Economic Growth

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A suggested solution to the problem set at the exam June 7, 2004

Four hours. No auxiliary material

1. Solution to Problem 1

For convenience we repeat the basic relations:

$$Y = cL + G + I,$$

$$\dot{K} = I - \delta K,$$

$$\frac{G}{Y} = \bar{g}, \tag{*}$$

$$\left[\tau(ra + w) + \tau_{\ell}\right] L = G, \tag{GBC}$$

$$Y_i = AK_i^{\alpha}(GL_i)^{1-\alpha}, \quad 0 < \alpha < 1, A > 0, \quad i = 1, 2, ..., M.$$

a) The decision problem of firm i is:

$$\max_{K_i, L_i} \Pi_i = AK_i^{\alpha}(GL_i)^{1-\alpha} - RK_i - wL_i,$$

where $R = r + \delta$. FOCs:

$$\partial\Pi_i/\partial K_i = \alpha AK_i^{\alpha-1}(GL_i)^{1-\alpha} - R = 0, \tag{FOC1}$$

$$\partial\Pi_i/\partial L_i = (1 - \alpha)AK_i^{\alpha}(GL_i)^{-\alpha}G - w = 0. \tag{FOC2}$$

From (FOC1) we find

$$k_i \equiv K_i/L_i = (\frac{\alpha A}{R})^{1/(1-\alpha)}G = (\frac{\alpha A}{r + \delta})^{1/(1-\alpha)}G. \tag{1.1}$$

b) Equilibrium at factor markets implies

$$\sum_i K_i = K, \quad \text{and} \tag{1.2}$$

$$\sum_i L_i = L. \tag{1.3}$$

The cost-minimizing capital intensity is the same for all firms, as seen by (1.1); this is due to all firms having the same production function. In general equilibrium this capital intensity must be equal to $k \equiv K/L$, a predetermined variable from the supply side.

Since $y_i \equiv Y_i/L_i = Ak_i^\alpha G = Ak^\alpha G \equiv y$, aggregate output can be written

$$Y = \sum_i Y_i = \sum_i y_i L_i = y \sum_i L_i = yL = Ak^\alpha G^{1-\alpha} L. \tag{1.4}$$

Substituting $Y = G/\bar{g}$ from (*) gives

$$G/\bar{g} = Ak^\alpha G^{1-\alpha} L.$$

Solving for G we get

$$G = (\bar{g}AL)^{1/\alpha} k. \tag{1.5}$$

Inserting into (1.4) gives

$$Y = Ak^\alpha \left[(\bar{g}AL)^{1/\alpha} k \right]^{1-\alpha} L = A^{1/\alpha} (\bar{g}L)^{(1-\alpha)/\alpha} kL \equiv \bar{A}K, \tag{1.6}$$

where, for convenience, we have introduced the constant

$$\bar{A} \equiv A^{\frac{1}{\alpha}} (\bar{g}L)^{\frac{1-\alpha}{\alpha}}.$$

c) In view of $k_i = k$, (1.1) gives

$$R = r + \delta = \alpha A (G/k)^{1-\alpha}. \tag{1.7}$$

Substituting (1.5) into (1.7) gives

$$r = \alpha A^{\frac{1}{\alpha}} (\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta \equiv \alpha \bar{A} - \delta, \tag{1.8}$$

which is a constant. By (FOC2),

$$\begin{aligned} w &= (1-\alpha)Ak_i^{\alpha}G^{1-\alpha} \\ &= (1-\alpha)Ak^{\alpha}\left[(\bar{g}AL)^{1/\alpha}k\right]^{1-\alpha} && \text{(from (1.5))} \\ &= (1-\alpha)A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}}k \\ &\equiv (1-\alpha)\bar{A}k. \end{aligned}$$

d) The representative household solves

$$\begin{aligned} \max_{(c_t)_{t=0}^{\infty}} U_0 &= \int_0^{\infty} \frac{c_t^{1-\theta}-1}{1-\theta}e^{-\rho t}dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= (1-\tau)ra_t + (1-\tau)w_t - \tau_\ell - c_t, \qquad a_0 \quad \text{given}, && (1.9) \\ \lim_{t\to\infty} a_t e^{-(1-\tau)rt} &\geq 0. && \text{(NPG)} \end{aligned}$$

