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# Perspectives on learning-by-doing and learning-by-investing

As a follow-up on Lecture Note 8, this lecture note contains the following sections:

1. Learning by doing.
2. Learning by investing.
3. The size of the learning parameter.
4. Disembodied vs. embodied technical change.
5. Taking stock.
6. Weak and strong scale effects.
7. Static comparative advantage vs. dynamics of learning by doing.

The growth rate of any time-dependent variable $z > 0$ is written $g_z \equiv \dot{z}/z$.

## 1 Learning by doing

The term *learning by doing* refers to the hypothesis that accumulated work experience, especially repetition of the same type of action, improves workers' productivity and adds to technical knowledge.

A learning-by-doing model typically combines an aggregate CRS production function,

$$Y_t = F(K_t, T_t L_t), \tag{1}$$

with a learning function, for example,

$$\dot{T}_t = AY_t^{\lambda}, \qquad A > 0, 0 < \lambda \leq 1, \tag{2}$$

where $\lambda$ is a learning parameter (otherwise notation is standard). In Section 7 below, inspired by Krugman (1987) and Lucas (1988), we consider a two-sector model where each sector's productivity growth is governed by such a relationship (with $\lambda = 1$).

Another learning hypothesis is of the form

$$\dot{T}_t = AT_t^{\lambda} L_t^{\mu}, \qquad T_0 > 0 \text{ given}, A > 0, \lambda \leq 1, \mu > 0. \tag{3}$$

Here $B$ is just a constant that depends on measuring units, whereas both $\lambda$ and $\mu$ are learning parameters, reflecting the elasticities of learning wrt. the technology level and labor hours, respectively. The higher the number of human beings involved in production and the more time they spend in production, the more experience is accumulated. Sub-optimal ingredients in the production processes are identified and eliminated. The experience and knowledge arising in one firm or one sector is speedily diffused to other firms and other sectors in the economy (knowledge spillovers),and as a result the aggregate productivity level is increased.

Since hours spent, $L_t$, is perhaps a better indicator for "experience" than output, $Y_t$, specification (3) may seem more appealing than specification (2). In any event, to begin with we will concentrate on (3).

If the labor force is growing, $\lambda$ should be assumed strictly less than one, because with $\lambda = 1$ there would be a built-in tendency to forever faster growth, which does not seem plausible. In fact, $\lambda < 0$ can not be ruled out; that would reflect that learning becomes more and more difficult ("the easiest ideas are found first"). On the other hand, the case of "standing on the shoulders" is also possible, that is, the case $0 < \lambda \leq 1$, which is the case where learning becomes easier, the more is learnt already.

Sometimes the $L$ in (3) is replaced simply by the size of population. Then the interpretation is that "population breeds ideas", cf. Kremer (1993). But in many models labor force and population size are proportional, and then it does not matter whether we use the learning-by-doing interpretation or the population-breeds-ideas interpretation.

Arrow (1962) refers to the so-called Horndal effect (reported by Lundberg, 1961):

> "The Horndal-iron works in Sweden had no new investment (and therefore presumably no significant change in its methods of production) for a period of 15 years, yet productivity (output per man-hour) rose on the average close to 2 % per annum. We find again steadily increasing performance which can only be imputed to learning from experience."

Similar patterns of on-the-job productivity improvements have been observed in shipbuilding, airframe construction, and chemical industries. On the other hand, within a single production line there seems to be a tendency for this kind of productivity increases to gradually peter out, which suggests $\lambda < 0$ in (3). We may call this phenomenon "diminishing returns in the learning process": the potential for new learning gradually evens out as more and more learning has already taken place. But new products are continuously invented and the accumulated knowledge is transmitted to the production of these new products that start on a "new learning curve", along which there is initially "a large amount to be learned".[1] This combination of qualitative innovation and continuous productivity improvement through learning may at the aggregate level end up in a $\lambda \geq 0$ in (3).

In any case, whatever the sign of $\lambda$ at the aggregate level, with $\lambda < 1$, this model is capable of generating sustained endogenous per capita growth if the labor force is growing at a constant rate $n > 0$. Indeed, as in Lecture Note 8, there are two cases that are consistent with a balanced growth path (BGP for short) with positive per capita growth, namely the case $\lambda < 1$ combined with $n > 0$, and the case $\lambda = 1$ combined with $n = 0$.

In both cases we will consider a closed economy with $L_t = L_0 e^{nt}$, $n \geq 0$, and with capital accumulation according to

$$\dot{K}_t = I_t - \delta K_t = Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ given.} \tag{4}$$

### 1.1 The case: $\lambda < 1$

Let us concentrate on a BGP. What is the growth rate of $y \equiv Y/L$ along a BGP?

There are two steps in the calculation of this growth rate.

*Step 1.* From the balanced growth proposition (Lecture Note 6, p. 3), we know that not only is, by definition, $g_Y$ and $g_K$ then constant, but they are also the same so that $Y_t/K_t$ is constant over time. Owing to the CRS assumption, (1) implies that

$$1 = F(\frac{K_t}{Y_t}, \frac{T_t L_t}{Y_t}). \tag{5}$$

[1] A *learning curve* is a graph of estimated productivity (or its inverse, cf. Fig. 1 or Fig. 2 below) as a function of cumulative output or of time passed since production of the new product began at a given plant.

Since $Y_t/K_t$ is constant, $T_tL_t/Y_t \equiv T_t/y_t$ must be constant. This implies that

$$g_T = g_y = g_Y - n, \tag{6}$$

a constant.

*Step 2.* Dividing through by $T_t$ in (3), we get

$$g_T \equiv \frac{\dot{T}_t}{T_t} = AT_t^{\lambda-1}L_t^{\mu}.$$

Taking logs gives $\log g_T = \log A + (\lambda - 1)\log T + \mu \log L$. And taking the time derivative on both sides of this equation leads to

$$\frac{\dot{g}_T}{g_T} = (\lambda - 1)g_T + \mu n. \tag{7}$$

In view of $g_T$ being constant along a BGP, we have $\dot{g}_T = 0$, and so (7) gives

$$g_T = \frac{\mu n}{1-\lambda},$$

presupposing $\lambda < 1$. Hence, by (6),

$$g_y = \frac{\mu n}{1-\lambda}.$$

Under the assumption that $n > 0$, this per capita growth rate is positive, whatever the sign of $\lambda$. Given $n$, the growth rate is an increasing function of *both* learning parameters. Since a positive per capita growth rate can in the long run be maintained only if supported by $n > 0$, this is an example of *semi-endogenous growth* (as long as $n$ is exogenous).

