Economic Growth Exercise solutions

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Suggested solutions to (parts of) Problem VI.3 and VI.9

Problem VI.3

We found *under b)*

$$X = (\alpha A)^{\frac{1}{1-\alpha}} LN \equiv X_{SP} N, \tag{1}$$

and

$$\begin{aligned} \frac{\dot{N}}{N} &= \frac{\dot{X}}{X} = \frac{\dot{c}}{c} = \frac{1}{\theta}\left[\frac{1}{\eta}(1-\alpha)\frac{Y}{N} - \rho\right] \\ &= \frac{1}{\theta}\left[\frac{L}{\eta}(\frac{1}{\alpha} - 1)(\alpha A)^{\frac{1}{1-\alpha}} - \rho\right] \equiv \gamma_{SP}. \end{aligned} \tag{2}$$

In class we named the growth rate γ, but it is better to name it γ_{SP} (in order not to confuse it with the growth rate in the laissez-faire market economy with monopoly pricing). Warning: be careful whenever you see the expression X_{SP}, which in some contexts might refer to the X on the *left-hand* side of (1), but in others, as here, to this X *divided* by N, that is, the input of *each* of the intermediate goods j, $j = 1, 2, ..., N$.

d) The *maximized* Hamiltonian is

$$\begin{aligned} \hat{H}(N, \lambda, t) &\equiv \max_{c,X} H(N, c, X, \lambda, t) \\ &= \frac{(\lambda L/\eta)^{-\frac{1-\theta}{\theta}} - 1}{1-\theta} + \frac{\lambda}{\eta}\left[A(\alpha A)^{\frac{\alpha}{1-\alpha}} LN - (\alpha A)^{\frac{1}{1-\alpha}} LN - (\lambda L/\eta)^{-\frac{1}{\theta}} L\right] \\ &\equiv f(\lambda) + \frac{\lambda}{\eta}\left[(\alpha A)^{\frac{1}{1-\alpha}}(\frac{1}{\alpha} - 1)LN - g(\lambda)\right]. \end{aligned}$$

This function is concave in the state variable, N. To see whether our candidate solution satisfies Arrow's transversality condition, we consider the limiting value of $N_t \lambda_t e^{-\rho t}$. We have

$$N_t \lambda_t e^{-\rho t} = N_0 e^{\gamma_{SP} t} \lambda_0 e^{-\theta \gamma_{SP} t} e^{-\rho t} = N_0 \lambda_0 e^{[(1-\theta)\gamma_{SP} - \rho]t} \to 0 \text{ for } t \to \infty,$$

in view of the assumption $(1-\theta)\gamma_{SP} < \rho$. Thus, Arrow's sufficient conditions are satisfied by our candidate solution, which then *is* an optimal solution.

In the present case we could also have used Mangasarian's sufficient conditions, see Lecture Note 10. But Arrows' sufficient conditions are weaker than Mangasarian's and therefore, there *are* cases where one can only prove sufficiency using Arrow's conditions, not Mangasarian's.

e) Our solution to the SP problem features fully endogenous growth and there is no transitional dynamics. In view of (2) we have $\partial\gamma_{SP}/\partial L > 0$ and thus there is a scale effect on growth. Such a scale effect is no surprise, since we have an AK-style model where the growth generating activity is knowledge creation (knowledge is a non-rival good).

