Exercise problems for Advanced Macroeconomics

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Preface

This is a collection of exercise problems that have been used in recent years in the course Advanced Macroeconomics at the Department of Economics, University of Copenhagen.

For ideas as to the content of the exercises and for constructive criticism as well as assistance with data graphs I want to thank the instructors Mads Diness Jensen, Jeppe Druedahl, and Niklas Brønager. I am also grateful to previous students for challenging questions. No doubt, it is still possible to find obscurities. Hence, I very much welcome comments and suggestions of any kind regarding these exercises.

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Remarks on notation

For historical reasons, in some of the exercises the "level of technology" (assumed measurable along a single dimension) is denoted A, in others T.

Whether we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

In discrete-time models the time argument of a variable, x, appears always as a subscript, that is, as x_t. In continuous-time models, the time argument of a variable *may* appear as a subscript rather than in the more common form $x(t)$ (this is to save notation).

Chapter 1

Refresher on technology and firms

I.1 *Short questions* (answering requires only a few well chosen sentences and possibly a simple illustration)

a) Consider an economy where all firms' technology is described by the same neoclassical production function, $Y_i = F(K_i, L_i)$, $i = 1, 2, ..., N$, with decreasing returns to scale everywhere (standard notation). Suppose there is "free entry and exit" and perfect competition in all markets. Then a paradoxical situation arises in that no equilibrium with a finite number of firms (plants) would exist. Explain.

b) As an alternative to decreasing returns to scale at *all* output levels, introductory economics textbooks typically assume that the long-run average cost curve of the firm is decreasing at small levels of production and constant or increasing at larger levels of production. Express what this assumption means in terms of local "returns to scale".

c) Give some arguments for the presumption that the average cost curve is downward-sloping at small output levels.

d) In many macro models the technology is assumed to have constant returns to scale (CRS) with respect to capital and labor taken together. What does this mean in formal terms?

e) Often the *replication argument* is put forward as a reason to expect that CRS should hold in the real world. What is the replication argument? Do you find the replication argument to be a convincing argument for the assumption of CRS with respect to capital and labor? Why or why not?

f) Does the logic of the replication argument, considered as an argument about a property of technology, depend on the availability of the different inputs.

g) Robert Solow (1956) came up with a subtle replication argument for CRS w.r.t. the rival inputs at the aggregate level. What is this argument?

h) Suppose that for a certain historical period there has been something close to constant returns to scale and perfect competition, but then, after a shift to new technologies in the different industries, increasing returns to scale arise. What is likely to happen to the market form? Why?

I.2 Consider a firm with the production function $Y = AK^{\alpha}L^{\beta}$, where $A > 0$, $0 < \alpha < 1$, $0 < \beta < 1$.

a) Is the production function neoclassical?

b) Find the marginal rate of substitution at a given (K, L).

c) Draw in the same diagram three isoquants and draw the expansion path for the firm, assuming it is cost-minimizing and faces a given factor price ratio.

d) Check whether the four Inada conditions hold for this function?

e) Suppose that instead of $0 < \alpha < 1$ we have $\alpha \geq 1$. Check whether the function is still neoclassical?

I.3 Consider the production function $Y = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$.

a) Does the function imply constant returns to scale?

b) Is the production function neoclassical? *Hint:* after checking criterion (a) of the definition of a neoclassical production function in Lecture Notes, Section 2.1.1, you may apply claim (iii) of Section 2.1.3 together with your answer to a).

c) Given this production function, is capital an essential production factor? Is labor?

d) If we want to extend the domain of definition of the production function to include $(K, L) = (0, 0)$, how can this be done while maintaining continuity of the function?

I.4 Write down a CRS two-factor production function with Harrod-neutral technological progress look. Why is the assumption of Harrod-neutrality so popular in macroeconomics?

I.5 *Stocks versus flows.* Two basic elements in long-run models are often presented in the following way. The aggregate production function is described by

$$Y_t = F(K_t, L_t, A_t), \tag{*}$$

where Y_t is output (aggregate value added), K_t capital input, L_t labor input, and A_t the "level of technology". The time index t may refer to period t, that is, the time interval $[t, t+1)$, or to a point in time (the beginning of period t), depending on the context. And accumulation of the stock of capital in the economy is described by

$$K_{t+1} - K_t = I_t - \delta K_t, \tag{**}$$

where δ is an (exogenous and constant) rate of (physical) depreciation of capital, $0 \leq \delta \leq 1$. Evolution in employment (assuming full employment) is described by

$$L_{t+1} - L_t = nL_t, \qquad n > -1. \tag{***}$$

In continuous time models the corresponding equations are: (*) combined with

$$\begin{aligned} \dot{K}(t) &\equiv \frac{dK(t)}{dt} = I(t) - \delta K(t), \qquad \delta \geq 0, \\ \dot{L}(t) &\equiv \frac{dL(t)}{dt} = nL(t), \qquad n \text{ "free"}. \end{aligned}$$

a) At the theoretical level, what denominations (dimensions) should be attached to output, capital input, and labor input in a production function?

b) What is the denomination (dimension) attached to K in the accumulation equation?

c) Are there any consistency problems in the notation used in (*) vis-à-vis (**) and in (*) vis-à-vis (***)? Explain.

d) Suggest an interpretation that ensures that there is no consistency problem.

e) Suppose there are two countries. They have the same technology, the same capital stock, the same number of employed workers, and the same number of man-hours per worker per year. Country a does not use shift work, but country b uses shift work, that is, two work teams of the same size and the same number of hours per day. Elaborate the formula (*) so that it can be applied to both countries.

f) Suppose F is a neoclassical production function with CRS w.r.t. K and L. Compare the output levels in the two countries. Comment.

g) In continuous time we write aggregate (real) gross saving as $S(t) \equiv Y(t) - C(t)$. What is the denomination of $S(t)$?

h) In continuous time, does the expression $K(t) + S(t)$ make sense? Why or why not?

i) In discrete time, how can the expression $K_t + S_t$ be meaningfully interpreted?

I.6 The Solow growth model can be set up in the following way (discrete time version). A closed economy is considered. There is an aggregate production function,

$$Y_t = F(K_t, T_t L_t), \tag{1}$$

where F is a neoclassical production function with CRS, Y is output, K is capital input, T is the technology level, and L is the labor input. So TL is effective labor input. It is assumed that

$$\begin{aligned} T_t &= T_0(1+g)^t, \quad \text{where } g \geq 0, && (2) \\ L_t &= L_0(1+n)^t, \quad \text{where } n \geq 0. && (3) \end{aligned}$$

Aggregate gross saving is assumed proportional to gross aggregate income which, in a closed economy, equals real GDP, Y:

$$S_t = sY_t, \qquad 0 < s < 1. \tag{4}$$

Capital accumulation is described by

$$K_{t+1} = K_t + S_t - \delta K_t, \qquad \text{where } 0 < \delta \leq 1. \tag{5}$$

The symbols g, n, and s represent parameters and the initial values T_0, L_0, and K_0, are given (exogenous) positive numbers.

a) What kind of technical progress is assumed in the model?

b) To get a grasp of the evolution of the economy over time, derive a first-order difference equation in the (effective) capital intensity $\tilde{k} \equiv k/T \equiv K/(TL)$, that is, an equation of the form $\tilde{k}_{t+1} = \varphi(\tilde{k}_t)$.

From now on suppose F is Cobb-Douglas.

c) Construct a "transition diagram" in the $(\tilde{k}_t, \tilde{k}_{t+1})$ plane.

d) Examine whether there exists a unique and asymptotically stable (non-trivial) steady state.

e) There is another kind of diagram that is sometimes (especially in continuous time versions of the model) used to illustrate the dynamics of the economy, namely the "Solow diagram". It is based on writing the difference equation of the model on the form $\tilde{k}_{t+1} - \tilde{k}_t = \left(\psi(\tilde{k}_t) - a\tilde{k}_t\right) / [(1+g)(1+n)/s]$. For the case of the general production function (1), find the function $\psi(\tilde{k}_t)$ and the constant a. By drawing the graphs of the functions $\psi(\tilde{k}_t)$ and $a\tilde{k}_t$ in the same diagram, one gets a Solow diagram. Indicate by arrows the resulting evolution of the economy.

I.7 We consider the same economy as that described by (1) - (5) in Problem I.6.

a) Find the long-run growth rate of output per unit of labor, $y \equiv Y/L$.

b) Suppose the economy is in steady state up to and including period $t-1$ such that $\tilde{k}_{t-1} = \bar{\tilde{k}} > 0$ (standard notation). Then, at time t (the beginning of period t) an upward shift in the saving rate occurs. Illustrate by a transition diagram the evolution of the economy from period t onward.

c) Draw the time profile of $\ln y$ in the $(t, \ln y)$ plane.

d) How, if at all, is the level of y affected by the shift in s?

e) How, if at all, is the growth rate of y affected by the shift in s? Here you may have to distinguish between temporary and permanent effects.

f) Explain by words the economic mechanisms behind your results in d) and e).

g) As Solow once said (in a private correspondence with Amartya Sen[1]): "The idea [of the model] is to trace full employment paths, no more." What market form is theoretically capable of generating permanent full employment?

h) Even if we recognize that the Solow model only attempts to trace hypothetical time paths with full employment (or rather employment corresponding to the "natural" or "structural" rate of unemployment), the model has at least one important limitation. What is in your opinion that limitation?

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model has at least one important limitation. What is in your opinion that limitation?

I.8 *A more flexible specification of the technology than the Cobb-Douglas function.* Consider the CES production function[3]

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}, \qquad (*)$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$.

a) Does the production function imply CRS? Why or why not?

b) Show that (*) implies

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta}\left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}\left(\frac{Y}{L}\right)^{1-\beta}.$$

c) Express the marginal rate of substitution of capital for labor in terms of $k \equiv K/L$.

d) In case of an affirmative answer to a), derive the intensive form of the production function.

e) Is the production function neoclassical? *Hint:* a convenient approach is to focus on $\partial Y/\partial K$ expressed in terms of k and consider the cases $\beta < 0$ and $0 < \beta < 1$, separately; next use a certain symmetry visible in (*); finally use your answer to a).

f) Draw a graph of $y \equiv K/L$ as a function of k for the cases $\beta < 0$ and $0 < \beta < 1$, respectively. Comment and compare with a Cobb-Douglas function on intensive form, $y = Ak^{\alpha}$.[4]

g) Write down a CES production function with Harrod-neutral technical progress.