This model thus gives growth results somewhat similar to the results in Arrow's learning-by-investing model, cf. Lecture Note 8. In the two models the learning is an unintended by-product of the work process and construction of investment goods, respectively. And both models assume that knowledge spillovers across firms are fast. So there are positive externalities which may motivate government intervention.

**Different approaches to the calculation of long-run growth rates** Even within this semi-endogenous growth case, depending on the situation, different approaches to the calculation of long-run growth rates are appropriate. In Lecture Note 8, in the analysis of the Arrow case $\lambda < 1$, the point of departure in the calculation was the steady state property of Arrow's model that $\tilde{k} \equiv K/(TL)$ is a constant. But this point of departure presupposes that we have established a well-defined steady state in the sense of a stationary point of a complete dynamic system (in the Arrow model consisting of two first-order differential equations in $\tilde{k}$ and $\tilde{c}$). In the present case we are not in this situation because we have not specified how the saving in (4) is determined. This explains why above we have taken another approach to the calculation of the long-run growth rate. We simply assume balanced growth and ask what the growth rate must then be. If the technologies in the economy are specified in such a way as to allow only exogenous productivity growth or semi-endogenous productivity growth in the long run, this approach is usually sufficient to determine a unique growth rate.

Note also, however, that this latter feature is in itself an interesting and useful result. It tells us what the growth rate *must* be in the long run provided that the system converges to balanced growth in the long run. The growth rate will be the same, independently of the specification of the household sector, that is, it will be the same whether, for example, there is a Ramsey-style household sector or an overlapping generations set-up. And in the first case the growth rate will be the same whatever the size of the preference parameters (the rate of time preference and the elasticity of marginal utility of consumption). Moreover, only if economic policy affects the learning parameters (or the population growth rate), will the long-run growth rate be affected. Still, economic policy can *temporarily* affect economic growth and in this way affect the *level* of the long-run growth path.

## 1.2 The case $\lambda = 1$, $n = 0$

With $\lambda = 1$, the above growth rate formulas are no longer valid. But returning to (3), we have $g_T = AL_t^{\mu}$. Then, unless $n = 0$, growth will tend to rise forever, since we have $g_T = AL_0^{\mu}e^{\mu n t} \to \infty$ for $n > 0$. So we will assume $n = 0$. Thus, $g_T = AL_0^{\mu}$, a positive constant. Since both $A$ and $L$ are exogenous, it is *as if* the rate of technical progress, $g_T$, were exogenous. Yet, technical progress is generated by an internal mechanism. If the government by economic policy could affect $A$ or $L$, also $g_T$ would be affected. In any case, under balanced growth (5) holds again and so $T_t L_t / Y_t = T_t / y_t$ must be constant. This implies $g_y = g_T = AL^{\mu} > 0$. Consequently, positive per capita growth can be maintained forever without support of growth in any exogenous factor, that is, growth is *fully endogenous*.

As in the semi-endogenous growth case we can here determine the growth rate along a

BGP independently of how the household sector is described. And preference parameters do not affect the growth rate. The fact that this is so even in the fully endogenous growth case is due to the "law of motion" of technology making up a subsystem that is independent of the remainder of the economic system. This is a special feature of the "growth engine" (3). The simple alternative, (2), is very different in this respect and so is the learning-by-investing case, to which we turn below.

Before proceeding, a brief remark on the explosive case $\lambda > 1$ seems in place. If we imagine $\lambda > 1$, growth becomes explosive in the extreme sense that productivity, output, and consumption will tend to *infinity in finite time.* This is so even if $n = 0$. The argument is based on the mathematical fact that, given a differential equation $\dot{x} = x^a$, where $a > 1$ and $x_0 > 0$, the solution $x_t$ has the property that there exists a $t_1 > 0$ such that $x_t \to \infty$ for $t \to t_1$. We come back to the explosion issue in Lecture note 10.

# 2 Learning by investing

In the above framework the work process is a source of learning whether it takes place in the consumption or capital goods sector. This is *learning by doing* in a broad sense. If the source of learning is specifically associated with the construction of capital goods, it is still natural to say that learning by doing is present. But in this case one can add that the learning by doing takes the form of *learning by investing.* Accordingly, we classify *learning by investing* as a specific form of learning by doing.

There seems to be a presumption in the literature that from an empirical point of view, learning by investing is the most important form; ship-building and airframe construction are prominent examples. To the extent that the construction of capital equipment is based on more complex and involved technologies than is the production of consumer goods, we are also, intuitively, inclined to expect that the greatest potential for productivity increases through learning is in the investment goods sector. Yet, after the information-technology (IT) revolution, this traditional presumption is perhaps less compelling.

In any event, in the simplest version of the learning-by-investment hypothesis, (3) above is replaced by

$$T_t = \left( \int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{8}$$

where $I_s^n$ is aggregate *net* investment. The Arrow and Romer models, as described in

Lecture Note 8, correspond to the cases $0 < \lambda < 1$ and $\lambda = 1$, respectively.[2]

In this framework, where the "growth engine" depends on capital accumulation, it is only in the Arrow case that we can calculate the per-capita growth rate along a BGP without specifying anything about the household sector.

## 2.1 The Arrow case: $\lambda < 1$

We may apply the same two steps as above. Step 1 is then an exact replication of step 1 above. Step 2 turns out to be even simpler than above, because (8) immediately gives $\log T = \lambda \log K$ so that $g_T = \lambda g_K$, which substituted into (6) yields

$$g_T = \lambda g_K = g_y = g_Y - n = g_K - n.$$

From this follows, first,

$$g_K = \frac{n}{1-\lambda}, \tag{9}$$

and, second,

$$g_y = \frac{\lambda n}{1-\lambda}.$$

Alternatively, we may in this case condense the two steps into one by rewriting (5) in the form

$$\frac{Y_t}{K_t} = F(1, \frac{T_t L_t}{K_t}) = F(1, K_t^{\lambda-1} L_t),$$

by (8). Along the BGP, since $Y/K$ is constant, so must the second argument, $K_t^{\lambda-1} L_t$, be. It follows that

$$(\lambda - 1) g_K + n = 0,$$

thus confirming (9).

