Written Exam for the M.Sc. in Economics 2009-II

Economic Growth: Solutions

Master's Course

June 12, 2009

(4-hour closed book exam)

Problem 1

Consider a closed market economy with N profit maximizing firms, operating under perfect competition (N "large"). There is a representative household (family dynasty) with L members at time t. Assume $L = L_0 e^{nt}$, where n is constant, $n \geq 0$. Each household member supplies one unit of labour per time unit. Aggregate output is Y per time unit, and output is used for consumption, $C \equiv cL$, and investment in physical capital K, i.e., $Y = C + \dot{K} + \delta K$, where $\delta \geq 0$ is the rate of physical decay of capital. Variables are dated implicitly. The initial value $K_0 > 0$ is given. There is a perfect market for loans at the real rate of interest r. There is perfect foresight.

The production function for firm i ($i = 1, 2, ..., N$) is

$$Y_i = F(K_i, TL_i), \tag{1}$$

where F is neoclassical and has CRS. The variable T evolves according to

$$T = T_t = e^{xt} K_t^{\lambda}, \quad x \geq 0, 0 < \lambda \leq 1, \tag{2}$$

where x and λ are constants and $K_t = \sum_i K_{it}$. Each firm is small and takes K_t as not affected by its own behavior.

a. Briefly interpret (1) and (2).

b. In general equilibrium, determine r and the aggregate production function at time t.

c. Assume $x > 0$ and $\lambda < 1$. Determine the rate of growth of Y and $y \equiv Y/L$ under balanced growth. *Hint:* use the proposition about equivalence of balanced growth and constancy of certain key ratios.

d. Comment on the model in relation to different types of endogenous growth.

From now on, let $\lambda = 1$ and $x = n = 0$.

e. Assume that the representative household has infinite horizon, an instantaneous utility function with absolute elasticity of marginal utility equal to a constant $\theta > 0$ and a constant rate of time preference w.r.t. utility, $\rho > 0$. Let $F_1(1, L) > \delta + \rho$. Determine the equilibrium rate of growth of c, k ($\equiv K/L$) and y, respectively. In case you need to introduce a restriction on some parameters, do it.

f. Now, introduce a government that pursues two activities:(i) it pays a subsidy, s, to the firms so that their capital costs reduce to

$$(1 - s)(r + \delta)$$

per unit of capital per time unit; (ii) it finances this subsidy by a constant consumption tax τ. The government budget is always balanced. In particular, the subsidy is financed by a consumption tax and no other expenditures take place, that is

$$\tau c L = s(r + \delta)K$$

g. Could there be good economic reasons for such a subsidy? Comment.

h. Provide an analysis of whether there is a level of the subsidy rate such that the social planner's allocation can in principle be implemented.

Problem 2

Consider the following growth model for a closed economy with a government sector. Firm i employs the following technology:

$$Y_{it} = AK_{it}^{\alpha} L_{it}^{1-\alpha} \hat{G}_t^{\pi}, \quad 0 < \alpha < 1, \quad \pi > 0 \tag{1}$$

where A is a constant, K_{it} denotes the firm-specific capital stock, L_{it} total labor input in firm i, while

$$\hat{G}_t = \frac{G_t}{K_t^{\phi} L_t^{\delta}}, \quad 0 < \phi < \alpha < 1, \quad 0 < \delta < 1 \tag{2}$$

where G_t represents government investments in infrastructure, while K_t and L_t are the aggregate stock of private capital and the total labor force in the economy, respectively. Let r_t denote the real interest rate, and w_t the real wage. All markets are competitive, and the price of output is normalized to 1. For simplicity it is assumed that capital does not depreciate. G_t is financed by a wealth tax, levied on the households. The government balances the budget at all points in time, i.e. $G_t = \tau K_t$, where τ is the (time constant) wealth tax rate. Finally, the total size of the labor force is constant at all points in time, hence $L_t = L$.

a. Provide an interpretation of equation (2).

b. Solve the profit maximization problem for firm i, and proceed to show that the aggregate production function can be written as

