Economic Growth Lecture Note 17

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Stochastic erosion of monopoly power: An extension of the simple increasing variety model

In B & S's Section 6.2 the simple increasing variety model of their Section 6.1, which we may call Model I, is extended by adding stochastic erosion of monopoly power. Compared with Model I the only difference in the setup is that the duration of monopoly power over the commercial use of an invention is *limited* and *uncertain*. This lecture note gives my exposition of this model extension, which we will call Model II.

1 The sectors

The technology of the economy is the same as in Model I. In the *basic-goods sector* (Sector 1) firms combine labor and N different intermediate goods to produce a homogeneous output good. Firm i ($i = 1, 2, ..., M$) in the sector has the production function

$$Y_i = A{L_i}^{1-\alpha} \sum_{j=1}^{N} {x_{ij}}^{\alpha}, \qquad A > 0,\ 0 < \alpha < 1, \tag{1}$$

where Y_i, L_i, and x_{ij} denote output of the firm, labor input, and input of intermediate good j, respectively, where $j = 1, 2, ..., N$. Aggregate output per time unit in Sector 1 is $Y \equiv \sum_i Y_i$. This sector, as well as the labor market, operate under perfect competition.

The aggregate output of "basic goods" is used partly for consumption, C, partly for investment in R&D, R, and partly for replacing the intermediate goods used up in the production of Y. Hence, we have

$$Y = C + R + X, \tag{2}$$

where $X \equiv \sum_j \sum_i x_{ij}$.

In the *innovative sector*, Sector 2, there are two kinds of activities, Activity 2.a and Activity 2.b. The first activity is to supply already invented intermediate goods. Once the technical design, j, has been invented, the inventor can effortlessly transform any number of basic goods into the same number of intermediate goods of type j simply by pressing a button on a computer, thereby activating a computer code. That is,

$$\text{it takes } x \text{ units of the basic good to supply } x \text{ units of intermediate good } j. \tag{3}$$

For a limited period after the invention has been made, the inventor maintains monopoly power over the commercial use of the invention. The length of this period is uncertain, see below.

Activity 2.b in the innovative sector is R&D. New "technical designs" (blueprints) for making new specialized intermediate goods are invented. It takes an input of η units of basic goods, and nothing else, to make an invention. There is free entry to this innovative activity. Ignoring indivisibilities, the aggregate number of inventions (new technical designs) in the economy per time unit is

$$\dot{N} \equiv \frac{dN}{dt} = \frac{R}{\eta}, \qquad \eta > 0,\ \eta \text{ constant}, \tag{4}$$

where, as noted above, R is the aggregate R&D investment per time unit measured in terms of basic goods. After an invention has been made, the inventor enters Activity 2.a and begins selling the new intermediate good to firms in Sector 1.

2 Temporary monopoly

To begin with the inventor has a monopoly over the production and sale of the new intermediate good. This may be in the form of a more or less effective patent (free of charge) or by secrecy and concealment of the new technical design. But sooner or later imitators find out how to make very close substitutes. Concealment cannot succeed forever and with respect to patents, it may be difficult to codify exactly the technical aspects of an inventions, hence a patents does not give effective protection. Anyway, by law, patents are in practice usually only of limited duration, say 15 years.

There is uncertainty as to how long the monopoly position of an inventor lasts. We assume the erosion of monopoly power can be described by a Poisson process with a Poisson "arrival rate" $p > 0$, the same for all monopolies. The event that "arrives" is competition, i.e., cessation of the status as a monopolist. Independently of how long the monopoly position for firm j has been maintained, the probability that it breaks down in the next time interval of length Δt is approximately $p \cdot \Delta t$ for Δt "small". Or, if T denotes the remaining lifetime of the monopoly status of good j, then the probability that $T > \tau$ is $e^{-p\tau}$ for all $\tau > 0$. Further, the cessations of the different monopolies are stochastically independent.

The individual "entrepreneur" who invests η units of the basic good in making an invention finances the investment by issuing equity shares. So the households that buy these shares seem to face the risk that the monopoly profits on the R&D investment will only last for a very short time. But the model assumes that the uncertainty is *idiosyncratic,* that is, the stochastic event that a given firm looses its monopoly position in the near future is not correlated with other firms loosing their monopoly position and is in fact not correlated with anything else in the economy. Therefore, assuming the number of invented intermediates that still hold monopoly is always "large", share owners can eliminate the risk by diversifying their portfolio across many different monopoly firms.

