Economic Growth   Lecture Note 16

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Economic policy in the simple model with horizontal innovations

In an economy described by the simple increasing variety model in B&S, Chapter 6.1, it seems likely that Pareto-improving government intervention is possible. Generally, in models where a nonrival good (here technical knowledge) enters the scene, we should always be prepared that laissez-faire is not a good idea. And more specifically, in the increasing variety model the monopolist suppliers of specialized intermediate goods charge a price above the marginal cost of supplying these goods. This lowers demand for these goods compared to a situation with marginal cost pricing and static inefficiency seems likely. Moreover, since the incentive to invest in R&D depends on expected future profits, which in turn depend on the size of the markets for intermediate goods, we are inclined to expect there will be too little R&D in the economy.

To clarify the issue, we set up the social planner's problem, assuming that the criterion function of the social planner is the same as that of the representative household. Although time is continuous in the model, to save notation we date the time-dependent variables by a sub-script t instead of (t).

1 The social planner's problem

The dynamic problem faced by the social planner is to choose $(c_t, X_t)_{t=0}^{\infty}$ so as to:

$$\max U_0 = \int_0^{\infty} \frac{{c_t}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.} \tag{1}$$

$$c_t > 0, \ X_t \geq 0, \tag{2}$$

$$\dot{N}_t = \frac{1}{\eta}(Y_t - X_t - c_t L), \text{ where } Y_t = A{X_t}^{\alpha}(N_t L)^{1-\alpha} \text{ and } N_0 \text{ is given,} \tag{3}$$

$$N_t \geq 0 \text{ for all } t \geq 0. \tag{4}$$

The notation is: $c =$ per capita consumption, $X =$ aggregate input of specialized intermediate goods, $N =$ number of different intermediate goods types (varieties), N "large", $Y =$

output of the manufacturing (or "basic-goods") sector. In (3) indivisibilities are ignored and N is regarded as a continuous and differentiable function of time t. The remaining variables are constant parameters:

$\theta > 0$ is the (absolute) elasticity of marginal utility, $u'(c) = c^{-\theta}$. Thus, θ reflects aversion to consumption variation.

$\rho > 0$ is the pure rate of time preference. Thus, ρ reflects impatience.

$\eta > 0$ is the R&D cost (in real terms, i.e., in terms of manufacturing goods) per invention.

$L > 0$ is the constant size of population = labor force.

$A > 0$ is a constant, which depends on measurement units. For given measurement units, A can be interpreted as an index of total factor productivity (TFP), if N is given. Yet, in standard growth accounting it is rather $AN^{1-\alpha}$ that would be seen as TFP. Sound institutions and a high level of "social capital" (trust etc.) can contribute to a high A.

$\alpha \in (0, 1)$ is the elasticity of output wrt. input of intermediates.

A static efficiency condition lies behind that the aggregate production function is written as it is in (3). Indeed, static efficiency requires, among other things, that the marginal product of any kind of input is the same across the firms and equal to the marginal cost. For firm i in the manufacturing sector the model assumes

$$Y_i = A\left(\sum_{j=1}^{N} x_{ij}{}^{\alpha}\right) L_i^{1-\alpha}, \qquad i = 1, 2, ..., M. \tag{5}$$

Thus, a static efficiency requirement is that

$$\partial Y_i/\partial x_{ij} = \alpha A x_{ij}^{\alpha-1} L_i{}^{1-\alpha} = p, \qquad i = 1, 2, ..., \ M; \quad j = 1, 2, ..., N. \tag{6}$$

where p is the required common value, across firms, of the marginal product of intermediate goods, the same for all j, in view of marginal costs being the same. (One might here add a further static efficiency requirement, namely that p should equal 1, which is the marginal cost of supplying the intermediate good. In any case, this condition comes out as one of the first-order conditions in the dynamic problem, see below.)

From (6) follows

$$x_{ij} = (\alpha A/p)^{\frac{1}{1-\alpha}} L_i \equiv x_i, \quad j = 1, 2, ..., N. \tag{7}$$

Hence, (5) can be simplified to

$$Y_i = AN{x_i}^{\alpha}{L_i}^{1-\alpha} = AN(\frac{x_i}{L_i})^{\alpha}L_i. \tag{8}$$

Now, from (7), we get

$$\frac{x_i}{L_i} = (\alpha A/p)^{\frac{1}{1-\alpha}}, \tag{9}$$

which is the same for all i. Thus, summing x_i over all i, we get the aggregate use of intermediate good j :

$$X_j = \sum_i x_{ij} = \sum_i x_i = (\alpha A/p)^{\frac{1}{1-\alpha}} \sum_i L_i = (\alpha A/p)^{\frac{1}{1-\alpha}} L \equiv X_{SP}. \tag{10}$$