Whatever the approach to the calculation, the per capita growth rate is here tied down by the size of the learning parameter and the growth rate of the labor force.

## 2.2 The Romer case: $\lambda = 1$ and $n = 0$

In the Romer case the growth rate along a BGP cannot be determined until the saving behavior in the economy is modeled. Indeed, the knife-edge case $\lambda = 1$ opens up for many different per capita growth rates under balanced growth. Which one is "selected"

[2] In equation (2) of Lecture Note 8 there is a misprint. The equation should read exactly the same as (8) above.

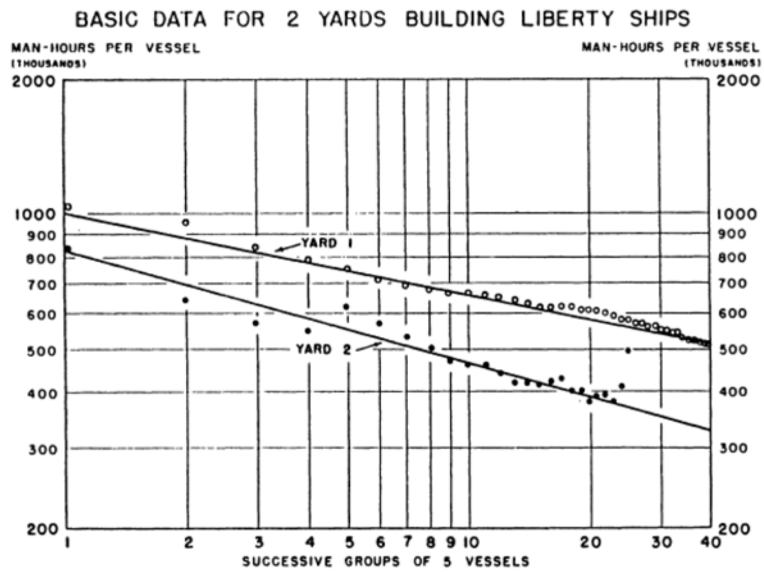

FIGURE 1.—Reductions in man-hours per vessel with increasing production. Merchant shipyards.

Figure 1:

by the economy depends on how the household sector is described. For a Ramsey setup with $n = 0$ the last part of Lecture Note 8 showed how the growth rate generated by the economy depends on the rate of time preference and the elasticity of marginal utility of consumption of the representative household. Not only is growth here *fully endogenous* in the sense that a positive per capita growth rate can be maintained forever without the support by growth in any exogenous factor. An economic policy that subsidizes investment can raise not only the productivity level, but also the productivity growth rate in the long run.

# 3 The size of the learning parameter

What is from an empirical point of view a plausible value for the learning parameter, $\lambda$? This question is important because the analysis of the models shows that quite different results emerges depending on whether $\lambda$ is close to 1 or considerably lower (fully endogenous growth or semi-endogenous growth). At the same time the question is difficult because $\lambda$ in the models is a parameter that represents the *aggregate* effect of learning in the single firms and industries and diffusion across firms and industries.

Like Lucas (1993), we will consider the empirical studies of on-the-job productivity

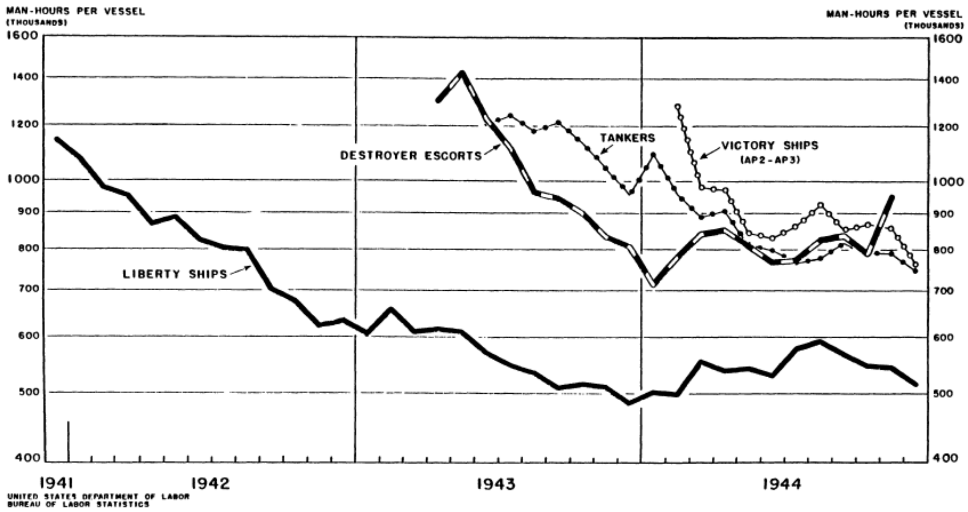


FIGURE 2.—Unit man-hour requirements for selected shipbuilding programs. Vessels delivered December 1941–December 1944.

Figure 2:

increases in ship-building by Searle (1945) and Rapping (1965). Both studies used data on the production of different types of cargo vessels during the second world war. Figures 1 and 2 are taken from Lucas' review article. For the vessel type called "Liberty Ships" Lucas cites the observation by Searle (1945):

> "the reduction in man-hours per ship with each doubling of cumulative output ranged from 12 to 24 percent."

Let us try to connect this observation to the learning parameter $\lambda$ in Arrow's and Romer's framework. We begin by considering firm $i$ which operates in the investment goods sector. We imagine that firm $i$'s equipment is unchanged during the observation period (as is understood in the above citation as well as the citation from Arrow (1962) in Section 1). Let firm $i$'s current output and employment be $Y_{it}$ and $L_{it}$, respectively. The current labor productivity is then $a_{it} = Y_{it}/L_{it}$. Let the firm's *cumulative* output be denoted $Q_{it}$. This cumulative output is a component of cumulative investment in society. At the micro-level the learning-by-investing hypothesis is the hypothesis that labor productivity is an increasing function of the firm's cumulative output, $Q_{it}$.