The current-value Hamiltonian is

$$H = \frac{c^{1-\theta}-1}{1-\theta} + \lambda\left[(1-\tau)(ra_t + w_t) - \tau_\ell - c_t\right],$$

where λ can be interpreted as the shadow price of financial wealth along the optimal path. First order conditions are:

$$\partial H/\partial c = c^{-\theta} - \lambda = 0, \text{ i.e., } c^{-\theta} = \lambda, \tag{1.10}$$

$$\partial H/\partial K = \lambda(1-\tau)r = \rho\lambda - \dot{\lambda}, \text{ i.e., } (1-\tau)r - \rho = -\dot{\lambda}/\lambda, \tag{1.11}$$

and the transversality condition is

$$\lim_{t\to\infty} a_t\lambda_t e^{-\rho t} = 0. \tag{TVC}$$

Log-differentiation w.r.t. t in (1.10) and inserting into (1.11) gives the Keynes-Ramsey rule for this model:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}((1-\tau)r - \rho) = \frac{1}{\theta}\left[(1-\tau)(\alpha\bar{A} - \delta) - \rho\right] \equiv \gamma, \tag{1.12}$$

where $\bar{A}$ is given in (1.6).

e) In view of (1.11) we have $\lambda_t = \lambda_0 e^{-[(1-\tau)r-\rho]t}$ so that (TVC) can be written

$$\begin{aligned} \lim_{t\to\infty} a_t \lambda_0 e^{-[(1-\tau)r-\rho]t} e^{-\rho t} &= 0 \Rightarrow \\ \lim_{t\to\infty} a_t e^{-(1-\tau)rt} &= 0, \end{aligned} \tag{TVC'}$$

since $\lambda_0 \neq 0$, by (1.10). The relevant discount rate is the *after*-tax rate of return, $(1-\tau)r$, the coefficient to a_t in (1.9).

f) The model implies a constant real rate of interest and a constant output-capital ratio, $\bar{A}$. Hence, the model belongs to the AK family, and from the theory of AK models we know that in equilibrium $\dot{k}/k$ and $\dot{y}/y$ are the same as $\dot{c}/c$, i.e., we have, from date zero,

$$\dot{k}/k = \dot{y}/y = \dot{c}/c = \frac{1}{\theta}((1-\tau)r - \rho) \equiv \gamma. \tag{1.13}$$

There is no transitional dynamics.

To ensure growth we assume $(1-\tau)r - \rho = (1-\tau)(\alpha\bar{A} - \delta) > \rho$, that is,

$$(1-\tau)(\alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta) > \rho. \tag{A1}$$

This requires that $\bar{g}$ is not "too small". On the other hand, to ensure bounded utility we assume

$$(1-\theta)\gamma < \rho. \tag{A2}$$

From the Keynes-Ramsey rule we have $(1-\tau)r = \theta\gamma + \rho$, so that the assumption A2 implies that

$$(1-\tau)r > \gamma,$$

i.e., the after-tax real rate of interest is higher than the GDP growth rate (this is a necessary condition for an equilibrium to exist in the model).

A more technical argument for the result in (1.13) is the following. We have

$$\dot{k} = \frac{\dot{K}}{L} = \frac{Y - G - C - \delta K}{L} = (1-\bar{g})y - c - \delta k$$
$$= \left[(1-\bar{g})A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta\right] k - c_0 e^{\gamma t} = \left[(1-\bar{g})\bar{A} - \delta\right] k - c_0 e^{\gamma t}.$$

This is a linear differential equation in k, and it has the solution (assuming $(1-\bar{g})\bar{A} - \delta > \gamma$)

$$k_t = (k_0 - \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma})e^{((1-\bar{g})\bar{A}-\delta)t} + \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma}e^{\gamma t}. \tag{1.14}$$

