Economic Growth Exercise solutions

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A suggested solution to Problem V.1

a) We solve the problem:

$$\max_{K^d, L^d} \Pi = F(K^d, hL^d) - \tilde{r}K^d - wL^d.$$

First-order conditions are

$$F_1(K^d, hL^d) - \tilde{r} = 0, \quad \text{(FOC1)}$$
$$F_2(K^d, hL^d) - \hat{w} = 0. \quad \text{(FOC2)}$$

In equilibrium, $K^d = K$ and $L^d = L$ so that (FOC1) and (FOC2) give

$$\tilde{r} = F_1(K, hL)$$
$$\hat{w} = F_2(K, hL),$$

respectively. In view of CRS, $F(K, hL) = hLF(\hat{k}, 1) \equiv hLf(\hat{k})$, where $f' = F_1 > 0, f'' < 0$.[1] Further, $F_2(K, hL) = f(\hat{k}) - f'(\hat{k})\hat{k}$. Hence, we get the solution

$$\tilde{r} = f'(\hat{k}), \quad (1)$$
$$\hat{w} = f(\hat{k}) - f'(\hat{k})\hat{k}. \quad (2)$$

b) From $h \equiv H/L$ and $\dot{H} = I_H - \delta H$ we have

$$\frac{\dot{h}}{h} = \frac{\dot{H}}{H} - n = \frac{I_H}{H} - (\delta + n), \quad \text{so that}$$
$$\dot{h} = \frac{I_H}{L} - (\delta + n)h \equiv i - (\delta + n)h. \quad (3)$$

Comment concerning B & S, Chapter 5: The result (3) shows that whenever we set up a human capital formation equation in the form $\dot{H} = I_H - \delta H$, where $H \equiv hL$, the conclusion (3) (with the $-nh$ term) is unavoidable. Thus, by writing $\dot{H} = I_H - \delta H$, one implicitly treats human capital as just another capital good, i.e., parallel to the way we treat physical capital. Therefore, the matter is not expressed in a precise way in the last

[1] In this solution I use B & S's hat ˆ and write $\hat{k}$ instead of my usual $\tilde{k}$.

line of my footnote 2 to the text in Problem Set V (starting with "Which of these ..."). The line should rather read: "Although it is not *directly* visible in Chapter 5 of B & S, the authors also here implicitly treat human capital in a way parallel to the treatment of physical capital" − which is problematic because human capital is not separable from the individuals who embody the human capital.

c) We consider the household problem: choose a path $(c_t, i_t)_{t=0}^{\infty}$ to maximize

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\begin{aligned}
U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}-1}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} & (4)\\
c_t &> 0,\ i_t \geq 0, & (5)\\
\dot{a}_t &= (r_t - n)a_t + \hat{w}_t h_t - c_t - i_t, \qquad a_0 \text{ given}, & (6)\\
\dot{h}_t &= i_t - (\delta + n)h_t, \qquad h_0 > 0 \text{ given}, & (7)\\
\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} &\geq 0, & (8)\\
h_t &\geq 0 \text{ for all } t. & (9)
\end{aligned}
$$

The household maximizes discounted utility. The pure rate of time preference (impatience) is ρ, but taking the possibly larger household size in the future into account, the effective rate of utility discount is the growth-corrected rate $\rho - n$, cf. (4). Instantaneous utility is of the CRRA type with (absolute) elasticity of marginal utility equal to the constant θ. Thus θ is a measure of the desire for consumption smoothing, $1/\theta$ being the intertemporal elasticity of substitution in consumption. The control variables of the household are per capita consumption, c_t, and per capita educational investment, i_t, none of which can be negative, cf. (5). There are two dynamic constraints, (6) and (7). If the household's financial wealth at time t is called A_t, then, by simple accounting,

$$\dot{A}_t = r_t A_t + w_t L_t - C_t - I_{Ht}, \qquad A_0 \text{ given}. \tag{10}$$

Differentiating $a_t \equiv A_t/L_t$ w.r.t. t and substituting (10) leads to the per capita financial wealth accumulation identity (6).

