Economic Growth Exercises

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Problem Set II

II.1 *Short questions*

a) Write down a Cobb-Douglas production function that allows non-neutral technical change.

b) "If and only if the production function is Cobb-Douglas with CRS and time-independent output elasticity w.r.t. capital, does the Solow model predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

II.2 *Speed of convergence (rate of adjustment)* Consider a Solow model with Cobb-Douglas production function but without technical progress (for simplicity). In standard notation the fundamental differential equation is

$$\dot{k} = sk^{\alpha} - (\delta + n)k, \qquad 0 < s < 1, \quad 0 < \alpha < 1, \ \delta + n > 0. \tag{*}$$

Let $k(0) > 0$ be arbitrary. As an indicator for the speed of convergence of the economy, we will find out how fast the capital-output ratio, k/y, converges towards its steady-state value.

Since (*) is a so-called Bernouilli differential equation (the right-hand side is the sum of a power function term and a linear term), we can find an explicit solution of (*):

a) Let $x \equiv k/y = k^{1-\alpha}$ and derive the differential equation for x.

b) Write down the solution of this differential equation. *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$, has the solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}.$$

c) Usually, the speed of convergence (SOC) of a converging variable is defined as the (proportionate) rate of decline of the distance to the steady-state value:

$$\text{SOC} = -\frac{d(x(t) - x^*)/dt}{x(t) - x^*}.$$

Find SOC in the present case. Comment.[1]

d) Suppose there is Harrod-neutral technical progress at the constant rate $g > 0$. What must the SOC of the capital-output ratio then be? Why? *Hint:* a quick answer is possible.

II.3 Consider an economy with aggregate production function

$$Y(t) = K(t)^{\alpha}(A(t)L)^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{1}$$

where Y is GDP, K capital input, A the technology level, and L labor input. It is assumed that L is time-independent, whereas the technology level grows at the constant rate $g > 0$. Gross saving is a constant proportion of gross income. The increase in capital per time unit is given by

$$\dot{K}(t) = S(t) - \delta K(t) \equiv Y(t) - C(t) - \delta K(t), \qquad \delta \geq 0,$$

where C is aggregate consumption. Suppose gross saving is always positive.

a) By the growth accounting method, calculate the TFP growth rate under balanced growth. *Hint:* with the given information we know something about Y/K under balanced growth.

b) What would the growth rate of Y and K be under balanced growth if g had been zero?

c) "In spite of the result from the growth accounting in a), with the assumed $g > 0$, technical progress in a sense explains *all* GDP growth." Comment.

II.4 *Short questions* We assume that a given selection of countries (considered, for simplicity, as closed economies) can be described by the Ramsey model for a closed economy with Harrod-neutral technical progress at a constant positive rate. For each country parameters and initial conditions are such that an equilibrium exists (standard notation).

[1] It can be shown that SOC of k, when k is close to k^*, is approximately the same as SOC for k/y. Similarly for SOC of y.

a) "The model predicts that for countries with the same technology (same F, A_0, g and δ), differences in per capita growth rates are only temporary and due to the transitional dynamics." True or false? Comment.

b) "The model predicts that for countries with the same technology, differences in per capita income are only temporary and due to the transitional dynamics." True or false? Comment.

c) "The Ramsey model predicts that for countries with similar structural characteristics, the further away from its steady state a country is, the higher is its per capita growth rate." True or false? Comment.

II.5 *Short questions* Consider the Ramsey model for a market economy with perfect competition.

a) Write down the dynamic budget constraint and the NPG condition for the representative household expressed in absolute terms (not per capita terms).

b) Derive the corresponding dynamic budget constraint and NPG condition expressed in per capita terms.

c) "Only if the production function is Cobb-Douglas with CRS and time-independent output elasticity w.r.t. capital, does the Ramsey model predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

d) Are the predictions of the Ramsey model (with exogenous Harrod-neutral technical progress) consistent with Kaldor's "stylized facts"? Give a reason for your answer.

II.6 *Short questions*

a) Can a path *below* the saddle path in the $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

b) Can a path *above* the saddle path in the $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Ramsey model? Why or why not?

c) Answer questions b) and c) now presuming that we are dealing with the solution of the problem from the point of view of a social planner in the Ramsey model.

d) "If and only if the production function is Cobb-Douglas, does the Ramsey model predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

e) Are predictions based on the Ramsey model (with exogenous Harrod-neutral technical progress) consistent with Kaldor's stylized facts? Why or why not?

f) In what sense does the Ramsey model imply a more concise theory of the long-run rate of return than do, e.g., the Solow model and the Diamond OLG model?

g) Briefly, assess the theory of the long-run rate of return implied by the Ramsey model. That is, mention what you regard as strengths and weaknesses of the theory.

II.7 Consider a standard Solow model for a closed economy with perfect competition. The rate of Harrod-neutral technical progress is 1.8 percent per year, the rate of population growth is 0.5 percent per year, capital depreciates at the rate 0.6 per year, and in steady state the share of labor income in steady state is 2/3.

a) Find the approximate speed of convergence for $\tilde{k}$ and the approximate half-life for the distance to the steady-state value, $\tilde{k}^*$. *Hint:* given the production function on intensive form, $f(\tilde{k})$, in a neighborhood of the steady state, the speed of adjustment is approximately $(1 - \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)})(g + n + \delta)$.

b) Comment on the result you have got in relation to your knowledge of direct estimates of empirical adjustment speeds.

c) What is the doubling-time of income per capita implied by the model?

d) What is the long-run per capita growth rate implied by the model?

e) Suppose the economy is in steady state. Then, for some extraneous reason, the saving rate is increased to a new constant level. Illustrate graphically what happens. Comment on *what* happens to the growth rate of $y \equiv Y/L$ temporarily and in the long run – and *why* it happens.

