\text{SOC}_{\tilde{k}} \equiv -\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \approx (1 - \varepsilon(\tilde{k}^*))(\delta + g + n) \equiv \beta(\tilde{k}^*), \qquad (*)$$

where $\varepsilon(\tilde{k})$ is the output elasticity w.r.t. capital, evaluated at the effective capital intensity $\tilde{k}$.

a) Show that (*) implies

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t}.$$

Hint: when a variable $x > 0$ has a constant growth rate, γ, its time path (in continuous time) is $x(t) = x(0)e^{\gamma t}$ (this follows from the hint to II.2b) in Problem Set II with $a = -\gamma$ and $b = 0$).

b) By a first-order Taylor approximation, show that $\log \tilde{k} \approx \log \tilde{k}^* + (\tilde{k} - \tilde{k}^*)/\tilde{k}^*$.

c) Find the approximate speed of convergence for $\log \tilde{k}$, $\text{SOC}_{\log \tilde{k}}$.

For a variable $x > 0$, let $g_x \equiv \dot{x}/x$.

d) Show that

$$g_{\tilde{k}} \approx -\beta(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*).$$

Hint: use your result in c).

Define $y \equiv Y/L \equiv \tilde{y}A$.

e) Show that $g_y = g + \varepsilon(\tilde{k})g_{\tilde{k}}$.

f) Show that

$$g_y \approx g - \beta(\tilde{k}^*)\varepsilon(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*). \qquad (**)$$

Define $y^*(t) \equiv f(\tilde{k}^*)A(t)$, where $f(\tilde{k}) \equiv F(\tilde{k}, 1)$.

g) Interpret $y^*(t)$. Draw the time profile of $\log y^*(t)$. In the same diagram, draw illustrating time profiles of $\log y(t)$ for the cases $\tilde{k}(0) < \tilde{k}^*$ and $\tilde{k}(0) > \tilde{k}^*$, respectively.

h) Show that

$$\log y - \log y^* \approx \varepsilon(\tilde{k}^*)(\log \tilde{k} - \log \tilde{k}^*). \qquad (***)$$

Hint: start from the left-hand side of (***), apply the principle in b), use that $\tilde{y}$ is a function of $\tilde{k}$, apply a first-order Taylor approximation on this function, and apply again the principle in b).

i) Based on (**) combined with (***), write down a Barro-style growth regression equation. *Hint:* a discrete-time approximation of g_y can be based on the principle in b).

III.2 This problem presupposes that you have already solved Problem III.1. The setup is exactly the same as in the introduction to Problem III.1.

a) "The SOC for $\log \tilde{k}$ must equal the SOC for $(\log \tilde{k} - \log \tilde{k}^*)$. True or false? Why? *Hint:* recall the general definition of SOC for a converging variable x.

b) Find the approximate speed of convergence for the vertical distance $\log y - \log y^*$ in your graph from g) of Problem III.1. *Hint:* use your result in a) together with (***) and the conclusion in c) of Problem III.1.

III.3 *The general version of Michael Kremer's subsistence economy model*[1] Consider a pre-industrial economy described by:

$$Y_t = A_t^{\sigma} L_t^{\alpha} Z^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, \tag{1}$$

$$\dot{A}_t = \lambda A_t^{\varepsilon} L_t, \qquad \lambda > 0, \varepsilon \leq 1, \tag{2}$$

$$L_t = \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, \tag{3}$$

where Y is aggregate output, A the level of technical knowledge, L the labor force (= population), and Z the amount of land (fixed). Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene.

a) Interpret the equations, including $\bar{y}$.

From now, let $Z = 1$.

b) Show that the dynamics of the model reduces to the "law of motion":

$$\dot{A} = \hat{\lambda} A^{\varepsilon + \frac{\sigma}{1-\alpha}}, \qquad \text{where } \hat{\lambda} \equiv \lambda \varphi^{\frac{1}{1-\alpha}}.$$

c) Define $\mu \equiv \varepsilon + \frac{\sigma}{1-\alpha}$ and solve the differential equation $\dot{A} = \hat{\lambda} A^{\mu}$. *Hint:* for $\mu = 1$, apply the hint in a) of Problem III.1; for $\mu \neq 1$, consider the implied differential equation for $x \equiv A^{1-\mu}$.

d) Show that "growth acceleration" ($\dot{A}/A$ rising over time) arises if and only if $\varepsilon > 1 - \frac{\sigma}{1-\alpha}$.

e) Show that the "growth acceleration" in this model takes a very dramatic form.

f) For fixed $\alpha = \bar{\alpha}$, illustrate in the (σ, ε) plane the region leading to "growth acceleration". Need ε be positive for this to happen?

g) If the parameters are such that the economy belongs to the region mentioned in f), we can conclude something about the sustainability of the Malthusian regime as described by the model. What can we conclude?

[1] This builds on Kremer, QJE 1993, §I-II. Acemoglu, 2009, p. 113-114, discusses two special cases of the model.

III.4 It is preferable to solve Problem III.3 before this problem. Consider a Solow-Malthus model of a subsistence economy:

$$\begin{aligned} Y_t &= K_t^{\varepsilon} L_t^{\alpha} Z^{1-\varepsilon-\alpha}, \qquad 0 < \varepsilon < \varepsilon + \alpha < 1, \\ \dot{K}_t &= sY_t, \qquad 0 < s < 1, \\ L_t &= \frac{Y_t}{\bar{y}} \equiv \varphi Y_t, \qquad \bar{y} > 0, \end{aligned}$$

where Y is aggregate output, K input of physical capital, L the labor force (= population), and Z the amount of land (fixed). For simplicity, we ignore capital depreciation. Time is continuous and it is understood that a kind of Malthusian population mechanism is operative behind the scene.

a) Interpret the equations, including $\bar{y}$. The exponents to K, L, and Z sum to one. What could be the argument for this?

From now, let $Z = 1$.

b) Derive the law of motion of the economy.

c) Does this model lead us to predict that the economy must sooner or later transcend the Mathusian subsistence regime? Why or why not? *Hint:* consider the behavior of $\dot{K}/K$ for $t \to \infty$.

d) Comment.

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