I.9 *A potential source of permanent productivity growth* (this exercise presupposes that f) of Problem I.8 has been solved). Consider a Solow-type growth model, cf. Problem I.6. Suppose the production function is a CES function as in (*) of Problem I.8. Let $\beta \in (0,1)$, $\alpha^{1/\beta}A > (n+\delta)/s$, and ignore technical progress.

[3] CES stands for Constant Elasticity of Substitution.

[4] This function can in fact be shown to be the limiting case of the CES function (in intensive form) for $\beta \to 0$.

a) Express y/k in terms of k, where $k \equiv K/L$ and $y \equiv Y/L$.

b) For a given k_0, illustrate the dynamic evolution of the economy by a "modified Solow diagram", i.e., a diagram with k on the horizontal axis and sy/k on the vertical axis.

c) Find the asymptotic value of the growth rate of k for $t \to \infty$. Comment.

d) What is the asymptotic value of the growth rate of y for $t \to \infty$.

e) The model displays a feature that may seem paradoxical in view of the absence of technical progress. What is this feature and why is it not paradoxical after all, given the assumptions of the model?

I.10 An important aspect of macroeconomic analysis is to pose good questions in the sense of questions that are concise, interesting, and manageable. If we set aside an hour or so in one of the lectures or class exercises, what question would you suggest should be discussed?

Chapter 2

Public debt and fiscal sustainability

Borrowed from Jeppe Druedahl:

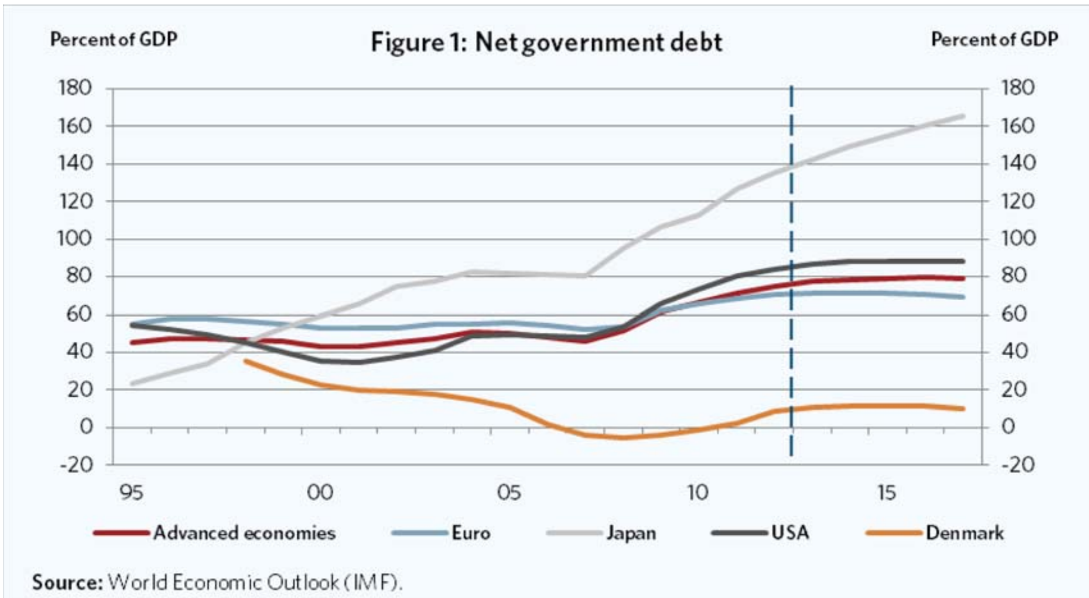


Figure 2.1: Some background material for this section of the exercises.

II.1 Consider the government budget in a small open economy (SOE) fully integrated in the world market for goods and financial capital. Time is discrete, the period length is one year, and there is no uncertainty. Let g and n be non-negative constants and let

$$
\begin{aligned}
Y_t &= Y_0(1+g)^t(1+n)^t = \text{ real GDP,} \\
G_t &= \text{ real government spending on goods and services,} \\
T_t &= \text{ real net tax revenue } (= \text{ gross tax revenue} - \text{transfer payments}), \\
B_t &= \text{ real public debt at the start of period } t, \\
r_t &= \text{ real interest rate in the SOE} = \text{world market real interest rate.}
\end{aligned}
$$

We assume that any government budget deficit is exclusively financed by issuing debt (and any budget surplus by redeeming debt).

a) Interpret g and n.

b) Suppose the current inflation rate in the SOE equals π_t. Given this inflation rate and given r_t, what is the level of the nominal interest rate, i_t? You should provide the exact formula, not an approximation. Let $r_t = 0.03$ per year and $\pi_t = 0.02$. What is i_t exactly? Instead, let $r_t = 0.04$ per year and $\pi_t = 0.15$ (as in many countries in the aftermath of the second oil crisis 1979-80). What is i_t exactly? Compare with the result you get from the standard approximative formula.

c) Returning to variables in real terms, write down the real budget deficit and an equation showing how B_{t+1} is determined.

From now on assume that $r_t = r$, a constant. Consider a scenario with $B_0 > 0$, $1 + r > (1 + g)(1 + n)$, and $T_t/Y_t = \tau$, a positive constant less than one.

d) What does government solvency mean and what does fiscal sustainability mean?

e) Find the maximum constant G/Y which is consistent with fiscal sustainability (as evaluated on the basis of the expected evolution of the debt-GDP ratio). *Hint:* the difference equation $x_{t+1} = ax_t + b$, where a and b are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = b/(1 - a)$.

II.2 Consider a small open economy (SOE) facing a constant real interest rate $r > 0$, given from the world market for financial capital. We ignore business cycle fluctuations and assume that real GDP, Y_t, grows at a constant exogenous rate $g_Y > 0$. We assume $g_Y < r$.

Time is discrete. Further notation is:

$$\begin{aligned} G_t &= \text{real government spending on goods and services,} \\ T_t &= \text{real net tax revenue } (= \text{ gross tax revenue} - \text{transfer payments}), \\ GBD_t &= \text{real government budget deficit,} \\ B_t &= \text{real public debt (all short-term) at the start of period } t. \end{aligned}$$

Assume that any government budget deficit is exclusively financed by issuing debt (and any budget surplus by redeeming debt).

a) Write down the dynamic identity relating the increase in B to the level of GBD.

Suppose that $B_0 > 0$ and $G_t = \gamma Y_t$, $t = 0, 1, \ldots$, where $0 < \gamma < 1$. Define the "net tax burden" as $\tau_t \equiv T_t / Y_t$.

b) Find the minimum net tax burden, $\bar{\tau}$, which, if maintained, is consistent with fiscal sustainability (as evaluated on the basis of the expected evolution of the debt-GDP ratio). *Hint:* different approaches are possible; one of these focuses on the debt-income ratio and uses the fact that a difference equation $x_{t+1} = ax_t + b$, where a and b are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = b/(1-a)$.

c) How does $\bar{\tau}$ depend on r and g_Y, respectively? Comment.

II.3 Consider a budget deficit rule saying that $\lambda \cdot 100$ percent of the interest expenses on public nominal debt, D, plus the primary budget deficit must not be above $\alpha \cdot 100$ percent of nominal GDP, PY, where Y is real GDP, growing at a constant rate, $g_Y > 0$, and P is the GDP deflator. So the rule requires that

$$\lambda i D_t + P_t(G_t - T_t) \leq \alpha P_t Y_t, \qquad (*)$$

where $\lambda > 0$, $\alpha > 0$, and

$$\begin{aligned} G_t &= \text{real government spending on goods and services,} \\ T_t &= \text{real net tax revenue,} \\ i &= (1+r)(1+\pi) - 1, \text{ where } r \text{ is the real interest rate,} \\ \pi &= \frac{P_t - P_{t-1}}{P_{t-1}} = \text{the inflation rate, a given non-negative constant.} \end{aligned}$$

a) Is the deficit rule of the Stability and Growth Pact in the EMU a special case of (*)? Comment.

b) Let $b_t \equiv D_t/(P_{t-1}Y_t)$. Derive the law of motion (difference equation) for b_t, assuming the deficit ceiling is always binding. *Hint:* $\text{GBD}_t = iD_t + P_t(G_t - T_t)$.

Suppose λ is such that $0 < 1 + (1 - \lambda)i < (1 + \pi)(1 + g_Y)$.

c) For an arbitrary $b_0 > 0$, find the time path of b. Briefly comment. *Hint:* the difference equation $x_{t+1} = ax_t + c$, where a and c are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = c/(1 - a)$.

d) How does a rise in λ affect the long-run debt-income ratio? Comment.

e) Let the steady-state value of b be denoted b^* and assume $b_0 > b^*$. Illustrate the time path of b in the (t, b) plane. Comment.

f) How does b^* depend on α? Comment.

g) How does b^* depend on π? Comment.

h) What could the motivation for having $\lambda < 1$ be? Comment.

II.4 *Short questions*

a) What is meant by the No-Ponzi-Game condition of the government?

b) The No-Ponzi-Game condition of the government and the intertemporal budget constraint of the government are closely related. In what sense?

c) "A given fiscal policy is sustainable if and only if it maintains compliance with the intertemporal budget constraint of the government." True or false? Briefly discuss.

d) In the absence of uncertainty and credit frictions, if $r < g_Y$, a government can run a permanent debt rollover without experiencing solvency difficulties (standard notation). Briefly explain.

e) How is the inequality in d) modified in the presence of uncertainty and credit frictions?

II.5 *The Ricardian equivalence issue.* What is meant by Ricardian equivalence? Under the assumption of rational expectations and at most a "weak" bequest motive, overlapping generations models refute Ricardian equivalence. How?

