Economic Growth, June 2006.
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A suggested solution to the problem set at the exam in Economic Growth, June 7, 2006

Four hours. No auxiliary material

1. Solution to Problem 1 (10 %)

a) We say that a given collection of countries show σ *convergence* with respect to a given measure of dispersion if this measure of dispersion, applied to income per capita or output per worker across the countries, declines systematically over time. On the other hand, the countries show σ *divergence* if the dispersion increases systematically over time.

A relevant measure of dispersion is for example the standard deviation of *relative* income per capita (or of $\log y$ where $y \equiv Y/L$). The standard deviation of income per capita as such is not a very relevant measure. This is because income per capita is trending upwards and the standard deviation is not a scale-free measure.

The reported data shows a tendency for world income per capita relative to income per capita in the US 1952-96 to increase over time, i.e., the ratio

$$\frac{GDP_{world}/L_{world}}{GDP_{US}/L_{US}}$$

tends to increase. Although this tendency is not an unambiguous indication of σ convergence, it is at least consistent with σ convergence. More detailed data across the countries in the world (Sala-i-Martin 2002) do in fact show a (weak) tendency to σ convergence, at least in the last 20 years, when countries' per-capita income is *weighted by population size* as a fraction of world population. Without such weighting, data for countries in the world show slight σ divergence (this is because the developments in small slow-growing countries in, e.g., Africa and Latin America then count equally much as the development in fast-growing large countries in Asia, such as India and China).

b) Let country i have a per capita production function $y_i = f(k_i, T_i)$, where k_i is the capital intensity and T_i is the technology level of country i. As explanations of the observed σ convergence (for countries weighted by population size) one could imagine the following:

1. *Solow-type transition dynamics.* If countries have similar structural characteristics (i.e., they have access to the same technology and they share the same parameters s, δ and n), but different initial conditions, then they would be converging towards the same steady state ($y_i \to \hat{y}^* T$ for $t \to \infty$) and therefore show σ convergence. There are two kinds of problems with such an explanation: a) The countries of the world are generally not closed economies, but parts of an international economic system. b) The countries of the world are far from having similar structural characteristics. (An analogue argument goes through if we think of Ramsey-type transition dynamics and replace the Solow parameter s by the two Ramsey parameters ρ and θ).

2. *Factor movements across countries and regions.* Factors tend to move to regions where they get the highest remuneration.

3. *Technological catching up.* In general countries do not have access to the same technology. It takes time for technology to diffuse across countries. After the second world war, the economies of the world have generally become more and more open economies (less restrictions on trade and capital movements). This promotes technological catching-up which can be described in the following way. Let T_i be the technology level of country i,

$i = 1, 2, ..., N$. Let T_L be the technology level of the world leader (after the second world war the US). Suppose $T_L = T_L(0)e^{xt}$, where $x > 0$ is a constant. Then one way of formulating the catching-up hypothesis is:

$$\frac{\dot{T}_i}{T_i} = x + \xi_i(\frac{T_L}{T_i} - 1), \qquad \xi_i > 0.$$

The parameter ξ_i is sometimes called the learning capacity of country i and is generally assumed to depend on the "quality" of institutions and the level of human capital in country i. This can be called "strong catching up".

Bernard and Jones (1996) consider a weaker form of catching up:

$$\frac{\dot{T}_i}{T_i} = \xi_i \frac{T_L}{T_i},$$

where $\xi_L = x$. Suppose country i has $\xi_i < \xi_L$ and is initially far behind the leader so that $T_L/T_i > \xi_L/\xi_i > 1$. Then initially $\dot{T}_i/T_i = \xi_i T_L/T_i > \xi_L = x$, hence T_L/T_i is declining for a while. When T_L/T_i reaches ξ_L/ξ_i, however, we get $\dot{T}_i/T_i = x = \dot{T}_L/T_L$ although still $T_L/T_i > 1$. Thus in this case country i never catches up fully with the leader. On the other hand, if $\xi_i \geq \xi_L$, then country i tends to catch up fully and may even overtake the current leader and itself become a new leader.