As long as inventor j (firm j in Sector 2) is still a monopolist, the earned profit per time unit is

$$\pi_j = (P_j - 1)X_j(P_J) = (\frac{1}{\alpha} - 1)X^m \equiv \pi^m, \tag{5}$$

where $P_j = 1/\alpha$ is the monopoly price (the basic good is our numeraire) and the corresponding sales of intermediate good j are

$$X_j(P_j) = \sum_i x_{ij} = (\alpha A/P_j)^{\frac{1}{1-\alpha}} \sum_i L_i = \left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} L = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L \equiv X^m, \tag{6}$$

cf. Lecture Note 15. We call X^m the *monopoly supply.* The corresponding total revenue per time unit is $(1/\alpha) \cdot X^m$ and the total cost is $1 \cdot X^m$, resulting in (5).

As just described, however, sooner or later inventor j loses the monopoly. When this happens, intermediate good j faces competition and its price is driven down to the competitive market price = marginal cost = 1. Since also *average* cost is 1, profits vanish. The aggregate sales of intermediate good j are then

$$X_j(P_j) = X_j(1) = (\alpha A)^{\frac{1}{1-\alpha}} L \equiv X^c > X^m, \tag{7}$$

where X^c will be called the *competitive supply.* The inequality in (7) follows from $\alpha^{\frac{1}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}}$, in view of $\alpha^{\frac{1}{1-\alpha}} < 1$, which in turn follows from $0 < \alpha < 1$. Economically, the inequality in (7) reflects that the demand depends negatively on the price, which is lower under competition.

3 The aggregate production function

Firm i ($i = 1, 2, ..., M$) in Sector 1 has the production function (1) and demands, per time unit,

$$x_{ij} = (\alpha A)^{\frac{1}{1-\alpha}} P_j^{-\frac{1}{1-\alpha}} L_i$$

units of intermediate good j ($j = 1, 2, ..., N$), cf. Lecture Note 15. In view of production and cost symmetry, firm i chooses the same amount of each intermediate supplied under monopolistic conditions and the same amount of each competitive intermediate. That is,

$$x_{ij} = \begin{cases} x_i^m = (\alpha^2 A)^{\frac{1}{1-\alpha}} L_i \text{ if } j \text{ is still a monopoly (so that } P_j = \frac{1}{\alpha}), \\ x_i^c = (\alpha A)^{\frac{1}{1-\alpha}} L_i \text{ if } j \text{ is no longer a monopoly (so that } P_j = 1). \end{cases}$$

Substituting into (1), we can write output by firm i in Sector 1 as

$$Y_i = A \left[N^m (x_i^m)^\alpha + N^c (x_i^c)^\alpha \right] L_i^{1-\alpha} = A \left[(N - N^c) \left(\frac{x_i^m}{L_i} \right)^\alpha + N^c \left(\frac{x_i^c}{L_i} \right)^\alpha \right] L_i. \quad (8)$$

where N^c is the number of intermediates that have become competitive at the considered point in time and N^m is the number of intermediates that are still supplied under monopolistic conditions. For each t we have

$$N(t) = N^m(t) + N^c(t). \quad (9)$$

So there are *two* stock variables in the model and therefore scope for transitional dynamics, as we will see soon.

Cost minimization implies that all firms in Sector 1 choose the same input ratios. So x_i^m / L_i and x_i^c / L_i will be the same for all i, hence equal to the corresponding aggregate ratios. Consequently,

$$\begin{aligned} \frac{x_i^m}{L_i} &= \frac{\sum_i x_i^m}{\sum_i L_i} = \frac{X^m}{L} = \left(\alpha^2 A\right)^{\frac{1}{1-\alpha}}, \quad \text{and} \\ \frac{x_i^c}{L_i} &= \frac{\sum_i x_i^c}{\sum_i L_i} = \frac{X^c}{L} = (\alpha A)^{\frac{1}{1-\alpha}} > \left(\alpha^2 A\right)^{\frac{1}{1-\alpha}}, \end{aligned}$$

where we have used (6) and (7), respectively.