In view of (1.14), the transversality condition (TVC') is satisfied if and only if

$$c_0 = \left[(1-\bar{g})\bar{A} - \delta - \gamma\right] k_0. \tag{1.15}$$

Indeed, multiplying through in (1.14) by $e^{-(1-\tau)rt}$ gives

$$k_t e^{-(1-\tau)rt} = (k_0 - \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma})e^{\left[(1-\bar{g})\bar{A}-\delta-(1-\tau)r\right]t} + \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma}e^{-((1-\tau)r-\gamma)t}, \tag{1.16}$$

where, since $(1-\tau)r > \gamma$, the last term approaches zero for $t \to \infty$. Hence, satisfying (TVC') requires the first term to vanish for $t \to \infty$. Assuming $\bar{g}$ is not "too large", $(1-\bar{g})\bar{A} - \delta \geq (1-\tau)r > \gamma$ so that the term in square brackets in (1.15) is positive (otherwise equilibrium in the economy is impossible). Therefore, for the first term in (1.16) to vanish for $t \to \infty$, (1.15) must hold.

Inserting (1.15) into (1.14) gives

$$k_t = \frac{c_0}{(1-\bar{g})\bar{A} - \delta - \gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, from date zero, k grows at the same rate as c, the rate γ. Since, by (1.6), $y = Ak$, y does the same.

g) In addition to the standard results for strictly endogenous growth models (like $\partial\gamma/\partial\rho < 0, \partial\gamma/\partial\theta < 0$) we get

$$
\begin{aligned}
\frac{\partial \gamma}{\partial \bar{g}} &> 0, \qquad \text{(the government expenditure is productive)} \\
\frac{\partial \gamma}{\partial \tau} &< 0, \qquad \text{(the tax implies lower after-tax rate of return)} \\
\frac{\partial \gamma}{\partial L} &= \frac{(1-\tau)\alpha}{\theta}\frac{\partial \bar{A}}{\partial L} = \frac{(1-\tau)\alpha}{\theta} A^{\frac{1}{\alpha}} \bar{g}^{\frac{1-\alpha}{\alpha}} L^{\frac{1}{\alpha}-2} > 0.
\end{aligned}
$$

There is a scale effect on the growth rate. This is because of the assumption that the productive public service is a pure public good (nonrival). This always implies economies of scale, and in a model with strictly endogenous growth it implies a scale effect on the per capita growth rate.

h) In view of $a = k$ in equilibrium and $G = \bar{g}Y = \bar{g}\bar{A}K$, the government budget constraint can be written

$$
[\tau(rk + w) + \tau_\ell] L = G = \bar{g}\bar{A}K. \tag{1.17}
$$

We have

$$
rk + w = (\alpha\bar{A} - \delta)k + (1-\alpha)\bar{A}k = (\bar{A} - \delta)k.
$$

Hence, with $\tau_\ell = 0$, (1.17) gives

$$
\tau(\bar{A} - \delta)kL = \bar{g}\bar{A}K
$$

or

$$
\tau = \frac{\bar{g}\bar{A}}{\bar{A} - \delta}. \tag{1.18}
$$

We thus see that it is possible to fix τ at a constant level such that the government budget is balanced for all $t \geq 0$ in spite of $\tau_\ell = 0$. (The exceptionally good answer checks that this tax policy is viable. Viability requires

$$
\begin{aligned}
\frac{\bar{g}\bar{A}}{\bar{A} - \delta} &< 1, \text{ i.e.,} \\
\bar{g}A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} &< A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta, \\
(1-\bar{g})A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} &> \delta,
\end{aligned}
$$

saying again that $\bar{g}$ should neither be "too little" or "too large".)

i) The aggregate production function is $Y = Ak^{\alpha}G^{1-\alpha}L = AK^{\alpha}G^{1-\alpha}L^{1-\alpha}$ so that

$$\frac{\partial Y}{\partial G} = (1-\alpha)AK^{\alpha}G^{-\alpha}L^{1-\alpha} = (1-\alpha)\frac{Y}{G}.$$

The net gain by increasing G marginally is

$$\frac{\partial(Y-G)}{\partial G} = (1-\alpha)\frac{Y}{G} - 1 \gtreqless 0 \text{ for } \frac{G}{Y} \lesseqgtr 1-\alpha.$$

Hence, $\bar{g} = G/Y = 1-\alpha$ is required for static efficiency.