Chapter 3

More about budget deficits and public debt

III.1 Consider a small open economy facing an exogenous constant real interest rate r. Suppose that at time 0 government debt is $B_0 > 0$. GDP is denoted Y_t and grows at the constant rate $g_Y < r$. Assume government spending, G_t, satisfies $G_t = \gamma Y_t$ and that net tax revenue, T_t, satisfies $T_t = \tau Y_t$, where γ and τ are positive constants and $t = 0, 1, 2, \ldots$.

a) What is the minimum size of the primary budget surplus as a share of GDP required for satisfying the government's intertemporal budget constraint as seen from time 0 (the beginning of period 0)? Derive your result by two different methods, that is, by using first the debt arithmetic method focusing on the dynamics of the debt-income ratio and next the method based on the intertemporal government budget constraint.

b) What key condition in the setup is it that ensures that both methods are appropriate and give the same result?

III.2 *A budget deficit rule.* Let time be continuous and suppose that money financing of budget deficits never occurs. Consider a budget deficit rule saying that the *nominal* budget deficit must never be above $\alpha \cdot 100$ per cent of *nominal* GDP, PY, $\alpha > 0$, that is, the requirement is

$$\dot{D} \leq \alpha PY, \qquad (*)$$

where $\dot{D} \equiv dD/dt$ (given $D = D(t)$ is nominal government debt) and $P = P(t)$ is a price index, whereas $Y = Y(t)$ is real GDP.

a) Is the deficit rule in the SGP of the EMU a special case of this? Why or why not?

b) Suppose the deficit rule (*) is always *binding* for the economy we look at. Derive the implied long-run value, b^*, of the debt-income ratio $b \equiv D/(PY)$, assuming a non-negative, constant inflation rate π (just a symbol for a constant, not necessarily the mathematical constant 3.14159...) and a positive constant growth rate, g_Y, of GDP. *Hint:* the differential equation $\dot{x} + ax = k$, where a and k are constants, $a \neq 0$, has the solution $x_t = (x_0 - x^*)e^{-at} + x^*$, where $x^* = k/a$.

c) Let the time unit be one year, $\pi = 0.02$, and $g_Y = 0.03$ for the SGP of the EMU. Calculate the value of b^*. Comment.

III.3 *Short questions*

a) Briefly describe what a cyclically adjusted budget deficit rule is.

b) "When $r > g_Y$ (standard notation), the No-Ponzi-Game condition of the government is both a necessary and sufficient condition for government solvency.

III.4 *When does the dynasty model imply Ricardian equivalence?* Consider a small open economy, SOE, with perfect mobility of goods and financial capital across borders, but no mobility of labour. Domestic and foreign financial claims are perfect substitutes. The real rate of interest at the world financial market is a constant, r. Time is discrete. People live for two periods, as young and as old. As young they supply one unit of labour inelastically. As old they do not work. As in the Barro dynasty model we consider single-parent families with a bequest motive. Each parent belonging to generation t has $1+n$ descendants, $n < r$ and n constant. There is perfect competition on all markets, no uncertainty, and no technical progress. Notation is

$$\begin{aligned} L_t &= \text{ number of young in period } t, \\ \tilde{T}_t &= \text{ real gross tax revenue in period } t, \\ \sigma_t &= \tilde{T}_t/L_t = \text{ a lump-sum tax levied on the young in period } t, \\ B_t &= \text{ real government debt at the start of period } t. \end{aligned}$$

In every period each old receives the same pension payment, π, from the government. From time to time the government runs a budget deficit (surplus) and in such cases the deficit is financed by bond issue (withdrawal). That is,

$$B_{t+1} - B_t = rB_t + \pi L_{t-1} - \tilde{T}_t,$$

where B_0, L_0, and π are given (until further notice, π is constant). Thus, the pension payments are, along with interest payments on government debt, the only government expenses. The government always preserves solvency in the sense that sooner or later tax revenue is adjusted to satisfy the intertemporal government budget constraint (more about this below).

The representative young individual
An individual belonging to generation t chooses saving, s_t, and bequest, b_{t+1}, to each of the descendants so as to maximize

$$U_t = \sum_{i=0}^{\infty}(1+\bar{R})^{-i}\left[u(c_{1t+i}) + \frac{1}{1+\rho}u(c_{2t+i+1})\right] \qquad (*)$$

s.t.

$$\begin{aligned} c_{1t} + s_t &= w_t - \sigma_t + b_t, \\ c_{2t+1} + (1+n)b_{t+1} &= (1+r)s_t + \pi, \qquad b_{t+1} \geq 0, \end{aligned}$$

and taking into account the optimal responses of the descendants. Here $1+\bar{R} \equiv (1+R)/(1+n)$, where $R > n \geq 0$ (both R and n constant). Also $\rho > -1$ is constant. The period utility function u satisfies the No Fast assumption and $u' > 0, u'' < 0$. Negative bequests are forbidden by law.

a) How comes that the preferences of the single parent can be expressed as in (*)?

b) Derive the first-order conditions for the decision problem, taking into account that two cases are possible, namely that the constraint $b_{t+1} \geq 0$ is binding and that it is not binding. Interpret the first-order conditions.

Suppose it so happens that $R = r$ and that, at least for a while, circumstances are such that the agents are at an interior solution (i.e., $b_{t+1} > 0$). We define a steady state of this economy as a path along which c_{1t} and c_{2t} do not change over time.

c) Is the economy in a steady state? Why or why not? *Hint:* combine the first-order conditions and use that $R = r$.

The link between the intertemporal budget constraint of the government and that of the dynasty

As seen from the beginning of period t the intertemporal government budget constraint is:

$$\begin{aligned}
\sum_{i=0}^{\infty} \pi L_{t+i-1}(1+r)^{-i-1} &= \sum_{i=0}^{\infty} \tilde{T}_{t+i}(1+r)^{-i-1} - B_t \Rightarrow && \text{(i)} \\
L_t \sum_{i=0}^{\infty} \sigma_{t+i} \frac{(1+n)^i}{(1+r)^{i+1}} &= L_t \sum_{i=0}^{\infty} \frac{\pi}{1+n} \frac{(1+n)^i}{(1+r)^{i+1}} + B_t \Rightarrow && \text{(ii)} \\
L_t \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}} \left[\sigma_{t+i} - \frac{\pi}{1+n} \right] &= B_t. && \text{(iii)}
\end{aligned}$$

d) Briefly explain in economic terms what each row here expresses.

e) The intertemporal budget constraint of the representative dynasty is

$$L_{t-1} \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}} [c_{2t+i} + (1+n)c_{1t+i}] = A_t + H_t,$$

where A_t is aggregate financial wealth in the economy and H_t is aggregate human wealth (after taxes):

$$H_t = L_t \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}} (w_{t+i} - \sigma_{t+i} + \frac{\pi}{1+n}).$$

Briefly explain in economic terms these two equations.

f) Suppose that in period $t+1$, π is increased (a little) to a higher constant level, before the bequest b_{t+1} is decided. Is the consumption path $(c_{2t+i}, c_{1t+i})_{i=1}^{\infty}$ affected? Why or why not?

g) Given π, suppose that for some periods there is a (small) tax cut so that $\tilde{T}_{t+i} < \pi L_{t+i-1} + rB_{t+i}$, that is, a budget deficit is run. Is the consumption path $(c_{2t+i}, c_{1t+i})_{i=0}^{\infty}$ affected? Why or why not?

Implications of $R > r$

Now suppose instead that $R > r$ (but still $r > n$) and that the economy is, at least initially, in steady state.

h) Will the bequest motive be operative? Why or why not?

i) Suppose π is increased (a little) to a higher level without σ_t being immediately adjusted correspondingly. Is resource allocation affected? Why or why not?

j) Given π, suppose a tax cut occurs so that for some periods a budget deficit is run. Is resource allocation affected? Why or why not?

k) In a few words relate the results of your analysis to the conclusions from other dynamic general equilibrium models you know of.

ℓ) In a few words assess the Barro model of infinitely-lived families linked through bequests.

III.5 Consider a small open economy (SOE) facing a real interest rate, r_t, given from the world market for financial capital. There is no cross-country mobility of labor. Under "normal circumstances" the following holds:

- aggregate employment, N, is at the "full employment" level, $N = (1 - \bar{u})L$, where $\bar{u}$ is the NAIRU and L is the aggregate labor supply, a given constant;
- real GDP, Y_t, equals its given trend level, $\bar{Y}_t$, which grows at a constant exogenous rate $g_Y > 0$ due to technical progress;
- $r_t = r$, where r is a constant and $r > g_Y$.

Time is discrete. Further notation is:

$$\begin{aligned}
G_t &= \text{real government spending on goods and services,} \\
T_t &= \text{real net tax revenue (= gross tax revenue − transfer payments),} \\
GBD_t &= \text{real government budget deficit,} \\
B_t &= \text{real public debt (zero coupon one period bonds) at start of period } t.
\end{aligned}$$

Assume that any government budget deficit is exclusively financed by issuing debt (and any budget surplus by redeeming debt).

a) Write down two equations showing how GBD_t and B_{t+1}, respectively, are determined by variables indexed by t. Also write down an equation indicating how B_{t+1} is related to GBD_t.

Suppose that $B_0 > 0$ and $G_t = \gamma \bar{Y}_t$, $t = 0, 1, \ldots$, where $0 < \gamma < 1$. Define the "net tax burden" as $\tau_t \equiv T_t / Y_t$.

b) Find the minimum constant net tax burden, $\hat{\tau}$, which is consistent with fiscal sustainability. *Hint:* different approaches are possible; one focuses on the debt-income ratio and uses the fact that a difference equation $x_{t+1} = a x_t + c$, where a and c are constants, $a \neq 1$, has the solution $x_t = (x_0 - x^*)a^t + x^*$, where $x^* = c/(1 - a)$.

c) How does $\hat{\tau}$ depend on $r - g_Y$ and $b_0 \equiv B_0/Y_0$, respectively?

d) Suppose the government for some reason (economic or political) can not raise the net tax burden above some threshold value, $\bar{\tau}$, and can not decrease the $G_t/\bar{Y}_t$ below some value, $\bar{\gamma}$. Find the maximum value, $\bar{r}$, of the interest rate consistent with a non-accelerating debt-income ratio. How does $\bar{r}$ depend on b_0? This dependency tells us why for some countries a high debt-income ratio is problematic. Explain.