More interestingly, γ_{SP} is higher than the growth rate in the laissez-faire market economy with infinitely-lived patents. The reason is that in the laissez-faire market economy with monopoly pricing the monopoly markup on marginal costs implies too low demand for each kind of the specialized intermediate goods, namely the demand

$$X_m = (\alpha A)^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L = \alpha^{\frac{1}{1-\alpha}} X_{SP} < X_{SP},$$

cf. Lecture Note 17. Therefore, from a social point of view, too little of these goods are used. This also results in too little remuneration of the R&D activity, which invents new specialized intermediate goods. Consequently, there is too little incentive to do R&D.

f) The inefficiency is in this model due solely to monopoly pricing of intermediate goods and can be overcome by a subsidy, for example of the kind suggested, as we shall see. The production subsidy to firms, $i = 1, 2, ..., M$, in the basic-goods sector should be such that we end up with

$$\frac{\partial Y_i}{\partial x_{ij}} = MC = 1. \tag{3}$$

Given the subsidy, s, when firm i maximizes its profit under perfect competition, we have

$$(1+s)\frac{\partial Y_i}{\partial x_{ij}} = p_j = \frac{1}{\alpha}, \quad \text{for } j = 1, 2, ..., N. \tag{4}$$

Combining this with (3), we get $1 + s = 1/\alpha$ or

$$s = \frac{1}{\alpha} - 1. \tag{5}$$

g) With this value of s (3) holds and so

$$\frac{\partial Y_i}{\partial x_{ij}} = AL_i^{1-\alpha}\alpha x_{ij}^{\alpha-1} = 1,$$

which gives

$$x_{ij} = (\alpha A)^{\frac{1}{1-\alpha}} L_i \equiv x_i \quad \text{for } j = 1, 2, ..., N.$$

Hence,

$$X_j \equiv \sum_{i=1}^{M} x_{ij} = (\alpha A)^{\frac{1}{1-\alpha}} \sum_{i=1}^{M} L_i = (\alpha A)^{\frac{1}{1-\alpha}} L, \quad j = 1, 2, ..., N, \tag{6}$$

which is identical to X_{SP}, i.e., the social planner's solution for X_j.

The profit obtained per time unit by firm j is now

$$\pi_j = (\frac{1}{\alpha} - 1)(\alpha A)^{\frac{1}{1-\alpha}} L \equiv \pi \qquad \text{for } j = 1, 2, ..., N.$$

In general equilibrium with innovation, $V = \eta$ and thus the no-arbitrage condition,

$$\frac{\pi + \dot{V}}{V} = r,$$

implies

$$r = \pi/\eta = \frac{L}{\eta}(\frac{1}{\alpha} - 1)(\alpha A)^{\frac{1}{1-\alpha}},$$

a constant. Now, by the Keynes-Ramsey rule of the households,

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left[\frac{L}{\eta}(\frac{1}{\alpha} - 1)(A\alpha)^{\frac{1}{1-\alpha}} - \rho\right]. \tag{7}$$

This is exactly the social planner's growth rate γ_{SP}.

By (6),

$$X \equiv \sum_{i=1}^{M} X_j = (A\alpha)^{\frac{1}{1-\alpha}} NL, \tag{8}$$

which equals the social planner's solution for X. Substituting into the given formula for Y yields

$$\begin{aligned} Y &= AX^{\alpha}(NL)^{1-\alpha} = A\left[(A\alpha)^{\frac{1}{1-\alpha}} NL\right]^{\alpha} (NL)^{1-\alpha} \\ &= A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} LN \equiv A_{SP} N. \end{aligned}$$

Since the decentralized economy now has $X = X_{SP}$, the aggregate production function in the basic-goods sector is also the same as that implied by the social planner's allocation.

Could the *financing* of the subsidy create problems? Not if lump-sum taxation is available. In the context of the present representative agent model, lump-sum taxes would naturally take the form of a per capita tax, τ_t, imposed on all households. With a balanced government budget, tax revenue equals government spending for all t:

$$\tau_t L = sY_t = (\frac{1}{\alpha} - 1)Y_t = (\frac{1}{\alpha} - 1)A_{SP}N_t.$$

The tax τ_t will enter the dynamic budget identity of the household in the following way:

$$\dot{a}_t = ra_t + w_t - \tau_t - c_t, \qquad a_0 \text{ given.}$$

The consumption path of the household would still satisfy the Keynes-Ramsey rule (7).