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} \hat{G}_t^{\pi}$$

c. What would π need to fulfill so that the model can exhibit fully endogenous growth? Assume the restriction just derived holds. The representative agent maximizes discounted utility from consumption. More specifically, the problem is

$$\begin{aligned}
& \max_{\{c_t\}_{t=0}^{\infty}} \int_0^{\infty} \frac{c_t^{1-\theta}-1}{1-\theta} e^{-\rho t} dt, \quad \theta > 0, \quad \rho > 0 \\
& \text{s.t.} \\
c_t & \geq 0 \\
\dot{k}_t & = (r-\tau) k_t + w_t - c_t, \quad k_0 \text{ given} \\
\lim_{t\to\infty} k_t e^{-\int_{s=0}^{t} r_s ds} & \geq 0
\end{aligned}$$

where the wealth of the representative agent equals the capital/labor ratio, $k_t \equiv K_t/L$.

d. (i) Solve the consumer's problem and derive the growth rate of GDP per capita. (ii) Explain why the tax rate, τ, is related to the growth rate in the manner suggested by the formula.

e. The growth rate depends on the size of the labor force: is it possible to impose a restriction on certain parameters so as to eliminate scale effects in the present model, while preserving endogenous growth?

Problem 3

a. *In brief, virtually all the R&D-based models in the literature share a prediction of "scale effects": if the level of resources devoted to R&D- measured, say, by the number of scientists engaged in R&D- is doubled, then the per capita growth rate of output should also double, at least in the steady state. Empirically, of course, such a prediction receives little support.* Discuss this statement from Jones (Journal of Political Economy, 1995), making clear whether he refers to strong or weak scale effects, and suggest possible mechanisms to overcome the rise of these counterfactual effects in R&D-based models of economic growth.

b. Comment on the models of endogenous growth proposed by Arrow (Review of Economic Studies, 1962) and Romer (Journal of Political Economy, 1986) in relation to the concepts of fully endogenous growth and semi-endogenous growth. In addition, discuss in which case taxes and subsidies may have long-run growth effects, as compared to the case in which they can only exert level effects.

Solution to Problem 1

a. $T_t = e^{xt} K_t^{\lambda}$ characterizes the level of technology. The production technology displays learning-by-investing, which in turn reflects quick knowledge spillovers channeled by the factor K_t^{λ}, where λ indexes the speed of these spillovers. Moreover, T_t partly captures technical progress at an exogenously postulated rate x. Exogenous technical progress can be interpreted as having a residual impact which cannot be otherwise explained by endogenous effects.

b. From now on we suppress the time index when not needed for clarity. Under perfect competition the maximization of firm-specific profits, $\Pi_i = F(K_i, TL_i) - (r+\delta)K_i - wL_i$, leads to the following first-order conditions

$$\begin{aligned} \partial\Pi_i/\partial K_i &= F_1(K_i, TL_i) - (r+\delta) = 0, \\ \partial\Pi_i/\partial L_i &= F_2(K_i, TL_i)T - w = 0. \end{aligned} \tag{3}$$

Behind (3) is the presumption that each firm is small relative to the economy as a whole (N "large"), so that each firm's investment has a negligible effect on the aggregate capital stock. Since F is homogeneous of degree one, by Euler's theorem F_1 is homogeneous of degree zero. Thus, we can write (3) as

$$F_1(k_i, T) = r + \delta, \tag{4}$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (4) determines k_i uniquely.

From (4) follows that the chosen k_i will be the same for all firms. In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labour, respectively (both pre-determined). It follows that the chosen capital intensity, k_i, satisfies

$$k_i = k \equiv \frac{K}{L}, \qquad i = 1, 2, ..., N. \tag{5}$$

As a consequence we can interpret (4) as *determining* the equilibrium interest rate:

$$r = F_1(k, T) - \delta. \tag{6}$$

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, T)L_i = \sum_i F(k, T)L_i \quad \text{(by (1) and (5))} \\ &= F(k, T)\sum_i L_i = F(k, T)L = F(K, TL), \end{aligned} \tag{7}$$

where we have used the fact that F is homogeneous of degree one.

c. Assume that $x > 0$ and $\lambda < 1$.