Now consider an alternative scenario. In period $t = -1$ the SOE is hit by a huge negative demand shock and gets into a substantial recession (henceforth denoted a slump) with Y_{-1} far below $\bar{Y}_{-1}$. In response the government decides an "expansionary fiscal policy" instead of "laissez-faire", where:

- "laissez-faire" means maintaining $G_t = \gamma\bar{Y}_t$, $t = 0, 1, \ldots;$
- "expansionary fiscal policy" entails a discretionary increase in G of size ΔG, beginning in period 0 and maintained during the slump to stimulate economic activity, that is, $G_t = \gamma\bar{Y}_t + \Delta G$, where ΔG is a positive constant.

Let the tax and transfer rules in the economy imply that net tax revenue in period 0 is given by the function $T = T(Y)$; thus, $T_0 = T(Y_0)$. Assume that under the current slump conditions marginal net tax revenue is $T'(Y)$ $= 0.50$ whereas the spending multiplier is $\partial Y/\partial G = 1.5$.

e) For a given r_0 and given $\Delta G > 0$, find expressions for the effect of the expansionary fiscal policy on GBD_0 and B_1, respectively, in comparison with laissez-faire?

f) For a given r_1, and assuming that both $\partial Y/\partial G$ and $T'(Y)$ are approximately the same in period 1 as in period 0, find an expression for the effect of the expansionary fiscal policy on B_2 in comparison with laissez-faire?

Suppose the slump is over in period 2 and onwards whereby $G_t = \gamma Y_t$, $t = 2, 3, \ldots$. Suppose further that compared with the expansionary fiscal policy, laissez-faire during the slump would have implied not only higher unemployment, but also more people experiencing *long-term* unemployment. As a result some workers would have become de-qualified and in effect be driven out of the effective labor force. Suppose the loss in "full employment" output from period 2 and onwards implied by laissez-faire is ΔY per period, where ΔY is a positive constant.[1] Finally, let the ensuing loss in net tax revenue be $\tau \cdot \Delta Y$ per period, where τ is a positive constant (possibly close to $\hat{\tau}$ from b)).

g) With $r_t = r_2$, $t = 2, 3, \ldots$, and given ΔG and τ, find an expression for the value of ΔY required for the expansionary fiscal policy to "pay for itself" in period 2 and onwards in the sense that the averted loss in net tax revenue exactly offsets the extra interest payments?

h) Given $\tau = 0.29$, $r_1 = 0.01$, and $r_2 = 0.03$, answer again g). Comment.

[1] It is theoretically possible that ΔY is more or less constant for a long time because two offsetting effects are operative. Because of technical progress the loss of output per lost worker is growing over time. On the other hand the pool of long-term unemployed generated by the slump will over time be a decreasing share of the labor force due to exit by the old and entrance by young people in the labor force.

Chapter 4

Overlapping generations in discrete and continuous time

IV.1 *Interdependence across generations, fortuitous inheritance, and income distribution*

a) In the simple Diamond OLG model without technical progress, if not in reality, all are born with zero financial wealth, the same work ability, and the same work willingness − within generations as well as across generations. Nevertheless, as long as the economy has not reached a steady state, the members of different generations get different labor incomes. Why?

b) In the model, through what channel does the behavior of one generation affect the economic conditions for the next generation?

We now extend the model by adding uncertainty about the time of death. We also assume that people may live *three* periods (childhood is ignored). But they always work only in the first two. Individual labor supply is inelastic and equals one unit of labor in each of the two periods. All people survive the two first periods of life, but there is a probability $p \in (0, 1)$ of dying before the end of the third period. Since in period analysis events happen either at the beginning or the end of the period, we have to assume that an "early" death occurs immediately after retirement.

Suppose families are single-parent families: for each parent in generation t there are $1 + n$ children and these belong to the next generation. Any financial wealth left by a person who just died is inherited equally by the $1 + n$ children.

For members of generation t the probability of staying alive three periods is thus $1 - p$. Suppose each individual "born" at time t (the beginning of period t)[1] maximizes expected utility, $U_t = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) + (1-p)(1+\rho)^{-2}u(c_{3t+2})$.

c) Given the pure rate of time preference ρ, in what direction does a decrease in p affect the effective rate of time preference for a middle-aged person?

d) Suppose that there are no life annuity markets, and that the young knows the inheritance (positive or zero) before deciding the saving in the first period of life. Assume there is a constant real interest rate, r. For a young belonging to generation t whose parent dies at the end of period t with financial wealth z, the period budget constraints are

$$
\begin{aligned}
c_{1t} + s_t &= w_t + \frac{1}{1+n}z, \\
c_{2t+1} + a_{t+2} &= w_{t+1} + (1+r)s_t, \\
c_{3t+2} &= (1+r)a_{t+2}.
\end{aligned}
$$

Explain. Interpret a_{t+2} : is it saving in period $t+1$ or what?

e) Suppose that for some unexplained reason all members of generation $t-1$ happens to have the same financial wealth at the time of retirement. Yet, after one period an inegalitarian distribution of wealth within generations tends to arise although all individuals have the same rate of time preference. Explain in a few words why.

At a certain point in time a competitive market for private life annuities arises. Then before retiring middle-aged individuals place part or all their financial wealth in life annuity contracts issued by the life insurance companies. These use the deposits to buy capital goods which are rented out to the production firms. Next period the production firms pay back a risk-free return, $1+r$, per unit of account invested. At the same time, the insurance companies distribute their holdings (with interest) to their *surviving* depositors in proportion to their initial deposits.

f) Suppose that the insurance companies have no operating costs. Their aim is to maximize expected profit. Then, given free entry and exit, in equilibrium what will expected profit in the annuity industry be?

g) In equilibrium, how much will each surviving depositor receive per unit of account initially deposited?

[1] As usual we identify period t with the time interval $[t, t+1)$. This is the timing convention normally used in growth and business cycle theory in discrete time.

h) Suppose somebody claimed: "The middle-aged individuals will choose to hold all their financial wealth in the form of life annuities." True or false? Why?

i) What will the wealth distribution within generations in the long run look like?

IV.2 Consider an individual's saving problem in Blanchard's "perpetual youth" model (standard notation):

$$\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} U_0 &= \int_0^{\infty} (\ln c_t) e^{-(\rho+m)t} dt \quad \text{s.t.} \\
c_t &\geq 0, \\
\dot{a}_t &= (r_t + m)a_t + w_t - c_t, \qquad \text{where } a_0 \text{ is given,} \\
\lim_{t\to\infty} a_t e^{-\int_0^t (r_s+m)ds} &\geq 0.
\end{aligned}$$

a) Briefly interpret the objective function, the constraints, and the parameters.

b) Derive first the first-order conditions and the transversality condition, next the Keynes-Ramsey rule.

c) Derive the full solution to the problem, i.e., find the consumption function. *Hint:* combine the Keynes-Ramsey rule with strict equality in the intertemporal budget constraint.

d) How will a rise in the interest rate level affect current consumption and saving? Comment in terms of the Slutsky effects.

IV.3 *Short questions*

a) State with your own words what the No-Ponzi-Game condition says.

b) The No-Ponzi-Game condition belongs to problems with an infinite horizon. What is the analogue condition for a problem with finite horizon?

c) In the consumption/saving problem of a household, is the household's transversality condition a constraint in the maximization problem or does it express a property of the solution to the problem?

d) Write down the perfect foresight transversality condition of a household with infinite horizon. In fact there are three ways of writing it. Indicate all three.

e) State with your own words what the transversality condition says in each of the three versions.

IV.4 *Demography and the rate of return* The Blanchard OLG model for a closed economy is described by the two differential equations

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + b - m)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{1}$$

$$\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho - g\right]\tilde{c}_t - b(\rho + m)\tilde{k}_t, \tag{2}$$

and the condition that for any fixed pair (v, t_0), where $t_0 \geq 0$ and $v \leq t_0$,

$$\lim_{t\to\infty} a_{v,t} e^{-\int_{t_0}^{t}(f'(\tilde{k}(s))-\delta+m)ds} = 0. \tag{3}$$

Notation: $\tilde{k}_t \equiv K_t/(T_t N_t)$ and $\tilde{c}_t \equiv C_t/(T_t N_t) \equiv c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, N_t is population, and T_t is the technology level, all at time t. $f(\cdot)$ is a production function on intensive form, satisfying $f(0) = 0$, $f' > 0$, $f'' < 0$, and the Inada conditions. Finally, $a_{v,t}$ is financial wealth of an individual born at time v and still alive at time t. The remaining symbols stand for parameters and we assume all these are strictly positive. Furthermore, $\rho \geq b - m \geq 0$.

a) Briefly interpret the three above equations, including the parameters.

b) Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive initial value of $\tilde{k}$. Can the divergent paths be excluded? Why or why not?

In the last more than one hundred years the industrialized countries have experienced a gradual decline in the three demographic parameters m, b, and n. Indeed, m has gone down, thereby increasing life expectancy, $1/m$. Also $n \equiv b - m$ has gone down, hence b has gone even more down than m. The question is what effect on the long-run interest rate, r^*, we should expect? Below you are asked to give a "rough answer" based on stepwise curve shifting (comparative analysis) in the phase diagram. In this context it is convenient to consider n and m as the basic parameters and $b \equiv n + m$ as a derived one. So in (1) and (2) substitute $b \equiv n + m$.

c) How does a lower n affect the position of the $\dot{\tilde{k}} = 0$ locus? Illustrate by a new phase diagram. Comment.

d) Given n, how does a lower m affect the position of the $\dot{\tilde{c}} = 0$ locus? Illustrate in the phase diagram. Comment.

e) Given m, how does a lower $b = n + m$ affect the position of the $\dot{\tilde{c}} = 0$ locus? Illustrate in the phase diagram. Comment. *Hint for an explanation:* Sign the effect of the lower b on the proportion of young people in the population, then on the ratio H/A, next on the ratio $[C/(\rho + m)]/A = (A+H)/A$ (standard notation), and finally on $S/A \equiv (Y - C)/A$.

f) What is your overall conclusion as to the sign of the effect of the demographic change on r^*?

g) The method of analysis has a limitation which explains the proviso hinted at by the expression "rough answer" above. What is this limitation?

