Economic Growth, June 2008.
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A suggested solution to the problem set at the exam in Economic Growth, June 13, 2008

(4-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the characteristics that matter for economic growth.

1. Solution to Problem 1 (10 %)

a) Not true. There is not convergence with respect to the standard deviation of income per capita. That would require the forces of convergence to be so strong as to produce a narrowing of the *absolute* distance between the EU countries. But the forces of convergence are *not* so strong.

Income per capita is trending upwards and the standard deviation is not a scale-free measure. What is true, at least until recently, is that the 12 old EU member countries tend to experience income convergence in a *relative* sense: the standard deviation of the *logarithm* of income per capita across the countries diminishes over time (or, more or less equivalently, the coefficient of variation diminishes over time).

b) The Alesina-Rodrik model is an attempt at explaining the presumed negative cross-country correlation between inequality and growth. The explanation

[1] The solution below contains *more* details and more precision than can be expected at a four hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

suggested by their model is called the "fiscal policy approach" and is based on two mechanisms:

(1) The *political mechanism*. High inequality leads to strong pressure, by the majority in the population, for redistribution through some form of progressive taxation.

(2) The *economic mechanism*. A high tax rate on capital income leads to a low after-tax rate of return on saving. This results in low aggregate investment and thereby low growth.

South Korea and the Philippines were in 1960 much alike in many respects of importance for growth, for example initial level of income per capita, size of population, degree of urbanization, shares of agriculture and manufacturing in the total economy, and educational level. But there was one respect in which the two countries differed a lot, namely the degree of income inequality as measured by the Gini coefficient, the Philippines having a very high Gini and South Korea a quite low Gini. Hence, according to the combined effect of the mechanisms (1) and (2), we should expect South Korea to have a higher per capita growth rate than the Philippines, as the data shows. Thus, applied to the difference in growth performance of South Korea and the Philippines, the fiscal policy explanation fits well at the theoretical level.

A problem with the explanation is, however, that when the two mechanisms are tested separately (as by Perotti, 1996), the political mechanism, (1), is not supported by the data. Perotti argues for an alternative explanation of the negative cross-country correlation between inequality and growth, emphasizing that (a) very unequal societies tend to be politically and socially unstable, which is reflected in low rates of investment, thus hampering growth; and (b) more equal societies have higher rates of investment in education, which promotes growth.

c) True. The congestion implies negative externalities and to reduce these, a welfare-maximizing government will use taxes that *dampen* economic activity. This allows higher initial productivity and consumption, but implies a forever lower growth rate (in the reduced-form AK model of Barro and Sala-i-Martin) in such a way that, due to the discounting of future consumption, a welfare-improvement is obtained.

2. Solution to Problem 2 (50 %)

For convenience, we here repeat the dynamic problem faced by the social planner. The problem is to choose $(c_t, X_t)_{t=0}^{\infty}$ so as to:

$$\max U_0 = \int_0^{\infty} \frac{c_t{}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.}$$

$$c_t > 0, \;\; X_t \geq 0,$$

$$\dot{N}_t = \frac{1}{\eta}(Y_t - X_t - c_t L), \;\text{ where } Y_t = A X_t{}^{\alpha}(N_t L)^{1-\alpha}, \; N_0 \text{ given}, \qquad (*)$$

$$N_t \geq 0 \text{ for all } t \geq 0. \qquad (**)$$

The notation is: $c =$ per capita consumption, $X =$ aggregate input of specialized intermediate goods, $N =$ number of different intermediate goods types (N "large", N can also be seen as a measure of the level of technical knowledge), and $Y =$ output of the manufacturing sector.

a) θ is the (absolute) elasticity of marginal utility, $u'(c) = c^{-\theta}$. Thus, θ reflects aversion to consumption variation.

ρ is the pure rate of time preference. Thus, ρ reflects impatience.

η is the R&D cost (in real terms, i.e., in terms of manufacturing goods) per invention.

L is the constant size of population = labor force.

