Economic Growth. Lecture Note 10.

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Productive government services

Below we give a detailed account of the analytical derivations in the two models with productive public services, discussed in B & S, § 4.4. In the first model, which is sometimes called the Barro growth model (after Barro, 1990), the productive public service is thought of as completely *non-rival.* An economic good is called nonrival if its use by one agent does not prevent or limit its use by other agents. All members of society can receive the benefit of the same good. In the second model the service is *partially rival* in the sense that *congestion* problems may arise.

1 The pure public goods model (B & S, § 4.4.1)

In general it is most natural to think of productive government services as coming from a *stock* of public capital (infrastructure, general technical knowledge etc.). Then the associated government spending does not directly deliver the service, but represents gross investment building up and maintaining the public capital stock. But in Barro's model these things are modelled in a simplified way. There is no stock of public capital. The productive public service is a *flow* variable coming directly from current government spending. We may think of a gratis technical information service on TV or the internet. Another example would be software that is downloadable from the internet and provided by the government free of charge. Face-to-face teaching in schools would not be a good example because that form of teaching is to a large extent rival (there are limits on the size of the audience in a lecture room). But TV- or internet-transmitted lectures would be an example.

Strictly speaking such government services are not pure public goods, although the headline of this section suggests otherwise (as does the corresponding headline in B & S, p. 220). Recall that a pure public good is defined as a good that is not only non-rival but also non-excludable. A good is *non-excludable* if the supplier or the owner of the good cannot, technologically or legally, charge a fee for its use. But it is possible to demand payment for TV- or internet-transmitted services. National defence, rule by law, and basic science are examples of pure public goods.

Whatever the degree of excludability or lack of excludability, what is important for the growth model to be considered is the non-rival character of the productive public service and the fact that it is provided free of charge.

1.1 The firms

In contrast to the learning-by-investing model of B & S, Section 4.3, here a Cobb-Douglas specification of the production function is introduced from the beginning. We consider a closed economy with a large number of firms, $i = 1, 2, ..., N$. There is perfect competition in output and labor markets. Firm i produces according to

$$Y_{it} = AK_{it}^{\alpha}(G_t L_{it})^{1-\alpha}, \qquad A > 0,\ 0 < \alpha < 1, \tag{1}$$

where G_t is the amount per time unit of the productive public service. GDP or aggregate output is $Y_t = \sum_i Y_{it}$. From national income accounting we have

$$Y_t = c_t L + G_t + I_t,$$

where I_t is gross investment and L is the population size which equals the size of the labor force and is assumed constant over time (in order to avoid a forever rising growth rate, cf. the strong scale effect discussed below). In (1) we see that the *total* amount of the public service is available to *each* firm (or each worker). Therefore, for a given G_t, the total benefit from the service is proportional to the number of users. This is a manifestation of the nonrival character of the service.[1]

From now, when not needed for clarity, the timing of the variables is suppressed for notational convenience. Assume there are no taxes on firms. Letting $\hat{r}$ denote capital costs (i.e., $\hat{r} = r + \delta$), the decision problem of firm i is:

$$\max_{K_i, L_i} \Pi_i = AK_i^{\alpha}(GL_i)^{1-\alpha} - \hat{r}K_i - wL_i.$$

FOCs:

$$\partial \Pi_i / \partial K_i = \alpha A K_i^{\alpha-1}(GL_i)^{1-\alpha} - \hat{r} = 0, \tag{FOC1}$$

$$\partial \Pi_i / \partial L_i = (1-\alpha) A K_i^{\alpha}(GL_i)^{-\alpha} G - w = 0. \tag{FOC2}$$

[1] If the public services were rival, we should imagine that each worker had access to $g_t \equiv G_t/L$ units per time unit of the service so that $Y_{it} = AK_{it}^{\alpha}(g_t L_{it})^{1-\alpha}$.

From (FOC1) we find

$$k_i \equiv K_i/L_i = (\alpha A/\hat{r})^{1/(1-\alpha)}G. \tag{2}$$

We see that all firms choose the same capital intensity (this is what makes aggregation easy).

1.2 The government

We assume that the government chooses G such that G/Y is constant over time:

$$G = \bar{g}Y, \qquad 0 < \bar{g} \leq 1 - \alpha. \tag{3}$$

The upper bound on $\bar{g}$ is imposed to ensure existence of general equilibrium. The public expenditure is financed by a *lump-sum* tax, τ, such that

$$\tau L = G. \tag{4}$$

There are no other government expenditures than G.

1.3 General equilibrium and the implied aggregate production function

Equilibrium in the factor markets implies

$$\sum_i K_i = K, \quad \text{and} \tag{5}$$

$$\sum_i L_i = L, \tag{6}$$

where K and L are the supplies of capital and labor at time t. Since all firms choose the same capital intensity, the chosen capital intensity must in general equilibrium equal the pre-determined $k \equiv K/L$ from the supply side. Since $y_i \equiv Y_i/L_i = Ak_i^\alpha G^{1-\alpha} = Ak^\alpha G^{1-\alpha} \equiv y$, aggregate output can be written

$$Y = \sum_i Y_i = \sum_i y_i L_i = y \sum_i L_i = yL = Ak^\alpha G^{1-\alpha} L. \tag{7}$$