2. Solution to Problem 2 (30 %)

a) We solve the problem:

$$\max_{K^d, L^d} \Pi = F(K^d, hL^d) - RK^d - wL^d.$$

First-order conditions are

$$\begin{aligned} F_1(K^d, hL^d) - R &= 0, && \text{(FOC1)} \\ F_2(K^d, hL^d)h - w &= 0. && \text{(FOC2)} \end{aligned}$$

In equilibrium, $K^d = K$ and $L^d = L$ so that (FOC1) and (FOC2) give

$$\begin{aligned} R &= F_1(K, hL) \\ w &= F_2(K, hL)h, \end{aligned}$$

respectively. In view of CRS, $F(K, hL) = hLF(\hat{k}, 1) \equiv hLf(\hat{k})$, where $f' = F_1 > 0, f'' < 0$. Further, $F_2(K, hL) = f(\hat{k}) - f'(\hat{k})\hat{k}$. Hence, we get the solution

$$R = f'(\hat{k}), \tag{2.1}$$

$$w \equiv \hat{w}h = \left[f(\hat{k}) - f'(\hat{k})\hat{k}\right] h. \tag{2.2}$$

b) We consider the household problem: choose a path $(c_t, i_t)_{t=0}^{\infty}$ to maximise

$$U_0 = \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \qquad \text{s.t.} \tag{2.3}$$

$$c_t \geq 0, \; i_t \geq 0, \tag{2.4}$$

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t - i_t, \qquad a_0 \text{ given}, \tag{2.5}$$

$$\dot{h}_t = i_t - (\delta + n)h_t, \qquad h_0 > 0 \text{ given}, \tag{2.6}$$

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0, \tag{2.7}$$

$$h_t \geq 0 \text{ for all } t. \tag{2.8}$$

The household maximizes discounted utility. The pure rate of time preference (impatience) is ρ, but taking the possibly larger household size in the future into account, the effective rate of utility discount is the growth-corrected rate $\rho - n$, cf. (2.3). Instantaneous utility is of the CIES type with (numerical) elasticity of marginal utility equal to the constant θ. Thus θ is a measure of the desire for consumption smoothing, $1/\theta$ being the intertemporal elasticity of substitution in consumption. The control variables of the household are per-capita consumption, c_t, and per-capita educational investment, i_t, none of which can be negative, cf. (2.4). There are two dynamic constraints, (2.5) and (2.6). If the household's financial wealth at time t is called A_t, then, by simple accounting,

$$\dot{A}_t = r_t A_t + w_t L_t - C_t - I_{Ht}, \qquad A_0 \text{ given}. \tag{2.9}$$

Differentiating $a_t \equiv A_t/L_t$ w.r.t. t and substituting (2.9) leads to the per-capita financial wealth accumulation constraint (2.5). In a similar way, from $\dot{H}_t = I_{Ht} - \delta H_t$ and $h_t \equiv H_t/L_t$ we get the per-capita human capital accumulation constraint (2.6). In terms of aggregate financial wealth the standard No-Ponzi-Game condition (implying a constraint on how fast the family's net debt is allowed to grow in the long run) would read

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0.$$

Inserting $A_t \equiv a_t L_t = a_t L_0 e^{nt}$, this gives (2.7), ignoring the unimportant positive constant L_0. Finally, whereas in principle we can have $a_t < 0$ (implying a positive net debt), human capital is by definition constrained to be non-negative as expressed by (2.8).