The per capita human capital accumulation constraint (7) was derived under b) above and reflects that the model is based on the "human capital parallel to physical capital approach". The constraint implies that to sustain a certain level of average human capital in society, the required per capita educational investment, $i = I_H/L$, is higher, the higher is the population growth rate. This has the natural interpretation that higher n means more newcomers (the young) to educate.

In terms of aggregate financial wealth the standard No-Ponzi-Game condition (implying a constraint on how fast the family's net debt is allowed to grow in the long run) would read

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0.$$

Inserting $A_t \equiv a_t L_t = a_t L_0 e^{nt}$, this gives (8), ignoring the unimportant positive constant L_0. Finally, whereas in principle we can at any time have $a_t < 0$ (implying a positive net debt), human capital is by definition constrained to be non-negative as expressed by (9).

d) The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} + \lambda_1\left[(r-n)a + \hat{w}h - c - i\right] + \lambda_2\left[i - (\delta+n)h\right],$$

where λ_1 and λ_2 are the shadow prices of per-capita financial wealth and per-capita human capital, respectively, along the optimal path. An interior solution satisfies the first order conditions:

$$\begin{aligned}
\partial\mathcal{H}/\partial c &= c^{-\theta} - \lambda_1 = 0, \quad \text{i.e., } c^{-\theta} = \lambda_1, && (11)\\
\partial\mathcal{H}/\partial i &= -\lambda_1 + \lambda_2 = 0, \quad \text{i.e., } \lambda_2 = \lambda_1, && (12)\\
\partial\mathcal{H}/\partial a &= \lambda_1(r-n) = -\dot{\lambda}_1 + (\rho-n)\lambda_1, \quad \text{i.e.,} \\
-\dot{\lambda}_1/\lambda_1 &= r - n - (\rho - n) = r - \rho, && (13)\\
\partial\mathcal{H}/\partial h &= \lambda_1\hat{w} - \lambda_2(\delta+n) = -\dot{\lambda}_2 + (\rho-n)\lambda_2, \quad \text{i.e.,} \\
-\dot{\lambda}_2/\lambda_2 &= \lambda_1\hat{w}/\lambda_2 - (\delta+n) - (\rho-n) = \lambda_1\hat{w}/\lambda_2 - (\delta+\rho), && (14)
\end{aligned}$$

and the transversality conditions:

$$\begin{aligned}
\lim_{t\to\infty} a_t\lambda_{1t}e^{-(\rho-n)t} &= 0, && (\text{TVC}_1)\\
\lim_{t\to\infty} h_t\lambda_{2t}e^{-(\rho-n)t} &= 0. && (\text{TVC}_2)
\end{aligned}$$

That is, on the margin, according to (11), income must be equally valuable in its two uses, consumption or saving. Similarly, on the margin, according to (12), non-leisure time must be equally valuable in its two uses, work or education. Moreover, (13) and (14) tell how the shadow prices of the two assets must move over time in the optimal plan. Finally, (TVC_1) and (TVC_2) ensure that none of the assets are over-accumulated.

e) Log-differentiating (11) w.r.t. t gives $-\theta\dot{c}/c = \dot{\lambda}_1/\lambda_1$. We substitute (13) into this and get, after ordering,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho), \tag{15}$$

which is the Keynes-Ramsey rule.

f) From (12) follows $\dot{\lambda}_2/\lambda_2 = \dot{\lambda}_1/\lambda_1$ which together with (13) and (14) implies $\lambda_1 \hat{w}/\lambda_2 - (\delta + \rho) = r - \rho$. By (12) this yields

$$\hat{w}_t - \delta = \frac{\hat{w}_t - \delta}{1} = r_t. \tag{16}$$

This is a no-arbitrage relationship saying that along an interior optimal path the household is indifferent between placing the marginal unit of saving in a financial asset yielding the rate of return r or in education to obtain one more unit of human capital. The last alternative gives an extra labour income gross-of-human-capital depreciation equal to $\hat{w}$ (which is the real wage per unit of human capital). The net-of-depreciation return on that alternative is then $\hat{w} - \delta$. This explains (16).