Now we can express G and Y in terms of pre-determined variables and parameters only. Indeed, inserting (7) into (3) yields $G = \bar{g}Ak^\alpha G^{1-\alpha}L$. Solving for G, we find

$$G = (\bar{g}AL)^{1/\alpha}k. \tag{8}$$

By (7), the aggregate production function can now be written

$$\begin{aligned} Y &= Ak^{\alpha}G^{1-\alpha}L = Ak^{\alpha}\left[(\bar{g}AL)^{1/\alpha}k\right]^{1-\alpha}L \\ &= A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}}kL \equiv \bar{A}K, \end{aligned} \tag{9}$$

where, for convenience, we have introduced the constant

$$\bar{A} \equiv A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}}. \tag{10}$$

Using that $k_i = k$ in (2) gives

$$\hat{r} = \alpha A(G/k)^{1-\alpha} = \alpha A\left[(\bar{g}AL)^{1/\alpha}\right]^{1-\alpha} = \alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}},$$

where the second equality follows from (8). Thus, since $\hat{r} = r + \delta$,

$$r = \alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta \equiv \alpha\bar{A} - \delta \equiv \bar{r}. \tag{11}$$

We see that the aggregate production function is of AK form and the equilibrium real interest rate is constant over time. We now embed this in a Ramsey-style household sector and so we get a reduced-form AK Ramsey model.

1.4 The households

Each member of the representative household supplies inelastically one unit of labor per time unit, so that our measure for population, L, also measures labor supply. With a_t denoting real financial wealth per capita in the household, its intertemporal decision problem is:

$$\begin{aligned} \max U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}-1}{1-\theta}e^{-\rho t}dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= \bar{r}a_t + w_t + \tau_t - c_t, \qquad a_0 \quad \text{given}, \\ \lim_{t\to\infty} a_t e^{-\bar{r}t} &\geq 0. \end{aligned} \tag{NPG}$$

Since taxes, τ, are lump-sum, the first-order conditions lead to the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\bar{r}-\rho) = \frac{1}{\theta}(\alpha\bar{A}-\delta-\rho) \equiv \gamma. \tag{12}$$

In addition the household's optimal plan will satisfy the transversality condition that the (NPG) condition holds with equality.

The constant real interest rate implies that consumption grows, from date zero, at a constant rate, γ. Note that γ, via (10) and (11), is an increasing function of $\bar{g}$.

2 Dynamics

To ensure growth we assume $\bar{A} > (\delta + \rho)/\alpha$, i.e.,

$$A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} > \delta + \rho; \tag{A1}$$

this requires that $\bar{g}$ is not too small. On the other hand, to ensure bounded utility we assume

$$\rho > (1-\theta)\gamma, \tag{A2}$$

which (if $\theta < 1$) requires that g, hence $\bar{g}$, is not too high (here is then an implicit further constraint on $\bar{g}$, in addition to that in (3)). From the Keynes-Ramsey rule (12) we have $r = \rho + \theta\gamma$, so that the assumption (A2) is equivalent with

$$r > \gamma, \tag{13}$$

i.e., the real interest rate is higher than the GDP growth rate (as in all representative agent models with infinite time horizon, this is a necessary condition for an equilibrium to exist).

2.1 Movement of k and y over time

How does capital intensity and output per capita move over time? We have

$$\begin{aligned}\dot{k} &= \frac{\dot{K}}{L} = \frac{Y - G - C - \delta K}{L} = (1-\bar{g})y - c - \delta k \\ &= (1-\bar{g})A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}}k - \delta k - c_0 e^{\gamma t} = \left[(1-\bar{g})\bar{A} - \delta\right]k - c_0 e^{\gamma t}.\end{aligned}$$

This is a linear differential equation in k. Its solution (cf. Appendix to LN 4) is

$$k_t = \left(k_0 - \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma}\right)e^{\left[(1-\bar{g})\bar{A}-\delta\right]t} + \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma}e^{\gamma t}, \tag{14}$$

presupposing

$$(1-\bar{g})\bar{A} - \delta \neq \gamma. \tag{15}$$

The transversality condition of the household can be written

$$\lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

We now show that, in view of (14) and the parameter restriction in (3) below, this transversality condition is satisfied if and only if

$$c_0 = \left[(1-\bar{g})\bar{A} - \delta - \gamma\right]k_0. \tag{16}$$

Indeed, multiplying through in (14) by $e^{-\bar{r}t}$ we get

$$k_t e^{-\bar{r}t} = (k_0 - \frac{c_0}{(1-\bar{g})\bar{A}-\delta-\gamma})e^{\left[(1-\bar{g})\bar{A}-\delta-\bar{r}\right]t} + \frac{c_0}{(1-\bar{g})\bar{A}-\delta-\gamma}e^{-(\bar{r}-\gamma)t}, \tag{17}$$

where, in view of (13), the last term approaches zero for $t \to \infty$. In view of the constraint on $\bar{g}$ in (3), we have $1-\bar{g} \geq \alpha$, hence $(1-\bar{g})\bar{A}-\delta \geq \bar{r}$, by (11). Combining this with (13), we see that (15) holds and that c_0 in (16) is positive as it should be.[2] And since $\bar{g} \leq 1-\alpha$, the constant coefficient to t in the first exponential function in (17) is nonnegative. Hence, satisfying (TVC) requires the first term to vanish. This is the same as requiring (16) to hold.