From the assumption of balanced growth:

$$g_Y = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{T}}{T} = x + \lambda\frac{\dot{K}}{K} + \frac{\dot{L}}{L} = \lambda\frac{\dot{K}}{K} + n + x.$$

Therefore, $g_Y = g_K = \frac{x+n}{\lambda-1}$. As to the per capita growth rate, g_y:

$$y = \frac{Y}{L} \Longrightarrow g_y = g_Y - n = \frac{x+n}{\lambda-1} - n = \frac{x+\lambda n}{\lambda-1}.$$

d. Growth has an endogenous component $\left(\frac{\lambda n}{\lambda-1}\right)$. Since either $x > 0$ or $n > 0$ are required to induce growth, semi-endogenous growth is generated.

From now on, let $\lambda = 1$ and $x = n = 0$.

e. The household sector is described by the standard Ramsey framework with a finite number of infinitely-lived dynasties. These are all alike and have CRRA instantaneous utility with parameter $\theta > 0$. This leads to the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(\underbrace{F_1(1,L) - \delta}_{\bar{r}} - \rho) \equiv \gamma, \tag{8}$$

which is also constant "from the beginning". To ensure positive growth we assume

$$F_1(1,L) - \delta > \rho \tag{A2'}$$

And to ensure bounded intertemporal utility it is assumed that

$$\rho > (1-\theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A1"}$$

Solving the linear differential equation (8) gives

$$c_t = c_0 e^{\gamma t}, \tag{9}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by using the transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

Notice that the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1,L)K - cL - \delta K,$$

or, in per-capita terms:

$$\dot{k} = [F(1,L) - \delta]\, k - c_0 e^{\gamma t}. \tag{10}$$

Provided that (A1") holds, $F(1,L) - \delta - \gamma > F(1,L) - \delta - \bar{r} = F(1,L) - F_1(1,L) = F_2(1,L)L > 0$: the first equality is due to $\bar{r} = F_1(1,L) - \delta$, while the second one comes from the fact that since F is

homogeneous of degree 1, Euler's theorem implies $F(1,L) = F_1(1,L) \cdot 1 + F_2(1,L)L > F_1(1,L) > \delta$, according to (A2'). As indicated in the appendix, the solution of a general linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $a \neq 0$ and $h \neq -a$, is

$$x(t) = \left(x(0) - \frac{c}{a+h}\right) e^{-at} + \frac{c}{a+h} e^{ht}. \tag{11}$$

Thus the solution to (10) is

$$k_t = \left(k_0 - \frac{c_0}{F(1,L) - \delta - \gamma}\right) e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1,L) - \delta - \gamma} e^{\gamma t}. \tag{12}$$

To check the transversality condition we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= \left(k_0 - \frac{c_0}{F(1,L) - \delta - \gamma}\right) e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L) - \delta - \gamma} e^{(\gamma-\bar{r})t} \\ &\to \left(k_0 - \frac{c_0}{F(1,L) - \delta - \gamma}\right) e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \to \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A1"). But $\bar{r} = F_1(1,L) - \delta < F(1,L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma)k_0. \tag{13}$$

If c_0 is less than this, there will be over-saving and the TVC is violated. If c_0 is higher than this, both the TVC and the NPG are violated. Inserting the solution for c_0 into (12), we get

$$k_t = \frac{c_0}{F(1,L) - \delta - \gamma} e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional
dynamics)

f. The answer is yes, as there is a positive externality that can in principle be internalized by a suitable policy strategy.

h. The social planner faces the aggregate production function $Y_t = F(1,L)K_t$ or, in per capita terms, $y_t = F(1,L)k_t$. The social planner's problem is to choose $(c_t)_{=0}^{\infty}$ so as to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &> 0, \\ \dot{k}_t &= F(1,L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, & (14) \\ k_t &\geq 0 \text{ for all } t > 0. & (15) \end{aligned}$$

The current-value Hamiltonian is

$$H(k, c, \eta, t) = \frac{c^{1-\theta} - 1}{1-\theta} + \eta \left[F(1,L)k - c - \delta k\right],$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labour. The necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \tag{16}$$

$$\frac{\partial H}{\partial k} = \eta(F(1,L) - \delta) = -\dot{\eta} + \rho\eta. \tag{17}$$

While the transversality condition reads as

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \tag{18}$$

which must be satisfied by an interior optimal solution. This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really a solution. Log-differentiating w.r.t. t in (16) and combining with (17) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1,L) - \delta - \rho) \equiv \gamma_{SP}. \tag{19}$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal product of capital is $\partial y_t/\partial k_t = F(1,L) > F_1(1,L)$.