A is a positive constant, which depends on measurement units. For given measurement units, A can be interpreted as an index of total factor productivity

(TFP), if N is given. Yet, in standard growth accounting it is rather $AN^{1-\alpha}$ that would be seen as TFP.

α is the elasticity of output wrt. input of intermediates.

b) A static efficiency condition lies behind that the aggregate production function takes the form (*). Indeed, static efficiency requires, among other things, that the marginal product of any kind of input is the same across the firms and equal to the marginal cost. For firm i in the manufacturing sector we have the production function

$$Y_i = A\left(\sum_{j=1}^{N} x_{ij}{}^{\alpha}\right) L_i^{1-\alpha}, \qquad i = 1, 2, ..., M. \tag{***}$$

Thus, a static efficiency requirement is that

$$\partial Y_i/\partial x_{ij} = \alpha A x_{ij}^{\alpha-1} L_i{}^{1-\alpha} = p, \qquad i = 1, 2, ..., \ M; \quad j = 1, 2, ..., N. \tag{2.1}$$

where p is the required common value, across firms, of the marginal product of intermediate goods, the same for all j, in view of marginal costs being the same. (One might here add a further static efficiency requirement, namely that p should equal 1, which is the marginal cost of supplying the intermediate good. In any case, this condition comes out as one of the first-order conditions in the dynamic problem, see below.) From (2.1) follows

$$x_{ij} = (\alpha A/p)^{\frac{1}{1-\alpha}} L_i \equiv x_i, \quad j = 1, 2, ..., N. \tag{2.2}$$

Hence, (***) can be simplified to

$$Y_i = ANx_i{}^{\alpha} L_i{}^{1-\alpha} = AN(\frac{x_i}{L_i})^{\alpha} L_i. \tag{2.3}$$

Now, from (2.2), we get

$$\frac{x_i}{L_i} = (\alpha A/p)^{\frac{1}{1-\alpha}}, \tag{2.4}$$

which is the same for all i. Thus, summing x_i over all i, we get the aggregate use of intermediate good j :

$$X_j = \sum_i x_{ij} = \sum_i x_i = (\alpha A/p)^{\frac{1}{1-\alpha}} \sum_i L_i = (\alpha A/p)^{\frac{1}{1-\alpha}} L \equiv X_{SP}. \tag{2.5}$$

Comparing with (2.4) we see that

$$\frac{x_i}{L_i} = \frac{X_{SP}}{L},$$

which substituted into (2.3) gives

$$Y_i = AN(\frac{X_{SP}}{L})^{\alpha} L_i.$$

Now, summing over all i yields

$$\begin{aligned} Y &= \sum_{i=1}^{M} Y_i = AN(\frac{X_{SP}}{L})^{\alpha} L = AX_{SP}^{\alpha} L^{1-\alpha} N & (2.6) \\ &= A(NX_{SP})^{\alpha} N^{1-\alpha} L^{1-\alpha} = AX^{\alpha}(NL)^{1-\alpha}, & (2.7) \end{aligned}$$

where X is the total input of intermediate goods, $X = NX_{SP}$. This concludes the demonstration that static efficiency implies (2.7) which is the same as (*).

c) The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda \frac{1}{\eta}(Y - X - cL), \qquad \text{where } Y = AX^{\alpha}(NL)^{1-\alpha}.$$

Here λ is the shadow price of knowledge (N) along the optimal path. An interior solution satisfies the first-order conditions:

$$\begin{aligned} \partial\mathcal{H}/\partial c &= c^{-\theta} - \frac{\lambda}{\eta} L = 0, \quad \text{i.e., } c^{-\theta} = \frac{\lambda}{\eta} L, & (2.8) \\ \partial\mathcal{H}/\partial X &= \frac{\lambda}{\eta}(\frac{\partial Y}{\partial X} - 1) = 0, \quad \text{i.e., } \frac{X}{Y} = \alpha, & (2.9) \\ \partial\mathcal{H}/\partial N &= \frac{\lambda}{\eta}\frac{\partial Y}{\partial N} = -\dot{\lambda} + \rho\lambda, \quad \text{i.e., } -\frac{\dot{\lambda}}{\lambda} = \frac{1}{\eta}(1-\alpha)\frac{Y}{N} - \rho. & (2.10) \end{aligned}$$

We guess that also the transversality condition

$$\lim_{t\to\infty} N_t \lambda_t e^{-\rho t} = 0 \tag{TVC}$$

is necessary for optimality.