Note that a constant consumption tax would work as a lump-sum tax in this model, where there is no utility of leisure.

All in all, the allocation in the decentralized equilibrium, with the production subsidy $s = \frac{1}{\alpha} - 1$ financed by with lump-sum taxes, is identical to that of the social planner.

Problem VI.9 For convenience we repeat the equations of the model:

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t L_{Yt})^{1-\alpha}, \qquad 0 < \alpha < 1, && (9) \\ \dot{K}_t &= Y_t - c_t L_t - \delta K_t, \qquad \delta \geq 0, && (10) \\ \dot{A}_t &= \mu A_t^{\varphi} L_{At}, \qquad \mu > 0, \varphi < 1, && (11) \\ L_{Yt} + L_{At} &= L_t, && (12) \\ L_t &= L_0 e^{nt}, \; n > 0, \; \text{constant.} && (13) \end{aligned}$$

a) In view of $\varphi < 1$ in the "growth engine", we have an indication that this is not an AK-style model, but rather a semi-endogenous growth model (where aspects of diminishing returns are present). Then, as suggested by the hint, instead of first finding the real interest rate in equilibrium (as we do in AK-style models), we take a simple growth-accounting approach. Dividing through by A_t in (11) gives

$$\frac{\dot{A}}{A} \equiv g_A = \mu A^{\varphi - 1} L_A. \tag{14}$$

Presupposing $g_A > 0$, log-differentiating w.r.t. t gives

$$\frac{\dot{g}_A}{g_A} = (\varphi - 1) g_A + g_{L_A}. \tag{15}$$

In balanced growth g_A and g_{L_A} are constant so that $\dot{g}_A = 0$ and $g_{L_A} = n$. Then (15) gives

$$g_A = \frac{n}{1 - \varphi}. \tag{16}$$

b) Log-differentiating w.r.t. t in the aggregate production function gives

$$g_Y = \alpha g_K + (1 - \alpha)(g_A + g_{L_Y}). \tag{17}$$

In view of the closed-economy assumption (10), we have under balanced growth $g_Y = g_K$ and $g_{L_Y} = n$. Then (17) gives

$$\begin{aligned} g_Y &= \alpha g_Y + (1-\alpha)(g_A + n) \quad \text{or} \\ g_Y &= g_A + n. \end{aligned}$$

Thereby,

$$g_y = g_Y - n = g_A = \frac{n}{1-\varphi}. \tag{18}$$

Remark. Thus, for answering these simple balanced growth questions a simple "growth-accounting approach" works well. If we were asked to completely solve the model (with Ramsey households), including finding the transitional dynamics, the approach would be to first derive the complete system of differential equations like we did in the standard Ramsey model in B & S, Chapter 2. Then one finds that the dynamics are described by a *four*-dimensional dynamic system (in contrast to the standard Ramsey model which has two-dimensional dynamics). Characterizing the solution to that four-dimensional system is possible, but outside the confines of this course.

c) Defining $C \equiv cL$, under balanced growth $g_C = g_Y$ and so

$$g_c = g_C - n = g_Y - n = \frac{n}{1-\varphi}.$$

d) We consider an R&D subsidy which increases $s_A \equiv L_A/L$. Since the model is saddle-point stable, the economy converges to a balanced growth path in the long run with growth rate g_y given by (18).

1. No, a higher s_A will not affect g_y in the long run, since (18) shows that g_y only depends on n and φ, not on s_A. A higher s_A will temporarily increase the growth rate of A and may temporarily increase also the growth rate of y. But the fact that $\varphi < 1$ (diminishing returns to knowledge) in the growth engine makes it impossible to maintain the higher growth rate in A forever. This is like in a Solow model where an increase in the saving rate raises the growth rate only temporarily due to the falling marginal product of capital.