Implementation of the social planner's solution in the market economy

Returning to the market economy, we assume there is a government with only two activities: (i) it pays a subsidy, s, to the firms so that their capital costs are reduced to

$$(1-s)(r+\delta)$$

per unit of capital per time unit; (ii) it finances this subsidy by a constant consumption tax τ. The government budget is always balanced. Let us first find the size of s needed to establish the SP

allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-s)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-s)(r+\delta) \quad \forall\ i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-s)(r+\delta),$$

where $k \equiv K/L$, which is pre-determined. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k, K)}{1-s} - \delta = \frac{F_1(1, L)}{1-s} - \delta, \tag{20}$$

where we have appealed once again to the Euler's theorem. It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the

production technology, namely $\partial Y/\partial K - \delta = F(1, L) - \delta$. If we can obtain $r = F(1, L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied

by the technology. The required s thus satisfies

$$r = \frac{F_1(1, L)}{1-s} - \delta = F(1, L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1, L)}{F(1, L)}.$$

It remains to find the required consumption tax rate τ. The tax revenue will be $\tau c L$, and the *required* tax revenue is

$$T = s(r+\delta)K = [F(1, L) - F_1(1, L)]\, K.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{T}{cL} = \frac{F(1, L) - F_1(1, L)}{c/k} = \frac{F(1, L) - F_1(1, L)}{F(1, L) - \delta - \gamma_{SP}}, \tag{21}$$

where we have used that the proportionality between c and k holds for all $t \geq 0$. Substituting (19) into (21), the solution for τ can be written as

$$\tau = \frac{\theta\, [F(1, L) - F_1(1, L)]}{(\theta - 1)(F(1, L) - \delta) + \rho}.$$

Therefore, the required tax rate on consumption is a constant. As such, it does not distort the consumption/saving decision at the margin. In addition, as the tax has no other uses than financing the

subsidy, and households are the ultimate owners of the firms, the tax is eventually "paid back" to the households. A policy (s, τ) which in a decentralized system induces the SP allocation is called a

first-best policy.

Solution to Problem 2

a. The formulation captures public goods that are subject to congestion (such as queues on the highway, overloaded phone networks etc). Hence, in order to obtain increasing productivity, G will need to rise relative to the "demand for use", which is assumed to be proportional to K and L.

b. Each firm faces the following maximization problem

$$\{K_i, L_i\} = \arg\max \Pi_i = AK_i^{\alpha} L_i^{1-\alpha} \hat{G}_t^{\pi} - rK_i - wL_i$$

which yields the following first-order conditions (where firms take $\hat{G}_t^{\pi}$ as given, given that they are "small" relative to the size of the model economy under scrutiny):

$$\begin{aligned} r &= \alpha \frac{Y_i}{K_i} = \alpha A k_i^{\alpha-1} \hat{G}_t^{\pi} \\ w &= (1-\alpha) \frac{Y_i}{L_i} = (1-\alpha) A k_i^{\alpha} \hat{G}_t^{\pi} \end{aligned}$$

We can combine these expressions to obtain

$$\frac{r}{w} = \frac{\alpha A k_i^{\alpha-1} \hat{G}_t^{\pi}}{(1-\alpha) A k_i^{\alpha} \hat{G}_t^{\pi}} \Rightarrow k_i = \frac{\alpha w}{(1-\alpha) r} = k.$$

Therefore, we can take the following steps (by appealing to the same conditions as in Problem 1.b):

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i AK_i^{\alpha} L_i^{1-\alpha} \hat{G}_t^{\pi} = A\hat{G}_t^{\pi} \sum_i k_i^{\alpha} L_i \\ &= A\hat{G}_t^{\pi} k^{\alpha} \sum_i L_i = A\hat{G}_t^{\pi} k^{\alpha} L = A\hat{G}_t^{\pi} K^{\alpha} L^{1-\alpha}, \end{aligned}$$

and get the aggregate production function.