g) With $Y = AK^\alpha (hL)^{1-\alpha}$, (1) gives

$$\tilde{r}_t = \alpha A \hat{k}_t^{\alpha-1}.$$

Placing the marginal unit of saving on the loan market gives the rate of return r and placing it in physical capital gives the (net) rate of return$\tilde{r} - \delta$. Hence, in equilibrium,$\tilde{r} - \delta = r$ so that

$$r_t = \alpha A \hat{k}_t^{\alpha-1} - \delta, \tag{17}$$

where $\hat{k}_t \equiv K_t/H_t \equiv K_t/(h_t L_t)$ is predetermined.

h) For an interior solution to obtain, the no-arbitrage condition (16) must hold. In view of (2), this implies

$$\hat{w} = f(\hat{k}) - f'(\hat{k})\hat{k} = A\hat{k}^\alpha - \alpha A \hat{k}^{\alpha-1}\hat{k} = (1-\alpha)A\hat{k}^\alpha = r + \delta = f'(\hat{k}) = \alpha A \hat{k}^{\alpha-1}.$$

From this follows

$$\hat{k} \equiv \frac{K}{H} = \frac{\alpha}{1-\alpha} \equiv \hat{k}^*. \tag{18}$$

It is assumed that parameters are such that $\dot{c}/c > 0$ and U_0 is bounded. By (17) and (18), (15) implies

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\alpha A (\frac{\alpha}{1-\alpha})^{\alpha-1} - \delta - \rho) = \frac{1}{\theta}(\alpha^\alpha (1-\alpha)^{1-\alpha} A - \delta - \rho) \equiv \gamma, \tag{K-R}$$

a constant. Hence, we have $\gamma > 0$. A condition ensuring that U_0 is bounded is the assumption $(1-\theta)\gamma < \rho - n$.

Suppose that initially $K_0/H_0 > \hat{k}^*$. Then human capital is relatively scarce and the marginal rate of return on investing in education is higher than on investing in physical capital. Hence, for a while the economy invests only in human capital. This results in a falling K/H. When K/H reaches the level $\hat{k}^*$, the phase of complete specialization ends. From now on the economy invests in both human and physical capital in such proportions as to maintain the efficient ratio $\hat{k}^*$. Similarly, if initially $K_0/H_0 < \hat{k}^*$, there will be a phase of complete specialization in physical capital investment, until the efficient ratio $\hat{k}^*$ is obtained. In both cases, in the long run (indeed after some finite period of time) the economy will be in steady state and behave in an AK-style manner.

i) With $\delta = 0$ (for simplicity) (16) is replaced by

$$\frac{\hat{w}_t - \delta}{1-s} = r_t.$$

That is, with $\delta = 0$ (for simplicity) we have

$$\begin{aligned}(1-\alpha)A\hat{k}^{\alpha} &= (1-s)\alpha A\hat{k}^{\alpha-1} \quad \text{so that} \\ \hat{k} &= \frac{(1-s)\alpha}{1-\alpha} \equiv \hat{k}^{*\prime}.\end{aligned}$$

The r appearing in the K-R rule is now

$$r^* = f'(\hat{k}^{*\prime}) = \alpha A\hat{k}^{*\prime\alpha-1} = \alpha A\left(\frac{(1-s)\alpha}{1-\alpha}\right)^{\alpha-1},$$

which is higher than the old. This higher return on saving, whether in financial wealth or human capital, implies in *this* model (which is a fully endogenous growth model) a higher long-run growth rate in that (K-R) is replaced by

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\alpha A(\frac{(1-s)\alpha}{1-\alpha})^{\alpha-1} - \delta - \rho) \equiv \gamma' > \gamma.$$

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