IV.5 *Productivity speed up* The basic model for this problem is the same as in Problem IV.4. Assume the economy has been in steady state until time 0. Then an unanticipated shift in g to a higher positive level occurs. Hereafter everybody rightly expects g to remain at this new level forever.

a) What happens to the real interest rate on impact? Comment.

b) Illustrate by a phase diagram the evolution of the economy for $t \geq 0$. There might be different possibilities to consider. Comment.

c) What happens to the real interest rate in the long run? Comment.

d) Compare two closed economies, A and B, that can be described by this model and have the same production function, the same g and ρ, the same initial conditions, K_0, T_0, and N_0, and the same n. The only difference is that country B for some reason has a higher health level and therefore lower m than country A (and lower b, since n is the same). "Country B will in the long run experience a higher level of labor productivity, Y/N, than country A". True or false? Why?

IV.6 *Short questions*

a) Make a list of motives for individual saving. Are some of these motives more in focus in an OLG framework than in a Ramsey framework?

b) In standard long-run models with perfect competition (like Blanchard's OLG model with exponential retirement or the Ramsey model), the real rate of interest, r_t, and the real rental rate, $\tilde{r}_t$, for physical capital (i.e., a price on the market for capital services) may or may not coincide for all t. Give a necessary and sufficient condition that they coincide.

IV.7 *Short questions.*

a) What is the *golden rule* capital intensity?

b) A steady-state capital intensity can be in the "dynamically efficient" region or in the "dynamically inefficient" region. What is meant by "dynamically efficient" and "dynamically inefficient"? Give a simple characterization of the two regions.

c) Compare some long-run properties of the Blanchard OLG model with the corresponding long-run properties of the Ramsey model. *Hint:* For example, think of the long-run interest rate and/or the possibility of dynamic inefficiency.

d) The First Welfare Theorem states that, given certain conditions, any competitive equilibrium (Walrasian equilibrium) is Pareto optimal. Give a list of circumstances that each tend to obstruct the needed conditions and thus make the conclusion untrue.

IV.8 *Short questions.* Consider the Blanchard OLG model with Harrod-neutral technical progress at rate g.

a) Can a path *below* the saddle path in $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Blanchard OLG model? Why or why not?

b) Can a path *above* the saddle path in $(\tilde{k}, \tilde{c})$ space be precluded as an equilibrium path with perfect foresight in the Blanchard OLG model? Why or why not?

IV.9 *Short questions (functional income distribution, stylized facts, rate of return).*

a) "If and only if the production function is Cobb-Douglas, does the Blanchard OLG model predict that the share of labor income in national income is constant in the long run." True or false? Give a reason for your answer.

b) Are predictions based on the Blanchard OLG model (with exogenous Harrod-neutral technical progress) consistent with Kaldor's stylized facts? Why or why not?

c) Suppose we want a concise economic theory giving the long-run level of the average rate of return in the economy as an explicit or implicit function of only a few parameters and/or exogenous variables. Does the Blanchard OLG model give us such a theory? Why or why not?

d) Briefly, assess the theory of the long-run rate of return implied by the Blanchard OLG model compared with that of the Ramsey model. That is, mention what you regard as strengths and weaknesses of the Blanchard theory.

IV.10 *Short questions*

a) What does Barro's dynasty model conclude about the hypothesis of *Ricardian equivalence*?

b) What does Blanchard's OLG model conclude about the hypothesis of *Ricardian equivalence*?

c) What is the basic reason that the two models lead to different conclusions in this regard?

IV.11 *Short questions*

a) "Considering the different Slutsky effects, the consumption function of the individual in the Blanchard OLG model (with logarithmic instantaneous utility) is such that a higher tax on interest income lowers current consumption." True or false? Why?

b) "When the real interest rate remains above the GDP growth rate of the economy, then the NPG condition for the government is a necessary and sufficient condition for fiscal sustainability." True or false? Comment.

IV.12 *Some quotations.*

a) Two economists – one from MIT and one from Chicago – are walking down the street. The MIT economist sees a 100 dollar note lying on the sidewalk and says: "Oh, look, what a fluke!". "Don't be silly, obviously it is false", laughs the Chicago economist, "if it wasn't, someone would have picked it up". Discuss in relation to the theoretical concepts of arbitrage and equilibrium.

b) A riddle asked by Paul Samuelson (Nobel Prize winner 1970): A physicist, a chemist, and an economist are stranded on an island, with nothing to eat. A can of soup washes ashore. The physicist says "let us smash the can open with a rock". The chemist says "let us build a fire and heat the can first". Guess what the economist says?

Chapter 5

More applications of the OLG model. Long-run aspects of fiscal policy

V.1 Consider a small open economy where domestic and foreign financial claims are perfect substitutes and there is perfect mobility of financial capital, but no mobility of labor. The real interest rate in the world financial market is a positive constant r. The dynamics of the economy are described (at least for some time) by the differential equation

$$\begin{aligned} \dot{\tilde{a}}_t &= (r-g-n)\tilde{a}_t + \tilde{w}^* \frac{b}{\lambda+b} - \tilde{c}_t, \qquad \text{where} \\ \tilde{c}_t &= (\rho+m)(\tilde{a}_t + \tilde{h}^*), \qquad \tilde{h}^* = \frac{b\tilde{w}^*}{(\lambda+b)(r+\lambda+m-g)}. \end{aligned}$$

Notation: $\tilde{a} \equiv A/(TN) \equiv a/T$, $\tilde{c} \equiv C/(TN) \equiv c/T$, $A =$ national wealth, $T =$ technology level, $N =$ population, $C =$ aggregate consumption, and $\tilde{w}^*$ is the real wage per unit of effective labor. The following parameters are strictly positive: r, g, λ, b, m; the remaining are non-negative.

a) Briefly interpret the model, including the parameters.

Assume $r > \rho + g + b$.

b) Draw a phase diagram in $(\tilde{a}, \dot{\tilde{a}})$ space as well as $(\tilde{a}, \tilde{c})$ space. Illustrate in the diagram the path the economy follows for $t \geq 0$, given the initial condition: $\tilde{a}_0 > -\tilde{h}^*$. Comment.

c) Sign the long-run current account surplus of the country. *Hint:* in the balance of payments accounting the current account surplus equals the increase in net foreign assets (whether this increase is positive or negative).

d) Suppose that at some point in time an unanticipated shift in the world interest rate occurs. If we imagine that this happens against the background of an international financial turmoil like the one in 2008-2009, what sign should we expect the shift to have? Why?

e) Assume agents rightly expect the new interest rate level to last for a long time. Draw a phase diagram illustrating the effects of the shift. *Hint:* how is $\tilde{h}^*$ affected? Comment.

f) Comment on the long-run development of the economy.

g) Briefly relate to the evolution of the Chinese economy since 1980.

V.2 The Blanchard OLG model for a closed economy is described by the two differential equations

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + b - m)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{1}$$

$$\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho - g\right]\tilde{c}_t - b(\rho + m)\tilde{k}_t, \tag{2}$$

and the condition that for any fixed pair (v, t_0), where $t_0 \geq 0$ and $v \leq t_0$,

$$\lim_{t\to\infty} a_{v,t} e^{-\int_{t_0}^{t}(f'(\tilde{k}(s))-\delta+m)ds} = 0. \tag{3}$$

Notation: $\tilde{k}_t \equiv K_t/(T_tL_t)$ and $\tilde{c}_t \equiv C_t/(T_tL_t) \equiv c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, L_t is population = labor supply, and T_t is the technology level, all at time t. Finally, f is a production function on intensive form, satisfying $f(0) = 0$, $f' > 0$, $f'' < 0$, and the Inada conditions. The remaining symbols, except $a_{v,t}$, stand for parameters and we assume all these are strictly positive; $a_{v,t}$ is financial wealth at time t of a person born at time v. Furthermore, $\rho \geq b - m \geq 0$.

a) Briefly interpret (1), (2), and (3), including the five parameters.

b) Draw a phase diagram and illustrate the path the economy will follow, given some arbitrary positive initial value of $\tilde{k}$. Can the divergent paths be excluded? Why or why not?

c) Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then an unanticipated technology shock occurs so that T_0 is replaced by $T_0' > T_0$. After this shock everybody rightly expects T to grow forever at the same rate, g, as before.

d) Illustrate by the phase diagram (or a new one) what happens to $\tilde{k}$ and $\tilde{c}$ on impact, i.e., immediately after the shock, and in the long run.

e) What happens to the rate of return on impact and in the long run?

f) Why is the sign of the impact effect on the real wage ambiguous (at the theoretical level) as long as f is not specified further?[1]

g) What happens to the real wage in the long run?

V.3 *Fiscal sustainability.* Consider the government budget in a small open economy (SOE) with perfect mobility of financial capital, but no mobility of labor. The real rate of interest at the world financial market is a positive constant r. Time is continuous. Let

$$
\begin{aligned}
Y_t &= \text{GDP at time } t, \\
G_t &= \text{government spending on goods and services at time } t, \\
T_t &= \text{net tax revenue (gross tax revenue} - \text{transfer payments) at time } t, \\
B_t &= \text{public debt at time } t.
\end{aligned}
$$

All variables are in real terms (i.e., measured with the output good as numeraire). Taxes and transfers are lump-sum. Assume there is no uncertainty and that the budget deficit is exclusively financed by debt issue (no money financing).

a) Write down an equation describing how the budget deficit and the increase per time unit in public debt are linked.

Suppose Y grows at a constant rate equal to $g + n$, where g is the rate of (Harrod-neutral) technical progress and n is the growth rate of the labor force (= employment). Suppose $r > g + n > 0$. Assume $T_t = \tau Y_t$ and $G_t = \gamma Y_t$, where τ and γ are constant over time, $0 < \gamma < 1$. Let initial debt, B_0, be positive.

[1] Remark: for "empirically realistic" aggregate production functions (having elasticity of factor substitution larger than elasticity of production w.r.t. capital) the impact effect on the real wage *is* positive, however.

b) Find the minimum initial primary surplus S_0 required for fiscal sustainability. *Hint:* one possible approach is to derive an expression for $\dot{b}_t$, where $b_t \equiv B_t/Y_t$; another approach is based on the fact that $\int_0^\infty e^{-at}dt = 1/a$ for a given constant $a \neq 0$.

c) Suppose $\tau > \gamma$. Is debt explosion possible?

d) How does S_0 depend on the growth-corrected interest rate?

