Economic Growth Exercises

03.02.2011. Christian Groth

Problem Set I. A refresher on basic concepts. The Solow model in continuous time[1]

I.1 In the last four decades China has had very high growth in real GDP per capita, cf. Table 1. Answer questions a), b), and c) presupposing that the growth performances of China and the U.S. continue to be like what they have been 1980-2007.

a) How many years does it take for China's GDP per capita to be doubled? You should explain your method.

b) How many years does it take for GDP per capita in the U.S. to be doubled?

c) How long time, reckoned from 2007, will it take for China to catch up with the US? You should explain your method.

d) Do you find it likely that the actual course of events will be (approximately) like that? Why or why not?

Table 1. GDP per capita in USA and China 1980 - 2007 (I$ in 2005 Constant Prices)

country	year	rgdpch
United States	1980	24537.41
United States	2007	42886.92
China	1980	1133.21
China	2007	7868.28

Source: PWT 6.3. Note: For China the Version 2 data series is used.

[1] Given a production function $Y = F(K, AL)$, at least in this problem set we use the notation $y \equiv Y/L$, $k \equiv K/L$, $\tilde{y} \equiv Y/(AL)$, and $\tilde{k} \equiv K/(AL)$. Acemoglu uses y and k in the same way, but he defines $\hat{y} \equiv Y/(AL)$ and $k \equiv K/(AL)$.

The "level of technology" will sometimes be denoted A (as above), sometimes T, depending on convenience.

I.2 In a popular magazine on science the data in Table 2 was reported:

Table 2. World income per capita relative to income per capita in the US: 1952-96

Year	*Percent*
1952	13.0
1962	13.3
1972	13.0
1982	13.8
1992	15.1
1996	17.7

Source: Knowledge, Technology, & Policy 13, no. 4, 2001, p. 52.
Note. Countries' per capita income are weighted by population as a fraction of the world population.

a) Briefly, discuss this data relative to concepts of income convergence and divergence and relative to your knowledge of the importance of weighting by population size.

b) What is meant by the terms unconditional (or absolute) income convergence and conditional convergence?

c) Give a short list of mechanisms that could in principle explain the data above.

I.3 *Stocks versus flows.* We consider a closed economy. Initially, let time be discrete. Two basic elements in growth models are often presented in the following way. The aggregate production function is described by

$$Y_t = F(K_t, L_t, A_t), \tag{*}$$

where Y_t is output, K_t capital input, L_t labor input, and A_t the "level of technology". The time index t may refer to period t, that is the time interval $[t, t+1)$, or to a point in time, depending on the context. Accumulation of the stock of capital is described by

$$K_{t+1} - K_t = Y_t - C_t - \delta K_t, \tag{**}$$

where δ is an (exogenous) rate of (physical) depreciation of capital, $0 \leq \delta \leq 1$.

In continuous time models the corresponding equations are typically written

$$Y(t) = F(K(t), L(t), A(t)),$$

and

$$\dot{K}(t) \equiv \frac{dK(t)}{dt} = Y(t) - C(t) - \delta K(t), \qquad \delta \geq 0.$$

a) At the theoretical level, what denominations (dimensions) should be attached to output, capital input, and labor input in a production function?

b) What is the denomination attached to K in the accumulation equation?

c) Is there any consistency problem in the notation used in (*) vis-a-vis (**)? Explain.

d) Suggest an interpretation that ensures that there is no consistency problem.

e) Suppose there are two countries. They have the same technology, the same capital stock, and the same number of man-hours per worker per year. Country A does not use shift work, but country B uses shift work, two work teams per day. Adapt the formula (*) so that it can be applied to both countries.

f) Suppose F is a neoclassical production function with CRS wrt. K and L. Compare the output levels in the two countries. Comment.

g) In continuous time we write aggregate gross saving as $S(t) \equiv Y(t) - C(t)$. What is the denomination of $S(t)$.

h) In continuous time, does the expression $K(t) + S(t)$ make sense? Why or why not?

i) In discrete time, how can the expression $K_t + S_t$ be meaningfully interpreted?