Comparing with (9) we see that

$$\frac{x_i}{L_i} = \frac{X_{SP}}{L},$$

which substituted into (8) gives

$$Y_i = AN(\frac{X_{SP}}{L})^{\alpha}L_i.$$

Now, summing over all i yields

$$\begin{aligned} Y &= \sum_{i=1}^{M} Y_i = AN(\frac{X_{SP}}{L})^{\alpha}L = AX_{SP}^{\alpha}L^{1-\alpha}N & (11)\\ &= A(NX_{SP})^{\alpha}N^{1-\alpha}L^{1-\alpha} = AX^{\alpha}(NL)^{1-\alpha}, & (12) \end{aligned}$$

where X is the total input of intermediate goods, $X = NX_{SP}$. This concludes the demonstration that static efficiency implies (12) which is the same as the specification in (3).

2 Solving the problem

The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} + \lambda\frac{1}{\eta}(Y - X - cL), \qquad \text{where } Y = AX^{\alpha}(NL)^{1-\alpha}.$$

Here λ is the shadow price of knowledge, N, along the optimal path. An interior solution satisfies the first-order conditions:

$$\begin{aligned} \partial\mathcal{H}/\partial c &= c^{-\theta} - \frac{\lambda}{\eta}L = 0, \quad \text{i.e., } c^{-\theta} = \frac{\lambda}{\eta}L, & (13)\\ \partial\mathcal{H}/\partial X &= \frac{\lambda}{\eta}(\frac{\partial Y}{\partial X} - 1) = 0, \quad \text{i.e., } \frac{X}{Y} = \alpha, & (14)\\ \partial\mathcal{H}/\partial N &= \frac{\lambda}{\eta}\frac{\partial Y}{\partial N} = -\dot{\lambda} + \rho\lambda, \quad \text{i.e., } -\frac{\dot{\lambda}}{\lambda} = \frac{1}{\eta}(1-\alpha)\frac{Y}{N} - \rho. & (15) \end{aligned}$$

We guess that also the transversality condition

$$\lim_{t\to\infty} N_t \lambda_t e^{-\rho t} = 0 \tag{TVC}$$

is necessary for optimality.

Interpretation. On the margin, according to (13), income must be equally valuable in its two uses, consumption or R&D investment. Similarly, by (14), the marginal input of intermediates must satisfy that $\partial(Y-X)/\partial X = \partial Y/\partial X - 1 = 0$ or $\partial Y/\partial X = 1 = MC$. Thus, our p (from above) is 1. Moreover, (15) tells us that in the optimal plan, the no-arbitrage condition

$$\frac{\frac{\lambda}{\eta}\frac{\partial Y}{\partial N} + \dot{\lambda}}{\lambda} = \rho$$

must hold. Finally, (TVC) ensures that the asset, which is here knowledge, N, is not over-accumulated.

From (14) and (12) follows

$$X = \alpha A X^{\alpha}(NL)^{1-\alpha} = N(\alpha A)^{\frac{1}{1-\alpha}} L = N X_{SP}, \tag{16}$$

which is consistent with (10), in view of $p = 1$. Log-differentiating (13) wrt. t gives $-\theta\dot{c}/c = \dot{\lambda}/\lambda$, and by substituting (15) we get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\frac{1}{\eta}(1-\alpha)\frac{Y}{N} - \rho\right] = \frac{1}{\theta}\left[\frac{L}{\eta}(\frac{1}{\alpha}-1)(\alpha A)^{\frac{1}{1-\alpha}} - \rho\right] \equiv \gamma_{SP}. \tag{17}$$

This is the form taken by the Keynes-Ramsey rule in this problem.

It remains to characterize the path of N, X, and Y in our candidate solution. As reflected in (17), the marginal rate of return in the social planner's allocation is

$$r_{SP} = \frac{\partial Y}{\partial(\eta N)} = \frac{1}{\eta}\frac{\partial Y}{\partial N} = \frac{1}{\eta}(1-\alpha)\frac{Y}{N} = \frac{1}{\eta}(\frac{1}{\alpha}-1)(\alpha A)^{\frac{1}{1-\alpha}} L = \frac{1}{\eta}(\frac{1}{\alpha}-1)X_{SP}, \tag{18}$$

which is a constant. Moreover, by (11) and (10), with $p = 1$, the optimized aggregate production function is

$$Y = A X_{SP}^{\alpha} L^{1-\alpha} N = A((\alpha A)^{\frac{1}{1-\alpha}} L)^{\alpha} L^{1-\alpha} N = \frac{1}{\alpha}(\alpha A)^{\frac{1}{1-\alpha}} L N = \frac{1}{\alpha} X_{SP} N, \tag{19}$$

showing that output is proportional to "knowledge capital", here N. Thus, also the social planner's economy is of AK-style. From our general knowledge of AK-style models we know that the transversality condition (TVC) can only be satisfied if c_0 is chosen such that $\dot{N}/N = \dot{c}/c = \gamma_{SP}$, already from the beginning.[1] And then, by (16) and (19), also $\dot{X}/X = \gamma_{SP}$ and $\dot{Y}/Y = \gamma_{SP}$, respectively.