Inserting (16) into (14) gives

$$k_t = \frac{c_0}{(1-\bar{g})\bar{A}-\delta-\gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, from date zero, k grows at the same constant rate as c, the rate γ. Since, by (9), $y = \bar{A}k$, y does the same. There are no transitional dynamics. This is also what we should expect in view of the model implying a constant real interest rate and a constant output-capital ratio, $\bar{A}$.

2.2 Interesting results

The model exhibits fully endogenous growth. In addition to the standard results for fully endogenous growth models (like $\partial\gamma/\partial\rho < 0, \partial\gamma/\partial\theta < 0$), we get from (12) and (10)

$$\frac{\partial\gamma}{\partial\bar{g}} = \frac{\alpha}{\theta}\frac{\partial\bar{A}}{\partial\bar{g}} = \frac{1-\alpha}{\theta}A^{\frac{1}{\alpha}}L^{\frac{1-\alpha}{\alpha}}\bar{g}^{\frac{1}{\alpha}-2} > 0,$$

indicating that the productive public service promotes growth. The model also generates a *strong scale effect:*

$$\frac{\partial\gamma}{\partial L} = \frac{\alpha}{\theta}\frac{\partial\bar{A}}{\partial L} = \frac{1-\alpha}{\theta}A^{\frac{1}{\alpha}}\bar{g}^{\frac{1-\alpha}{\alpha}}L^{\frac{1}{\alpha}-2} > 0.$$

This scale effect on growth comes about because the productive public service is assumed to be *nonrival.* A higher G raises total factor productivity and a given G can *costlessly* be spread to additional users. Indeed, the nonrival character of G implies that per capita output depends on the total amount of the productive service, G, not on the per capita amount, G/L. In other words, the *per capita cost* of raising total factor productivity by a given amount is a decreasing function of population size.

[2] Here we see the role of the constraint on $\bar{g}$ in (3). Without this constraint there is a risk that c_0 in (16) would be negative which is tantamount to non-existence of equilibrium. The intuition is that a too large G/Y would imply too little gross investment to deliver the capital accumulation needed to sustain the high growth in consumption warranted by the high $\bar{r}$ implied by a high G/Y.

2.3 Digression: the corresponding semi-endogenous growth model

The above results are due to the strong assumption that labor efficiency is proportional to G. That is, there are no diminishing returns to the productivity-enhancing effect of a higher G. This is the reason that the model becomes a fully endogenous growth model.

Now, consider a more general specification of firm i's production function:

$$Y_{it} = AK_{it}^{\alpha}(G_t^{\lambda}L_{it})^{1-\alpha}, \qquad A > 0, 0 < \alpha < 1, 0 < \lambda \leq 1.$$

We can derive the aggregate production function in a similar way as above. We get

$$Y = \left(A\bar{g}^{\lambda(1-\alpha)}K^{\alpha}L^{1-\alpha}\right)^{\frac{1}{1-\lambda(1-\alpha)}}.$$

Taking growth rates on both sides gives

$$g_Y = \frac{\alpha}{1-\lambda(1-\alpha)}g_K + \frac{1-\alpha}{1-\lambda(1-\alpha)}n,$$

where we allow $n \geq 0$. Assuming $\lambda < 1$, we get under balanced growth, where $g_Y = g_K$,

$$g_Y = \frac{n}{1-\lambda} \text{ and } g_y = g_Y - n = \frac{\lambda n}{1-\lambda}.$$

Hence, $\partial g_y/\partial \bar{g} = 0 = \partial g_y/\partial L_0$. A rise in $\bar{g}$ no longer has a permanent positive effect on productivity growth (only a temporary positive effect). And the strong scale effect has been replaced by a weak scale effect. This change in the results are due to the diminishing returns to the productivity-enhancing effect of a higher G implied by a $\lambda < 1$. The fact that there is still a scale effect, although weak, is the natural implication of the productive public service being nonrival.

3 The social planner's solution

We now return to the knife-edge case $\lambda = 1$, on which B & S focus their analysis.

So far the level of $\bar{g}$ has been arbitrary. But can we say something about an optimal level of $\bar{g}$? To find out we study the social planner's problem.

To ensure static efficiency in production the social planner will dictate (as did the market mechanism) the same marginal productivities of capital and labor in all firms, hence the same capital intensity in all firms. So again the aggregate production will be as in (7), namely

$$Y_t = Ak_t^{\alpha}G_t^{1-\alpha}L = AK_t^{\alpha}(G_tL)^{1-\alpha}. \tag{18}$$

The social planner faces the dynamic optimization problem:

$$
\begin{aligned}
&\max_{(c_t, G_t)} U_0 = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.} \\
&\qquad c_t \geq 0,\ G_t \geq 0, \\
&\qquad \dot{K}_t = AK_t^\alpha (G_t L)^{1-\alpha} - G_t - c_t L - \delta K_t, \qquad K_0 > 0 \text{ given}, \\
&\qquad K_t \geq 0.
\end{aligned} \tag{19}
$$