c) The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda_1 \left[(r-n)a + \hat{w}h - c - i\right] + \lambda_2 \left[i - (\delta + n)h\right],$$

where λ_1 and λ_2 are the shadow prices of per-capita financial wealth and per-capita human capital, respectively, along the optimal path. An interior solution satisfies the first order conditions:

$$\begin{aligned}
\partial\mathcal{H}/\partial c &= c^{-\theta} - \lambda_1 = 0, \quad \text{i.e., } c^{-\theta} = \lambda_1, && (2.10)\\
\partial\mathcal{H}/\partial i &= -\lambda_1 + \lambda_2 = 0, \quad \text{i.e., } \lambda_2 = \lambda_1, && (2.11)\\
\partial\mathcal{H}/\partial a &= \lambda_1(r-n) = -\dot{\lambda}_1 + (\rho - n)\lambda_1, \quad \text{i.e.,} \\
-\dot{\lambda}_1/\lambda_1 &= r - n - (\rho - n) = r - \rho, && (2.12)\\
\partial\mathcal{H}/\partial h &= \lambda_1\hat{w} - \lambda_2(\delta + n) = -\dot{\lambda}_2 + (\rho - n)\lambda_2, \quad \text{i.e.,} \\
-\dot{\lambda}_2/\lambda_2 &= \lambda_1\hat{w}/\lambda_2 - (\delta + n) - (\rho - n) = \lambda_1\hat{w}/\lambda_2 - (\delta + \rho), && (2.13)
\end{aligned}$$

and the transversality conditions:

$$\begin{aligned}
\lim_{t\to\infty} a_t\lambda_{1t}e^{-(\rho-n)t} &= 0, && (\text{TVC}_1)\\
\lim_{t\to\infty} h_t\lambda_{2t}e^{-(\rho-n)t} &= 0. && (\text{TVC}_2)
\end{aligned}$$

That is, on the margin, according to (2.10), income must be equally valuable in its two uses, consumption or saving. Similarly, on the margin, according to (2.11), non-leisure time must be equally valuable in its two uses, work or education. Moreover, (2.12) and (2.13) tell how the shadow prices of the two assets must move over time in the optimal plan. Finally, (TVC_1) and (TVC_2) ensure that none of the assets are over-accumulated.

d) Log-differentiating (2.10) w.r.t. t gives $-\theta\dot{c}/c = \dot{\lambda}_1/\lambda_1$. We substitute (2.12) into this and get, after ordering,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho), \tag{2.14}$$

which is the Keynes-Ramsey rule.

From (2.11) follows $\dot{\lambda}_2/\lambda_2 = \dot{\lambda}_1/\lambda_1$ which together with (2.12) and (2.13) implies $\lambda_1\hat{w}/\lambda_2 - (\delta + \rho) = r - \rho$. By (2.11) this yields

$$\hat{w}_t - \delta = \frac{\hat{w}_t - \delta}{1} = r_t. \tag{2.15}$$

This is a no-arbitrage relationship saying that along an interior optimal path the household is indifferent between placing the marginal unit of saving in a financial asset yielding the rate of return r or in education to obtain one more unit of human capital. The last alternative gives an extra labour income gross-of-human-capital depreciation equal to $\hat{w}$ (which is the real wage per unit of human capital). The net-of-depreciation return on that alternative is then $\hat{w} - \delta$. This explains (2.15).

e) With $Y = AK^{\alpha}(hL)^{1-\alpha}$, (2.1) gives

$$R_t = \alpha A\hat{k}_t^{\alpha-1}.$$

Placing the marginal unit of saving on the loan market gives the rate of return r and placing it in physical capital gives the (net) rate of return $R - \delta$. Hence, in equilibrium, $R - \delta = r$ so that

$$r_t = \alpha A\hat{k}_t^{\alpha-1} - \delta, \tag{2.16}$$

where $\hat{k}_t \equiv K_t/H_t \equiv K_t/(h_t L_t)$ is predetermined.

f) For an interior solution to obtain, the no-arbitrage condition (2.15) must hold. In view of (2.2), this implies

$$\hat{w} = f(\hat{k}) - f'(\hat{k})\hat{k} = A\hat{k}^\alpha - \alpha A\hat{k}^{\alpha-1}\hat{k} = (1-\alpha)A\hat{k}^\alpha = r + \delta = f'(\hat{k}) = \alpha A\hat{k}^{\alpha-1}.$$