Interpretation. On the margin, according to (2.8), income must be equally valuable in its two uses, consumption or R&D investment. Similarly, by (2.9), the marginal input of intermediates must satisfy that $\partial(Y-X)/\partial X = \partial Y/\partial X - 1 = 0$ or $\partial Y/\partial X = 1 = MC$. Thus, our p (from above) is 1. Moreover, (2.10) tells us that in the optimal plan, the no-arbitrage condition

$$\frac{\frac{\lambda}{\eta}\frac{\partial Y}{\partial N} + \dot{\lambda}}{\lambda} = \rho$$

must hold. Finally, (TVC) ensures that the asset, which is here knowledge, N, is not over-accumulated.

From (2.9) and (2.7) follows

$$X = \alpha A X^{\alpha}(NL)^{1-\alpha} = N(\alpha A)^{\frac{1}{1-\alpha}} L = N X_{SP}, \tag{2.11}$$

which is consistent with (2.5), in view of $p = 1$. Log-differentiating (2.8) wrt. t gives $-\theta \dot{c}/c = \dot{\lambda}/\lambda$, and by substituting (2.10) we get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\frac{1}{\eta}(1-\alpha)\frac{Y}{N} - \rho\right] = \frac{1}{\theta}\left[\frac{L}{\eta}(\frac{1}{\alpha} - 1)(\alpha A)^{\frac{1}{1-\alpha}} - \rho\right] \equiv \gamma_{SP}. \tag{2.12}$$

This is the form taken by the Keynes-Ramsey rule in this problem.

It remains to characterize the path of N, X, and Y in our candidate solution. As reflected in (2.12), the marginal rate of return is

$$r_{SP} = \frac{\partial Y}{\partial(\eta N)} = \frac{1}{\eta}\frac{\partial Y}{\partial N} = \frac{1}{\eta}(1-\alpha)\frac{Y}{N} = \frac{1}{\eta}(\frac{1}{\alpha} - 1)(\alpha A)^{\frac{1}{1-\alpha}} L = \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_{SP}, \tag{2.13}$$

which is a constant. Further, by (2.6) and (2.5), with $p = 1$, the optimized aggregate production function is

$$Y = A X_{SP}^{\alpha} L^{1-\alpha} N = A((\alpha A)^{\frac{1}{1-\alpha}} L)^{\alpha} L^{1-\alpha} N = \frac{1}{\alpha}(\alpha A)^{\frac{1}{1-\alpha}} L N = \frac{1}{\alpha} X_{SP} N, \tag{2.14}$$

showing that output is proportional to "knowledge capital", here N. Thus, our model is an AK-style model. From our general knowledge of AK-style models, we know that the transversality condition (TVC) can only be satisfied if c_0 is chosen such that $\dot{N}/N = \dot{c}/c = \gamma_{SP}$, already from the beginning.[2] And then, by (2.11) and (2.14), also $\dot{X}/X = \gamma_{SP}$ and $\dot{Y}/Y = \gamma_{SP}$, respectively. This completes the characterization of our candidate solution.

d) The reason that $\partial Y/\partial X = 1$ is an optimality condition is that the input X should be increased up to the point where its marginal product equals the marginal cost of supplying the input. And this marginal cost is 1, as can be seen from (*).

e) To ensure a bounded utility integral, we need the restriction

$$(1-\theta)\gamma_{SP} < \rho, \tag{A1}$$

with γ_{SP} given in (2.12). To ensure positive growth ($\gamma_{SP} > 0$), we need

$$r_{SP} > \rho, \tag{A2}$$

where r_{SP} is given in (2.13).