In figures 1 and 2 the dependent variable is not directly labor productivity, but its inverse, namely the required man-hours per unit of output, $m_{it} = L_{it}/Y_{it} = 1/a_{it}$. Fig. 1

suggests a log-linear relationship between this variable and the cumulative output:

$$\log m_{it} = \alpha - \beta \log Q_{it}. \tag{10}$$

Then the required man-hours per unit of output declines over time in the following way as cumulative output rises,

$$m_{it} = \frac{e^{\alpha}}{Q_{it}^{\beta}}.$$

Or labor productivity rises over time in this way:

$$a_{it} = \frac{1}{m_{it}} = e^{-\alpha} Q_{it}^{\beta}.$$

So, specifying the relationship by a power function, as in (8), makes sense.

Now, let $t = t_1$ be a fixed point in time. Then, (10) becomes

$$\log m_{it_1} = \alpha - \beta \log Q_{it_1}.$$

Let $t_2$ be the later point in time where cumulative output has been doubled. Then at time $t_2$ the required man-hours per unit of output has declined to

$$\log m_{it_2} = \alpha - \beta \log Q_{it_2} = \alpha - \beta \log(2Q_{it_1}).$$

Hence,

$$\log m_{it_1} - \log m_{it_2} = -\beta \log Q_{it_1} + \beta \log(2Q_{it_1}) = \beta \log 2. \tag{11}$$

Lucas' citation above from Searle amounts to a claim that

$$0.12 < \frac{m_{it_1} - m_{it_2}}{m_{it_1}} < 0.24. \tag{12}$$

By a first-order Taylor approximation we have $\log m_{it_2} \approx \log m_{it_1} + (m_{it_2} - m_{it_1})/m_{it_1}$. Hence, $(m_{it_1} - m_{it_2})/m_{it_1} \approx \log m_{it_1} - \log m_{it_2}$. Substituting this into (12) gives, approximately,

$$0.12 < \log m_{it_1} - \log m_{it_2} < 0.24.$$

Combining this with (11) gives $0.12 < \beta \log 2 < 0.24$ so that

$$0.17 = \frac{0.12}{\log 2} < \beta < \frac{0.24}{\log 2} = 0.35.$$

Rapping (1965) finds by a more rigorous econometric approach $\beta$ in the vicinity of 0.26. Solow (1997) refers to data on airframe building. This data suggests $\beta = 1/3$.

How can this be translated into a guess on the "aggregate" parameter $\lambda$ in (8)? This is not an easy question and the subsequent remarks are very tentative. First of all, the potential for both internal and external learning seems to vary a lot across different industries. Second, the amount of spillovers can not simply be added to the $\beta$ above, since they are already partly included in the estimate of $\beta$. Even theoretically, the role of experience in different industries cannot simply be added up because to some extent there is redundancy due to *overlapping* experience and sometimes the learning in other industries is of limited relevance. Given that we are interested in an upper bound for $\lambda$, a "guestimate" is that the spillovers matter for the final $\lambda$ at most the same as $\beta$ so that $\lambda \leq 2\beta$.[3]

As a conclusion, a $\lambda$ higher than about 2/3 may be considered implausible and this speaks for the Arrow case of semi-endogenous growth rather than the Romer case of fully endogenous growth, at least as long as we think of learning as the sole source of productivity growth. Another point is that to the extent learning is internal, we should expect at least some firms to internalize the phenomenon in its optimizing behavior. Although the learning is far from fully excludable, it takes time for others to discover and imitate technical and organizational improvements. Many macro models ignore this and treat all learning by doing as a 100 percent externality, which seems an exaggeration.

A further issue is to what extent learning by investing takes the form of *disembodied* or *embodied* technical change.

# 4 Disembodied vs. embodied technical change

Arrow's and Romer's model build on the idea that the *source* of learning is primarily experience in the investment goods sector. But both models assume that the learning, via knowledge spillovers across firms, provides an engine of productivity growth in essentially *all* sectors of the economy and a firm can benefit from recent technical advances irrespective of whether its equipment is new or old. This takes us to the distinction between disembodied and embodied technical change.

[3]For more elaborate studies of empirical aspects of learning by doing and learning by investing, see Irwin and Klenow (1994), Thornton and Thompson (2001), and Greenwood and Jovanovic (2001). Studies finding that the quantitative importance of spillovers is significantly smaller than required by the Romer case include Englander and Mittelstadt (1988) and Benhabib and Jovanovic (1991).

## 4.1 Disembodied technical change

*Disembodied technical change* occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again (1) and (3). When the $K_t$ appearing in (1) refers to the total, historically accumulated capital stock, then the interpretation is that the higher technology level generated in (3) or (8) results in higher productivity of *all* labor, independently of the vintage of the capital equipment with which this labor is combined. Thus also firms with old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.

## 4.2 Embodied technical change

In contrast, we say that technical change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technical progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.

One way to formally represent embodied technical progress is to write capital accumulation in the following way,

$$\dot{K}_t = T_t I_t - \delta K_t, \tag{13}$$

where $I_t$ is gross investment in period $t$ and $T_t$ measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising $T_t$ so that a given level of investment gives rise to a greater and greater addition to the capital stock, $K$, measured in constant efficiency units. Even if technical change does not directly appear in the production function, that is, even if for instance (1) is replaced by $Y_t = F(K_t, L_t)$, the economy may in this manner still experience a rising standard of living.

Embodied technical progress is likely to result in a steady decline in the price of capital