From this follows

$$\hat{k} \equiv \frac{K}{H} = \frac{\alpha}{1-\alpha} \equiv \hat{k}^*. \tag{2.17}$$

It is assumed that parameters are such that $\dot{c}/c > 0$ and U_0 is bounded. By (2.16) and (2.17), (2.14) implies

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\alpha A(\frac{\alpha}{1-\alpha})^{\alpha-1} - \delta - \rho) = \frac{1}{\theta}(\alpha^\alpha(1-\alpha)^{1-\alpha}A - \delta - \rho) \equiv \gamma,$$

a constant. Hence, we have $\gamma > 0$. A condition ensuring that U_0 is bounded is the assumption $(1-\theta)\gamma < \rho - n$.

Suppose that initially $K_0/H_0 > \hat{k}^*$. Then human capital is relatively scarce and the marginal rate of return on investing in education is higher than on investing in physical capital. Hence, for a while the economy invests only in human capital. This results in a falling K/H. When K/H reaches the level $\hat{k}^*$, the phase of complete specialization ends. From now on the economy invests in both human and physical capital in such proportions as to maintain the efficient ratio $\hat{k}^*$. Similarly, if initially $K_0/H_0 < \hat{k}^*$, there will be a phase of complete specialization in physical capital investment, until the efficient ratio $\hat{k}^*$ is obtained. In both cases, in the long run (indeed after some finite period of time) the economy will be in steady state and behave in an AK-style manner.

3. Solution to Problem 3 (50 %)

For convenience, the basic equations of the model are repeated here:

$$Y_i = AL_i{}^{1-\alpha}\sum_{j=1}^{N}(x_{ij})^\alpha, \qquad A > 0,\ 0 < \alpha < 1,\ i = 1, 2, ..., M. \tag{3.1}$$

$$\dot{N} = \mu R, \qquad \mu > 0. \tag{3.2}$$

$$Y \equiv \sum_{i=1}^{M} Y_i = C + R + X, \tag{3.3}$$

where $C \equiv cL$ and $X \equiv \sum_j \sum_i x_{ij}$.

a) As long as inventor j (firm j in Sector 2) is still a monopolist, the earned profit per unit of time is

$$\pi_j = (\frac{1}{\alpha} - 1)X^m \equiv \pi^m, \tag{3.4}$$

where

$$X^m = LA^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}}. \tag{3.5}$$

Explanation: With the basic good as the numeraire, the monopoly price (profit-maximizing price) is $1/\alpha$, because the monopolist faces a demand function with price elasticity $-1/(1-\alpha)$. At the price $P_j = 1/\alpha$, aggregate demand for input good j is as in (3.5). Hence, total revenue is $(1/\alpha) \cdot X^m$ and total cost is $1 \cdot X^m$, resulting in (3.4).

A more detailed explanation (not necessary) could be based on an explicit derivation of, first, the demand function, second, the monopoly price. Firm i in Sector 1 solves the problem:

$$\max_{L_i, (x_{ij})_{j=1}^N} \Pi_i = A{L_i}^{1-\alpha} \sum_{j=1}^{N} (x_{ij})^\alpha - wL_i - \sum_{j=1}^{N} P_j x_{ij}.$$

FOCs are:

$$\partial \Pi_i / \partial L_i = \partial Y_i / \partial L_i - w = (1-\alpha) A L_i^{-\alpha} \sum_{j=1}^{N} (x_{ij})^\alpha - w = 0, \tag{3.6}$$

$$\partial \Pi_i / \partial x_{ij} = \partial Y_i / \partial x_{ij} - P_j = \alpha A {L_i}^{1-\alpha} x_{ij}^{\alpha-1} - P_j = 0, \quad j = 1, 2, ..., N. \tag{3.7}$$

(3.7) gives the demand

$$x_{ij} = L_i (\alpha A)^{\frac{1}{1-\alpha}} P_j^{-\frac{1}{1-\alpha}}, \quad j = 1, 2, ..., N. \tag{3.8}$$

Hence, aggregate demand for input good j is

$$X_j^d = \sum_i x_{ij} = L\,(\alpha A)^{\frac{1}{1-\alpha}}\,P_j^{-\frac{1}{1-\alpha}} \equiv X_j(P_j),$$

where we have used $\sum_i L_i = L$.