f) The Hamiltonian is a sum of concave functions and is therefore itself jointly concave in (N, c, X). This confirms condition 1. Constraint (**) confirms condition 2. Finally, our candidate solution is constructed so as to satisfy condition 3. An explicit proof of this goes as follows (but is not necessary). Our candidate solution gives

$$N_t\lambda_t e^{-\rho t} = N_0 e^{\gamma_{SP}t}\lambda_0 e^{\theta\gamma_{SP}t}e^{-\rho t} = N_0\lambda_0 e^{[(1-\theta)\gamma_{SP}-\rho]t} \to 0 \text{ for } t \to \infty,$$

in view of (A1). This is exactly the transversality condition (TVC).

[2] A detailed proof of this can be based on solving the differential equation (*), given $c_t = c_0 e^{\gamma_{SP}}$.

Thus, our candidate solution satisfies Mangasarian's sufficient conditions. It follows that our candidate solution *is* an optimal solution. We shall call it the *SP allocation*.

g) In the laissez-faire market economy with monopoly pricing, the monopoly price, $1/\alpha$, implies too low demand for each type of intermediate good, namely the demand

$$X_m = (\alpha^2 A)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{1}{1-\alpha}} (\alpha A)^{\frac{1}{1-\alpha}} L = \alpha^{\frac{1}{1-\alpha}} X_{SP} < X_{SP},$$

cf. (2.5) with $p = 1/\alpha$. Therefore, from a social point of view, too little of these goods is used. This results in too little remuneration of the R&D activity, which invents new types of intermediate goods. Consequently, there is too little incentive to do R&D, and the growth rate becomes too small. Indeed, the rate of return on saving (investing in R&D) will be

$$r = \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_m < \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_{SP} = r_{SP}.$$

h) The subsidy to firms in the manufacturing sector should be such that we end up with

$$\frac{\partial Y_i}{\partial x_{ij}} = MC = 1. \tag{2.15}$$

Given the subsidy σ, when firm i maximizes its profit under perfect competition, we have

$$\frac{\partial Y_i}{\partial x_{ij}} = (1-\sigma)P_j = (1-\sigma)\frac{1}{\alpha}, \qquad \text{for } j = 1, 2, ..., N. \tag{2.16}$$

Combining this with (2.15), we get

$$\sigma = 1 - \alpha. \tag{2.17}$$

i) With the constant consumption tax rate, τ, the tax revenue is

$$T_t = \tau c_t L.$$

The required tax revenue to finance the aggregate government expenses on the subsidy is

$$T_t = \sum_{j=1}^{N_t} \sigma P_j X_j = (1-\alpha)\frac{1}{\alpha}\sum_{j=1}^{N_t} X_j = (1-\alpha)\frac{1}{\alpha}N_t X_{SP}.$$

Hence, the required tax rate is

$$\tau = (1-\alpha)\frac{N_t X_{SP}}{\alpha c_t L}. \tag{2.18}$$

It remains to determine $c_t L/N_t$. Dividing through by N_t in (*) gives

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\eta}\left(\frac{Y_t}{N_t} - \frac{X_t}{N_t} - \frac{c_t}{N_t}L\right) = \gamma_{SP},$$

as noted under c). Substituting (2.14) and (2.11), this yields

$$\frac{1}{\eta}\left(\frac{1}{\alpha}X_{SP} - X_{SP} - \frac{c_t L}{N_t}\right) = \gamma_{SP},$$

or

$$\frac{c_t L}{N_t} = (\frac{1}{\alpha} - 1)X_{SP} - \eta\gamma_{SP}.$$

Substituting into (2.18), we get

$$\tau = \frac{(1-\alpha)\frac{X_{SP}}{\alpha}}{(\frac{1}{\alpha}-1)X_{SP} - \eta\gamma_{SP}} = \frac{(1-\alpha)X_{SP}}{(1-\alpha)X_{SP} - \alpha\eta\gamma_{SP}} > 1.$$