As to the form of taxation, taxation of interest income is distorting. Indeed, by (1.7),

$$\begin{aligned}
(1-\tau)r &= (1-\tau)(\alpha(\frac{k}{G})^{\alpha-1} - \delta) \\
&= (1-\tau)(\alpha K^{\alpha-1}(GL)^{1-\alpha} - \delta) \\
&= (1-\tau)(\frac{\partial Y}{\partial K} - \delta) < \frac{\partial Y}{\partial K} - \delta
\end{aligned}$$

for $\tau > 0$ and $\frac{\partial Y}{\partial K} > \delta$. That is, the private return to saving is smaller than the social return.

Is a wage income tax τ_w a viable alternative? No, the required tax rate would satisfy

$$\begin{aligned}
\tau_w wL &= G = (1-\alpha)Y, \text{ i.e.,} \\
\tau_w &= \frac{(1-\alpha)Y}{wL} = \frac{(1-\alpha)Y}{(1-\alpha)\bar{A}kL} = 1!
\end{aligned}$$

hence, there would be no net income from working. A better alternative is a constant consumption tax τ_c :

$$\begin{aligned}
\tau_c cL &= G = (1-\alpha)Y, \text{ i.e.,} \\
\tau_c &= (1-\alpha)\frac{Y}{cL},
\end{aligned}$$

a constant in view of the AK structure of the model. Hence, this tax is non-distorting. This result is due to the fact that leisure does not enter the utility function in this model.

2. Solution to Problem 2

For convenience, the basic equations of the model are repeated here:

$$Y_i = A{L_i}^{1-\alpha}\sum_{j=1}^{N}(x_{ij})^{\alpha}, \qquad A > 0,\ 0 < \alpha < 1. \tag{2.1}$$

$$\dot{N} \equiv \frac{dN}{dt} = \beta R, \qquad \beta > 0,\ \beta \text{ constant.}$$

$$Y \equiv \sum_{i=1}^{M} Y_i = cL + R + X,$$

where $X \equiv \sum_j \sum_i x_{ij}$.

a) Inventor j (firm j in Sector 2) earns a profit, π_j, per unit of time:

$$\pi_j = (\frac{1}{\alpha} - 1)X^m \equiv \pi^m, \tag{2.2}$$

where

$$X^m = LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}. \tag{2.3}$$

With the basic good as the numeraire, the monopoly price (profit-maximizing price) is $1/\alpha$, because the monopolist faces a demand function with price elasticity $-1/(1-\alpha)$). At the price $1/\alpha$, demand for input good j by the firms in the basic-goods sector is as in (2.3). Hence, total revenue is $(1/\alpha) \cdot X^m$ and total cost is $1 \cdot X^m$, resulting in the profit (2.2).

b) The firm j in Sector 2 has market value

$$\begin{aligned} V(t) &= \int_t^{\infty} \pi_j(\tau)e^{-\int_t^{\tau} r(s)ds}d\tau \\ &= \pi^m \int_t^{\infty} e^{-\int_t^{\tau} r(s)ds}d\tau, \end{aligned}$$

namely present discounted value of future profits.

c) The cost of making $\dot{N}$ inventions is $R = \dot{N}/\beta$. The cost of making one invention is $1/\beta$. Hence, equilibrium with $\dot{N} > 0$ requires $V(t) = 1/\beta$. The general no-arbitrage condition is

$$\frac{\pi(t) + \dot{V}(t)}{V(t)} = r(t).$$

With $V(t) = 1/\beta$ and $\pi(t) = \pi^m$, this takes the form

$$\beta(\pi^m + 0) = r(t)$$

or

$$r(t) = \beta(\frac{1}{\alpha} - 1)X^m = \beta(\frac{1}{\alpha} - 1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} \equiv r,$$

a constant. Since the demand X^m per input good is a constant, the real rate of interest is constant over time, an indication that the model belongs to the AK family.

d) In the present model the Keynes-Ramsey rule takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left[\beta(\frac{1}{\alpha} - 1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - \rho\right] \equiv \gamma. \tag{2.4}$$

