Economic Growth. Exercises

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Problem set VI

VI.1 In 1960 per capita GDP in South Korea and Philippines were almost the same. Over the period 1960-1990 the average annual growth rate of per capita GDP was in South Korea 6.7 percent and in Philippines 1.5 percent. Give a brief account of alternative hypothetical explanations of this difference in the growth performance of the two countries.

VI.2 *On B & S's simple increasing variety model and a more robust alternative* First we consider the "simple increasing variety model" in the B & S text. Firm i ($i = 1, 2, ..., M$) in the manufacturing sector has the production function

$$Y_i = A \left(\sum_{j=1}^{N} {x_{ij}}^{\alpha} \right) L_i^{1-\alpha}, \qquad A > 0, \ 0 < \alpha < 1. \tag{1}$$

Here Y_i, L_i, and x_{ij} denote output of the firm, labour input and input of intermediate good j, respectively ($j = 1, 2, ..., N$; N "large").

a) In equilibrium the symmetry in (1) and the fact that the prices of intermediate goods are all set by monopoly firms at the same level $p = 1/\alpha$, induce firm i to choose $x_{ij} = x_i$ for all j. Explain by a few well-chosen sentences why this is so. Next derive the implied result:
$$Y_i = AN{x_i}^{\alpha} L_i^{1-\alpha}. \tag{2}$$

b) A general feature of increasing variety models is the hypothesis that "variety is productive" or, with a broader formulation, "there are gains by specialization". Is this hypothesis consistent with the equation (2)? Yes or no? Explain.

Let the aggregate input of intermediate goods in the manufacturing sector and the aggregate output in the sector be denoted X and Y, respectively. Thus, $X = \sum_i \sum_j x_{ij}$ and $Y = \sum_i Y_i$.

c) Write down an expression for the value added in the manufacturing sector. Comment.

The model leads to the following expression for the aggregate production function in manufacturing at time t :

$$Y_t = AX_t^{\alpha}(N_t L)^{1-\alpha}, \qquad A > 0, 0 < \alpha < 1, \tag{3}$$

where L is the constant labour force, $L > 0$.

d) Already this formula gives a hint that the model is (technologically) capable of generating fully endogenous growth. Briefly explain.

The model implies a relation of the form

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\eta}\left[A\left(\frac{X_t}{N_t}\right)^{\alpha} L^{1-\alpha} - \frac{X_t}{N_t} - \frac{c_t L}{N_t}\right], \qquad \eta > 0, \tag{4}$$

where c is per capita consumption.

e) Briefly explain how this relation is derived and interpret the parameter η.

f) "The expression (4) indicates that a social planner can always establish a positive constant growth rate of N." True or false? Comment.

In more general increasing variety models[1] (1) is replaced by

$$Y_i = AN^{\beta}(CES_i)^{\alpha} L_i^{1-\alpha}, \qquad A > 0, \ \beta > 0, 0 < \alpha < 1, \tag{5}$$

where the parameter β reflects "gains to specialization" and CES_i is a CES aggregate[2] of the quantities $x_{i1}, ..., x_{iN}$:

$$CES_i \equiv N\left(N^{-1}\sum_{j=1}^{N} x_{ij}{}^{\sigma}\right)^{\frac{1}{\sigma}}, \qquad 0 < \sigma < 1, \tag{6}$$

(This is the "CRS definition" of a CES aggregate in that the right-hand side of (6) has CRS with respect to the inputs $x_{i1}, ..., x_{iN}$; in B & S, p. 286, footnote 2, also appears a CES aggregate, but without this convenient CRS property which opens up for "gains to specialization" to appear explicitly *outside* the CES index as in (5).) Again, in equilibrium, because of symmetry and the fact that the prices of intermediate goods will all be set at the same level $p = 1/\sigma$, firm i chooses $x_{ij} = x_i$, for all j.

[1] For example Jones (AER, 2002) and Alvarez-Pelaez and Groth (2005).

[2] CES = Constant Elasticity of Substitution.

g) "The B & S specification (1)-(2) is a special case of (5)-(6), namely the case $\sigma = \alpha$ and $\beta = 1 - \alpha$." True or false? Comment.

By the method described in Lecture Note 17 it can be shown that the aggregate production function in manufacturing in the general case is

$$Y_t = AN_t^{\beta} X_t^{\alpha} L^{1-\alpha}.$$

h) Suppose $\beta < 1 - \alpha$. Comment on the likely capability of this model to generate one or another kind of endogenous growth.