c. To generate endogenous growth we need constant returns to the reproducible factor of production, i.e. capital. Since the balanced budget implies $\tau K_t = G_t$ (this can be deduced from the budget constraint of the representative household along with the fact that the size of the labor force — and population (competitive markets) — is L), we may write the aggregate production function:

$$Y = A\left(\frac{\tau K_t}{K_t^{\phi} L_t^{\delta}}\right)^{\pi} K^{\alpha} L^{1-\alpha} = A\tau^{\pi} K^{(1-\phi)\pi+\alpha} L^{1-\alpha-\pi\delta}.$$

Therefore, the following restriction is required:

$$(1-\phi)\pi + \alpha = 1$$

or, equivalently:

$$\pi = \frac{1-\alpha}{1-\phi}.$$

Imposing this restriction leads to the following AK-type production function:

$$Y = A\tau^{\frac{1-\alpha}{1-\phi}} L^{\frac{(1-\phi-\delta)(1-\alpha)}{1-\phi}} K = \widetilde{A}(\tau) K.$$

d. (i) The Hamiltonian for our maximization problem reads as:

$$H(k, c, \eta, t) = \frac{c^{1-\theta} - 1}{1 - \theta} + \lambda \left[(r - \tau) k + w - c \right]$$

where $\lambda = \lambda_t$ is the adjoint variable associated with the state variable, which is capital per unit of labour. Necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \lambda = 0, \text{ i.e., } c^{-\theta} = \lambda, \tag{22}$$

$$\frac{\partial H}{\partial k} = \lambda(r - \tau) = -\dot{\lambda} + \rho\lambda. \tag{23}$$

We guess that also the transversality condition,

$$\lim_{t \to \infty} k_t \lambda_t e^{-\rho t} = 0. \tag{24}$$

Log-differentiating w.r.t. t in (22) and combining with (23) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r - \tau - \rho) \equiv \gamma. \tag{25}$$

Now, since the model is of the AK-variety it follows that the model exhibits balanced growth. Accordingly, if $\gamma > 0$ (which will be adhered to), then all endogenous variables grow at the same rate γ. Therefore, to obtain the growth rate of GDP per capita we need to substitute for the equilibrium real rate of return:

$$r = \alpha \frac{Y_i}{K_i} = \alpha \widetilde{A}(\tau).$$

Hence, the growth rate is:

$$\gamma = \frac{1}{\theta} \left(\alpha \widetilde{A}(\tau) - \tau - \rho \right) = \frac{1}{\theta} \left(\alpha A \tau^{\frac{1-\alpha}{1-\phi}} L^{\frac{(1-\phi-\delta)(1-\alpha)}{1-\phi}} - \tau - \rho \right).$$

(ii) Since it is assumed that $\phi < \alpha$, it follows that $\pi < 1$. Therefore, the relationship between γ and τ is hump-shaped. Specifically, growth is maximizaed at the point

$$\tau^* = \arg\max \gamma$$

The first order condition for this maximization problem is

$$\alpha \frac{1-\alpha}{1-\phi} A \tau^{\frac{1-\alpha}{1-\phi} - 1} L^{\frac{(1-\phi-\delta)(1-\alpha)}{1-\phi}} - 1 = 0.$$

Thus (assuming that the second order condition is satisfied):

$$\tau^* = \left(\alpha \frac{1-\alpha}{1-\phi} A L^{\frac{(1-\phi-\delta)(1-\alpha)}{1-\phi}} \right)^{\frac{1}{1 - \frac{1-\alpha}{1-\phi}}}.$$

The reason for the hump-shaped association is simple. On the one hand, when taxes are raised this allows for more productive government investments, which increase the marginal productivity of capital, and in turn growth. On the other hand, increasing taxes will exert a distortive effect on the desire to save. This has a negative impact on growth. If $\tau < \tau^*$ the former effect dominates, whereas the latter dominates when $\tau > \tau^*$.

e. In general, the scale impact is ambiguous, since it depends on the relative size of ϕ and δ. In order to eliminate scale effects, we need to either modify households' saving behavior, or alternatively ensure that $\frac{\partial MP_k}{\partial L} = 0$. The former solution is not feasible under the present microfoundations for consumers' behavior (as perfect altruism is reflected). But the latter holds (in the aggregate, but not for individual firms) if:

$$1 - \phi - \delta = 0 \Leftrightarrow \delta = 1 - \phi.$$

Under this restriction, congestion from the size of population exactly works to offset the otherwise present tendency for MP_k to rise along with the labor force.

