arrangements the No-Ponzi-Game condition precludes.

The terminal optimality condition, known as a *transversality condition*, can be shown[9] to be

$$\lim_{t\to\infty}(1+\rho)^{-(t-1)}u'(c_{t-1})a_t = 0.$$

9.3 Transition to continuous time analysis

In the formulation of a model we have a choice between putting the model in period terms or in continuous time. In the former case, denoted period analysis or discrete time analysis, the run of time is divided into successive periods of equal length, taken as the time-unit. We may index the periods by $i = 0, 1, 2, ...$ Thus, in period analysis financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given,}$$

where s_i is (net) saving in period i.

Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t + \Delta t)$ refers to financial wealth at time $t + \Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t+\Delta t)-a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$.

Now, *keep the time unit unchanged*, but let the length of the time interval $[t, t + \Delta t)$ approach zero, i.e., let $\Delta t \to 0$. When a is a differentiable function of t, we have

$$\lim_{\Delta t\to 0}\frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{a(t+\Delta t)-a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *time derivative of* a at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given,} \tag{9.21}$$

where $s(t)$ is the saving flow (saving intensity) at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t + 1) - a(t) \approx s(t)$.

[9] The proof is similar to that given in Chapter 8, Appendix C.

As time unit choose one year. Going back to discrete time, if wealth grows at a constant rate g per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \qquad i = 0, 1, 2, \dots . \tag{9.22}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period, we have

$$a_i = a_0(1+\frac{g}{n})^i. \tag{9.23}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (9.23) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0\left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$. Thus $m \to \infty$. In the limit there is continuous compounding and we get

$$a(t) = a_0 e^{gt}, \tag{9.24}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1+1/m)^m \simeq 2.7182818285....$

The formula (9.24) is the continuous-time analogue to the discrete time formula (9.22) with annual compounding. A geometric growth factor, $(1+g)^i$, is replaced by an exponential growth factor, e^{gt}, and this growth factor is valid for any t in the time interval $(-\tau_1, \tau_2)$ for which the growth rate of a equals the constant g (τ_1 and τ_2 being some positive real numbers).

We can also view the formulas (9.22) and (9.24) as the solutions to a difference equation and a differential equation, respectively. Thus, (9.22) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (9.24) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$, a continuous function of t. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.25}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (9.25) is called the *accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.[10]

[10] Sometimes the accumulation factor with time-dependent growth rate is written in a different way, see Appendix B.

Compound interest and discounting in continuous time

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{9.26}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (9.25) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time, we use compound discounting. We reverse the accumulation formula and go from the compounded or terminal value to the present value, $a(t_0)$. Similarly, given a consumption plan $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt}dt, \tag{9.27}$$

presupposing a constant interest rate, r. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (9.27) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau}dt.$$

In (9.27) $c(t)$ is discounted by $e^{-rt} \approx (1+r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (9.19) where it is c_{t-1} that is discounted by $(1+r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t-1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

The range for particular parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given,} \tag{9.28}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in period analysis, now $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (9.28) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$, and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations*. What is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these elementary units we can form *composite measurement units*. Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t) + I(t)$ or $K(t) + \dot{K}(t)$ are thus illegitimate. But one can write $K(t + \Delta t) \approx K(t) + (I(t) - \delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* (or saving *density*), at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0) \cdot \Delta t$, cf. the hatched area in Fig. 9.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

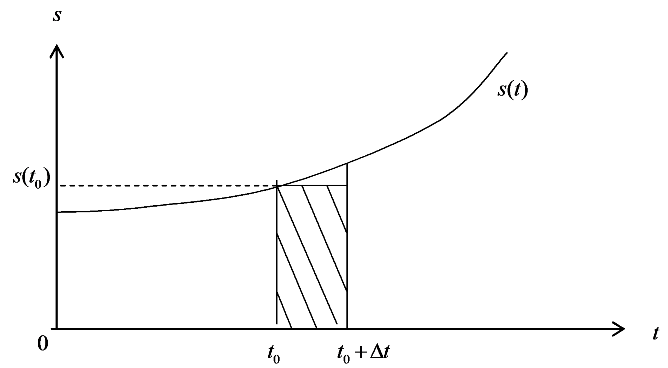

Figure 9.1: With Δt *small* the integral of $s(t)$ from t_0 to $t_0 + \Delta t \approx$ the hatched area.

The choice between discrete and continuous time formulation

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for period analysis rather than continuous time modelling. And the fact that economic actors often think, decide, and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. Moreover, as for instance Allen (1967) argued, it can hardly be said that the *mass* of economic actors think and decide with the same time distance between successive decisions and actions. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. In addition, centralized asset markets equilibrate very fast and respond almost immediately to new information. For such markets a formulation in continuous time seems a good approximation.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{9.29}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1+\alpha)x_t$, where $1+\alpha < 0$. The solution is $x_t = (1+\alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1+\alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise (together with divergence if $\alpha < -2$) in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

This potential problem can always be avoided, however, by choosing a sufficiently *short* period length in the discrete time model. The solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (9.29), the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (9.29). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. Yet, everything else equal, the clearer distinction between stocks and flows in continuous time than in discrete time speaks for the former. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.[11]

9.4 Maximizing discounted utility in continuous time

9.4.1 The saving problem in continuous time

In continuous time the analogue to the intertemporal utility function, (9.3), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{9.30}$$

In this context it is common to name the utility flow, u, the *instantaneous utility function*. We still assume that $u' > 0$ and $u'' < 0$. The analogue in continuous time to the intertemporal budget constraint (9.20) is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{9.31}$$

[11] In the latter case, Nobel laureate Robert C. Merton argues in favor of a continuous time formulation (Merton, 1975).

where, as before, a_0 is the historically given initial financial wealth, while h_0 is the given human wealth,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.32}$$

The household's problem is then to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (9.31).

Infinite time horizon Transition to infinite horizon is performed by letting $T \to \infty$ in (9.30), (9.31), and (9.32). In the limit the household's, or dynasty's, problem becomes one of choosing a plan, $(c(t))_{t=0}^\infty$, which maximizes

$$U_0 = \int_0^\infty u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.33}$$

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by letting T in (9.32) approach ∞. With an infinite horizon there may exist technically feasible paths along which the integrals in (9.30), (9.31), and (9.32) go to ∞ for $T \to \infty$. In that case maximization is not well-defined. However, the assumptions we are going to make when working with infinite horizon will guarantee that the integrals converge as $T \to \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high.

Generally we define a person as *solvent* if she is able to meet her financial obligations as they fall due. Each person is considered "small" relative to the economy as a whole. As long as all agents in an economy with a perfect loan market remain "small", they will in general equilibrium remain solvent if and only if their net debt does not exceed the present value of future primary saving.[12] Denoting by d_0 net debt at time 0, i.e., $d_0 \equiv -a_0$, the solvency requirement as seen from time 0 is

$$d_0 \leq \int_0^\infty (w(t) - c(t))e^{-\int_0^t r(\tau)d\tau}dt,$$

where the right-hand side is the present value of future primary saving. By the definition in (9.32), we see that this requirement is identical to the intertemporal budget constraint (IBC) which consequently expresses solvency.

[12] By *primary* saving is meant the difference between current non-interest income and current consumption, where non-interest income means labor income and transfers after tax.