VI.3 *First-best solution in the simple increasing variety model* Consider the "social planner" in an economy described by B & S's simple increasing variety model (standard notation). The social planner's criterion function is the same as that of the representative household. The social planner faces the problem to choose $(c_t, X_t)_{t=0}^{\infty}$ so as to:

$$\begin{aligned}
\max U_0 &= \int_0^{\infty} \frac{c_t{}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.} \\
& c_t > 0, \;\; X_t \geq 0, \\
& \dot{N}_t = \frac{1}{\eta}(Y_t - X_t - c_t L), \;\; \text{where } Y_t = AX_t{}^{\alpha}(N_t L)^{1-\alpha} \text{ and } N_0 \text{ is given,} \\
& N_t \geq 0 \text{ for all } t \geq 0. \qquad (*)
\end{aligned}$$

a) Briefly, interpret the four elements in the problem, including the parameters.

b) Solve the problem, assuming existence of an interior optimal solution.

c) Write down required parameter restrictions for existence of a solution with growth.

d) Check whether your candidate solution satisfies Arrow's sufficient conditions for an optimal solution. *Hint:* the relevant conditions are:

   1. The maximized Hamiltonian (i.e., the Hamiltonian with the controls replaced by their maximizing values expressed in terms of the adjoint variable and the state variable, assuming these maximizing values exist) should be concave in the state variable for all t.

2. Given the non-negativity constraint (*) on the state variable, the transversality condition

$$\lim_{t\to\infty} N_t \lambda_t e^{-\rho t} = 0,$$

where λ_t is the adjoint variable in the (current-value) Hamiltonian, should hold.

e) Comment on your candidate solution.

f) Suppose the economy is decentralized (in the standard way) and that innovative entrepreneurs have access to infinitely-lived patents (gratis). There is a government which wishes to implement the social planner's solution by use of a lump-sum tax to finance a production subsidy to firms in the basic-goods sector at rate s, i.e., firm i gets sales revenue (using standard notation)

$$(1+s)Y_i = (1+s)AL_i^{1-\alpha}\sum_{i=1}^{M} x_{ij}^{\alpha}, \qquad A > 0, 0 < \alpha < 1.$$

Derive the required value of the subsidy rate s.

g) Test whether this policy can implement the social planner's solution.

VI.4 *Stochastic erosion of monopoly power.* Here we extend the simple increasing variety model by introducing uncertainty as to how long the monopoly position of an inventor lasts.

a) Sooner or later inventor j looses the monopoly (patent protection is only temporary, imitators find out how to make close substitutes). When this happens, intermediate good j becomes competitive, i.e., it is supplied in the amount

$$X^c = L(\alpha A)^{\frac{1}{1-\alpha}}.$$

Explain this result and comment on the size relation between X^c and X^m.

b) Suppose the erosion of monopoly power can be described by a Poisson process. That is, if T denotes the remaining lifetime of monopoly j, then the probability that $T > z$ is e^{-pz}, where $p > 0$ is a given Poisson intensity (the same for all monopolies). Further, the cessations of the different monopolies are stochastically independent. N is "large" so that by holding shares in many different firms, the households face no risk. The market value of monopoly j at time t can be written

$$\begin{aligned} V(t) &= E_t \int_t^\infty \pi_j(\tau) e^{-\int_t^\tau r(s)ds} d\tau & (*) \\ &= \pi^m \int_t^\infty e^{-\int_t^\tau (r(s)+p)ds} d\tau, & (**) \end{aligned}$$

where r is the rate of interest on safe loans. Explain this result, either just in words or by deriving it formally.

c) Determine r and the growth rate of c in equilibrium with $\dot{N} > 0$. *Hint:* In equilibrium with $\dot{N} > 0$, $V(t)$ satisfies a simple relation. This can be combined with the no-arbitrage condition (or Fisher equation), which with this kind of uncertainly is $(\pi^m + \dot{V} - pV)/V = r$.

d) Let N^c denote the number of intermediates that have become competitive and let $N^m \equiv N - N^c$. It can be shown that N^c/N approaches a constant $(N^c/N)^*$ over time. This constant is an increasing function of p, and so is Y in equilibrium. Explain the intuition behind these two features. Combined with the result from c), in what sense does this illustrate a classical dilemma of patent legislation?

VI.5 Consider a closed market economy with L utility maximizing households. Each household supplies inelastically one unit of labour per time unit. There are two production sectors, the "basic-goods sector" and the "innovative sector". For convenience we call the two sectors Sector 1 and Sector 2, respectively. There is no physical capital in the economy. Households' financial wealth consists of shares in monopoly firms in Sector 2, which supplies specialized intermediate goods. These goods are input in Sector 1, where the firms operate under perfect competition. Also the labour market has perfect competition. All firms are profit maximizers. Generally variables are dated implicitly. A dot over a variable denotes the time derivative.