From the theory of AK models we know that in equilibrium the state variable, here N, grows at the same rate as consumption, i.e., we have, from date zero,

$$\frac{\dot{N}}{N} = \frac{\dot{c}}{c} = \gamma.$$

To ensure that the equilibrium path considered is really one with $\dot{N} > 0$ we need the parameter restriction

$$\beta(\frac{1}{\alpha} - 1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} > \rho. \tag{A1}$$

To ensure bounded utility we need, in addition, the restriction

$$(1 - \theta)\gamma < \rho \tag{A2}$$

with γ given in (2.4).

From (2.4) we have:

$\partial\gamma/\partial A = \frac{\beta L}{\theta\alpha} A^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} > 0$. Higher factor productivity $\Rightarrow$ higher return on saving $\Rightarrow$ more saving at the aggregate level (the negative substitution effect and wealth effect on consumption dominates the positive income effect) $\Rightarrow$ more investment in R&D.

$\partial\gamma/\partial\beta = \frac{1}{\theta}\left(\frac{1}{\alpha} - 1\right) LA^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} > 0$. Higher productivity of R&D investment induces more R&D investment.

$\partial\gamma/\partial L = \frac{1}{\theta}\left(\frac{1}{\alpha} - 1\right) \beta A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} > 0$. A larger population L implies lower per capita cost $1/(\beta L)$ associated with producing new technical knowledge. Since knowledge is a non-rival good this strengthens the incentive to do R&D. In this model the result is a higher growth rate. This is the controversial "strong" scale effect, typical for innovation-based growth models with strictly endogenous growth.

e) In view of production and cost symmetry ($p_j = 1/\alpha$ for all j), firm i in Sector 1 chooses the same amount of each input good, that is, $x_{ij} = x_i$ for $j = 1, 2, ..., N$. Hence, (2.1) can be simplified to

$$Y_i = A{L_i}^{1-\alpha} N{x_i}^{\alpha} = AN(\frac{x_i}{L_i})^{\alpha} L_i. \tag{2.5}$$

Cost minimization implies that all firms in Sector 2 chooses the same input ratio x_i/L_i, implying

$$\frac{x_i}{L_i} = \frac{\sum x_i}{\sum L_i} = \frac{X^m}{L}.$$

Thence, by (2.5),

$$\begin{aligned} Y &= \sum_{i=1}^{M} Y_i = AN(\frac{X^m}{L})^{\alpha} \sum_{i=1}^{M} L_i \\ &= AN^{1-\alpha}(NX^m)^{\alpha} L^{1-\alpha} \\ &= AX^{\alpha}(NL)^{1-\alpha}, \end{aligned} \tag{2.6}$$

in view of $X \equiv \sum_j \sum_i x_{ij} = NX^m$. We see that the "growth engine" has CRS w.r.t. producible inputs, X and N. Therefore, from a technological point of view, the model is capable of generating endogenous growth.

f) Yes, if in contrast to (A1),

$$\rho \geq \beta(\frac{1}{\alpha} - 1)X^m,$$

then impatience is so large that an equilibrium with growth cannot exist.

3. Solution to Problem 3A

We denote employment in the basic-goods sector by L', that is, $L' = (1-s)L$. The parameter restriction $s < \alpha/(1+\alpha)$, mentioned in question b), turns out to be important for question c).

a) Inventor j (firm j in Sector 2) solves the problem:

$$\begin{aligned} \max_{p_j} \ (1-\tau)\pi_j &= (1-\tau)(p_j - 1)X_j \quad \text{s.t.} \\ X_j &= L'(\frac{\alpha A}{p_j})^{1/(1-\alpha)}. \end{aligned}$$