The current-value Hamiltonian is

$$
H = \frac{c^{1-\theta} - 1}{1-\theta} + \mu \left(\overbrace{AK^\alpha (GL)^{1-\alpha}}^{Y} - G - cL - \delta K \right),
$$

where the co-state variable μ can be interpreted as the shadow price of capital along the optimal path. An interior solution must satisfy the first-order conditions

$$
\partial H / \partial c = c^{-\theta} - \mu L = 0, \text{ i.e., } c^{-\theta} = \mu L, \tag{FOC1}
$$

$$
\partial H / \partial G = \mu (\frac{\partial Y}{\partial G} - 1) = 0, \text{ i.e., } \frac{\partial Y}{\partial G} = (1-\alpha)\frac{Y}{G} = 1 \text{ or } \frac{G}{Y} = 1 - \alpha, \tag{FOC2}
$$

$$
\partial H / \partial K = \mu (\frac{\partial Y}{\partial K} - \delta) = \rho \mu - \dot{\mu}, \text{ i.e., } \frac{\partial Y}{\partial K} - \delta = \rho - \frac{\dot{\mu}}{\mu}. \tag{FOC3}
$$

Our conjecture is that the transversality condition

$$
\lim_{t \to \infty} K_t \mu_t e^{-\rho t} = 0 \tag{TVC$_{sp}$}
$$

is also a necessary optimality condition. This guess will be of help in finding a candidate solution. Having found a candidate solution, we can appeal to Mangasarian's theorem on *sufficient* conditions, cf. Lecture Note 8, p. 12, to ensure that our candidate solution *is* really a solution.

Log-differentiation wrt. t in (FOC1) and inserting into (FOC3) gives the usual Keynes-Ramsey rule,

$$
\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\frac{\partial Y}{\partial K} - \delta - \rho), \tag{20}
$$

where

$$
\frac{\partial Y}{\partial K} = \alpha A (\frac{G}{K})^{1-\alpha} L^{1-\alpha} = \alpha \frac{Y}{K}. \tag{21}
$$

According to (FOC2), optimality requires $\partial Y / \partial G = 1$. This is not surprising. The (real) cost of increasing G by one unit is one (i.e., one output unit less will be available for other uses) and the real benefit is $dY = (\partial Y / \partial G) dG = (\partial Y / \partial G) \cdot 1$. Therefore, the optimality condition, $MB = MC$, requires $dY = 1$, that is, $\partial Y/\partial G = 1$. To put it differently: a necessary condition for $Y - G$ to be maximized is that $dY = dG$. Since $\partial Y/\partial G = (1-\alpha)Y/G$, the implication is that $G = (1-\alpha)Y$ or $\bar{g}_{SP} = 1-\alpha$. Inserting this into (18) and solving for Y gives

$$Y = A^{1/\alpha}\left[(1-\alpha)L\right]^{(1-\alpha)/\alpha} K \equiv \tilde{A}K. \tag{22}$$

In the above analysis of the market economy we assumed $0 < \bar{g} \leq 1-\alpha$, which implies $\bar{g} \leq \bar{g}_{SP}$. Correspondingly,

$$\bar{A} = A^{1/\alpha}(\bar{g}L)^{(1-\alpha)/\alpha} \leq \tilde{A}, \tag{23}$$

with strict equality if and only if $\bar{g} = 1-\alpha$. If the government in the market economy chooses $\bar{g} = 1-\alpha$, then $\bar{A}$ from the market economy is the same as $\tilde{A}$ in the social planner's solution.

We can now write (19) as

$$\dot{K} = (1 - \frac{G}{Y})Y - cL - \delta K = \alpha Y - cL - \delta K = \alpha\tilde{A}K - cL - \delta K. \tag{24}$$

Inserting (22) into (21) gives

$$\frac{\partial Y}{\partial K} = \alpha\tilde{A}. \tag{25}$$

From (20) we get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\alpha\tilde{A} - \delta - \rho) \equiv \gamma_{SP}, \tag{26}$$

a constant. Note that it would be wrong to conclude from (22) that the marginal product of capital at the aggregate level equals $\tilde{A}$. The fact is that (22) is just a "reduced form" where G is not visible. But behind the relation in (22) lies the presumption that G/Y is constant, i.e., that G is increased along with Y. Indeed, in (22) the increase in Y associated with an increase in K is partly the effect of the implicit increase in G following the increase in K. The true marginal product of capital at the aggregate level is as given by (25) which is smaller than $\tilde{A}$.

Comparing the growth rate, γ, in the market economy with γ_{SP}, we see that (23) implies

$$\gamma \leq \gamma_{SP},$$

with strict equality if and only if $\bar{g} = 1-\alpha$. It follows that the parameter restriction (A1) ensures $\gamma_{SP} > 0$. On the other hand, to ensure bounded utility we now need

$$\rho > (1-\theta)\gamma_{SP}, \tag{A2'}$$

which, for $\theta < 1$, is more demanding than (A2).

Given (26), we may write (24) as

$$\dot{K} = (\alpha\tilde{A} - \delta)K - c_0 L e^{\gamma_{SP} t}. \tag{27}$$

From this and the transversality condition (TVC$_{sp}$), we can derive (by a similar procedure as above for the market economy) that the required initial consumption level, c_0, will be such that

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{c}}{c} = \gamma_{SP}. \tag{28}$$

In fact, instead of going through a detailed derivation, we may simply refer to the fact that the social planner's system belongs to the AK family (constant marginal product of capital at the aggregate level and constant output-capital ratio). Then, for a Ramsey set-up, it is a general result that k and y must from date zero grow at the same rate as c (there are no transitional dynamics). From this general knowledge we can find the initial consumption level, c_0. Indeed, combining (27) and (28) gives

$$\gamma_{SP} = \frac{\dot{K}_t}{K_t} = \alpha\tilde{A} - \delta - \frac{c_t}{k_t}.$$

Solving for c_t gives $c_t = (a\tilde{A} - \delta - \gamma_{SP})k_t$ so that

$$c_0 = (a\tilde{A} - \delta - \gamma_{SP})k_0. \tag{29}$$

In view of (A2'), $\rho + \theta\gamma_{SP} > \gamma_{SP}$ and since, from (26), $\rho + \theta\gamma_{SP} = \alpha\tilde{A} - \delta$, the proportionality factor in (29) is strictly positive (otherwise no solution to the social planner's problem would exist).