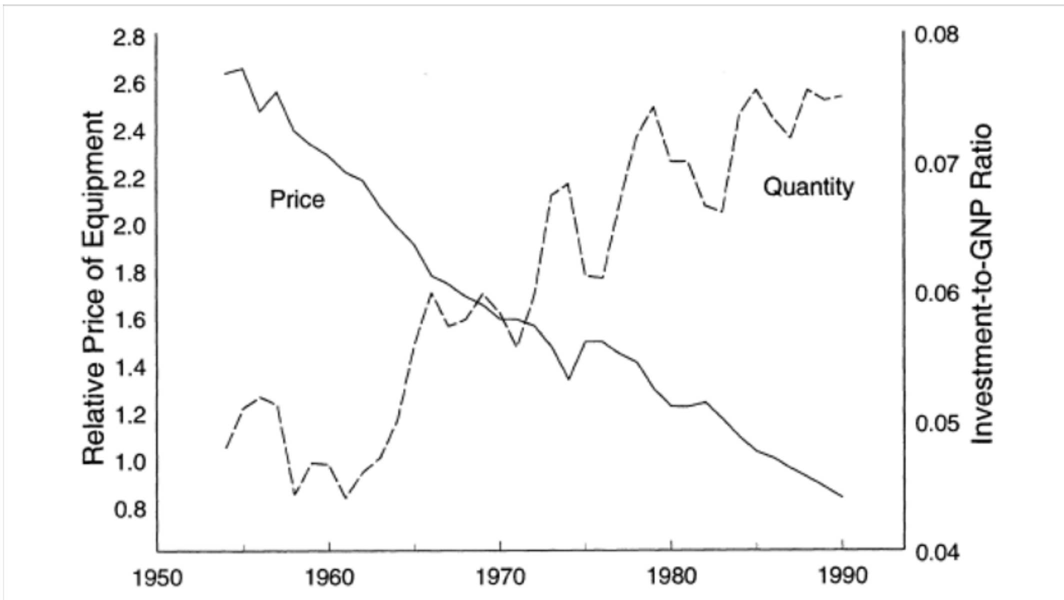


Figure 3: Relative price of equipment and equipment investment-to-GNP ratio. Source: Greenwood, Hercowitz, and Krusell (1997).

equipment relative to the price of consumption goods. This is what we see in the data. For the U.S. Greenwood et al. (1997) find that the relative price, $p$, of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990, cf. Fig. 3.[4] As Fig. 3 also shows, over the same period there has been a secular rise in the ratio of new equipment investment to GNP. Moreover, the correlation between de-trended $p$ and de-trended $I/GDP$ is $-0.46$. Greenwood et al. interpret this as evidence that technical advances have made equipment less expensive, triggering increases in the accumulation of equipment both in the short and long run. The authors also estimate that embodied technical change explains 60% of the growth in output per man hour.

## 4.3 Embodied technical change and learning by investing

So far nothing has been said about the source of increases in $T$ in (13). But a popular hypothesis is that the source is learning by investing. This learning may take the form (8) above. An alternative hypothesis, closer to both intuition and the original article by

[4]The relative price index in Fig. 3 is based on the book by R. Gordon (1990), which was an attempt to correct available price indices for equipment by better taking into account quality improvements in new equipment.

Arrow (Arrow 1962), is:

$$T_t = A\left(\int_{-\infty}^{t} I_s ds\right)^{\lambda}, \qquad A > 0,\ 0 < \lambda \leq \bar{\lambda}, \tag{14}$$

where $I_s$ is *gross* investment at time $s$. When we combine (14) with a Cobb-Douglas production function,

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \tag{15}$$

the upper bound, $\bar{\lambda}$, for the learning parameter, introduced to avoid explosive growth, is simply $\bar{\lambda} = (1-\alpha)/\alpha$.

Contrary to the integral based on net investment in (8), the integral in the learning hypothesis (14) does not allow an immediate translation into an expression in terms of the accumulated capital stock. But (14) gives rise to a tractable differential equation in $T$. Indeed, taking the time derivative on both sides in (14) gives

$$\dot{T}_t = A\lambda\left(\int_{-\infty}^{t} I_s ds\right)^{\lambda-1} I_t = \tilde{A} T_t^{(\lambda-1)/\lambda} I_t, \qquad \tilde{A} \equiv A^{1/\lambda}\lambda. \tag{16}$$

### 4.3.1 The case $\lambda < (1-\alpha)/\alpha$

Suppose $\lambda < (1-\alpha)/\alpha$. Using (16) together with (13), (15), and $I = Y - C$, one finds, under balanced growth with $s = I/Y$ constant and $0 < s < 1$,

$$g_K = \frac{(1-\alpha)(1+\lambda)n}{1-\alpha(1+\lambda)}, \tag{17}$$

$$g_T = \frac{\lambda}{1+\lambda} g_K, \tag{18}$$

$$g_Y = \frac{1}{1+\lambda} g_K, \tag{19}$$

$$g_y = g_Y - n = \frac{\alpha\lambda n}{1-\alpha(1+\lambda)}, \tag{20}$$

cf. Appendix A. So, if $n > 0$, there is semi-endogenous growth.

Let us assume there is perfect competition in all markets. Since $T$ capital goods can be produced at the same minimum cost as one consumption good, the equilibrium price, $p$, of one capital good in terms of the consumption good must equal the inverse of $q$, that is, $p = 1/T$. With the consumption good be the numeraire, let the rental rate in the market for capital services be denoted $R$ and the real interest rate in the market for loans be denoted $r$. Ignoring uncertainty, we have the no-arbitrage condition

$$\frac{R(t) - (\delta p(t) - \dot{p}(t))}{p(t)} = r(t), \tag{21}$$

where $\delta p(t) - \dot{p}(t)$ is the true economic depreciation of the capital good per time unit. Since $p = 1/T$, (18) and (17) indicate that along a BGP the relative price of capital goods will be declining according to

$$g_p = -\frac{(1-\alpha)\lambda n}{1-\alpha(1+\lambda)}.$$

Note that $g_K > g_Y$ along the BGP. Is this a violation of Proposition 1 of Lecture Note 6? No, that proposition presupposes that capital accumulation occurs according to the standard equation (4), not (13). And although $g_K$ differs from $g_Y$, the output-capital ratio in *value* terms, $Y/(pK)$, is constant along the BGP. In fact, the BGP complies entirely with Kaldor's stylized facts if we interpret "capital" as the value of capital, $pK$.

The formulas (17) and (20) display that $\alpha(1+\lambda) < 1$ is needed to avoid explosive growth if $n > 0$. This inequality is equivalent with $\lambda < (1-\alpha)/\alpha$ and confirms that the upper bound, $\bar{\lambda}$, in (14) equals $(1-\alpha)/\alpha$. With $\alpha = 1/3$, this upper bound is 2. The bound is thus no longer 1 as in the simple learning-by-investing model of Section 2. The reason is twofold, namely partly that now $T$ is formed via cumulative gross investment instead of net investment, partly that the role of $T$ is to strengthen capital formation rather than the efficiency of production factors in aggregate final goods produce.