Using this result in (8), we find

$$\begin{aligned}
Y &= \sum_{i=1}^{M} Y_i \\
&= A\left[(N-N^c)(\frac{X^m}{L})^\alpha + N^c(\frac{X^c}{L})^\alpha\right]\sum_{i=1}^{M} L_i = A\left[(N-N^c)\left(\alpha^2 A\right)^{\frac{\alpha}{1-\alpha}} + N^c(\alpha A)^{\frac{\alpha}{1-\alpha}}\right] L \\
&= A(\alpha A)^{\frac{\alpha}{1-\alpha}}\left[(N-N^c)\alpha^{\frac{\alpha}{1-\alpha}} + N^c\right] L = A(\alpha A)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} N\left[1-\frac{N^c}{N}+\frac{N^c}{N}\alpha^{\frac{-\alpha}{1-\alpha}}\right] L \\
&= A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}} LN\left[1+\frac{N^c}{N}(\alpha^{\frac{-\alpha}{1-\alpha}}-1)\right]. \qquad (10)
\end{aligned}$$

Aggregate output is seen to depend on N^c/N. If the dynamics are such that N^c/N tends to a constant, then Y will tend to be proportional to a produced input, N. Therefore, the model is likely to be capable of generating fully endogenous growth, driven by R&D. We come back to this below.

Note that the result in (10) can be written

$$Y = Y^m\left[1+\frac{N^c}{N}(\alpha^{\frac{-\alpha}{1-\alpha}}-1)\right] > Y^m, \qquad (11)$$

where $Y^m \equiv A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}} LN$ is the equilibrium output level in case of permanent monopoly power, that is, the equilibrium output level in Model I. In that model we have $p = 0$, hence $N^c = 0$. But with erosion of monopoly power, we have $N^c > 0$ and so a fraction of the intermediate goods are supplied at a price equal to marginal cost, and the tendency to keep back the supply of these goods is reduced. This enhances productivity and we get $Y > Y^m$ in (10) (since $\alpha^{\frac{-\alpha}{1-\alpha}} > 1$ in view of $0 < \alpha < 1$).

4 The no-arbitrage condition under uncertainty

The market value of monopoly j at time t is the present discounted value of expected future profits

$$V_j(t) = \int_t^\infty E_t\left[\pi_j(\tau)\right] e^{-\int_t^\tau r(s)ds} d\tau, \qquad (12)$$

where $\pi_j(\tau)$ is the profit obtained at time τ, now a stochastic variable as seen from time $t < \tau$:

$$\pi_j(\tau) = \begin{cases} \pi^m \text{ if firm } j \text{ is still a monopolist at time } \tau, \\ \qquad 0 \text{ if not.} \end{cases}$$

The real rate of interest, r, on safe loans is the relevant discount rate in the calculation of $V_j(t)$ because, under the assumed idiosyncratic uncertainty, households can eliminate any risk by holding shares in many different monopoly firms.

Expected profit at time τ as seen from time t is

$$E_t\pi_j(\tau) = \pi^m e^{-p(\tau-t)} + 0 \cdot (1 - e^{-p(\tau-t)}) = \pi^m e^{-p(\tau-t)}. \tag{13}$$

Substituting into (12) we get

$$V_j(t) = \pi^m \int_t^\infty e^{-\int_t^\tau (r(s)+p)ds} d\tau \equiv V(t), \tag{14}$$

the same for all goods j that at time t still retain monopoly, cf. (6.45) in B & S.[1] This expression gives the market value of a monopoly firm on so-called certainty-equivalent form. We look at the monopoly profit stream as if it were permanent, but discount it at an effective discount rate, $r(s) + p$. The effective discount rate includes the conditional probability, p, that the monopoly status breaks down in the time interval $(\tau, \tau + 1]$, given it is retained up to time τ.

By differentiating (14) wrt. t, using Leibniz' formula,[2] we get

$$\frac{\pi^m(t) + \dot{V}(t)}{V(t)} = r(t) + p, \tag{15}$$

where $\dot{V}(t)$ is the increase per time unit in the market value of the monopoly firm, conditional on its monopoly position remaining in place. This formula constitutes one way of writing the no-arbitrage condition for the investors and is analogue to the no-arbitrage condition in the certainty case where $p = 0$.

Alternatively we may derive the no-arbitrage condition (15) without appealing to Leibniz' formula, which may not be part of the reader's standard math tool box. This will perhaps be a more intuitive approach. Let

$z(t) \equiv$ the firm's earnings in the time interval $(t, t + \Delta t)$, given that the firm is still a monopolist at time t.