I.4 *Short questions* (answering requires only a few well chosen sentences).

a) Consider an economy where all firms' technology is described by the same neoclassical production function, $Y_i = F(K_i, L_i)$, $i = 1, 2, ..., N$, with decreasing returns to scale everywhere (standard notation). Suppose there is "free entry and exit" and perfect competition in all markets. Then a paradoxical situation arises in that no equilibrium with a finite number of firms (plants) would exist. Explain.

b) In the Solow model as in many other macro models, the technology is assumed to have constant returns to scale (CRS) with respect to capital and labor taken together. Often the so-called *replication argument* is put forward as a reason to expect that CRS should hold in the real world. What is the replication argument?

Do you find an appeal to the replication argument to be a convincing argument for the assumption of CRS with respect to capital and labor? Why or why not?

c) Does the validity of the replication argument, considered as an argument about a property of technology, depend on the availability of the different inputs.

d) Suppose that for a certain historical period there has been something close to constant returns to scale and perfect competition, but then, after a shift to new technologies in the different industries, increasing returns to scale arise. What is likely to happen to the market form? Why?

I.5 The Solow growth model in continuous time can be set up in the following way. A closed economy is considered. There is an aggregate production function,

$$Y(t) = F(K(t), T(t)L(t)), \tag{1}$$

where F is a neoclassical production function with CRS, Y is output, K is capital input, T is the technology level, and L is labor input. It is assumed that

$$T(t) = T(0)e^{gt}, \qquad T(0) = T_0, \qquad g \geq 0, \tag{2}$$

$$L(t) = L(0)e^{nt}, \qquad L(0) = L_0, \qquad n \geq 0. \tag{3}$$

Aggregate gross saving, S, is assumed proportional to gross aggregate income which, in a closed economy, equals real GDP, Y:

$$S(t) = sY(t), \qquad 0 < s < 1. \tag{4}$$

Capital accumulation is described by

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t), \qquad 0 < \delta \leq 1, \tag{5}$$

where δ is the rate of (physical) depreciation of capital. Finally, from national income accounting,

$$S(t) \equiv Y(t) - C(t). \tag{6}$$

The symbols g, n, s, and δ represent parameters (given constants) and the initial values T_0, L_0, and K_0, are given (exogenous) positive numbers.

a) What kind of technical progress is assumed in the model?

b) Let $\tilde{y} \equiv y/L \equiv Y/(TL)$. Let the production function in intensive form be denoted f. Derive f from F. Sign f' and f''. From the given information, can we be sure that $f(0) = 0$? Why or why not?

c) To get a grasp of the evolution of the economy over time, derive a first-order differential equation in the (effective) capital intensity, $\tilde{k} \equiv k/T \equiv K/(TL)$, that is, an equation of the form $\dot{\tilde{k}} = \varphi(\tilde{k})$. *Hint:* $\dot{\tilde{k}}/\tilde{k} = \dot{k}/k - \dot{T}/T = \dot{K}/K - \dot{L}/L - g$.[2]

d) If in c) you were able to write $\varphi(\tilde{k})$ on the form $\varphi(\tilde{k}) = \psi(\tilde{k}_t) - a\tilde{k}_t$, where a is a constant, you are one the right track. Draw a "Solow diagram version 1", that is, a diagram displaying the graphs of the functions $\psi(\tilde{k}_t)$ and $a\tilde{k}_t$ in the $(\tilde{k}, \tilde{y})$ plane. You may, at least initially, draw the diagram such that the two graphs cross each other.

e) Suppose there exists a (non-trivial) steady state, $\tilde{k}^* > 0$. Indicate $\tilde{k}^*$ in the diagram. Can there be more than one (non-trivial) steady state? Why or why not?