Suppose instead that B_0 is negative.

e) Is debt explosion possible?

f) Answer question b) again. Comment.

g) Answer question d) again. Comment.

V.4 Consider a Blanchard OLG model for a closed economy with a public sector, public debt, and lump-sum taxation. The dynamics of the economy are described by the differential equations

$$\dot{C}_t = (F_K(K_t, L) - \delta - \rho)C_t - m(\rho + m)(K_t + B_t), \tag{1}$$
$$\dot{B}_t = [F_K(K_t, L) - \delta]\, B_t + G - T_t, \qquad B_0 > 0 \text{ given}, \tag{2}$$
$$\dot{K}_t = F(K_t, L) - \delta K_t - C_t - G, \qquad K_0 > 0 \text{ given}, \tag{3}$$

the condition

$$\lim_{t\to\infty} B_t e^{-\int_0^t [F_K(K_s,L)-\delta]ds} \leq 0, \tag{4}$$

and a requirement that households satisfy their transversality conditions. Here, C_t is aggregate private consumption, K_t is physical capital, L is population = labor supply, B_t is public debt, G is government spending on goods and services, T_t is net tax revenue (= gross tax revenue−transfer payments), and F is an aggregate neoclassical production function with constant returns to scale and satisfying the Inada conditions. The other symbols stand for parameters and all these are positive; L and G are positive constants. A dot over a variable denotes the derivative w.r.t. time t.

a) Briefly interpret the equations (1) - (3) and the weak inequality (4), including the parameters.

b) Assuming $B_0 > 0$ and a balanced budget for all $t \geq 0$, construct a phase diagram and illustrate the path the economy follows, for the given K_0. It is understood that G and B_0 are "modest" relative to the production possibilities of the economy, given this K_0. Comment on the phase diagram.

c) Suppose that two countries, country I and country II, are well described by the model in b). The countries are similar at time $t = 0$, except that they differ w.r.t. B_0 and possibly also T_0 (but they have the same K_0). Comment on the implied long-run differences between the countries.

d) Suppose country I has been in its steady state until time $t_0 > 0$. Then, suddenly fiscal policy shifts such that $T_t = \bar{T}$ where $\bar{T}$ is a constant which is smaller than the tax revenue in the old steady state. Define what is meant by fiscal policy being sustainable. Is the fiscal policy $(G, \bar{T})$ sustainable? Why or why not? *Hint:* there may be different approaches; one approach uses that if a is a positive constant, then $\int_{t_0}^{\infty} e^{-a(t-t_0)}dt = 1/a$.

e) Suppose that at time $t_1 > t_0$, where $t_1 - t_0$ is relatively large, taxation in country I again changes such that for $t \geq t_1$ the government budget is balanced. Construct a phase diagram to illustrate the path that the economy follows for $t \geq t_1$. Illustrate by graphical time profiles the evolution of T_t, B_t, C_t, and K_t for $t \geq 0$. Comment.

V.5 *Welfare arrangements and fiscal sustainability in the "ageing society"* (from the exam Jan. 2005). Consider a small open economy (henceforth SOE) with a government sector. For simplicity, assume:

1. Perfect mobility of goods and financial capital across borders.

2. Domestic and foreign financial claims are perfect substitutes.

3. No labor mobility across borders.

4. No uncertainty.

5. Perfect competition on all markets.

There is at the world market for financial capital a constant (real) rate of interest $r > 0$. The SOE has (adult) population equal to N and a labor force equal to L, where both N and L are constant. Due to retirement we have $L < N$. The technology of the representative firm is given by

$$Y_t = F(K_t, E_t L) \equiv E_t L f(\tilde{k}_t),$$

where $\tilde{k}_t \equiv K_t/(E_t L)$, F is a neoclassical production function with CRS, and Y_t and K_t are output and capital input, respectively. The whole labor force is employed. We treat time t as continuous, and the time unit is one year. The symbol E_t represents a technology factor ("E" for "efficiency of labor") growing at the constant rate $g > 0$, that is, $E_t = e^{gt}$, by choosing measurement units such that $E_0 = 1$. There are no capital adjustment costs. The rate of physical capital depreciation is $\delta \geq 0$ and is constant. Firms maximize profit.

a) Find an expression showing how the capital intensity $\tilde{k}$ chosen by the firm is determined. Comment.

b) Show how the equilibrium real wage w_t is determined and that it can be written $w_t = w_0 e^{gt}$.

Let G_t denote government spending on goods and services. Suppose G_t is primarily eldercare including health services. Specifically, assume

$$G_t = \gamma(N - L)w_t, \qquad \gamma > 0,$$

where the factor of proportionality, γ, is a constant. Let X_t denote transfer payments including pensions. Assume

$$X_t = \alpha w_t(N - L), \qquad 0 < \alpha < 1,$$

where α is the "degree of compensation", a constant. Further, let $\tilde{T}_t$ denote gross tax revenue and assume

$$\tilde{T}_t = \tau(w_t L + X_t), \qquad 0 < \tau < 1,$$

where the tax rate τ is constant (capital income taxation, consumption taxes etc. are ignored). Finally, let B_t denote real public debt and assume that any budget deficit (whether positive or negative) is exclusively financed by changes in B (no money financing). Initial debt, B_0, is positive. This level of debt as well as γ are assumed of "moderate" size, allowing a $\tau \in (0, 1)$ to be consistent with a balanced budget.

c) Write down an equation describing how the budget deficit and the increase per time unit in public debt are linked.

d) Determine the primary surplus, S_t, and its growth rate. How does S_t depend on L?

Assume $r > g$. Let $\bar{S}_0$ denote the minimum size of the initial primary surplus consistent with fiscal sustainability.

e) Find $\bar{S}_0$. What is the sign of $\bar{S}_0$? Comment. *Hint:* if a is a positive constant, then $\int_{t_0}^{\infty} e^{-a(t-t_0)}dt = \frac{1}{a}$.

From now, suppose $S_0 = \bar{S}_0$.

f) Find τ. Comment.

g) Determine the path over time of the debt-income ratio $b_t \equiv B_t/Y_t$. Illustrate the time profile of b_t in a diagram. Comment. *Hint:* the differential equation $\dot{x} + ax = c$, where a and c are constants, $a \neq 0$, has the solution $x_t = (x_{t_0} - x^*)e^{-a(t-t_0)} + x^*$, where $x^* = \frac{c}{a}$.

Suppose that, in analogy with the Blanchard OLG model with age-dependent labor supply,

$$L = \frac{m}{\lambda + m}N,$$

where λ is a constant "retirement rate" (prescribed by law), and m is a constant "death rate", so that $1/m$ is a rough indicator of "life expectancy", i.e., expected life time (as adult). As a crude representation of the much debated supposed increase in life expectancy of future generations, imagine that the government at time $t_0 > 0$ becomes aware that from time $t_1 = t_0+35$ years, life expectancy for a young person just entering the labor force will be $1/m'$ instead of $1/m$, where $m' < m$ (of course, in the real world this demographic change will not be a once for all change, but a gradual change, but for simplicity this is ignored). Population size remains equal to the constant N.

h) In a diagram draw the time profile of $\ln S_t$ as it would be in case there is no change in fiscal policy. Is the current fiscal policy sustainable? *Hint:* consider either the present discounted value of future primary surpluses as seen from time t_1 or the time path of the debt-income ratio.

Let τ' denote the minimum size of the (constant) tax rate required for fiscal sustainability from time t_1, assuming γ and α to be unchanged for ever and no change in taxation before time t_1 (Policy I).

i) Find τ'. Determine the sign of $\tau' - \tau$. Comment.

Now assume instead that at time t_0 the government decides to incur a budget surplus (including interest payments) until time t_1 such that the debt-income ratio in the time interval (t_0, t_1) gradually falls according to

$$\dot{b}_t = -c,$$

where c is a positive constant large enough such that at time t_1 one has $b = 0$. The plan is to accomplish this not by changing τ, but by temporary and gradual adjustments of γ and/or α.

j) Find the required value of c. *Hint:* if $\dot{x} = a$, a constant, then $x_t = x_{t_0} + \int_{t_0}^{t} \dot{x}_\tau d\tau = x_{t_0} + a(t - t_0)$.

Further, the plan is, for $t \geq t_1$, to let γ and α be back at their pre t_0 level and to let τ take the minimum value, τ'', now needed to obtain fiscal sustainability from time t_1 (Policy II).

k) Find τ''. Determine the sign of $\tau'' - \tau'$. Comment.

Suppose that at time t_0 an alternative policy is proposed, namely to let τ, γ, and α stay at their pre t_0 level forever and at time t_1 adjust λ such that fiscal sustainability is obtained (Policy III).

l) Find the required λ.

m) Assuming $\tau'' > \tau$, compare Policy II and Policy III w.r.t. the implied intergenerational "burden" and "benefit" distributions.

V.6 *Short questions.*

a. In many fiscal policy models total of government expenditure on goods and services is (implicitly or explicitly) public consumption. Can you imagine a class of richer models modifying the conclusion as to the consequences for future generations of government expenditure and budget deficits? Briefly explain.

b. Consider a small open economy with perfect mobility of goods and financial capital, but no mobility of labor. Can you imagine an aggregate production function, with three inputs, that would allow a lump-sum-tax financed increase in public capital to increase private wealth formation?

c. "Public debt is a strain on future generations and should always be avoided or at least be reduced as fast as possible." Discuss.

V.7 Consider the article on government debt by Elmendorf and Mankiw (1999). In their footnote 12, p. 1635, they present a formula for the long-run multiplier on capital w.r.t. the level of government debt in a closed economy. The formula, which is based on the Blanchard OLG model, indicates a negative long-run multiplier. Population growth, technical change, and retirement are ignored.

a) Using our standard notation (i.e., replacing their W with K^*, their p with m, and their θ with ρ), show by phase diagram analysis that the negative sign is correct. Comment. *Hint:* set up the model as in Lecture Notes, Ch. 13.

b) Derive Elmendorf and Mankiw's formula, using our standard notation. Their formula appears different from the corresponding formula in Lecture Notes. Is there a real difference or are the two formulas equivalent? Comment.

c) In their footnote 11, p. 1635, Elmendorf and Mankiw present a formula for the long-run multiplier on national wealth w.r.t. the level of government debt in a small open economy. Is the absolute value of the long-run multiplier smaller or larger than that for the closed economy? Why?