2. We have

$$y \equiv \frac{Y}{L} = \frac{Y}{L_Y}\frac{L_Y}{L} = \frac{Y}{L_Y}(1-s_A) = \hat{k}^{\alpha} A(1-s_A), \tag{19}$$

where $\hat{k} \equiv K/(AL_Y)$. We consider s_A as fixed by policy. Under balanced growth one can infer stocks from flows. Indeed, from (14) and (16) follows

$$\mu A^{\varphi-1}L_A = \frac{n}{1-\varphi},$$

implying

$$A_t = \left(\frac{n}{\mu(1-\varphi)}\right)^{\frac{1}{\varphi-1}} L_{At}^{\frac{1}{1-\varphi}} = \left(\frac{n}{\mu(1-\varphi)}\right)^{\frac{1}{\varphi-1}} (s_A L_t)^{\frac{1}{1-\varphi}}.$$

Substituting into (19) gives

$$y_t = \hat{k}^{\alpha}\left(\frac{n}{\mu(1-\varphi)}\right)^{\frac{1}{\varphi-1}} (s_A L_0 e^{nt})^{\frac{1}{1-\varphi}}(1-s_A). \tag{20}$$

In balanced growth $\hat{k}$ is a constant, which is independent of s_A (this is shown under e) below). We see that the path for y_t depends on s_A and thus the answer is: yes, policy has long-run level effects.

Note that the effect on levels is of ambiguous sign. Defining

$$z \equiv s_A{}^{\frac{1}{1-\varphi}}(1-s_A),$$

we see that

$$\begin{aligned}
\frac{\partial z}{\partial s_A} &= (1-s_A)\frac{1}{1-\varphi}s_A^{\frac{1}{1-\varphi}-1} - s_A^{\frac{1}{1-\varphi}} \\
&= \frac{s_A^{\frac{1}{1-\varphi}-1}}{1-\varphi}\left[1-s_A-(1-\varphi)s_A\right] \\
&= \frac{s_A^{\frac{1}{1-\varphi}-1}}{1-\varphi}\left[1-(2-\varphi)s_A\right] \gtreqless 0 \text{ for } s_A \lesseqgtr \frac{1}{2-\varphi}.
\end{aligned}$$

Thus, if s_A is not "too high", an increase in s_A will have a positive level effect on y via the productivity-enhancing effect of more knowledge creation. But if s_A is already quite high, L_Y must be low and therefore have a high marginal product. This high marginal product is the opportunity cost of increasing s_A and dominates the benefit of a higher s_A, when $s_A > 1/(2-\varphi)$.

e) That s_A under balanced growth is independent of L, follows from the formulas in Jones, 1995, p. 769. That also $\hat{k}$ under balanced growth is independent of L, follows from the Keynes-Ramsey rule in the following way. First, from the aggregate production function we have

$$\frac{Y}{K} = K^{\alpha-1}(AL_Y)^{1-\alpha} \equiv \hat{k}^{\alpha-1}. \tag{21}$$

With r denoting the real interest rate, using the household's Keynes-Ramsey rule we have, under balanced growth,

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left(\alpha^2\frac{Y}{K} - \delta - \rho\right) = \frac{n}{1-\varphi},$$

and therefore

$$\frac{Y}{K} = \frac{1}{\alpha^2}(\delta + \rho + \frac{\theta n}{1-\varphi}),$$

which is independent of L. It now follows from (21) that $\hat{k}$ is independent of L. Now (20) gives

$$\frac{\partial y_t}{\partial L_0} > 0.$$

So the answer is: yes, there is a scale effect on levels in the model.

Correction to Groth (2006)

In footnote 9 on p. 7 in Groth (2006) X is defined as $X = A(A^{-1}\sum_{i=1}^{N} x_i^{\varepsilon})^{1/\varepsilon}$, where $0 < \varepsilon < 1$. But this should read

$$X = N(N^{-1}\sum_{i=1}^{N} x_i^{\varepsilon})^{1/\varepsilon}.$$

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