The monopolist chooses P_j so as to maximize profits subject to the demand curve $X_j(P_j)$. Let $P_j(X_j)$ denote the maximum price at which the amount X_j can be sold. Then the profit maximizing P_j is the price at which $MR = MC$. We have $MC = 1$. From $TR = P_j(X_j)X_j$ we find

$$\begin{aligned} MR &= dTR/dX_j = P_j + X_j dP_j/dX_j \\ &= P_j(1 + \frac{X_j}{P_j} dP_j/dX_j) = P_j(1-(1-\alpha)) = P_j\alpha = MC = 1 \\ &\Rightarrow \; P_j = \frac{1}{\alpha}. \end{aligned}$$

b) When intermediate good j has become competitive, it is supplied in the amount $X^c = L(\alpha A)^{\frac{1}{1-\alpha}}$. Explanation: competition implies market price = marginal cost = 1 so that X^c = aggregate demand = $X_j(P_j) = X_j(1) = L\,(\alpha A)^{\frac{1}{1-\alpha}}$. Intuitively, since the demand depends negatively on the price, we should expect $X^m < X^c$. This also holds true, since $\alpha^{\frac{2}{1-\alpha}} = (\alpha^{\frac{1}{1-\alpha}})^2 < \alpha^{\frac{1}{1-\alpha}}$, in view of $\alpha^{\frac{1}{1-\alpha}} < 1$, which follows from $0 < \alpha < 1$.

c) The market value of monopoly j at time t is the present discounted value of expected future profits

$$V(t) = E_t \int_t^{\infty} \pi_j(\tau) e^{-\int_t^{\tau} r(s)ds} d\tau, \tag{3.9}$$

where

$$\pi_j(\tau) = \begin{cases} \pi^m \text{ if firm } j \text{ is still a monopolist at time } \tau, \\ 0 \text{ if not.} \end{cases}$$

The real rate of interest, r, on safe loans is the relevant discount rate because, by holding shares in many different monopoly firms, the household faces no risk. The expression (3.9) can be simplified:

$$\begin{aligned} V(t) &= E_t \int_t^\infty \pi_j(\tau) e^{-\int_t^\tau r(s)ds} d\tau = \int_t^\infty (E_t \pi_j(\tau)) e^{-\int_t^\tau r(s)ds} d\tau \\ &= \int_t^\infty (\pi^m e^{-p(\tau-t)} + 0 \cdot (1 - e^{-p(\tau-t)})) e^{-\int_t^\tau r(s)ds} d\tau \\ &= \pi^m \int_t^\infty e^{-\int_t^\tau (r(s)+p)ds} d\tau, \end{aligned}$$

where the "effective" rate of discount is $r(s) + p$.

d) The cost of making $\dot{N}$ inventions is $R = \dot{N}/\mu$. The cost of making one invention is $1/\mu$. Hence, equilibrium with $\dot{N} > 0$ requires $V(t) = 1/\mu$. As indicated by the hint, the described form of uncertainty implies the no-arbitrage condition

$$\frac{\pi^m + \dot{V}(t) - pV(t)}{V(t)} = r(t).$$

In view of $V(t) = 1/\mu$, this gives

$$\begin{aligned} r(t) &= \mu\pi^m - p = \mu(\frac{1}{\alpha} - 1)X^m - p \\ &= \mu(\frac{1}{\alpha} - 1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - p \equiv r, \end{aligned}$$

a constant.