V.8 *A fiscal sustainability gap indicator with the Danish economy in mind, October 2005.*[2] Consider the government budget in a small open economy (SOE) with perfect mobility of financial capital, but no mobility of labor. The real rate of interest at the world capital market is a positive constant $r > g + n > 0$, where g is a constant rate of (Harrod-neutral) technical progress and n is a constant rate of growth of the labor force. The aggregate production function, $Y = F(K, TL)$, has CRS. Time is continuous. Let

$$\begin{aligned} Y_t &= \text{GDP}, \\ G_t &= \text{government spending on goods and services including} \\ &\quad\ \text{elder care and health services}, \\ X_t &= \text{transfer payments including public pensions}, \\ \tilde{T}_t &= \text{gross tax revenue}, \\ B_t &= \text{public debt}, \end{aligned}$$

[2] This problem is based on a suggestion by instructor Mads Diness Jensen, October 2005.

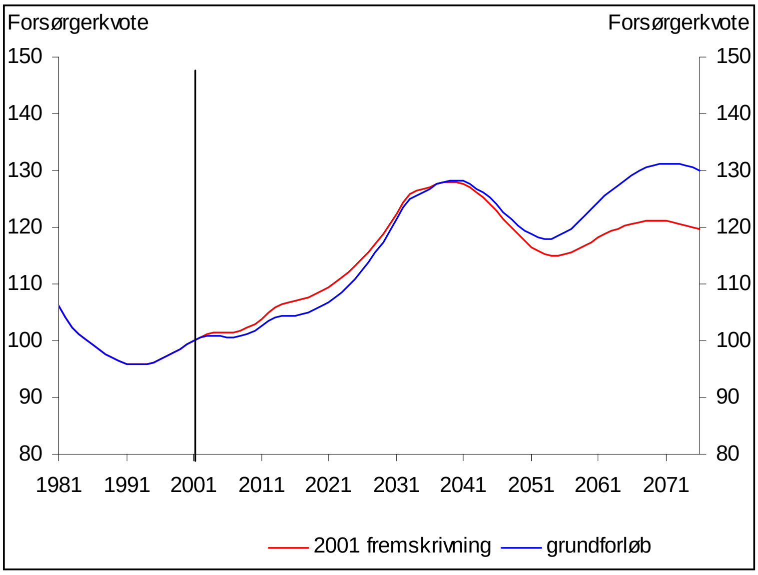


Figure 5.1: Elderly dependency ratio, Denmark. Simulation based on DREAM. Source: The Danish Welfare Commission, 2005.

all at time t and in real terms (i.e., measured with the output good as numeraire). Assume the future is known with certainty and that budget deficits are exclusively financed by debt issue (no money financing). Initial public debt, B_0, is positive.

a) Ignoring business cycle fluctuations, what is the growth rate of GDP? Write down the time path of Y_t.

b) Write down an expression for the real primary budget surplus, S_t, at time t.

c) Write down a condition that the time path of S_t must satisfy, as seen from time 0, for fiscal policy to be solvent. Translate this to a condition for the time path of the primary surplus-income ratio, $s_t \equiv S_t/Y_t$. It is convenient to introduce $\tilde{r} \equiv r - (g+n)$.

Let $\tau_t \equiv \tilde{T}_t/Y_t$, $x_t \equiv X_t/Y_t$ and $\gamma_t \equiv G_t/Y_t$. Suppose G_t grows at the same rate as Y_t.

d) What can we conclude about the time path of γ_t?

It is well-known that many industrialized countries feature an increasing elderly dependency ratio due to longer life span and lower fertility. Figure

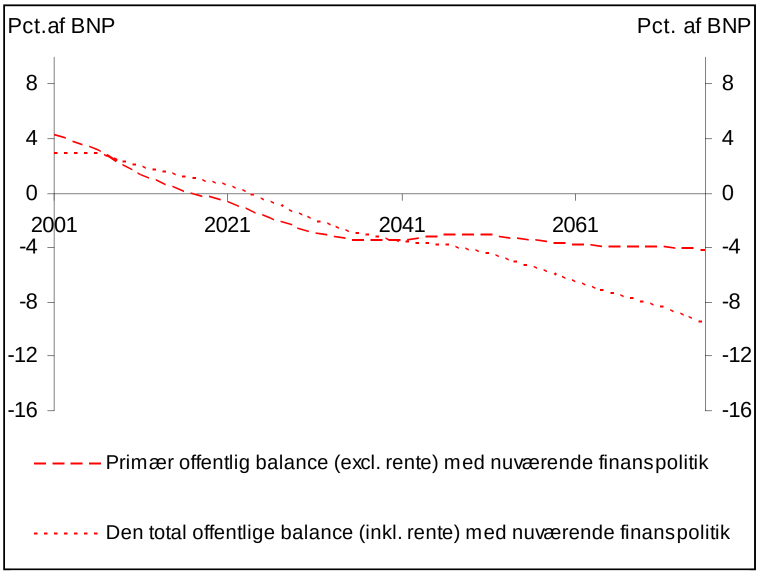


Figure 5.2: Primary budget deficit and total budget deficit if current fiscal policy is maintained. Denmark. Simulation based on DREAM. Source: The Danish Welfare Commission, 2005.

5.1 shows this for Denmark. Figure 5.2 shows projected paths of the primary budget deficit and the total budget deficit in Denmark if current (2005) fiscal policy rules, including welfare arrangements, are maintained. A rough formalization of this expected development is:

$$\tau_t = \tau_0 - (1 - e^{-\mu t})\Delta\tau, \qquad (*)$$

$$x_t = x_0 + (1 - e^{-\mu t})\Delta x, \qquad (**)$$

where $\Delta\tau$ and Δx are some positive numbers such that $\Delta\tau + \Delta x > s_0$, and μ (the adjustment speed) is positive.

e) Interpret. Find a formula showing the movement of s_t over time and find the limiting value of s_t for $t \to \infty$ if there is no change in fiscal policy rules. Illustrate the time profile of s_t in a diagram.

Numerical projections for Denmark indicate that the present discounted value of the future primary surpluses with unchanged fiscal policy is not very far from zero. Assume it is exactly zero.

f) Is current (2005) fiscal policy, which we may call $\mathcal{P}$, sustainable? Why or why not?

Suppose a suggested new policy design, $\mathcal{P}'$, implies that the path of G_t remains unchanged, but the path $(\tau_t, x_t)_{t=0}^{\infty}$ is replaced by the path $(\tau'_t, x'_t)_{t=0}^{\infty}$ with time profiles

$$\begin{aligned} \tau'_t &= \tau'_0 - (1 - e^{-\mu t})\Delta\tau, \\ x'_t &= x'_0 + (1 - e^{-\mu t})\Delta x, \end{aligned}$$

g) Write down an expression for the primary surplus-income ratio at time t according to the new policy $\mathcal{P}'$.

h) Find the minimum initial primary surplus-income ratio, $\bar{s}'_0$, required for the fiscal policy $\mathcal{P}'$ to be sustainable as seen from time 0. *Hint:* $\int_0^{\infty} e^{-at}dt = 1/a$ for any constant $a \neq 0$.

As a sustainability gap indicator at time 0 we choose $gap_0 \equiv \bar{s}'_0 - s_0$.

i) Illustrate the gap in the diagram from question e). How does gap_0 depend on:

1. the debt-income ratio at time 0?
2. the adjustment speed μ?
3. the spending-income ratio, γ.
4. the growth-corrected interest rate presupposing Δx, $\Delta\tau$ and μ are independent of the growth-corrected interest rate?

Hint: If no unambiguous answer as to the sign of the effect can be given, write down a criterion in the form of an inequality on which the sign depends. Comment.

j) In fact an increase in the interest rate *is* likely to affect $\Delta\tau$, namely by reducing $\Delta\tau$ partly through the higher tax revenue from postponed taxation of labor market pensions and partly through the induced increase in wealth accumulation, which implies higher future tax revenue; there are also potential counteracting factors such as a possible increase in tax deductibility due to increased interest payments. Can this matter for the conclusion to i.3)? Comment.

V.9 Consider the government budget in a small open economy. Time is continuous, the time unit is one year, and there is no uncertainty. Let g and n be non-negative constants and let

$$\begin{aligned} Y_t &= Y_0 e^{(g+n)t} = \text{ real GDP}, \\ G_t &= \text{ real government spending on goods and services}, \\ T_t &= \text{ real net tax revenue (= gross tax revenue -- transfer payments)}, \\ B_t &= \text{ real public debt}, \\ r &= \text{ real interest rate, a constant.} \end{aligned}$$

Assume that the budget deficit is exclusively financed by issuing debt.

a) Write down an equation showing how the increase in B_t per time unit is determined.

Consider a scenario with $B_0 > 0$, $r > g+n \geq 0$, and $T_t/Y_t = \tau$, a positive constant less than one.

b) Find the maximum constant G/Y which is consistent with fiscal sustainability. *Hint:* the differential equation $\dot{x} + ax = k$, where a and k are constants, $a \neq 0$, has the solution $x_t = (x_0 - x^*)e^{-at} + x^*$, where $x^* = k/a$.

Consider another scenario, where there is a deficit rule saying that $\lambda \cdot 100$ per cent of the interest expenses on public debt plus the primary budget deficit must not be above $\alpha \cdot 100$ per cent of nominal GDP, i.e.

$$\frac{\lambda i D_t + P_t(G_t - T_t)}{P_t Y_t} \leq \alpha, \qquad (*)$$

where $0 < \lambda \leq 1$, $\alpha > 0$, and

$$\begin{aligned} D_t &= \text{ nominal public debt}, \\ P_t &= \text{ price level}, \\ i &= r + \pi = \text{ nominal interest rate}, \\ \pi &= \text{ the inflation rate which we assume constant and non-negative.} \end{aligned}$$

c) Is the deficit rule of the SGP of the EMU a special case of (*)? Comment.

d) Let $b_t \equiv B_t/Y_t$. Derive the law of movement (differential equation) for b_t, assuming the deficit rule is always binding.