In view of (A2), we can be sure that the denominator is positive, since (A2) implies $\gamma_{SP} < \rho + \theta\gamma_{SP} = \frac{1}{\eta}(\frac{1}{\alpha} - 1)X_{SP}$, from (2.12), so that

$$\alpha\eta\gamma_{SP} < (1-\alpha)X_{SP}.$$

There are alternative ways of writing the solution for τ :

$$\tau = \frac{\theta(1-\alpha)X_{SP}}{\theta(1-\alpha)X_{SP} - (\alpha(\frac{1}{\alpha}-1)X_{SP} - \alpha\eta\rho)} = \frac{\theta(1-\alpha)X_{SP}}{(\theta-1)(1-\alpha)X_{SP} + \alpha\eta\rho}.$$

The fiscal policy (σ, τ) establishes $X_j = X_{SP}$ for all j and so the "right" growth rate, γ_{SP}, is ensured. The policy is thus sufficient to establish the SP allocation (although, possibly, a consumption tax above 100% may not be popular).

j) A theoretically interesting aspect of the model is that it describes productivity growth as coming about through purposeful decisions by firms in search for monopoly profits on innovations. A theoretically weak aspect is that the model relies on very special functional forms. In particular, the model contains the following arbitrary parameter links. The elasticity of substitution between intermediates in (***) is $1/(1-\alpha)$ and at the same time $1-\alpha$ is the production elasticity wrt. labor as well as wrt. knowledge. These knife-edge conditions lie behind the reduced-form AK structure of the model.

It is also theoretically unsatisfactory that the model implies a strong scale effect (larger population implies higher growth rate). Therefore, the model can not allow population growth without a forever rising per capita growth rate. This feature is empirically a failure. The industrialized part of the world economy has had population growth of, say, $\frac{1}{2}$-1% per year for a century, but per capita growth rates have been essentially stationary.

3. Solution to Problem 3 (30 %)

For convenience, we here repeat the basic equations:

$$Y = AK^{\alpha}(\pi L)^{1-\alpha}, \qquad A > 0, \quad 0 < \alpha < 1, \tag{3.1}$$

$$Y = C + I, \tag{3.2}$$

With $g_x \equiv \dot{x}/x$, log-differentiation wrt. t in (3.1) gives

$$g_Y = \alpha g_K + (1-\alpha)(g_\pi + g_L). \tag{3.3}$$

In view of the closed-economy assumption, we have under balanced growth $g_Y = g_K$, so that (3.3) gives $g_Y = g_\pi + g_L$. Defining $y \equiv Y/L$, it follows that

$$g_y = g_Y - g_L = g_\pi. \tag{3.4}$$

In Model 1, $g_L = n \geq 0$, a constant,

$$I = I_K + I_H, \tag{3.5}$$

$$\begin{aligned} \dot{K} &= I_K - \delta K, && (3.6) \\ \dot{H} &= I_H - \delta H, && (3.7) \end{aligned}$$

and

$$\pi = h^\beta, \qquad 0 < \beta \leq 1, \tag{3.8}$$

where $h \equiv H/L$.

a) We say that *endogenous growth* occurs if there is sustained per capita growth generated through some *internal* mechanism (in contrast to exogenous technology growth). In assessing whether Model 1 is technologically capable of generating endogenous growth, one may use either a short argument (based on general knowledge) or a longer argument.

