Economic Growth

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A suggested solution to Problem IV.1

For convenience we repeat the basic relations:

$$Y = cL + G + I,$$

$$\dot{K} = I - \delta K,$$

$$G = \bar{g}Y, \tag{*}$$

$$\left[\tau(ra + w) + \tau_\ell\right] L = G, \tag{GBC}$$

$$Y_i = AK_i^\alpha (GL_i)^{1-\alpha}, \quad 0 < \alpha < 1, A > 0, \quad i = 1, 2, ..., M.$$

a) (*) indicates that G is a productive government service, affecting productivity. Since the productivity of every worker depends on the total of G (not the per capita amount, G/L), G is completely nonrival. From (GBC) we see there is no fee for using G. Thus, G appears in this economy as a pure public good.

b) When the firms maximize profits under perfect competition, they end up choosing the same capital intensity, $k_i \equiv K_i/L_i$. Then, in equilibrium, $k_i = K/L \equiv k$ and output per unit of labor is the same for all firms. This allows finding, first, the aggregate production function $Y = \bar{A}K$, second, the level of G in terms of parameters and predetermined variables. One of firm i's first-order conditions equates the firm's marginal product of capital to the capital costs, $r + \delta$. This first-order condition allows us to find r in equilibrium.

Remark. As the question is formulated, the algebraic derivations are not necessary. *If* they were, you could either proceed as in Lecture Note 12, Section 1, or you could find Y before you find G. This last approach is shown here. We have $y_i = Ak_i^\alpha G^{1-\alpha} = Ak^\alpha G^{1-\alpha} \equiv y$, from which follows

$$Y = \sum_i Y_i = \sum_i y_i L_i = y \sum_i L_i = yL = Ak^\alpha G^{1-\alpha} L = Ak^\alpha (\bar{g}Y)^{1-\alpha} L.$$

Solving for Y gives

$$Y = A^{1/\alpha} k \bar{g}^{(1-\alpha)/\alpha} L^{1/\alpha} = A^{1/\alpha} (\bar{g}L)^{(1-\alpha)/\alpha} kL \equiv \bar{A}K, \tag{1}$$

where

$$\bar{A} \equiv A^{\frac{1}{\alpha}} (\bar{g}L)^{\frac{1-\alpha}{\alpha}}.$$

One of firm i's first-order conditions is $\partial \Pi_i / \partial K_i = \alpha A K_i^{\alpha-1} (GL_i)^{1-\alpha} - (r + \delta) = 0$, so that, since $k_i = k$,

$$r = \alpha A k^{\alpha-1} G^{1-\alpha} - \delta. \tag{2}$$

From (*) and (1) we get

$$G = \bar{g}Y = \bar{g} A^{1/\alpha} (\bar{g}L)^{(1-\alpha)/\alpha} kL = (\bar{g}AL)^{1/\alpha} k. \tag{3}$$

Substituting this into (2) gives

$$r = \alpha A k^{\alpha-1} \left[(\bar{g}AL)^{1/\alpha} k \right]^{1-\alpha} - \delta = \alpha A^{\frac{1}{\alpha}} (\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta \equiv \alpha \bar{A} - \delta \equiv \bar{r}. \tag{4}$$

c) The representative household solves

$$\max_{(c_t)_{t=0}^{\infty}} U_0 = \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.}$$

$$c_t > 0,$$

$$\dot{a}_t = (1-\tau)\bar{r} a_t + (1-\tau) w_t - \tau_\ell - c_t, \qquad a_0 \quad \text{given}, \tag{5}$$

$$\lim_{t \to \infty} a_t e^{-(1-\tau)\bar{r}t} \geq 0. \tag{NPG}$$

The current-value Hamiltonian is

$$H = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda \left[(1-\tau)(\bar{r} a_t + w_t) - \tau_\ell - c_t \right],$$

where λ can be interpreted as the shadow price of financial wealth along the optimal path. First-order conditions are