### 4.3.2 The case $\lambda = (1-\alpha)/\alpha$ and $n = 0$

When $\lambda = (1-\alpha)/\alpha$, we have $\alpha(1+\lambda) = 1$ and so the growth formulas (17) and (20) no longer hold. But the way that (18) and (19) are derived (see Appendix A) ensures that these two equations remain valid along a BGP. Given $\lambda = (1-\alpha)/\alpha$, (18) reduces to $g_T = (1-\alpha)g_K$, which is equivalent with

$$T_t = BK_t^{1-\alpha}$$

along a BGP ($B$ is some positive constant). From (16) we then have

$$g_T = \frac{\dot{T}_t}{T_t} = \tilde{A}T_t^{-1/\lambda}I_t = \tilde{A}T_t^{-\alpha/(1-\alpha)}I_t = \tilde{A}B^{-\alpha/(1-\alpha)}K_t^{-\alpha}I_t = \tilde{A}B^{-\alpha/(1-\alpha)}K_t^{-\alpha}sY_t,$$

considering a BGP with $s = I/Y$ constant. Substituting (15) into this, we get

$$g_T = \tilde{A}B^{-\alpha/(1-\alpha)}K_t^{-\alpha}sK_t^{\alpha}L^{1-\alpha} = \tilde{A}B^{-\alpha/(1-\alpha)}sL^{1-\alpha}. \tag{22}$$

We see that if $n > 0$, there is a tendency to a forever increasing growth rate. To avoid this, let us, like Romer above, assume $n = 0$.

The striking feature is that (22) shows that the saving rate, $s$, matters for the growth rate of $T$, hence also of $K$ and $Y$, along a BGP. As in the Romer case of the disembodied learning-by-investing model, the growth rate along a BGP cannot be determined until the saving behavior in the economy is modeled. The considered knife-edge case, $\lambda = (1-\alpha)/\alpha$ combined with $n = 0$, opens up for many different per capita growth rates under balanced growth. Which one is "selected" by the economy depends on how the household sector is described. In a Ramsey setup with $n = 0$ one can show that the growth rate under balanced growth depends negatively on the rate of time preference and the elasticity of marginal utility of consumption of the representative household. And not only is growth in this case *fully endogenous* in the sense that a positive per capita growth rate can be maintained forever without the support by growth in any exogenous factor. An economic policy that subsidizes investment can increase not only the productivity level, but also the productivity growth rate in the long run.

If instead $\alpha > 1/(1+\beta)$, we get a tendency to explosive growth − infinite output in finite time − a not plausible scenario.

# 5 Taking stock

Recall our notation: $y \equiv Y/L$ and $g_y \equiv \dot{y}/y$. In this course I use the general definitions:

*Endogenous growth* is present if there is a positive long-run per capita growth rate (i.e., $g_y > 0$) and the source of this is some internal mechanism in the model (in contrast to exogenous technology growth).

*Fully endogenous growth* (sometimes called *strictly endogenous growth*) is present if there is a positive long-run per capita growth rate and this occurs without the support by growth in any exogenous factor (for example exogenous growth in the labor force).

Thus for example the Romer model of learning by investing features fully endogenous growth. The technical reason for this is the assumption that the learning parameter, $\lambda$, is such that there are constant returns to capital at the aggregate level. We get $g_y > 0$, constant, and, in a Ramsey set-up, results like $\partial g_y/\partial\rho < 0$ and $\partial g_y/\partial\theta < 0$, that is, preference parameters matter for long-run growth. This suggests, at least at the theoretical level, that taxes and subsidies, by affecting incentives, may have effects on long-run growth (cf. Lecture Note 8 and B & S, p. 217-18). On the other hand, a fully-endogenous growth model *need* not have this implication. We saw an example of this in Section 1, where the "law of motion" of technology makes up a subsystem that is independent of the remainder of the economic system.

In any case, fully endogenous growth is technologically possible if and only if there are non-diminishing returns (at least asymptotically) to the producible inputs in the growth-generating sector(s).

*Semi-endogenous growth* is present if growth is endogenous but exponential growth can not be sustained without the support by growth in some exogenous factor (for example exogenous growth in the labor force).

For example, the Arrow model of learning by investing features semi-endogenous growth. The technical reason for this is the assumption that the learning parameter $\lambda < 1$, which implies diminishing returns to capital at the aggregate level. Along a BGP we get

$$g_y = \frac{\lambda n}{1-\lambda}.$$

If and only if $n > 0$, can a positive $g_y$ be maintained forever. The key role of population growth derives from the fact that although there are diminishing returns to capital at the aggregate level, there are increasing returns to scale wrt. capital *and* labor. For the increasing returns to be sufficiently exploited to generate exponential growth, population growth is needed. Note that in this case $\partial g_y/\partial\rho = 0 = \partial g_y/\partial\theta$, that is, preference parameters do not matter for *long-run* growth (only for the *level* of the growth path). This suggests that taxes and subsidies do not have *long-run* growth effects. Yet, in Arrow's model and similar semi-endogenous growth models economic policy can have important long-run *level* effects.

Strangely enough, in their entire book B & S do not call much attention to the distinction between fully endogenous growth and semi-endogenous growth. Rather, they tend to use the term endogenous growth as synonymous with what we (and others) call fully endogenous growth. Although the semi-endogenous growth cases are emphasized a lot by other authors, B & S for some reason tend to ignore them.

## 6 Weak and strong scale effects

As underlined in B & S, Section 4.3.5, Romer's learning-by-investing hypothesis (where the learning parameter equals 1) implies a problematic (strong) scale effect. When embedded in a Ramsey set-up the model generates a time path along which

$$g_y = g_k = g_c = \frac{1}{\theta}(F_1(1, L) - \delta - \rho).$$

From this follows not only standard results for fully endogenous growth models, such as

$$\frac{\partial g_y}{\partial \rho} < 0, \qquad \frac{\partial g_y}{\partial \theta} < 0,$$

but also[5]

$$\frac{\partial g_y}{\partial L} = \frac{1}{\theta} F_{12}(1, L) > 0. \tag{23}$$

This is because in this model the rate of return, $F_1(1, L) - \delta$, depends (positively) on $L$. Interpreting the size ("scale") of the economy as measured by the size, $L$, of the labor force, we call such an effect a *scale effect.* To distinguish it from another kind of scale effect, it is useful to name it a *scale effect on growth* or a *strong scale effect.*

Scale effects can be of a less dramatic form. In this case we speak of *scale effect on levels* or a *weak scale effect.* This form arises when the learning parameter is less than 1. Thus, in Arrow's model of learning-by-investing (cf. Lecture Note 8) with learning parameter $\lambda < 1$, the steady state growth rate is

$$g_y = g_k = g_c = \frac{\lambda n}{1 - \lambda}.$$

This growth rate is independent of the *size* of the economy. Consequently, in Arrow's model there is no strong scale effect. There is, however, a scale effect on *levels* in the sense that along a steady state growth path,

$$\frac{\partial y_t}{\partial L_0} > 0. \tag{24}$$

This says the following. Suppose we consider two closed economies characterized by the same parameters, including the same $n$.[6] The economies differ only wrt. initial size of the labor force. Suppose both economies are in steady state. Then, according to (24), the economy with the larger labor force has, for all $t$, larger output per unit of labor. The

[5] Here we use that a neoclassical production function $F(K, TL)$ with CRS satisfies the complementarity condition $F_{12} > 0$.