The *short argument* (which is sufficient). Substitute (3.8) into (3.1) to get

$$\begin{aligned} Y &= AK^\alpha(h^\beta L)^{1-\alpha} = AK^\alpha h^{\beta(1-\alpha)}L^{1-\alpha} = AK^\alpha(\frac{H}{L})^{\beta(1-\alpha)}L^{1-\alpha} \\ &= AK^\alpha H^{\beta(1-\alpha)}L^{1-\alpha-\beta(1-\alpha)} = AK^\alpha H^{\beta(1-\alpha)}L^{(1-\beta)(1-\alpha)}. \end{aligned}$$

Case 1: $\beta = 1$. Then $Y = AK^\alpha H^{1-\alpha}$, so that the "growth engine" has CRS wrt. producible inputs, K and H. Hence, the model is technologically capable of generating *fully* endogenous growth. *Case 2:* $\beta < 1$. Then the "growth engine" has DRS wrt. producible inputs, K and H, and it has CRS wrt. $K, H,$ *and* L. Hence, neither fully endogenous nor semi-endogenous growth can be generated.

The reason that not even semi-endogenous growth is possible is the absence of IRS wrt. K, H, and L (which is due to the fact that no *non-rival* inputs are involved).

The *longer argument.* Consider a BGP, so that (3.4) holds, and we get

$$g_y = g_\pi = \beta g_h, \tag{3.9}$$

using (3.8). If it is technologically feasible to maintain g_h equal to a positive constant, endogenous growth is possible. Hence, we consider

$$\begin{aligned} g_h &= g_H - n = \frac{I_H}{hL} - \delta - n = s_H \frac{Y}{hL} - (\delta + n) \\ &= s_H \frac{AK^\alpha (h^\beta L)^{1-\alpha}}{hL} - (\delta + n) = s_H A k^\alpha h^{\beta(1-\alpha)-1} - (\delta + n), \end{aligned} \tag{3.10}$$

where $I_H = s_H Y$ and s_H is educational spending as a fraction of GDP. There are two cases.

Case 1: $\beta = 1$. Then (3.10) simplifies to

$$g_h = s_H A k^\alpha h^{-\alpha} - (\delta + n) = s_H A (\frac{k}{h})^\alpha - (\delta + n). \tag{3.11}$$

Within a finite time interval $[0, t_1)$, by investment specialization ($I_K > 0, I_H = 0$), k/h can be adjusted to a level, say ω, sufficiently high so that the RHS of (3.11) is positive for a sufficiently small s_H to leave room for both $I_K > 0$ and $C > 0$. Then, for $t \geq t_1$, let investment also in H take place, at rate $s_H > 0$, so as to maintain $k/h = \omega$ forever. Thus, *fully* endogenous growth occurs.

Case 2: $\beta < 1$. Then $\beta(1 - \alpha) - 1 < 0$, and

$$k^\alpha h^{\beta(1-\alpha)-1} = (\frac{k}{h})^\alpha h^{\beta(1-\alpha)-(1-\alpha)} = (\frac{k}{h})^\alpha h^{(\beta-1)(1-\alpha)}.$$

Now, even if k/h is kept constant, for a while, by net investment in both k and h, the growing h implies that $h^{\beta(1-\alpha)-1}$ decreases, and the RHS of (3.11) can not be maintained positive. Endogenous growth is not possible.

Model 2 shares equations (3.1), (3.2), (3.5), and (3.6) with Model 1, but replaces (3.7) and (3.8) by the Mincer hypothesis

$$\pi = e^{\psi s}, \quad \psi > 0, \tag{3.12}$$

where s is a measure of time spent in school by the "average citizen". In Model 2, $I_H \equiv 0$. With N denoting the population of age ≤ 65, Model 2 assumes $N \equiv N_t = N_0 e^{nt}$, and that, by proper choice of measurement units, the relationship

$$L = (1 - s)N \tag{3.13}$$

holds approximately (n still ≥ 0 and constant).

b) For Model 2, (3.4) gives

$$g_y = g_\pi \equiv \frac{\dot{\pi}}{\pi} = \frac{d \log \pi}{dt} = \psi \dot{s},$$

by (3.12). To maintain g_y equal to a positive constant would require $\dot{s}$ equal to a positive constant, which is impossible as s is a fraction. Thus, Model 2 is not technologically capable of generating endogenous growth.