In the same way as in Lecture Note 8, p. 12 we can now appeal to Mangasarian's theorem on *sufficient* conditions, to ensure that our candidate solution *is* really an optimal solution.

An interesting conclusion is that if the government in the market economy chooses $\bar{g} = 1 - \alpha$ (static efficiency), then not only is the market economy's growth rate the same as that chosen by the social planner, but the entire resource allocation is the same. This is due, however, to the unrealistic assumption of lump-sum taxation. If the public services were financed by an income tax (including taxation of capital income), then the market economy would get a growth rate lower than the optimal one in spite of $\bar{g} = 1 - \alpha$. This is because of the wedge that arises in this case between the social returns to saving, $\alpha\tilde{A} - \delta$, given in (25), and the private (after-tax) returns, $(1-\tau_r)\bar{r}$.[3] The capital income tax is a disincentive to save; and in a closed economy, this amounts to a disincentive to invest. In a fully endogenous growth model as this one this acts as a drag on growth even in the long run.

However, a constant consumption tax, τ_c, is a non-distortionary tax (since labor supply is inelastic) and could be used to finance the government spending G.

4 Productive public services with congestion (B & S, § 4.4.2)

The results are different if there are *congestion* problems associated with the productive public service, i.e., if it is partly *rival*. For example, optimal taxation will be different. And as B & S (p. 223) emphasize, the empirically problematic scale effect on growth will disappear.[4]

Let us assume the service is partly rival, but non-excludable. The combination of rivalry and non-excludability implies that a *free-access problem* arises. Infrastructure services are an example (in principle); think of the highways to Copenhagen during rush hours.[5] Due to the presence of congestion effects, a growing population can be allowed without implying a forever increasing growth rate. Hence, we assume that the representative household is of size L, where

$$L = L_0 e^{nt}, \qquad n \geq 0.$$

Each member of the household supplies inelastically one unit of labor per time unit, so that also labor supply grows at the constant rate $n \geq 0$. (B & S unnecessarily assume $n = 0$.)

We follow B & S and assume firm i has the production function

$$Y_i = Af(\frac{G}{Y})K_i, \tag{30}$$

where the *non-congestion* function f satisfies[6]

$$f(0) \geq 0, f' > 0, f'' < 0. \tag{31}$$

[3] See Exercise III.1.

[4] As we saw above, this scale effect on growth will of course also disappear if we maintain non-rivalry, but replace the knife-edge assumption $\lambda = 1$ by $0 < \lambda < 1$.

[5] I say "in principle", because taking infrastructure seriously would require modelling infrastructure as a *stock* variable, not as a flow variable as B & S do.

[6] An alternative way of representing congestion is sketched in the appendix.

There are non-diminishing returns to capital, even at firm level. This is a strong (and questionable) assumption which can, at best, only be maintained if we think of K_i as "broad capital", including either human capital or firm-specific knowledge capital, cf. B & S, § 5.1. (Or, if you want, you may think of K_i as only physical capital, and interpret the model as describing a thought experiment: a fully automatized economy.)

4.1 The market economy

Since $\partial Y_i/\partial K_i = Af(\frac{G}{Y})$, profit maximization under perfect competition implies

$$K_i \begin{cases} = \infty, \text{ if } Af(\frac{G}{Y}) > r + \delta, \\ \text{undetermined, if } Af(\frac{G}{Y}) = r + \delta, \\ = 0, \text{ if } Af(\frac{G}{Y}) < r + \delta. \end{cases} \tag{32}$$

Equilibrium in the factor market implies

$$\sum_i K_i = K,$$

i.e., demand is equal to supply. By (32), this requires (since $K > 0$) that the real rate of interest adjusts so that

$$r = Af(\frac{G}{Y}) - \delta. \tag{33}$$

The government chooses G such that G/Y is a constant, $\bar{g} \in (0, 1)$. There are no other government expenditures than G. The government budget is balanced. We postpone specification of how what taxation rule is used.

Households have infinite horizon and are described in the usual Ramsey way. The consumption/saving behavior leads to the Keynes-Ramsey rule, which in case of lump-sum taxation takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left(Af(\bar{g}) - \delta - \rho\right). \tag{34}$$

The aggregate production function becomes

$$Y = \sum_i Y_i = Af(\bar{g}) \sum_i K_i = Af(\bar{g})K. \tag{35}$$

Now let us consider the social planner's problem.