In case of no erosion of monopoly power the rate of interest would be $r' = \mu(\frac{1}{\alpha} - 1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} = r + p > r$. Because of the limited duration of monopoly when $p > 0$, the (expected) rate of return on investing in R&D is smaller than in the case of no erosion of monopoly power.

e) The households (all alike) choose $(c(t))_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c(t) &\geq 0, \\ \dot{a}(t) &= ra(t) + w(t) - c(t), \quad A(0) \text{ given}, \\ \lim_{t\to\infty} a(t)e^{-rt} &\geq 0, \end{aligned}$$

where $a(t)$ is per-capita financial wealth, i.e., $a(t) = N(t)V(t)/L$. From the first-order conditions we find the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left[\mu(\frac{1}{\alpha}-1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - p - \rho\right] \equiv \gamma_c. \tag{3.10}$$

We assume parameters are such that $\gamma_c > 0$; this requires that L is "large enough" (due to the - problematic - scale effect on growth displayed in (3.10)). In addition, to avoid unbounded utility, we assume $(1-\theta)\gamma_c < \rho$.

In case of no erosion of monopoly power, the equilibrium real rate of interest would have been $r' = r + p > r$ so that the consumption growth rate would have been

$$\gamma_c' = \frac{1}{\theta}\left[\mu(\frac{1}{\alpha}-1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - \rho\right] > \gamma_c.$$

Comment: In case of no erosion of monopoly power, monopoly profits are maintained for ever. We would then have $V > 1/\mu$ if the rate of interest were still equal to r. That would imply excess demand for finance of R&D expenditure. Thus, the rate of interest would be driven upwards until V (the PDV of future profits) were down at the level $1/\mu$. The higher interest rate would induce the required extra saving. And this would be reflected in consumption starting from a low level, but growing at a higher speed.

f) We shall explain the economic logic behind the result that output by firm i in Sector 1 can be written

$$Y_i = AL_i\left[(N-N^c)(\frac{x_i^m}{L_i})^{\alpha} + N^c(\frac{x_i^c}{L_i})^{\alpha}\right], \tag{3.11}$$

where N^c is the number of intermediate goods that have become competitive, so that $N^m \equiv N - N^c$ is the number of intermediate goods that are still supplied under monopolistic conditions. In view of production and cost symmetry, $P_j = 1/\alpha$ for all intermediates j supplied under monopolistic conditions, and $P_j = 1$ for all competitive intermediates. Hence, firm i in Sector 1 chooses the same amount of each input good of the first kind as well as the same amount of each input good of the second kind, i.e.,

$$x_{ij} = \begin{cases} x_i^m \text{ if } j \text{ is still a monopoly,} \\ x_i^c \text{ if not.} \end{cases}$$

Thus (3.1) can be simplified to

$$\begin{aligned} Y_i &= A L_i^{1-\alpha} \left[N^m (x_i^m)^\alpha + N^c (x_i^c)^\alpha \right] \\ &= A L_i \left[N^m \left(\frac{x_i^m}{L_i}\right)^\alpha + n^c \left(\frac{x_i^c}{L_i}\right)^\alpha \right]. \end{aligned} \tag{3.12}$$

With $N^m \equiv N - N^c$, this is the same as (3.11).

g) Cost minimization implies that all firms in Sector 1 choose the same input ratios x_i^m/L_i and x_i^c/L_i, respectively. Thus,

$$\begin{aligned} \frac{x_i^m}{L_i} &= \frac{\sum_i x_i^m}{\sum_i L_i} = \frac{X^m}{L}, \qquad \text{and} \\ \frac{x_i^c}{L_i} &= \frac{\sum_i x_i^c}{\sum_i L_i} = \frac{X^c}{L}. \end{aligned}$$