f) In the same or a new diagram, draw a "Solow diagram version 2", that is, a diagram displaying the graphs of the functions $\psi(\tilde{k}_t)/s$ and $a\tilde{k}_t/s$ in the $(\tilde{k}, \tilde{y})$ plane. At what value of $\tilde{k}$ will these two graphs cross?

g) Suppose capital is *essential*, that is, $F(0, TL) = 0$ for all TL. In terms of limiting values of f' for the capital intensity approaching zero and infinity, respectively, write down a necessary and sufficient condition for a (non-trivial) steady state to exist.

h) Is the steady state, $\tilde{k}^*$, globally asymptotically stable?

i) Find the long-run growth rate of output per unit of labor.

I.6 This problem is about exactly the same model as Problem I.5, the standard version of the Solow model.

a) Suppose the economy is in steady state until time t_0. Then an upward shift in the saving rate occurs. Illustrate by the Solow diagram the evolution of the economy from t_0 onward.

[2] Recall the following simple continuous time rule: Let $z = y/x$, where z, y, and x are differentiable functions of time t and positive for all t. Then $\dot{z}/z = \dot{y}/y - \dot{x}/x$, exactly. *Proof:* We have $\log z = \log y - \log x$. Now take the time derivative on both sides of the equation.

b) Draw the time profile of $\ln y$ in the $(t, \ln y)$ plane.

c) How, if at all, is the level of y affected by the shift in s?

d) How, if at all, is the growth rate of y affected by the shift in s? Here you may have to distinguish between transitory and permanent effects.

e) Explain by words the economic mechanisms behind your results in j) and k).

f) As Solow once said (in a private correspondence with Amartya Sen[3]): "The idea [of the model] is to trace full employment paths, no more." What market form is theoretically capable of generating permanent full employment?

g) Even if we recognize that the Solow model only attempts to trace hypothetical time paths with full employment (or rather employment corresponding to the "natural" or "structural" rate of unemployment), the model has several important limitations. What is in your opinion the most important limitations?

I.7 Set up a Solow model where, although there is no technical progress, sustained per capita growth occurs. Comment. *Hint:* a simple approach can be based on the production function $Y = BK^{\alpha}L^{1-\alpha} + AK$, where $A > 0, B > 0, 0 < \alpha < 1$. "Sustained per capita growth" is said to occur if $\lim_{t\to\infty} \dot{y}/y > 0$ or $\lim_{t\to\infty} \dot{c}/c > 0$ (standard notation).

I.8 Consider a closed economy with technology described by the aggregate production function

$$Y = F(K, L),$$

where F is a neoclassical production function with CRS and satisfying the Inada conditions, Y is output, K is capital input and L is labor input = labor force = population (there is always full employment). A constant fraction, s, of *net* income is saved (and invested). Capital depreciates at the constant rate $\delta > 0$.

a) Assuming a constant population growth rate n, derive the fundamental differential equation of the model and illustrate the dynamics by a phase diagram. Comment.

[3] *Growth Economics. Selected Readings*, edited by Amartya Sen, Penguin Books, Middlesex, 1970, p. 24.

b) Assume instead that the population growth rate n is a smooth function of per capita income, i.e., $n = n(y)$, where $y \equiv Y/L$. At very low levels of per capita income, n is zero, at higher per capita income, n is a hump-shaped function of y, and at very high levels of y, n tends to zero, that is, for some $\bar{y} > 0$ we have

$$n'(y) \gtreqless 0, \text{ for } y \lesseqgtr \bar{y}, \text{ respectively,}$$

whereas $n(y) \approx 0$ for y considerably above $\bar{y}$. Show that this may give rise to a dynamics quite different from that of the Solow model. Comment.

I.9 "The Cobb-Douglas production function has the property that, under technical progress, it satisfies all three neutrality criteria if it satisfies one of them." True or false? Explain why.

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