4.2 The social planner

Ex ante, the social planner need not choose G/Y constant. So the aggregate production function faced by the social planner should be written in the general form,

$$Y = \sum_i Y_i = Af(\frac{G}{Y}) \sum_i K_i = Af(\frac{G}{Y})K. \tag{36}$$

The decision problem is

$$\begin{aligned}
&\max_{(c_t, G_t)_{t=0}^{\infty}} \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho - n)t} dt \quad \text{s.t.} \\
c_t &\geq 0, \; G_t \geq 0, && (37) \\
\dot{K}_t &= Af(\frac{G_t}{Y_t})K_t - G_t - c_t L_t - \delta K_t, \quad \text{where } K_0 \text{ is given, and } Y_t \text{ is as in (36),} && (38) \\
K_t &\geq 0 \text{ for all } t \geq 0. && (39)
\end{aligned}$$

The current-value Hamiltonian is

$$H = \frac{c^{1-\theta} - 1}{1-\theta} + \mu \left(\overbrace{Af(\frac{G}{Y})K}^{Y} - G - cL - \delta K \right).$$

First-order conditions are

$$\begin{aligned}
\frac{\partial H}{\partial c} &= c^{-\theta} - \mu L = 0 \Rightarrow c^{-\theta} = \mu L \Rightarrow -\theta \frac{\dot{c}}{c} = \frac{\dot{\mu}}{\mu} + n, && \text{(FOC1)} \\
\frac{\partial H}{\partial G} &= \mu \left(\frac{\partial Y}{\partial G} - 1 \right) = 0 \Rightarrow \frac{\partial Y}{\partial G} = 1, && \text{(FOC2)} \\
\frac{\partial H}{\partial K} &= \mu \left(\frac{\partial Y}{\partial K} - \delta \right) = (\rho - n)\mu - \dot{\mu} \Rightarrow \frac{\partial Y}{\partial K} - \delta = \rho - n - \frac{\dot{\mu}}{\mu}. && \text{(FOC3)}
\end{aligned}$$

The conjectured necessary transversality condition is

$$\lim_{t \to \infty} K_t \mu_t e^{-(\rho - n)t} = 0. \tag{TVC*}$$

From (FOC1) and (FOC3) we get

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\frac{\partial Y}{\partial K} - \delta - \rho), \tag{K-R}$$

where $\partial Y / \partial K$ ($< Af(G/Y)$) is to be found.

To proceed we shall use (FOC2). But we cannot directly calculate $\partial Y / \partial G$ from (36), since Y appears on both sides. Instead we consider (36) as an equation of the form

$$Y = h(Y, G, K), \tag{40}$$

which defines Y as an implicit function of G and K,

$$Y = Y(G, K).$$

To find the partial derivatives of this implicit function we use implicit differentiation. To illustrate our procedure, consider (40). Taking the total differential on both sides, we get

$$dY = h_Y dY + h_G dG + h_K dK.$$

From this, by setting $dK = 0$, we find

$$\frac{\partial Y}{\partial G} = \frac{h_G}{1 - h_Y}.$$

And by setting $dG = 0$, we find

$$\frac{\partial Y}{\partial K} = \frac{h_K}{1 - h_Y}.$$

Applying this procedure to (36), we find

$$\begin{aligned} dY &= A\left[f\left(\frac{G}{Y}\right)dK + Kf'\left(\frac{G}{Y}\right)\frac{YdG - GdY}{Y^2}\right] \Rightarrow \\ \left[1 + AKf'\left(\frac{G}{Y}\right)\frac{G}{Y^2}\right]dY &= Af\left(\frac{G}{Y}\right)dK + \frac{AK}{Y}f'\left(\frac{G}{Y}\right)dG \Rightarrow \\ \left[1 + \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}\frac{G}{Y}\right]dY &= Af\left(\frac{G}{Y}\right)dK + \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}dG, \end{aligned} \tag{41}$$

since $AK/Y = 1/f(\frac{G}{Y})$, by (36). Hence, first we have[7]

$$\begin{aligned} \frac{\partial Y}{\partial G} &= \frac{f'\left(\frac{G}{Y}\right)/f\left(\frac{G}{Y}\right)}{1 + \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}\frac{G}{Y}} = \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right) + f'\left(\frac{G}{Y}\right)\frac{G}{Y}} = 1, \qquad \text{(by (FOC2))} \\ &\Rightarrow f'\left(\frac{G}{Y}\right) = f\left(\frac{G}{Y}\right) + f'\left(\frac{G}{Y}\right)\frac{G}{Y} \Rightarrow \\ \left(1 - \frac{G}{Y}\right)f'\left(\frac{G}{Y}\right) &= f\left(\frac{G}{Y}\right). \end{aligned} \tag{42}$$

Second, the marginal product of capital is

$$\begin{aligned} \frac{\partial Y}{\partial K} &= \frac{Af\left(\frac{G}{Y}\right)}{1 + \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}\frac{G}{Y}} = \frac{Af\left(\frac{G}{Y}\right)}{1 + \frac{1}{1 - G/Y}\frac{G}{Y}}. \qquad \text{(by (42))} \\ &= (1 - \frac{G}{Y})Af\left(\frac{G}{Y}\right). \end{aligned} \tag{43}$$

[7] There is a typo in (4.52) in B & S, p. 224. The factor L should be replaced by $1/L$.