h) Using this result together with (3.12) we get

$$
\begin{aligned}
Y &= \sum_{i=1}^{M} Y_i = A \sum_{i=1}^{M} L_i \left[(N - N^c)(\frac{X^m}{L})^\alpha + N^c(\frac{X^c}{L})^\alpha \right] \\
&= A \left[(N - N^c)(\frac{X^m}{L})^\alpha + N^c(\frac{X^c}{L})^\alpha \right] L \\
&= A \left[(N - N^c)(A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}})^\alpha + N^c(\alpha A)^{\frac{\alpha}{1-\alpha}} \right] L \\
&= A(\alpha A)^{\frac{\alpha}{1-\alpha}} \left[(N - N^c)\alpha^{\frac{\alpha}{1-\alpha}} + N^c \right] L \\
&= A(\alpha A)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} N \left[1 - \frac{N^c}{N} + \frac{N^c}{N} \alpha^{\frac{-\alpha}{1-\alpha}} \right] L \\
&= A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} LN \left[1 + \frac{N^c}{N}(\alpha^{\frac{-\alpha}{1-\alpha}} - 1) \right]. \qquad (3.13)
\end{aligned}
$$

Comment: Aggregate output is seen to depend on N^c/N. Note also that if N^c/N tends to a constant, then Y tends to be proportional to a produced "input", N. Therefore, the model is likely to be capable of generating (fully) endogenous growth, driven by R&D.

i) We shall explain the intuition behind the following three features: (i) over time N^c/N approaches a constant, $(N^c/N)^*$; (ii) $(N^c/N)^* = p/(\gamma_N + p)$, where $\gamma_N \equiv \dot{N}/N$; and (iii) $\gamma_N = \gamma_c$ in steady state.

On (i): When N^c is "small", $N^m = N - N^c$ is "large" and we have, by the law of large numbers,

$$\dot{N}^c \approx E\dot{N}^c = pN^m, \qquad (3.14)$$

which is "large" so that is N^c increases fast. Thus it seems likely that $N^c/N \to (N^c/N)^*$ for $t \to \infty$. (A more elaborate argument is given below.)

On (ii): The higher the "competition arrival rate" p, the higher should the fraction of competitive intermediates be. And the higher the growth rate, γ_N, of the number of different intermediates, the lower should the fraction of competitive intermediates be.

On (iii): In steady state $N^c/N = (N^c/N)^*$. Hence, in view of (3.13), Y is proportional to the produced "level of knowledge", N. Thus, in steady state, $\gamma_Y \equiv \dot{Y}/Y = \gamma_N$. Being a component of Y, consumption will grow at the same rate as Y in steady state, implying $\gamma_N = \gamma_c$.

A more elaborate argument (not necessary) for (i), (ii) and (iii) is the following. Let $u \equiv N^c/N$. Then,

$$\begin{aligned}
\frac{\dot{u}}{u} &= \frac{N\dot{N}^c - N^c\dot{N}}{N^2} = \frac{\dot{N}^c}{N} - u\gamma_N \\
&\approx p\frac{N - N^c}{N} - u\gamma_N \qquad \text{(by (3.14))} \\
&= p(1-u) - u\gamma_N = p - u(p + \gamma_N) \gtreqless 0 \text{ for } u \lesseqgtr \frac{p}{p+\gamma_N}.
\end{aligned}$$

Although γ_N is not constant outside steady state, this makes (i) plausible and proves (ii). As to (iii), by (3.13), in steady state

$$\begin{aligned}
Y &= \bar{A}N, \text{ where} \\
\bar{A} &\equiv A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}L\left[1 + \frac{p}{p+\gamma_N}(\alpha^{\frac{-\alpha}{1-\alpha}} - 1)\right].
\end{aligned} \tag{3.15}$$

At the same time, by (3.3),

$$\begin{aligned}
Y &= cL + R + X = cL + \frac{\dot{N}}{\mu} + N^m X^m + N^c X^c \\
&= cL + \gamma_N\frac{N}{\mu} + N\left[(1 - \frac{p}{p+\gamma_N})X^m + \frac{p}{p+\gamma_N}X^c\right],
\end{aligned}$$

where the term in square brackets is a constant in steady state. Combining this with (3.15), we see that also c must be proportional to N in steady state, thus proving (iii).