$$\partial H / \partial c = c^{-\theta} - \lambda = 0, \text{ i.e., } c^{-\theta} = \lambda, \tag{6}$$

$$\partial H / \partial K = \lambda (1-\tau) \bar{r} = \rho \lambda - \dot{\lambda}, \text{ i.e., } (1-\tau)\bar{r} - \rho = -\dot{\lambda}/\lambda, \tag{7}$$

and the necessary transversality condition (according to the standard formula) is

$$\lim_{t \to \infty} a_t \lambda_t e^{-\rho t} = 0. \tag{TVC}$$

Log-differentiation w.r.t. t in (6) and inserting into (7) gives the Keynes-Ramsey rule for this model:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}((1-\tau)\bar{r} - \rho) = \frac{1}{\theta}\left[(1-\tau)(\alpha\bar{A} - \delta) - \rho\right] \equiv \gamma, \tag{8}$$

where $\bar{A}$ is given above.

d) From (TVC) combined with (7) it is easily shown that the transversality condition implies

$$\lim_{t\to\infty} a_t e^{-(1-\tau)\bar{r}t} = 0.$$

Comment: if this was not satisfied, there would be individual oversaving (the (NPG) would be over-satisfied).

e) The model implies a constant real interest rate, $\bar{r}$, and a constant output-capital ratio, $\bar{A}$. Hence, the model belongs to the AK family, and from the theory of AK models we know that in equilibrium $\dot{k}/k$ and $\dot{y}/y$ are the same as $\dot{c}/c$. Thus, from date zero

$$\dot{k}/k = \dot{y}/y = \dot{c}/c = \frac{1}{\theta}((1-\tau)\bar{r} - \rho) \equiv \gamma. \tag{9}$$

There is no transitional dynamics.

To ensure growth we assume $(1-\tau)\bar{r} - \rho = (1-\tau)(\alpha\bar{A} - \delta) > \rho$, that is,

$$(1-\tau)(\alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta) > \rho. \tag{A1}$$

This requires that $\bar{g}$ is not too small. On the other hand, to ensure bounded utility integral we assume

$$(1-\theta)\gamma < \rho. \tag{A2}$$

From the Keynes-Ramsey rule we have $(1-\tau)r = \theta\gamma + \rho$, so that the assumption (A2) implies

$$(1-\tau)r > \gamma,$$

i.e., the after-tax real interest rate is higher than the GDP growth rate (this is a necessary condition for an equilibrium to exist in a representative agent model).

f) In addition to the standard results for fully endogenous growth models (like $\partial\gamma/\partial\rho < 0, \partial\gamma/\partial\theta < 0$) we get

$$\frac{\partial\gamma}{\partial L} = \frac{(1-\tau)\alpha}{\theta}\frac{\partial\bar{A}}{\partial L} = \frac{(1-\tau)\alpha}{\theta}A^{\frac{1}{\alpha}}\bar{g}^{\frac{1-\alpha}{\alpha}}L^{\frac{1}{\alpha}-2} > 0.$$

There is a scale effect on the growth rate. A combination of two things explains this. First, because of the assumption that the productive public service is a pure public good (nonrival), there are economies of scale. Second, the reason that these economies of scale not just have a *level* effect, but an effect on (long-run) growth, is the linearity assumption in (*), namely that the factor multiplied on L_i is G and not for example G^{φ} with $0 < \varphi < 1$. This second circumstance is the reason that we end with a reduced-form AK structure and thereby with a fully endogenous growth model in which the scale effect takes the form of a scale effect on growth.

g) In equilibrium in our closed economy $a = k$. Further, $G = \bar{g}Y = \bar{g}\bar{A}K$. We can therefore write the government budget constraint as

$$[\tau(rk + w) + \tau_{\ell}]\, L = G = \bar{g}\bar{A}K. \tag{10}$$

From firm i's standard first-order condition which equates the firm's marginal product of labor to the labor cost w (not shown above), we find

$$\begin{aligned} w &= (1-\alpha)Ak_i^{\alpha}G^{1-\alpha} \\ &= (1-\alpha)Ak^{\alpha}\left[(\bar{g}AL)^{1/\alpha}k\right]^{1-\alpha} \qquad \text{(from (3))} \\ &= (1-\alpha)A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}}k \\ &\equiv (1-\alpha)\bar{A}k. \end{aligned} \tag{11}$$