Firm i $(i = 1, 2, ..., M)$ in Sector 1 has the production function

$$Y_i = AL_i{}^{1-\alpha} \sum_{j=1}^{N} (x_{ij})^\alpha, \qquad A > 0,\ 0 < \alpha < 1.$$

Here Y_i, L_i and x_{ij} denote output of the firm, labour input and input of intermediate good j, respectively $(j = 1, 2, ..., N)$.

In Sector 2 R&D activity occurs. New "technical designs", that is, blueprints for making new specialized intermediate goods are invented. Ignoring indivisibility problems, we assume that the number of new technical designs invented in the economy per time unit can be written

$$\dot{N} = \mu R, \qquad \mu > 0, \ \mu \text{ constant},$$

where R denotes the aggregate R&D cost (per time unit) in terms of basic goods. For simplicity it is assumed that inventions can go in so many directions that the likelihood of different agents chasing and making the same invention is negligible.

After an invention has been made, the inventor begins supplying the new intermediate good. To begin with the inventor has a monopoly over the production and sale of the new good (say by concealment of the new technical design). But sooner or later imitators find out how to make very close substitutes (it is difficult to codify the technical aspects of the inventions, hence patents do not give effective protection and are in any case only of limited duration). There is uncertainty as to how long the monopoly position of an inventor lasts. We assume the erosion of monopoly power can be described by a Poisson process. That is, if T denotes the remaining lifetime of monopoly j, then the probability that $T > \tau$ is $e^{-p\tau}$, where $p > 0$ is a given Poisson "arrival rate" (the same for all monopolies). Further, the cessations of the different monopolies are stochastically independent. N is "large" so that by holding shares in many different firms, the households face no risk.

Aggregate output of basic goods, $Y \equiv \sum_{i=1}^{M} Y_i$, is used partly for consumption, $C \equiv cL$, partly for input in R&D activity and partly for input in the production of specialized intermediate goods. Once invented, an intermediate good of type j costs one basic good as input and nothing else (the same for all j). Hence we have

$$Y = C + R + X,$$

where $X \equiv \sum_j \sum_i x_{ij}$.

a) As long as inventor j (firm j in Sector 2) is still a monopolist, the earned profit per unit of time is $\pi_j = (\frac{1}{\alpha} - 1)X^m \equiv \pi^m$, where $X^m = LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}$. Explain this result (you don't have to derive it formally).

b) As described above, sooner or later inventor j loses the monopoly. When this happens, intermediate good j becomes competitive, i.e., it is supplied in the amount $X^c = L(\alpha A)^{\frac{1}{1-\alpha}}$. Explain this result and comment on the relative size of X^c and X^m.

The market value of monopoly j at time t can be written

$$V(t) = E_t \int_t^\infty \pi_j(\tau) e^{-\int_t^\tau r(s)ds} d\tau = \pi^m \int_t^\infty e^{-\int_t^\tau (r(s)+p)ds} d\tau,$$

where r is the real rate of interest on safe loans.

c) Explain this result, either just in words or by deriving it formally.

d) Find r in an equilibrium with $\dot{N} > 0$; compare with what r would be in case of no erosion of monopoly power. *Hint:* In equilibrium with $\dot{N} > 0$, $V(t)$ satisfies a simple relation. This can be combined with the no-arbitrage condition $(\pi^m + \dot{V} - pV)/V = r$.

Suppose the households, all alike, have an intertemporal utility function with infinite horizon and a constant rate of time preference $\rho > 0$. The instantaneous utility function has elasticity of marginal utility equal to a constant $\theta > 0$.

e) Find the rate of growth of c in an equilibrium with $\dot{N} > 0$ (an informal argument, based on your general knowledge, is acceptable). In case you need to introduce restrictions on some parameters in order to ensure positive growth and/or bounded utility, do it. Let the growth rate of c be denoted γ_c. Compare with what the growth rate of c would be in case of no erosion of monopoly power. Comment.

Let N^c denote the number of intermediates that have become competitive and let $N^m \equiv N - N^c$.

f) Output by firm i in Sector 1 can now be written

$$Y_i = AL_i \left[(N - N^c)(\frac{x_i^m}{L_i})^\alpha + N^c(\frac{x_i^c}{L_i})^\alpha \right].$$

What is the economic logic behind this result?

g) Further, $x_i^m/L_i = X^m/L$ and $x_i^c/L_i = X^c/L$. Why?

h) Find an expression for aggregate output of basic goods as a function of L, N and N^c. Comment.

i) It can be shown that N^c/N approaches a constant $(N^c/N)^*$ over time, that this constant is $p/(\gamma_N + p)$, where $\gamma_N \equiv \dot{N}/N$, and that $\gamma_N = \gamma_c$ in steady state. Briefly explain the intuition behind these three features.

j) How does the size of p affect steady state growth? Comment.

k) Use the answers to h) and i) to find the solution for aggregate output of basic goods in steady state. Compare with what output would be in case of no erosion of monopoly power. Comment.