Solution to Problem 3

a. Jones (1995) considers the scale effects prediction in the Romer/Grossman-Helpman/Aghion-Howitt models, which can be summarized by the following equations:

$$Y = K^{1-\alpha} (AL_y)^{\alpha} \tag{26}$$

$$\frac{\dot{A}}{A} = \delta L_A \tag{27}$$

where Y is output, A is knowledge, and K is capital. Labour is either used to produce output (L_y) or to search for new knowledge (L_A). From equation (27) it is clear that, as labour impacts on the knowledge rate of growth, Jones (1995) considers strong scale effects. Scale effects are a counterfactual implication of this class of models: despite the size of labour force has grown dramatically over the last 25-100 years, the average growth rate has been relative constant. A possible solution to this incongruence comes from an alternative way to introduce R&D in this class of models:

$$\frac{\dot{A}}{A} = \delta \frac{L_A}{L}$$

It turns out that this modeling device is not consistent with the microfoundations for R&D models developed by Romer et al. Also, it imposes that an economy with one unit of labour can produce as much TFP (total factor productivity)-growth as 1 million units of labour, which is not supported by the empirical literature.

Jones finally proposes a semi-endogenous version of the model which is more consistent with empirical findings. The model, unlike the AK-style and the R&D based models mentioned above, predicts that the growth rate is determined by parameters that are typically invariant to policy manipulations. On the one hand, as in the Solow-model, subsidies to R&D and to capital-accumulation have no long-run effects, but only affect the transition path. On the other hand, unlike the Solow-model, this model is endogenous in the sense that it derives from the pursuit of new technologies by rational, profit-maximizing agents.

b. Let $y \equiv Y/L$ and $\gamma_y \equiv \dot{y}/y$. We first report some definitions:

- *Endogenous growth* is present if there is a positive long-run per capita growth rate (i.e., $\gamma_y > 0$) and the source of this is some internal mechanism in the model (in contrast to exogenous technology growth).

- *Fully endogenous growth* (sometimes called *strictly endogenous growth*) is present if there is a positive long-run per capita growth rate and this occurs without the support of growth in some exogenous factor (for example exogenous growth in the labour force).

- *Semi-endogenous growth* is present if growth is endogenous but exponential growth can not be sustained without the support by growth in some exogenous factor (for example exogenous growth in the labour force).

The Arrow model of learning by investing features semi-endogenous growth. The technical reason for this is the assumption that the learning parameter $\lambda < 1$, which implies diminishing returns to capital at the aggregate level. Over the balanced growth path we have:

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \lambda \frac{\dot{K}}{K} + n,$$

so that

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda},$$

and thereby

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - n = \frac{\lambda n}{1-\lambda},$$

where n is the (constant) rate of growth in the labour force. Iff $n > 0$ we do have $\dot{y}/y > 0$ in the long run. The key role of population growth derives from the fact that although there are diminishing returns to capital at the aggregate level, there are increasing returns to scale with respect to capital and labour. For increasing returns to be sufficiently exploited so as to generate endogenous growth, population growth is needed. Note also that in this case $\partial\gamma_y/\partial\rho = 0 = \partial\gamma_y/\partial\theta$, which implies that preference parameters do not matter for long-run growth (only for the level of the growth path). This suggests that taxes and subsidies do not have long-run growth effects. Yet, in Arrow's model and similar semi-endogenous growth models scale as well as policies have long-run level effects.

Let us now consider the limiting case $\lambda = 1$. To many researchers this would look like an unrealistically high value of the learning parameter (see, e.g., Solow, 1997). To avoid a forever rising growth rate we have to add the restriction $n = 0$. These restrictions lead us to the case considered by Romer (1986). In turns out that this model generates fully endogenous growth, as positive long-run per capita growth occurs without the support of growth in some exogenous factor (recall that $n = 0$). In the fully endogenous growth case $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$, which means that preference parameters matter for long-run growth (and thus not only for the level of the growth path). This suggests that taxes and subsidies can have long-run growth effects. In any case, in this model there is an incentive for government intervention due to possibility to internalize the positive externality of private investment.