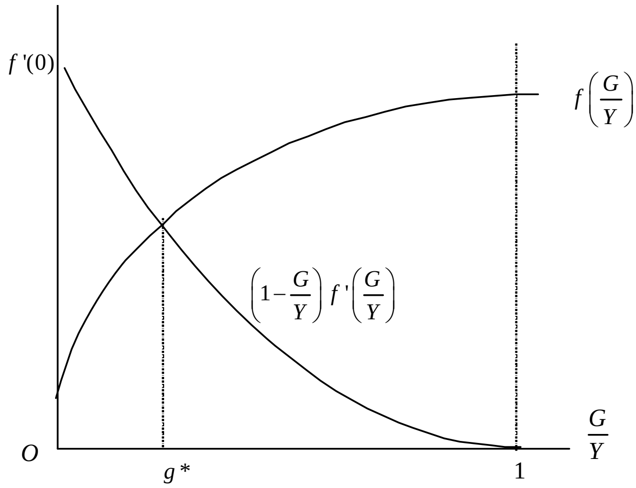

Figure 1: Determination of g^*.

Graphs of the left-hand side and the right-hand side of (42) are shown in Fig. 1. The unique crossing point gives the optimal G/Y, which we denote g^*. For $G/Y < g^*$, the high marginal product of G, and thus the productivity effect of a higher G, dominates the combined effect of the higher direct cost in the form of a higher G (which for given Y makes $Y - G$ smaller) and the congestion resulting from the higher Y. So in this case $dY > dG$, implying that a marginal increase in G pays. On the other hand, if $G/Y > g^*$, the marginal product of G, and thus the productivity effect of a higher G is dominated by the combined effect of the high direct cost in the form of a high G and the congestion effect of the resulting high Y and so $dY < dG$. Hence, in this case a marginal decrease in G pays. Notice that g^* is time independent (and independent of n).

With $G/Y = g^*$, (43) implies

$$\frac{\partial Y}{\partial K} = (1 - g^*)\, Af\,(g^*)\,, \tag{44}$$

and (K-R) becomes

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[(1 - g^*)\, Af\,(g^*) - \delta - \rho\right] \equiv \gamma_{SP}. \tag{45}$$

Again, since the marginal and average product of capital are constant,[8] we have an AK-style model and there will be no transitional dynamics. Right from date zero c, k, and y grow at the same constant rate, γ_{SP}. This rate is positive if and only if $(1 - g^*)\, Af\,(g^*) > \delta + \rho$, which we assume. Further, to ensure boundedness of the utility integral we assume

[8] $Y/K = Af(G/Y) = Af(g^*)$.

$(1-\theta)\gamma_{SP} < \rho - n$. (One should now check whether the above first-order conditions and the conjectured transversality condition are *sufficient* conditions for an optimal solution. One could probably use the Mangasarian conditions as in Lecture Note 8. I have not had time to check this.)

4.3 How to obtain the SP allocation in a decentralized economy

Under a balanced government budget, the required tax revenue is

$$T = G = g^* Y.$$

Let the tax revenue as a proportion of income be called τ. Then the required τ is

$$\tau \equiv \frac{T}{Y} = g^* \equiv \tau^*.$$

The interesting conclusion is that even if, in the market economy, the government chooses the "right" G/Y, i.e., $G/Y = g^*$, financing G with lump-sum taxes will *not* provide the social planner's allocation. Instead a *too high* growth rate is generated. This is seen by comparing (34) with (45). The explanation is that lump-sum taxation gives no correction for the negative externality implied by congestion.

In the decentralized equilibrium there is a tendency to a kind of overaccumulation because, with lump-sum taxes, the private agents do not internalize the congestion costs of higher production. The "private" marginal product of capital implied by (30) is $Af(g^*)$, whereas the social marginal product of capital, given in (44), is lower, namely $(1-g^*)\,Af(g^*)$. Thus, with lump-sum taxation the return to saving is too high. To put it differently, the cost of a too high level of saving is that, initially, consumption is too low in the sense that the initially forgone consumption is worth more in terms of current utility than the (discounted value of) the increase in future consumption that is obtained by higher saving and higher future growth.

In this model, internalization of the congestion costs requires a "distorting" tax. For example, a production tax at rate τ^* would do the job. Indeed, an extra cost (in addition to the direct production costs) should be imposed on a producer who raises Y_i. This extra cost should be enough to finance the extra government service needed to maintain the public services available to others, i.e., to keep G/Y constant. The required extra cost is $(G/Y)dY = g^*Af(g^*)$, cf. (44).

4.4 The relationship between τ and the per capita growth rate

Here we add an interesting observation, not mentioned in B & S.

Consider the case where the tax revenue as a proportion of income can be any given constant $\tau \in (0,1)$, not necessarily equal to g^*. Let G be determined by the tax revenue:

$$G = \tau Y.$$

Suppose τ takes the form of a production tax such that firm i's after-tax revenue from sales are $(1-\tau)Y_i$. Then, instead of (33) we get

$$r = (1-\tau)Af(\tau) - \delta,$$

and (34) is replaced by

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left[(1-\tau)Af(\tau) - \delta - \rho\right] \equiv \gamma(\tau).$$

Now, consider the problem: given that a production tax *must* be used, what level should it have to maximize growth? To derive an answer, we find the first-order condition by differentiating wrt. τ :

$$\begin{aligned} \gamma'(\tau) &= \frac{A}{\theta}\left[(1-\tau)f'(\tau) - f(\tau)\right] = 0 \Rightarrow \\ (1-\tau)f'(\tau) - f(\tau) &= 0. \end{aligned}$$

But this is exactly the same condition as (42) above, so that our candidate for a solution to the growth maximization problem is $\tau = \tau^*$. What about the second-order condition? We get

$$\gamma''(\tau) = \frac{A}{\theta}\left[(1-\tau)f''(\tau) - f'(\tau) - f'(\tau)\right] < 0,$$

by the assumptions on f in (31); Fig. 2 illustrates. It follows that $\tau = \tau^*$ *is* the solution, i.e., τ^* is that tax rate which maximizes growth in the decentralized equilibrium, given the requirement that a production tax has to be used. The explanation is that $\tau = \tau^*$ is required to ensure the "static efficiency" condition, $\partial Y / \partial G = 1$. Satisfying this condition is a "minimum" requirement for maximizing growth in *any* fully endogenous growth model with productive public services.