Hence,

$$rk + w = (\alpha\bar{A} - \delta)k + (1-\alpha)\bar{A}k = \bar{A}k - \delta k.$$

Given $\tau_{\ell} = 0$, (10) therefore gives

$$\tau(\bar{A} - \delta)kL = \bar{g}\bar{A}K$$

or

$$\tau = \frac{\bar{g}\bar{A}}{\bar{A} - \delta}. \tag{12}$$

We see it is possible to fix τ at a constant level such that the government budget is balanced for all $t \geq 0$ in spite of $\tau_{\ell} = 0$. We should also check whether this tax policy is viable. Viability requires

$$\begin{aligned} \frac{\bar{g}\bar{A}}{\bar{A} - \delta} &< 1, \text{ i.e.,} \\ \bar{g}A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} &< A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta, \\ (1-\bar{g})A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} &> \delta, \end{aligned} \tag{A3}$$

saying again that $\bar{g}$ should neither be "too little" or "too large".

h) The aggregate production function is $Y = Ak^{\alpha}G^{1-\alpha}L = AK^{\alpha}G^{1-\alpha}L^{1-\alpha}$ so that

$$\frac{\partial Y}{\partial G} = (1-\alpha)AK^{\alpha}G^{-\alpha}L^{1-\alpha} = (1-\alpha)\frac{Y}{G}.$$

The net gain by increasing G by one unit is approximately

$$\frac{\partial(Y-G)}{\partial G} = (1-\alpha)\frac{Y}{G} - 1 \gtreqless 0 \text{ for } \frac{G}{Y} \lesseqgtr 1-\alpha.$$

Hence, $\bar{g} = G/Y = 1-\alpha$ is required for static efficiency.

As to the form of taxation, taxation of income is not here a good idea. This is because income taxation is also taxation of interest income and this distorts the consumption-saving decision. Indeed,

$$\begin{aligned}(1-\tau)r &= (1-\tau)(\alpha(\frac{k}{G})^{\alpha-1} - \delta) \\ &= (1-\tau)(\alpha K^{\alpha-1}(GL)^{1-\alpha} - \delta) \\ &= (1-\tau)(\frac{\partial Y}{\partial K} - \delta) < \frac{\partial Y}{\partial K} - \delta\end{aligned}$$

for $\tau > 0$ and $\partial Y/\partial K > \delta$. That is, due to the income tax the private return to saving is smaller than the social return. (Note also that in the present model, an AK-style model, this has the strong effect that $\partial\gamma/\partial\tau < 0$.)

Is a pure labor income tax, τ_w, a viable alternative? No, the required tax rate would satisfy

$$\begin{aligned}\tau_w wL &= G = (1-\alpha)Y, \text{ i.e.,} \\ \tau_w &= \frac{(1-\alpha)Y}{wL} = \frac{(1-\alpha)Y}{(1-\alpha)\bar{A}kL} = 1!\end{aligned}$$

Hence, there would be no net income from working. A better alternative is a constant consumption tax τ_c :

$$\begin{aligned}\tau_c cL &= G = (1-\alpha)Y, \text{ i.e.,} \\ \tau_c &= (1-\alpha)\frac{Y}{cL}.\end{aligned}$$

Note that this consumption tax rate *is* constant in view of the AK structure of the model (implying that $\dot{c}/c = \dot{y}/y$). Hence, this tax is non-distorting – in the present model. The basic assumption in the present model which ensures this is that leisure does not enter the utility function. Then labor supply is inelastic and thus not distorted by the consumption tax.

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