Model 3 differs from Model 2 by regarding total factor productivity, A, in (3.1) as endogenous and by having a separate innovative sector with the invention production function:

$$\dot{A} = \mu A^\varphi L_A, \qquad \mu > 0, \varphi < 1, \tag{3.14}$$

where L_A is input of research labor. Consequently, L in (3.1) is replaced by L_Y, and (3.13) is replaced by

$$L_Y + L_A = L = (1 - s)N.$$

c) In view of $\varphi < 1$ in the "growth engine", we have an indication that this is not an AK-style model, but rather a semi-endogenous growth model (where aspects of diminishing returns are present). We take a simple growth-accounting approach. Dividing through by A_t in (3.14) gives

$$\frac{\dot{A}}{A} \equiv g_A = \mu A^{\varphi-1} L_A. \tag{3.15}$$

Presupposing $g_A > 0$, log-differentiating wrt. t gives

$$\frac{\dot{g}_A}{g_A} = (\varphi - 1)g_A + g_{L_A}. \tag{3.16}$$

In balanced growth, g_A and g_{L_A} are constant so that $\dot{g}_A = 0$ and $g_{L_A} = n$ (since $0 < L_A < N = N_0 e^{nt}$). Then (3.16) gives

$$g_A = \frac{n}{1-\varphi}. \tag{3.17}$$

Log-differentiating wrt. t in (3.1), the aggregate production function gives

$$g_Y = g_A + \alpha g_K + (1-\alpha)(g_\pi + g_{L_Y}). \tag{3.18}$$

Under balanced growth, we take s to be constant so that $g_\pi = 0$ and $g_L = g_N = n$. Further, $g_Y = g_K$ and $g_{L_Y} = n$. Then (3.18) gives

$$\begin{aligned} g_Y &= g_A + \alpha g_Y + (1-\alpha)n \quad \text{or} \\ g_Y &= \frac{g_A}{1-\alpha} + n. \end{aligned}$$

Thereby,

$$g_y = g_Y - g_L = g_Y - n = \frac{g_A}{1-\alpha} = \frac{n}{(1-\alpha)(1-\varphi)}. \tag{3.19}$$

d) Yes, (3.19) shows that Model 3 is technologically capable of generating endogenous growth, if and only if $n > 0$. Growth is *endogenous* in the sense that its source is an internal mechanism (accumulation of technical knowledge through R&D). It is "only" *semi*-endogenous growth, because growth in an exogenous factor (population) is needed to sustain positive per capita growth. The reason that population growth is "good" for growth in the model is that knowledge creation is the basic growth-driving mechanism and knowledge is a non-rival good, implying advantages of scale.

e) A subsidy to R&D may increase L_A/L which tends to have a permanent positive *level effect* on y (at least if L_A/L is not already high initially). Increased subsidies (or other forms of support) to education may increase s and thereby y, via increasing π. Defining, $B \equiv A^{1/(1-\alpha)}$ and $\hat{k} \equiv K/(B\pi L_Y)$, we have

$$\begin{aligned} Y &= AK^\alpha(\pi L_Y)^{1-\alpha} = K^\alpha(B\pi L_Y)^{1-\alpha} \equiv \hat{k}^\alpha B\pi L_Y, \quad \text{so that} \\ y &\equiv \frac{Y}{L} = \frac{L_Y}{L}\frac{Y}{L_Y} = \frac{L_Y}{L}\hat{k}^\alpha B\pi = \frac{L_Y}{L}\hat{k}^\alpha A^{1/(1-\alpha)}\pi. \end{aligned}$$

Given L_Y/L, $\hat{k}$, and the path followed by $A^{1/(1-\alpha)}$, a higher π implies a higher y-path.

f) A drawback of Model 1 and 2 is that they do not include endogenous technical change, which is by most researchers considered the key factor behind growth. Model 1 needs a knife-edge condition to be consistent with sustained growth. Model 2 is not at all consistent with sustained growth.