[6] Remember, in contrast to the Romer model, Arrow's model allows $n > 0$.

intuitive reason is that there are *knowledge spillovers* across firms. Indeed, the model assumes that the labor efficiency index, $T_t$, depends (positively) on *aggregate* cumulative (net) investment in that

$$T_t = K_t^{\lambda}.$$

Thus, through the spillovers, a given level of per capita investment increases labor productivity more in a larger economy (where $\dot{K}_t$ will be larger) than in a smaller economy. More generally, the fundamental background is that *technical knowledge is a non-rival good* − its use by one firm does not (in itself) limit the amount of knowledge available to other firms.[7]

To prove (24) note that along a steady state path

$$y_t \equiv \tilde{y}_t T_t = \tilde{y}^* T_t = f(\tilde{k}^*) T_t = f(\tilde{k}^*) K_t^{\lambda}, \tag{25}$$

where

$$K_t \equiv \tilde{k}_t T_t L_t = \tilde{k}^* T_t L_t = \tilde{k}^* K_t^{\lambda} L_t.$$

Solving this equation for $K_t$ gives

$$K_t = (\tilde{k}^* L_t)^{1/(1-\lambda)} = (\tilde{k}^* L_0 e^{nt})^{1/(1-\lambda)}.$$

Substituting this into (25), we get

$$y_t = f(\tilde{k}^*)(\tilde{k}^* L_0 e^{nt})^{\lambda/(1-\lambda)}, \tag{26}$$

from which follows

$$\frac{\partial y_t}{\partial L_0} = \frac{\lambda}{1-\lambda} f(\tilde{k}^*)(\tilde{k}^* e^{nt})^{\lambda/(1-\lambda)} L_0^{[\lambda/(1-\lambda)]-1} = \frac{\lambda}{1-\lambda}\frac{y_t}{L_0}, \tag{27}$$

since $\tilde{k}^*$ is independent of $L_0$. This confirms (24). The scale effect on $y_t$ also gives scope for higher per capita consumption the higher is $L_0$.

The scale effect on levels displayed by (27) is increasing in the learning parameter $\lambda$, everything else equal. When $\lambda = 1$ the scale effect is so powerful that it is transformed into a scale effect on the growth rate.

In an internationalized world with a lot of knowledge spillovers across boarders cross-country regression analysis is not the right framework for testing for scale effects, whether

[7] By patent protection or secrecy some aspects of technical knowledge are sometimes *excludable,* but that is another matter (cf. B & S, p. 24, footnote 1). Excludability is ignored in our simple learning-by-doing and learning-by-investing models.

on levels or the growth rate. The relevant scale variable is not the size of the country, but the size of the region to which the country belongs, perhaps the whole world. Since in the last century there has been no clear upward trend in per capita growth rates in spite of a growing world population (and up to now also a growing population in the industrialized part of the world separately), most economists do not believe in strong scale effects. But on the issue of weak scale effects the opinion is divided. For a discussion, see Jones (2005).

# 7 Static comparative advantage vs. dynamics of learning by doing

Here we give a *very short* summary of a basic idea in Krugman (1987) and Lucas (1988, Section 5).

## 7.1 A simple two-sector learning-by-doing model

We consider an isolated economy with two production sectors, sector 1 and sector 2, each producing its specific consumption good. Labor is the only input and labor supply $L$ is constant. There are many small firms in the two sectors. Aggregate output in the sectors are:

$$\begin{aligned} Y_{1t} &= T_{1t}L_{1t}, && (28) \\ Y_{2t} &= T_{2t}L_{2t}, && (29) \end{aligned}$$

where

$$L_{1t} + L_{2t} = L.$$

There is *sector-specific* learning by doing in the following form:

$$\begin{aligned} \dot{T}_{1t} &= B_1 Y_{1t}, \qquad B_1 \geq 0, && (30) \\ \dot{T}_{2t} &= B_2 Y_{2t}, \qquad B_2 \geq 0. && (31) \end{aligned}$$

Let the relative price of sector 2-goods in terms of sector-1 goods be called $p_t$ (i.e., we use sector-1 goods as numeraire). Let the hourly wage in terms of sector-1 goods be $w_t$. Assume firms maximize profits and that there is perfect competition in the goods and labor markets. Then, in general equilibrium with production in both sectors:

$$T_{1t} = p_t T_{2t} = w_t,$$

saying that the value of the marginal product of labor in each sector equals the wage. Hence,

$$p_t \frac{T_{2t}}{T_{1t}} = 1 \quad \text{or} \quad p_t = \frac{T_{1t}}{T_{2t}}, \tag{32}$$

saying that the relative price of the two goods is inversely proportional to the relative labor productivities in the two sectors. The demand side, which is not modelled here, will of course play a role for the final allocation of labor to the two sectors.

Log-differentiating (32) wrt. $t$ gives

$$\frac{\dot{p}_t}{p_t} = \frac{\dot{T}_{1t}}{T_{1t}} - \frac{\dot{T}_{2t}}{T_{2t}} = \frac{B_1 Y_{1t}}{T_{1t}} - \frac{B_2 Y_{2t}}{T_{2t}} = B_1 L_{1t} - B_2 L_{2t},$$

using (30) and (31). Thus,

$$\dot{p}_t = (B_1 L_{1t} - B_2 L_{2t}) p_t.$$

Assume sector 2 (say some industrial activity) is more disposed to learning-by-doing than sector 1 (say mining) so that $B_2 > B_1$. Consider for simplicity the case where at time 0 there is symmetry in the sense that $L_{10} = L_{20}$. Then, the relative price $p_t$ of sector-2 goods in terms of sector-1 goods will, at least initially, tend to diminish over time. The resulting substitution effect is likely to stimulate demand for sector-2 goods. Suppose this effect is large enough to ensure that $L_2 = Y_2/T_2$ never becomes lower than $B_1 L_1/B_2$. Then the scenario with $\dot{p} \leq 0$ is sustained over time and the sector with highest growth potential remains a substantial constituent of the economy. This implies sustained economic growth in the aggregate economy.