II.8 *(a positive technology shock)* Consider a Ramsey model for a closed economy. The model can be reduced to two differential equations

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{*}$$

$$\dot{\tilde{c}}_t = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g)\tilde{c}_t, \tag{**}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = 0. \tag{***}$$

Notation is: $\tilde{k}_t = K_t/(T_t L_t)$ and $\tilde{c}_t = C_t/(T_t L_t) = c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, and L_t is population = labor supply, all at time t. Further, T_t is a measure of the technology level and f is a production function on intensive form, satisfying $f' > 0$, $f'' < 0$, and the Inada conditions. The remaining symbols stand for parameters and all these are positive. Moreover, $\rho - n > (1-\theta)g$.

a) Briefly interpret the equations (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive $\tilde{k}_0$. Can the divergent paths be ruled out? Why or why not?

c) Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then for some external reason an unanticipated technology shock occurs so that T_0 is replaced by $T_0' > T_0$. After this shock everybody rightly expects T to grow forever at the same rate as before. We now study short- and long-run effects of this shock.

d) Illustrate by means of the phase diagram what happens to $\tilde{k}$ and $\tilde{c}$ on impact, i.e., immediately after the shock, and in the long run.

e) What happens to the real interest rate on impact and in the long run?

f) Why is the sign of the impact effect on the real wage ambiguous (at the theoretical level) as long as f is not specified further?[2]

g) Compare the real wage in the long run to what it would have been without the shock.

[2] Remark: for "empirically realistic" production functions (having elasticity of factor substitution larger than elasticity of production wrt. capital), the impact effect *is* positive, however.

h) Suppose $\theta = 1$. Why is the sign of the impact effect on per capita consumption ambiguous? *Hint:* $c = (\rho - n)(k + h)$.

i) Compare per capita consumption in the long run to what it would have been without the shock.

II.9 *(aggregate saving and the return to saving)* Consider a Ramsey model for a closed competitive market economy with public consumption, transfers, and capital income taxation. The government budget is always balanced. The model leads to the following differential equations (standard notation)

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - \tilde{\gamma} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{*}$$

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[(1-\tau_r)(f'(\tilde{k}) - \delta) - \rho - \theta g\right]\tilde{c}, \tag{**}$$

and the condition

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left[(1-\tau_r)(f'(\tilde{k}_s)-\delta)-g-n\right]ds} = 0. \tag{***}$$

All parameters are positive and it is assumed that $\rho > n$ and

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \frac{\rho + \theta g}{1-\tau_r} > n + g > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta.$$

The government controls $\tilde{\gamma}$, $\tau_r \in (0,1)$, and the transfers. Until further notice $\tilde{\gamma}$ and τ_r are kept constant over time and the transfers are continuously adjusted so that the government budget remains balanced.

a) Briefly interpret (*), (**), and (***), including the parameters.

b) Draw a phase diagram and illustrate the path that the economy follows, for a given $\tilde{k}_0 > 0$. Comment.

c) Is it possible for a steady state to exist without assuming f satisfies the Inada conditions? Why or why not?

d) Suppose the economy has been in steady state until time t_0. Then, suddenly τ_r is increased to a higher constant level. Illustrate by a phase diagram what happens in the short and long run. Give an economic interpretation of your result.

e) Does the direction of movement of $\tilde{k}$ depend on θ? Comment.

f) Suppose $\theta = 1$. It is well-known that in this case the substitution effect and the income effect on current consumption of an increase in the (after-tax) rate of return offset each other. Can we from this conclude that aggregate saving does not change in response to the change in fiscal policy? Why or why not? Add some economic intuition. *Hint* regarding the latter*:* when $\theta = 1$, $c_t = (\rho - n)(a_t + h_t)$, where

$$h_t \equiv \int_t^\infty (w_s + x_s) e^{-\int_t^s [(1-\tau_r)r_\tau - n]d\tau} ds;$$

here, x_s is per capita transfers at time s. Four "effects" are in play, not only the substitution and income effects.

II.10 *(command optimum)* Consider a Ramsey setup with CRRA utility and exogenous technical progress at the constant rate $g \geq 0$. Suppose resource allocation is not governed by market mechanisms, but by a "social planner" – by which is meant an "all-knowing and all-powerful" central authority. The social planner is not constrained by other limitations than those from technology and initial resources and can thus ultimately decide on the resource allocation within these confines.

The decision problem of the social planner is (standard notation):

$$\max_{(c_t)_{t=0}^\infty} U_0 = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} \tag{1}$$

$$c_t \geq 0, \tag{2}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, \tag{3}$$

$$\tilde{k}_t \geq 0 \qquad \text{for all } t \geq 0, \tag{4}$$

where $\delta + g > 0$ and $\theta > 0$ (in case $\theta = 1$, the expression $(c^{1-\theta} - 1)/(1-\theta)$ should be interpreted as $\ln c$). Assume $\rho - n > (1-\theta)g$ and that the production function satisfies the Inada conditions.

a) Briefly interpret the problem, including the parameters. Comment on the inequality $\rho - n > (1-\theta)g$.

b) Derive a characterization of the solution to the problem.

c) Compare the solution with the equilibrium path generated by a market economy described by a Ramsey model with perfect competition and with the same preferences and technology as above. Comment.

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