Model 3 does not have these drawbacks. It is, however, awkward that the human capital dependent productivity factor, π, does not appear in the invention production function. This should read

$$\dot{A} = \mu A^\varphi \pi L_A.$$

Even better, a congestion externality in the form of duplication of effort could be taken into account so that

$$\dot{A} = \mu A^\varphi (\pi L_A)^\lambda, \qquad 0 < \lambda < 1.$$

The empirical evidence supports the key role of technology and that technology differs a lot across countries. Model 1 is based on the assumption that the production function in educational activity is the same as in manufacturing, whereas the empirics say that educational activity is much more human capital intensive than manufacturing. The Mincer hypothesis in Model 2 and 3 is consistent with a large empirical literature on schooling and wages (supposed to reflect productivity). On the other hand, (3.12) ignores that the educational sector also needs *some* physical capital (buildings, computers etc.).

A potential empirical criticism of Model 3 is that the result (3.19) implies the prediction that a higher population growth rate is conducive to higher economic growth. This is not supported by cross-country regression analysis. A counter argument to this is that in view of cross-border technology diffusion, the relevant observation unit is not the single country, but a much larger region, perhaps the world economy. Yet it is possible that taking population density, congestion, and pollution into account, simple results like (3.19) will to some extent be changed.

4. Solution to Problem 4 (10 %)

a) In the model, labor productivity, π, is assumed proportional to the productive public service, G, such that, by appropriate measurement units, we can write $\pi = G$. Thus Barro & Sala-i-Martin write the production function of firm i as (standard notation)

$$Y_i = AK_i^{\alpha}(GL_i)^{1-\alpha}.$$

When combined with Ramsey households, the model leads to a scale effect on growth. But proportionality (linearity) between π and G is just one possibility. Perhaps more likely is a diminishing-returns relationship, say

$$\pi = G^{\beta}, \quad \beta < 1.$$

Then

$$Y_i = AK_i^{\alpha}(G^{\beta}L_i)^{1-\alpha}.$$

Now, there are diminishing returns to producible inputs (in that $\alpha+\beta(1-\alpha) < 1$), and the model does not any more predict a strong scale effect on growth.

b) Lucas assumes an external positive effect on productivity from the average level, $\bar{h}$, of human capital in a country. For an individual with a given amount, h', of human capital, the wage is thus higher in a country with high $\bar{h}$ than in a country with low $\bar{h}$. Since relatively rich countries tend to have high average human capital, $\bar{h}$, the wage you obtain, given your h', tends to be higher in rich countries than in poor countries. This gives people from poor countries an incentive to migrate to rich countries. And this is how Lucas explains the observed huge pressure for migration from poor to rich countries.

In Romer's and Jones' innovation-based growth models, it is instead differences in the level of technology across countries that, via an otherwise similar mechanism, induce pressure for migration.

c) In the "simple increasing variety model" considered in Problem 2, an R&D subsidy is, as we saw, *not* needed to implement the social planner's allocation. A subsidy to purchases of the monopolized intermediate goods was enough. This is because in that model there is a one-to-one relationship between the key static efficiency condition ($X_j = X_{SP}$) and the rate of return on investment in R&D. There is only one stock variable, the number of intermediate goods varieties, and there is no positive intertemporal externality from R&D activity to future productivity of R&D activity.

In most innovation-based increasing variety models, however, this is not so. The basic reason is that these models usually have more than one stock variable and also often contain a positive intertemporal externality from R&D activity. Then, typically, in addition to a subsidy to compensate for monopolist pricing, a subsidy to increase the incentive to invest in R&D is needed. Examples:

1. The model with stochastic erosion of monopoly power in B & S. Here there are *two* endogenous stock variables, the number of intermediate goods varieties still supplied by monopolists and the number of intermediate goods varieties supplied under competitive conditions.

2. Romer's 1990-model, which has a separate R&D sector with its own "production function", and which has two endogenous stock variables, physical capital and the level of knowledge. The same holds true in the extended versions in Jones (1995) and Alvarez & Groth (2005).

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