Now, suppose the country considered is a rather backward, developing country, which until time $t_0$ has been a closed economy (very high tariffs etc.). Then the country decides to open up for free foreign trade. Let the relative world market price of sector 2-goods be $\bar{p}$, which we for simplicity assume is constant. At time $t_0$ there are two alternative possibilities to consider:

Case 1: $\bar{p} > \frac{T_{1t_0}}{T_{2t_0}}$ (world-market price of good 2 higher than the opportunity cost of producing good 2). Then the country specializes fully in sector-2 goods. Since this is the sector with a high growth potential, economic growth is stimulated. The relative productivity level $T_{1t}/T_{2t}$ decreases so that the scenario with $\bar{p} > T_{1t}/T_{2t}$ remains. A virtuous circle of dynamics of learning by doing is unfolded and high economic growth is sustained.

Case 2: $\bar{p} < \frac{T_{1t_0}}{T_{2t_0}}$ (world-market price of good 2 lower than the opportunity cost of producing good 2). Then the country specializes fully in sector-1 goods. Since this is the sector with a low growth potential, economic growth is impeded or completely halted. The relative productivity level $T_{1t}/T_{2t}$ does not decrease. Hence, the scenario with $\bar{p} < T_{1t}/T_{2t}$ remains. Low or zero economic growth is sustained. The static comparative advantage in sector-1 goods remains and the country is locked in low growth.

If instead $\bar{p}$ is time-dependent, suppose $\dot{\bar{p}}_t < 0$ (by similar arguments as for the closed economy). Then the case 2 scenario is again self-sustaining.

The point is that there may be circumstances (like in case 2), where protection for a backward country is growth promoting (the infant industry argument).

## 7.2 The resource curse problem

The analysis above also suggests a mechanism that, along with others, may help explaining the so-called *resource curse* problem. This problem refers to the paradox that being abundant in natural resources may sometimes seem a drag for a country rather than a blessing. At least quite many empirical studies have shown a negative correlation between resource abundance and economic growth (see, e.g., Sachs and Warner 1995).

Consider a mining country with an abundance of natural resources in the ground. Empirically growth in total factor productivity in mining activity is relatively low. Interpreting this as reflecting a relatively low learning potential in the sector, the mining sector may be represented by sector 1 above. Given the abundance of natural resources, $T_{1t_0}$ is likely to be high relative to the productivity in the manufacturing sector, $T_{2t_0}$. So the country is likely to be in the situation described as case 2 above. As a result, economic growth may never get started.

## 7.3 Discussion

The way (30) and (31) are formulated, we have

$$\frac{\dot{T}_{1t}}{T_{1t}} = B_1 L_{1t}, \tag{33}$$

$$\frac{\dot{T}_{2t}}{T_{2t}} = B_2 L_{2t}, \tag{34}$$

by (28) and (29). Thus, the model implies scale effects on growth, that is, *strong* scale effects. An alternative specification introduces limits to learning-by-doing in the following way:

$$\dot{T}_{1t} = B_1 Y_{1t}^{\varphi_1}, \qquad \varphi_1 < 1,$$
$$\dot{T}_{2t} = B_2 Y_{2t}^{\varphi_2}, \qquad \varphi_2 < 1.$$

Then (33) and (34) are replaced by

$$\frac{\dot{T}_{1t}}{T_{1t}} = B_1 T_{1t}^{\varphi_1 - 1} L_{1t}^{\varphi_1}, \tag{35}$$
$$\frac{\dot{T}_{2t}}{T_{2t}} = B_2 T_{2t}^{\varphi_2 - 1} L_{2t}^{\varphi_2}. \tag{36}$$

Now the problematic strong scale effect has disappeared. At the same time, since $\varphi_1 - 1 < 0$ and $\varphi_2 - 1 < 0$, (35) and (36) show that growth peters out as long as the "diminishing returns" to learning-by-doing are not compensated by an increasing labor force or an element of exogenous technical progress. If $n > 0$, we get sustained growth of the semi-endogenous type as in the Arrow model of learning-by-investing.

# 8 Appendix

**A. Balanced growth in the embodied technical change model with investment-specific learning**

In this appendix the results (17), (18), (19), and (20) are derived. The model is:

$$Y = K^\alpha L^{1-\alpha}, \tag{37}$$
$$I = Y - C, \tag{38}$$
$$\dot{K} = TI - \delta K, \tag{39}$$
$$\dot{T} = \tilde{A} T^{(\lambda-1)/\lambda} I, \tag{40}$$
$$L = L_0 e^{nt}. \tag{41}$$

Consider a BGP. By definition, $Y, K$, and $C$ then grow at constant rates, not necessarily positive. With $s = I/Y$ constant and $0 < s < 1$, (37) gives

$$g_I = g_Y = \alpha g_K + (1-\alpha)n, \tag{42}$$

a constant. By (39), $g_K = T\frac{I}{K} - \delta$, showing that $TI/K$ is constant along a BGP. Hence,

$$g_T + g_I = g_K, \tag{43}$$

and so also $g_T$ must be constant. From (40) follows that $g_T = \tilde{A}T^{-1/\lambda}I$. Taking logs in this equation and differentiating wrt. $t$ gives

$$\frac{\dot{g}_T}{g_T} = -\frac{1}{\lambda}g_T + g_I = 0,$$

in view of constancy of $g_T$. Substituting into (43) yields $(1+\lambda)g_I = g_K$, which combined with (42) gives

$$g_K = \frac{(1-\alpha)(1+\lambda)n}{1-\alpha(1+\lambda)},$$

which is (17). In view of $g_T = \lambda g_I = \lambda g_Y = \lambda(g_y + n) = \lambda g_K/(1+\lambda)$, the results (18), (19), and (20) immediately follow.

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