[1] B & S write the left-hand sides of expressions like (12) and (14) as $E_tV(t)$, thereby interpreting $V(t)$ as a stochastic variable. In the present context, I prefer to let $V(t)$ have the same meaning as in the certainty case (Model I), namely the current market value of the firm, an observable variable, not a stochastic variable. The uncertainty is about profits in the future, not about the market value today of the uncertain future profit stream.

[2] See Appendix A.

There will be no arbitrage if the expected rate of return per time unit on shares in the monopoly firm equals the required rate of return which is the risk-free interest rate, $r(t)$. This amounts to the condition

$$\frac{\lim_{\Delta t \to 0} \frac{E_t z(t)}{\Delta t}}{V(t)} = r(t). \tag{16}$$

The firm's earnings $z(t)$ is a stochastic variable and its expected value as seen from time t is

$$E_t z(t) \approx (1 - p\Delta t)(\pi^m + \dot{V}(t))\Delta t + p\Delta t\,[-V(t)]. \tag{17}$$

Indeed, $V(t)$ is the capital loss in case the monopoly position ceases. And $p\Delta t$ is the approximate probability that this event occurs within the time interval $(t, t + \Delta t]$, given that it has not yet occurred at time t. Similarly, $1 - p\Delta t$ is the approximate probability that a monopoly position retained up to time t remains in force at least up to time $t + \Delta t$. And $\pi^m + \dot{V}(t)$ is the total return in that case. Now, (17) can be written:

$$\begin{aligned} E_t z(t) &\approx (\pi^m + \dot{V}(t))\Delta t - p(\pi^m + \dot{V}(t))(\Delta t)^2 - p\Delta t V(t) \\ &= (\pi^m - pV(t) + \dot{V}(t))\Delta t - p(\pi^m + \dot{V}(t))(\Delta t)^2 \Rightarrow \\ \frac{E_t z(t)}{\Delta t} &\approx \pi^m - pV(t) + \dot{V}(t) - p(\pi^m + \dot{V}(t))\Delta t \\ &\to \pi^m - pV(t) + \dot{V}(t) \text{ for } \Delta t \to 0. \end{aligned} \tag{18}$$

Hence, the no-arbitrage condition (16) can be written

$$\frac{\pi^m - pV(t) + \dot{V}(t)}{V(t)} = r(t). \tag{19}$$

Reordering, we see that this is the same condition as (15).

5 The equilibrium interest rate when $R\&D$ is active

The cost of making $\dot{N}$ inventions per time unit is $R = \eta\dot{N}$. The cost of making one invention is η. Hence, equilibrium with $\dot{N} > 0$ requires $V(t) = \eta$ and therefore also $\dot{V}(t) = 0$. Substituting into (19), we get

$$\begin{aligned} r(t) &= \frac{1}{\eta}\pi^m - p = \frac{1}{\eta}(\frac{1}{\alpha} - 1)X^m - p \\ &= (\frac{1}{\alpha} - 1)\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}}\frac{L}{\eta} - p \equiv r^* = r^m - p < r^m. \end{aligned} \tag{20}$$

Here are several things to observe. First, the equilibrium interest rate in our Model II is seen to be a constant, r^*. Second, in view of $p > 0$, $r^* < (\frac{1}{\alpha} - 1)\,(\alpha^2 A)^{\frac{1}{1-\alpha}}\, L/\eta = r^m$, where r^m is the equilibrium interest rate from Model I, that is, the case of permanent monopoly. Because of the limited duration of monopoly power in the present model, the expected rate of return on investing in R&D is smaller than in the case of no erosion of monopoly power.

The description of the household sector is as in Model I, but now per capita financial wealth is

$$a(t) = \frac{N^m(t)V(t)}{L},$$

because there are only $N^m(t) = N(t) - N^c(t)$ firms with positive market value, namely the the number of intermediate-goods firms that still supply under monopolistic conditions. The household's first-order conditions lead to the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r^* - \rho) = \frac{1}{\theta}\left[(\frac{1}{\alpha} - 1)\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} \frac{L}{\eta} - p - \rho\right] \equiv \gamma_c^* < \gamma_c^m, \tag{21}$$

where γ_c^m is the per capita consumption growth rate from Model I, the case of permanent monopoly. We assume parameters are such that $\gamma_c^* > 0$; this requires that the productivity of the economic system, as determined by A and L/η, is "large enough". In addition, to avoid unbounded utility, we assume $\rho > (1 - \theta)\gamma_c^*$.

(cont. on p. 9)