ℓ) The model - and some of the above results - illustrate dilemmas in antitrust policy and patent legislation. Explain.

VI.6 *Short questions*

a) Give a qualified guess about what happens in the simple increasing variety model, if the size of the representative dynastic family is $L_t = L_0 e^{nt}$, where $n > 0$ and constant, and where each family member supplies one unit of labour per time unit.

b) Briefly compare the results from B & S's simple increasing variety model to the results from other innovation-based endogenous growth models.

VI.7 *Innnovation-based growth and scale effects* In innovation-based growth models there is a tendency for scale effects to arise.

a) Define different kinds of scale effects.

b) Why is it that scale effect tends to arise in innovation-based growth models?

c) Give examples of different innovation-based growth models with different kinds of scale effects.

d) Briefly discuss scale effects on growth and on levels, respectively, in relation to empirical evidence.

e) Briefly describe the main idea in quality ladder models.

f) Can scale effects on growth arise in quality ladder models? Why or why not?

VI.8 *Short questions*

a) The empirical evidence suggests that market economies do too little R&D investment compared to the "optimal level" (as usually defined from the perspective of a representative household). Is the increasing variety model by Romer consistent with this? Is the quality ladder model? In both cases, say why or why not?

b) What would the answers to a) be if the empirical evidence suggested that market economies do too much R&D investment compared to the "optimal level"? Discuss.

c) We have in this course studied models with intermediates of increasing variety and increasing quality, respectively. Now consider another type of models, namely increasing-consumption-goods-variety models and increasing-consumption-goods-quality models. Such models exist. Suppose the labour force is constant. Then there is in these models typically no growth in the physical output per unit of labour. Do you think they might be considered growth models anyway? In both cases, say why or why not?

VI.9 Consider the Jones (1995) model for a closed economy. For simplicity we ignore the duplication externality. With standard notation the model is:

$$\begin{aligned} Y_t &= K_t^{\alpha}(A_t L_{Yt})^{1-\alpha}, \qquad 0 < \alpha < 1, \\ \dot{K}_t &= Y_t - c_t L_t - \delta K_t, \qquad \delta \geq 0, \\ \dot{A}_t &= \mu A_t^{\varphi} L_{At}, \qquad \mu > 0, \varphi < 1, \\ L_{Yt} + L_{At} &= L_t, \\ L_t &= L_0 e^{nt}, \; n > 0, \; \text{constant.} \end{aligned} \tag{*}$$

a) Find the growth rate of "knowledge", A, under balanced growth. *Hint:* See the hint to b).

b) Find the growth rate of $y \equiv Y/L$ under balanced growth. *Hint:* Since the model is not a fully endogenous growth model, the approach to the study of balanced growth is different and more simple than that needed for AK-style models. A good starting point is the growth accounting relation $g_Y = \alpha g_K + (1-\alpha)(g_A + g_{L_Y})$, where one can use the fact that under balanced growth in a closed economy $g_Y = g_K$.

c) Find the growth rate of c under balanced growth.

d) Suppose $s_A \equiv L_A/L$ can be increased by an R&D subsidy.

1. Will this affect the long-run growth rate? Comment. *Hint:* It can be shown that the model is saddle-point stable.
2. Will it affect levels under balanced growth? Comment. *Hint:* Find an expression for y in terms of $\hat{k} \equiv K/(AL_Y)$, s_A and A under balanced growth. Then find an expression for A in terms of L_A under balanced growth.

Suppose we at the microeconomic level have an increasing-variety set-up with monopolists as in the simple increasing-variety model of Lecture Note 17 (apart from specialized intermediate goods being replaced by specialized capital goods). Then it can be shown that the equilibrium real interest rate at time t equals $\alpha^2 Y/K$. This information is useful for the next question.

e) Is there a scale effect on levels in the model? Comment. *Hint:* From Jones (1995, p. 769) we have that s_A under balanced growth is independent of L. Show by use of the Keynes-Ramsey rule that also $\hat{k}$ under balanced growth is independent of L. Then the conclusion follows.

VI.10 *Short questions*

a) "A relatively homogeneous group of countries such as for example the EU countries tend to experience income convergence in the sense that the standard deviation of income per capita across the countries diminishes over time." True or not true as an empirical statement? Explain.

b) "Growth accounting pinpoints the ultimate sources of growth." True or false? Explain.

c) "If there is 'learning-by-investing', standard growth accounting tend to understate total factor productivity growth." True or not true? Explain.

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