Irrespective of the tax, the solution is again $p_j = 1/\alpha$. The smaller basic-goods sector implies that demand for input good j is now

$$X^m = L' A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}}. \tag{3.1}$$

The market value of firm j is

$$\begin{aligned} V(t) &= \int_t^\infty (1-\tau)\pi_j(s) e^{-\int_t^s r(u)du} ds \\ &= (1-\tau)(\frac{1}{\alpha} - 1)L' A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} \int_t^\infty e^{-\int_t^s r(u)du} ds, \end{aligned}$$

namely present discounted value of future profits. By the same reasoning as in c) and d) of Problem 2 we get

$$\begin{aligned}
r(t) &= (1-\tau)\beta(\frac{1}{\alpha}-1)X^m \equiv r, \\
\frac{\dot{N}}{N} &= \frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}((1-\tau)\beta(\frac{1}{\alpha}-1)X^m - \rho) \equiv \gamma,
\end{aligned}$$

b) Firm i in Sector 1 solves the problem:

$$\begin{aligned}
\max_{L_i, x_{i1},\ldots,x_{iN}} \Pi_i &= Y_i - wL_i - \sum_{j=1}^{N} p_j x_{ij} \text{ s.t.} \\
Y_i &= A{L_i}^{1-\alpha}\sum_{j=1}^{N}(x_{ij})^\alpha.
\end{aligned}$$

One of the first order conditions is

$$\frac{\partial Y_i}{\partial L_i} = (1-\alpha)\frac{Y_i}{L_i} = w. \tag{3.2}$$

As in e) of Problem 2, due to symmetry all firms in Sector 1 have the same Y_i/L_i so that

$$\frac{Y_i}{L_i} = \frac{\sum Y_i}{\sum L_i} = \frac{AX^\alpha(NL')^{1-\alpha}}{L'}, \qquad \text{cf. (2.6).}$$

Hence, in equilibrium, (3.2) gives

$$\begin{aligned}
w &= (1-\alpha)A(\frac{X}{L'})^\alpha N^{1-\alpha} \\
&= (1-\alpha)A(NA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}})^\alpha N^{1-\alpha} \qquad \text{(from (3.1))} \\
&= (1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}N.
\end{aligned}$$

Surprisingly, the equilibrium real wage is independent of L'. Given N, an increase in L' (decrease in s) affects w through two channels. First, the larger employment in Sector 1 implies, cet. par., a lower marginal product of labour so that w tends to become lower. This is the usual diminishing returns effect. Second, the larger employment induces more input of intermediate goods and this augments, cet. par., the marginal product of labour. This complementarity effect exactly offsets the diminishing returns effect.

c) The government budget is balanced when

$$\begin{aligned} wsL &= \tau\pi^m N, \qquad \text{that is, when} \\ (1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}NsL &= \tau(\frac{1}{\alpha}-1)(1-s)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}N \qquad \text{or} \\ \tau &= \frac{s}{(1-s)\alpha}. \end{aligned}$$

To ensure that this tax rate is less than 1 we need the parameter restriction $s < \alpha/(1+\alpha)$.

d) The size of the government sector can be measured by s. An increase in s impedes growth through two channels:

$$s \uparrow \Rightarrow \begin{cases} L' \downarrow \Rightarrow X^m \downarrow \Rightarrow r \downarrow \Rightarrow \gamma \downarrow, \\ \tau \uparrow \Rightarrow 1-\tau \downarrow \Rightarrow r \downarrow \Rightarrow \gamma \downarrow. \end{cases}$$

4. Solution to Problem 3B

This problem extends the analysis of Problem 2 in another direction.

a) When intermediate good j looses the monopoly status and becomes competitive, it is supplied in the amount

$$X^c = L(\alpha A)^{1/(1-\alpha)}. \tag{4.1}$$

This is explained in the following way. The demand for intermediate good j is

$$X_j = L(\frac{\alpha A}{p_j})^{1/(1-\alpha)}.$$

Inserting the competitive price p_j = marginal cost = 1 gives $X_j = X^c$ in (4.1). Since the monopoly price is larger than 1, $X^c > X^m$.