As (14) shows, $\partial Y/\partial X = 1$ is an optimality condition. The intuition behind this is that the input X should be increased up to the point where its marginal product equals the marginal cost of supplying the input. And this marginal cost is 1, as can be seen from (3).

To ensure positive growth ($\gamma_{SP} > 0$), we need

$$r_{SP} > \rho, \tag{A1}$$

where r_{SP} is given in (18). To ensure a bounded utility integral, we need the restriction

$$\rho > (1-\theta)\gamma_{SP}, \tag{A2}$$

with γ_{SP} given in (17).

Checking sufficient conditions. We may check whether our candidate solution *is* really an optimal solution by checking whether it satisfies the Mangasarian sufficient conditions. For a problem like this, with two control variables and one state variable, the Mangasarian sufficient conditions are:

1. 1. The Hamiltonian is jointly concave in the control and state variables.
   2. There is for all $t \geq 0$ a non-negativity constraint on the state variable.
   3. The candidate solution satisfies the transversality condition $\lim_{t\to\infty} N_t\lambda_t e^{-\rho t} = 0$, where λ_t is the adjoint variable in the current-value Hamiltonian and λ_t is non-negative for all $t \geq 0$.

We observe that our Hamiltonian is a sum of concave functions and is therefore itself jointly concave in (N, c, X). This confirms condition 1. Constraint (4) confirms condition 2. Finally, our candidate solution is constructed so as to satisfy condition 3. An explicit proof of this goes as follows. Our candidate solution gives

$$N_t\lambda_t e^{-\rho t} = N_0 e^{\gamma_{SP}t}\lambda_0 e^{\theta\gamma_{SP}t}e^{-\rho t} = N_0\lambda_0 e^{[(1-\theta)\gamma_{SP}-\rho]t} \to 0 \text{ for } t \to \infty,$$

in view of (A2). This is exactly the transversality condition (TVC).

Thus, our candidate solution satisfies Mangasarian's sufficient conditions. It follows that our candidate solution *is* an optimal solution. We shall call it the *SP allocation.*

[1] A detailed proof of this can be based on solving the differential equation (3), given $c_t = c_0 e^{\gamma_{SP}}$.

3 Comparing with laissez faire

In the laissez-faire market economy with infinitely-lived patents and monopoly pricing, the monopoly price, $1/\alpha$, implies too low demand for each type of intermediate good, namely the demand

$$X_m = (\alpha^2 A)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{1}{1-\alpha}} (\alpha A)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{1}{1-\alpha}} X_{SP} < X_{SP},$$

cf. (10) with $p = 1/\alpha$. Therefore, from a social point of view, too little of these goods is supplied and used. This results in too little remuneration of the R&D activity, which invents new types of intermediate goods. Consequently, there is too little incentive to do R&D, and the growth rate becomes too small. Indeed, the rate of return on saving (investing in R&D) will be

$$r = \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_m < \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_{SP} = r_{SP}.$$

4 Implementing the SP allocation

We imagine there is a government that attempts to obtain the social planner's allocation in a decentralized way. The government pays a subsidy at constant rate, σ, to purchases of intermediate goods such that the net price of intermediate good j is $(1-\sigma)P_j$, where $P_j = 1/\alpha$ is the price set by the monopolist supplier of good j. The government finances this subsidy by taxing consumption at the constant rate τ. Our question is: can this policy succeed?