j) $\partial\gamma_c/\partial p = -\frac{1}{\theta} < 0$. *Comment:* A higher p means shorter duration of monopoly power. Thus, turning the argument at e) round, the rate of interest is driven down until V (the PDV of future profits) is pushed up to the equilibrium level

$1/\mu$. The lower interest rate induces less saving which is reflected in less finance for R&D and a lower growth rate of the economy. Or more brief: the shorter duration of monopoly power implies less incentive to do R&D, hence the growth rate becomes smaller.

k) The solution for aggregate output of basic goods in steady state is given by (3.15). The case of no erosion of monopoly power, corresponds to $p = 0$, implying an aggregate output equal to $Y' = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} LN$. Since $\alpha^{\frac{-\alpha}{1-\alpha}} > 1$, with $p > 0$ aggregate output is higher than Y'.

Comment: The intuition is that with erosion of monopoly power the wedge between the average price of intermediate goods and the marginal cost of producing them becomes smaller. Then the demand for these input goods becomes higher (closer to the efficient level). This leads to higher aggregate output.

ℓ) The model illustrates the classical dilemmas of antitrust policy and patent legislation. Consider first antitrust policy. This kind of policy aims at increasing competition. In the present model this corresponds to a policy that increases p. This leads to a *static efficiency gain* due to less monopoly power. On the other hand, as the answer to j) indicates, a higher p implies less incentive to do R&D. It can be shown that even with $p = 0$, the incentive to do R&D is already too low in the sense that R&D and growth are below what a social planner would accomplish. The reason is the wedge between the price of intermediate goods and the marginal cost of producing them. This implies too little demand for intermediate goods, hence too little remuneration for innovation and supplying the resulting new intermediates. Thus, more competition leads to a kind of *dynamic efficiency loss.*

Next, consider patent legislation. In order to induce more R&D, a government might consider prolonging the duration of patents. This corresponds to a lower p. Thereby, more innovation and growth is in fact induced according to this model, that is, this policy leads to a *dynamic efficiency gain.* On the other hand, as the answer to k) indicates, the policy implies a *static efficiency loss* due to the distortionary effects of monopoly power.

It can be shown that to reach the social planner's solution, two policy instruments are needed. To diminish the monopolist distortion and encourage demand of monopolized intermediates, a subsidy at some rate s to the purchase of intermediate goods is required. To encourage R&D, a subsidy to R&D spending at some rate σ is also needed. By comparing with the social plannner's solution (not considered here), it is possible to find exact formulas for s and σ such that the social plannner's solution is reached.

4. Solution to Problem 4 (10 %)

a) Not true. In the Ramsey model, a country can be *above* its steady state, yet far away from it. In that case, the per capita growth rate would be relatively low. If in the statement "further away from" is replaced by "further below", the statement would be true.

b) True. The model by Kongsamut et al. (2001) considers an economy with three sectors, agriculture, manufacturing and services. A representative household maximizes discounted utility. There is a constant discount rate and an instantaneous utility function implying that the income elasticity of demand for agricultural products is below 1, the income elasticity for manufactured goods is equal to 1 and that for services above 1. There is exogenous labour-augmenting technical progress at the same rate in all three sectors. As economic development proceeds, the fraction of the labour force working in the first sector gradually declines and that in the third sector gradually grows. Similarly, the output share of the first sector declines and that of the third sector grows.

c) Not true. In the Mincerian approach to human capital formation an individual's human capital is assumed to be proportional to an exponential function where the *exponent* is proportional to time spent in education. Indeed, consistent with a large empirical literature on schooling and wages, the Mincerian rule is $h = Ae^{\psi \ell_h}$, where ℓ_h is the number of years spent in education by the individual, ψ is a constant $\in (0.05, 0.15)$ and A is just some positive constant which by proper choice of measurement units can be replaced by 1 (cf. Jones 2002).

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