b) We have $\Pr\{T > z\} = e^{-pz}$. The market value of monopoly j is the present discounted value of expected future profits, i.e.,

$$V(t) = E_t \int_t^\infty \pi_j(\tau) e^{-\int_t^\tau r(s)ds} d\tau, \tag{4.2}$$

where $\pi_j(\tau)$, as seen from time $t < \tau$, is a stochastic variable. Indeed,

$$\pi_j(\tau) = \begin{cases} \pi^m, & \text{if firm } j \text{ is still a monopoly at time } \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Now (4.2) can be written

$$\begin{aligned} V(t) &= \int_t^\infty e^{-\int_t^\tau r(s)ds} E_t \pi_j(\tau) d\tau \\ &= \int_t^\infty e^{-\int_t^\tau r(s)ds} (\pi^m e^{-p(\tau - t)} + 0) d\tau \\ &= \pi^m \int_t^\infty e^{-\int_t^\tau (r(s)+p)ds} d\tau, \end{aligned}$$

which is (**). Thus, the effect of uncertainty is to increase the "effective" rate of discount.

c) Equilibrium with $\dot{N} > 0$ requires $V(t) = 1/\beta$. The no-arbitrage condition is

$$\frac{\pi^m + 0 - p/\beta}{1/\beta} = r(t).$$

This gives

$$\begin{aligned} r(t) &= \beta\pi^m - p = \beta(\frac{1}{\alpha} - 1)X^m - p \\ &= \beta(\frac{1}{\alpha} - 1)LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - p \equiv r^*, \end{aligned}$$

a constant. Thence, from the Keynes-Ramsey rule,

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r^* - \rho) \equiv \gamma^* < \gamma.$$

The shorter duration of monopoly power implies a smaller incentive to do R&D, hence the growth rate becomes smaller than in the model with $p = 0$ in Problem 2. And even that growth rate is below the level that a social planner would accomplish.

d) $N^m + N^c = N$; N^c/N approaches a constant $(N^c/N)^*$ over time. It can be shown that

$$p \uparrow \Rightarrow (N^c/N)^* \uparrow, \quad \text{and} \tag{4.3}$$

$$p \uparrow \Rightarrow Y \uparrow . \tag{4.4}$$

Explanation of (4.3): Higher p implies, on average, lower duration of the position as a monopolist, hence the proportion of competitive intermediate goods becomes larger. This also explains (4.4): The implication of a larger proportion of competitive suppliers in Sector 2 is that the wedge between the average price of intermediate goods and the marginal cost of producing them becomes smaller, so that the input of these goods comes closer to the efficient level. This leads to higher output in Sector 1.

This illustrates the classical dilemma of patent legislation in the following sense. Shorter duration of patents corresponds to higher p, hence the economy gains in terms of static efficiency. On the other hand, as the answer to c) shows, higher p implies too little incentive to do R&D. Hence, the economy looses in terms of "dynamic efficiency".

5. Solution to Problem 4

a) Not true. It is only the standard deviation of *relative* income per capita (or of $\log y$) that diminishes over time. This is because income per capita is growing, and the standard deviation is not a scale-free measure.

b) Not true. It tends to overstate TFP growth. Let $g \equiv TFP$ growth rate.

Then

$$g \equiv \frac{\dot{Y}}{Y} - s_K \frac{\dot{K}}{K} - s_L \frac{\dot{L}}{L}.$$

In standard growth accounting s_K is measured by the share of capital income. When there is learning-by-investing, this share is lower than the true output elasticity w.r.t. capital.

c) The model of Problem 2 has the weakness that with $\dot{L}/L = n > 0$, the per capita growth rate becomes increasing over time - without limit. This counterfactual implication is due to the "strong" scale effect.

B & S's approach to the problem is to let the cost (in terms of basic goods) of making an invention be increasing with N. Then a steady state becomes consistent with growth in the labour force.

In the original Romer (1990) model the technologies for producing basic goods and inventions, respectively, are different. Essentially, Romer assumes, with a notation similar to that above,

$$\dot{N} = \beta N L_N, \tag{5.1}$$

where L_N is labour input in R&D. With L_Y denoting labour input in the production of basic goods, $L_N + L_Y = L$. This also leads to a strong scale effect and increasing per capita growth if labour supply is growing.

But Jones (1995) changes (5.1) to

$$\dot{N} = \beta N^{\varphi} L_N, \qquad \varphi < 1.$$

This eliminates the strong scale effect, and a steady state becomes consistent with growth in the labour force.

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