Let us first derive the required value of the subsidy rate σ. The subsidy to firms in the manufacturing sector should be such that we end up with

$$\frac{\partial Y_i}{\partial x_{ij}} = MC = 1. \tag{20}$$

Given the subsidy σ, when firm i maximizes its profit under perfect competition, we have

$$\frac{\partial Y_i}{\partial x_{ij}} = (1-\sigma)P_j = (1-\sigma)\frac{1}{\alpha}, \quad \text{for } j = 1, 2, ..., N. \tag{21}$$

Combining this with (20), we get

$$\sigma = 1 - \alpha. \tag{22}$$

With the constant consumption tax rate, τ, the tax revenue is

$$T_t = \tau c_t L.$$

The required tax revenue to finance the aggregate government expenses on the subsidy is

$$T_t = \sum_{j=1}^{N_t} \sigma P_j X_j = (1-\alpha)\frac{1}{\alpha}\sum_{j=1}^{N_t} X_j = (1-\alpha)\frac{1}{\alpha}N_t X_{SP}.$$

Hence, the required tax rate is

$$\tau = (1-\alpha)\frac{N_t X_{SP}}{\alpha c_t L}. \tag{23}$$

It remains to determine $c_t L/N_t$. Dividing through by N_t in (3) gives

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\eta}\left(\frac{Y_t}{N_t} - \frac{X_t}{N_t} - \frac{c_t}{N_t}L\right) = \gamma_{SP},$$

as noted above. Substituting (19) and (16), this yields

$$\frac{1}{\eta}\left(\frac{1}{\alpha}X_{SP} - X_{SP} - \frac{c_t L}{N_t}\right) = \gamma_{SP},$$

or

$$\frac{c_t L}{N_t} = (\frac{1}{\alpha} - 1)X_{SP} - \eta\gamma_{SP}.$$

Substituting into (23), we get

$$\tau = \frac{(1-\alpha)\frac{X_{SP}}{\alpha}}{(\frac{1}{\alpha}-1)X_{SP} - \eta\gamma_{SP}} = \frac{(1-\alpha)X_{SP}}{(1-\alpha)X_{SP} - \alpha\eta\gamma_{SP}} > 1.$$

In view of (A2), we can be sure that the denominator is positive, since (A2) implies $\gamma_{SP} < \rho + \theta\gamma_{SP} = \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_{SP}$, from (17), so that

$$\alpha\eta\gamma_{SP} < (1-\alpha)X_{SP}.$$

There are alternative ways of writing the solution for τ :

$$\tau = \frac{\theta(1-\alpha)X_{SP}}{\theta(1-\alpha)X_{SP} - (\alpha(\frac{1}{\alpha}-1)X_{SP} - \alpha\eta\rho)} = \frac{\theta(1-\alpha)X_{SP}}{(\theta-1)(1-\alpha)X_{SP} + \alpha\eta\rho}.$$

The fiscal policy (σ, τ) establishes $X_j = X_{SP}$ for all j and so the "right"growth rate, γ_{SP}, is ensured. The policy is thus sufficient to establish the SP allocation (although, possibly, a consumption tax above 100% may not be popular).

5 Concluding remarks

A theoretically interesting aspect of the model is that it describes productivity growth as coming about through purposeful decisions by firms in search for monopoly profits on innovations. A theoretically weak aspect is that the model relies on very special functional forms. In particular, the model contains the following arbitrary parameter links. The elasticity of substitution between intermediates in (5) is $1/(1-\alpha)$ and at the same time $1-\alpha$ is the production elasticity wrt. labor as well as wrt. knowledge. These knife-edge conditions lie behind the reduced-form AK structure of the model.

It is also theoretically unsatisfactory that the model implies a strong scale effect (larger population implies higher growth rate). Therefore, the model can not allow population growth without a forever rising per capita growth rate. This feature is empirically a failure. The industrialized part of the world economy has had population growth of, say, $\frac{1}{2}$-1% per year for a century, but per capita growth rates have been essentially stationary.

A peculiar feature of the "simple increasing variety model" considered here is that an R&D subsidy is *not* needed to implement the social planner's allocation. A subsidy to purchases of the monopolized intermediate goods was enough. This is because there is in the model a one-to-one relationship between the key static efficiency condition ($X_j = X_{SP}$) and the rate of return on investment in R&D. There is only one stock variable, the number of intermediate goods varieties, and there is no positive intertemporal externality from current R&D to future productivity of R&D.

In most innovation-based increasing variety models, however, this is not so. The basic reason is that these models usually have more than one stock variable and also often contain a positive intertemporal externality from R&D activity. Then, typically, in addition to a subsidy to compensate for monopolist pricing, a subsidy to increase the incentive to invest in R&D is needed. Examples:

1. The model with stochastic erosion of monopoly power in B & S. Here there are *two* endogenous stock variables, the number of intermediate goods varieties still supplied by monopolists and the number of intermediate goods varieties supplied under competitive conditions.

2. Romer's 1990-model, which has a separate R&D sector with its own "production function", and which has two endogenous stock variables, physical capital and the

level of knowledge. The same holds true in the extended versions in Jones (1995) and Alvarez & Groth (2005).

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