The intuition behind the hump-shaped growth curve in Fig. 2 is the following. On the one hand a higher tax rate allows for a higher G, which means more productive public services. Everything else equal, this tends to raise the marginal product of capital and

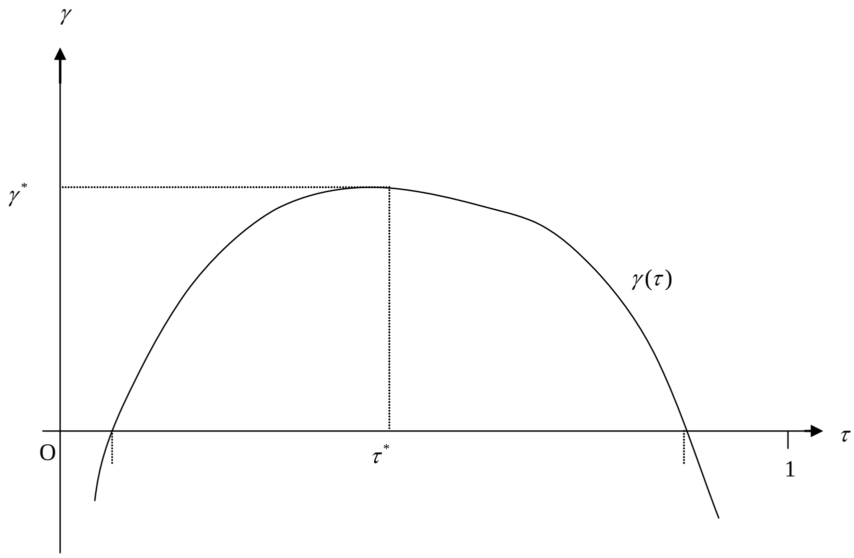

Figure 2:

increase the growth rate γ. But everything else is *not* equal. First, for a given Y, a higher G leaves less "surplus", $Y - G$, available for other purposes, including capital accumulation. Second, there are diminishing returns to a higher G, not only for the usual reason of a diminishing *direct* effect on Y, as envisioned by the fact that $f'' < 0$, but also because of the congestion effect of higher economic activity (here higher Y). Third, a higher G is here financed by a production tax and such a tax decreases firms' after-tax marginal product of capital and thereby the equilibrium real interest rate. This decreases the incentive to save and thereby the level of investment in the economy. In a fully endogenous growth model this decreases growth not only for a while, but permanently.

When τ (and therefore G) is low, the "direct" marginal product of G at the aggregate level is high, and this effect dominates the others. On the other hand, when τ (and therefore G) is high, the "direct" marginal product of G at the aggregate level is low, and then the other effects dominate.

Let us summarize. We have considered a model where production activity implies congestion, a negative externality. Hence, there should be a tax on production activity in some form or another. It need not be a production tax as considered here (in a more general setting it could be a fee on *using* the public service, for example a turnpike fee). In the present model it turns out that *if* a production tax is used to finance the public service, the level of this tax rate which is needed to accomplish the social planner's allocation is also the tax level which maximizes the per capita growth rate, *given that a production tax has to be used*.

This said, it should be added that it is because of the disputable assumption of a reduced-form AK structure, cf. (35), that it is possible to affect the per capita growth rate not only temporarily but permanently by the choice of a tax rate.

5 A final remark

Above we have, like B & S, modeled the productive public service as a *flow* directly related to the current public spending. Public infrastructure is different in the sense that is a *stock* and the associated service is proportional to this stock. A model of this requires that we introduce infrastructure as public capital, K_g. If this public capital is non-rival, firm i's production function would be

$$Y_{it} = F(K_{it}, L_{it}; K_{gt}).$$

Ignoring public consumption, the public spending, G_t, in the national accounting equation would then represent public investment so that $\dot{K}_{gt} = G_t - \delta_g K_{gt}$.[9]

6 Appendix

Since B & F let $f(G/Y)$ enter the way it does in (30) and $f' > 0$, the interpretation of $f(G/Y)$ is that it indicates the degree of *non-congestion*. We might instead want to introduce a positively-valued *congestion* function $h(Y/G)$, defined for $0 < Y/G < \infty$, where $h' < 0$, $h'' > 0$. Then (30) would be replaced by

$$Y_i = Ah(Y/G)K_i,$$

which would perhaps be a more intuitive way of looking at the matter.

Another hypothesis is that congestion of public services involves G in relation to K, rather than Y. This is considered in B & S, Exercise Problem 4.6, where firm i has the production function

$$Y_i = Af(\frac{G}{K})K_i.$$

[9] See, e.g., Nagatani et al. (SJE, 1993).