Suppose λ is such that $(1-\lambda)i < \pi + g + n$.

e) Find the time path of b.

f) Let the steady-state value of b be denoted b^* and assume $b_0 > b^*$. Will b explode or converge towards b^* over time? Comment.

g) How does b^* depend on λ? Comment.

Chapter 6

The q-theory of investment

VI.1 *A carbon tax and Tobin's q* We consider a small open economy (henceforth called SOE) with perfect mobility of financial capital but no mobility of labor. The SOE faces a constant real interest rate $r > 0$, given from the world market for financial capital. The technology of the representative firm is given by a neoclassical production function with constant returns to scale,

$$Y_t = F(K_t, L_t, M_t), \quad F_i > 0, F_{ii} < 0 \quad \text{for} \quad i = K, L, M.$$

Here Y is output gross of installation costs, imports, and physical capital depreciation, K is capital input, and L is labor input, whereas M is an imported fossil energy source, say oil. We also assume that the three inputs are *direct complements* in the sense that

$$F_{ij} > 0, \quad i \neq j.$$

The firm faces strictly convex capital installation costs and the installation cost function is homogeneous of degree one: $J_t = G(I_t, K_t) \equiv K_t g(I_t/K_t)$, $g(0) = g'(0) = 0$, $g'' > 0$.

In national accounting what is called Gross Domestic Output (GDP) is aggregate gross value added, i.e.,

$$GDP_t = Y_t - J_t - p_M M_t, \tag{1}$$

where $p_M > 0$ is the exogenous real price of oil which we treat as a shift parameter.

The labor force of the SOE is a constant $\bar{L}$. There is perfect competition in all markets. There is a tax, $\tau > 0$, on use of fossil energy (a "carbon tax"). In equilibrium with full employment the following holds:

$$M_t = M(K_t, (1+\tau)p_M), \qquad M_K > 0, M_{p_M} < 0. \tag{2}$$

We write the marginal product of capital ("Marginal Product of K") as a function $MPK(K_t, (1+\tau)p_M) \equiv F_K(K_t, \bar{L}, M(K_t, (1+\tau)p_M))$. It can be shown that

$$MPK_K < 0, MPK_{p_M} < 0. \tag{3}$$

a) Set up the value maximization problem of the representative firm and derive the first-order conditions.

b) Given the general information put up, briefly explain by words why GDP takes the form in (1), why the partial derivatives in (2) *must* have the shown signs, why we *must* have $MPK_{p_M} < 0$ as indicated in (3), and why, at first glance, the sign of MPK_K might seem ambiguous.[1]

The dynamics of the capital stock is given by

$$\dot{K}_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given}, \tag{4}$$

where $m(1) = 0$, $m' = 1/g''$, and δ is the capital depreciation rate whereas q is the shadow price of installed capital along the optimal path, satisfying the differential equation

$$\dot{q}_t = (r+\delta)q_t - MPK(K_t, (1+\tau)p_M) + g(m(q_t)) - m(q_t)(q_t - 1). \tag{5}$$

Moreover, a necessary transversality condition is

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0.$$

c) Briefly interpret (4) and (5): what is the economic "story" behind these equations?

d) Assuming F satisfies the Inada conditions, construct a phase diagram for the system (4) - (5). *Hint:* it can be shown that at least in a neighborhood of the steady state, the slope of the $\dot{q} = 0$ locus is negative.[2]

e) For an arbitrary $K_0 > 0$, indicate in the diagram the evolution of the pair (K_t, q_t) in general equilibrium. Does the convergent solution path satisfy the transversality condition? Is the solution to the model unique? *Hint:* it can be shown that the divergent solution paths violate the transversality condition.

[1] You do not have to explain why nevertheless $MPK_K < 0$ since it takes some steps to prove this. A proof is given in the appendix to Chapter 15 in the lecture notes.

[2] Indeed, the proof in Appendix E to Chapter 14 in Lecture Notes is also valid here.

f) Assume that until time $t_0 > 0$, the economy has been in its steady state. Then, unexpectedly the government raises the carbon tax to $\tau' > \tau$. Illustrate graphically what happens on impact and gradually over time. Comment on the effect of the tax rise on investment.

g) Illustrate in another figure the time profiles of K_t and q_t for $t \geq 0$. Briefly explain in words.

h) We now go a little outside the present simple model. Suppose, that a non-fossil energy source is available whose application requires a lot of specially designed capital equipment. Although before the tax rise this alternative technology has not been privately cost-efficient, after the tax rise it is. Moreover, the representative firm expects internal positive learning-by-doing effects by adopting the alternative technology. Briefly do an informal analysis of how the rise in the carbon tax could in this broader setup affect capital investment.

VI.2 *A convenient specification of the capital installation cost function.* In this exercise we consider an economy as described in Exercise VI.1, ignoring its last question. Assume

$$G(I, K) = \frac{1}{2}\beta\frac{I^2}{K}, \qquad \beta > 0.$$

Assume further that until time $t_0 > 0$, the economy has been in its steady state with carbon tax equal to τ. Then, unexpectedly the government raises the carbon tax to $\tau' > \tau$ at the same time as it introduces an investment subsidy σ, $0 < \sigma < 1$, so that to attain an investment level I, purchasing the investment goods involves a cost of $(1-\sigma)I$. The subsidy is financed by some tax not affecting firms' behavior (for example a constant tax on households' consumption).

a) Given τ', find the required σ in terms of $MPK(K^*, (1+\tau')p_M)$ such that the economy stays with its "old" steady-state capital stock, K^*. Comment.

b) How, if at all, is the steady-state value of q affected by the change in fiscal policy?

VI.3 *When capital installation costs are independent of the stock of already installed capital.* Consider a single firm with production function

$$Y_t = F(K_t, L_t),$$

where Y_t, K_t, and L_t are output, capital input, and labor input per time unit at time t, respectively, while F is a neoclassical production function with CRS and satisfying the Inada conditions. Time is continuous. The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad \delta > 0,$$

where I_t is gross investment per time unit at time t and δ is the capital depreciation rate. There is perfect competition in all markets and there is no uncertainty. The real interest rate faced by the firm is a positive constant r. Cash flow (in real terms) at time t is

$$R_t = F(K_t, L_t) - w_t L_t - I_t - G(I_t),$$

where w_t is the real wage and $G(I_t)$ is a capital installation cost function satisfying

$$G(0) = G'(0) = 0, \; G''(I) > 0.$$

(An example is $G(I) = (\beta/2)I^2$, $\beta > 0$.)

a) Set up the firm's intertemporal production and investment problem as a standard optimal control problem, given that the firm wants to maximize its market value.

b) Let the adjoint variable be denoted q_t. Derive the first-order conditions and state the necessary transversality condition, TVC. *Hint:* the TVC has the standard form for an infinite horizon optimal control problem with discounting.

c) What is the economic interpretation of q_t?

d) Show that the firm's chosen labor input is such that the capital-labor ratio at time t is a function of w_t. *Hint:* it is convenient to consider the production function on intensive form.

Suppose from now that $w_t = w$ for all $t \geq 0$, where w is a positive constant. Let the corresponding optimal capital-labor ratio be denoted $\bar{k}$.

e) The optimal investment level, I_t, can be written as an implicit function of q. Show this. Construct a phase diagram for the (K, q) dynamics, assuming that a steady state with $K > 0$ exists. Let the steady state value of K be denoted K^*. *Hint:* since the capital installation cost function is simpler than usual, the phase diagram may look somewhat different from the usual one.

f) For an arbitrary $K_0 > 0$, indicate in the diagram the movement of the pair (K_t, q_t) along the optimal path. In another diagram draw the time profiles of q_t, I_t, and K_t. Comment on why, in spite of the marginal productivity of capital in the steady state exceeding $r + \delta$, there is no incentive to increase K above K^*.

g) It is common to call K^* the "desired capital stock". Express the desired capital stock as an implicit function of r, δ, and w. How does the desired capital stock depend on r and w, respectively? Indicating the sign is enough. *Hint:* a simple approach can be based on curve shifting.

h) Show that optimal net investment, $I_t^n \equiv I_t - \delta K_t$, equals $\delta(K^* - K_t)$. In the obtained net investment rule you should recognize a principle from introductory macroeconomics. What is the name of this principle? Comment.

i) Let F be Cobb-Douglas with CRS and let $G(I) = (\beta/2)I^2$, $\beta > 0$. Find q^*, I, L, and K along the optimal path. *Hint:* the differential equation $\dot{x}(t) + ax(t) = b$ with $a \neq 0$ has the solution $x(t) = (x(0) - x^*)e^{-at} + x^*$, where $x^* = b/a$.

j) Evaluate the model.

VI.4 *Tobin's q and imperfect competition.* Consider a single firm supplying a differentiated good in the amount y_t per time unit at time t. The production function is

$$y_t = K_t^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{*}$$

where K_t and L_t are capital and labor input at time t, respectively.

The nominal wage and the nominal *general* price level in the economy faced by the firm are constant over time and exogenous to the firm. So the real wage is an exogenous positive constant, w. The demand, y^d, for the firm's output is perceived by the firm as given by

$$y^d = p^{-\varepsilon}\frac{Y}{n}, \qquad \varepsilon > 1, \tag{**}$$

where p is the price set in advance by the firm (as a markup on expected marginal cost), relative to the general price level in the economy, n is the given large number of monopolistically competitive firms in the economy, Y is the overall level of demand, and ε is the (absolute) price elasticity of demand. The interpretation is that the firm faces a downward sloping demand curve the position of which is given by the general level of demand, which is exogenous to the firm. We assume that within the time horizon relevant for the analysis, Y is constant and the firm keeps p fixed, possibly due to menu costs. Moreover, the analysis will ignore uncertainty.

The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad \delta > 0, \qquad K_0 > 0 \text{ given},$$

where I_t is gross investment per time unit at time t and δ is the capital depreciation rate. We assume that p is high enough to always be above actual marginal cost so that it always pays the firm to satisfy demand. Then cash flow at time t is

$$R_t = py^d - wL_t - I_t - G(I_t),$$

where $G(I_t)$ is a capital installation cost function satisfying

$$G(0) = G'(0) = 0, \; G''(I) > 0.$$

An example is $G(I) = (\beta/2)I^2$, $\beta > 0$.

a) To obtain $y_t = y^d$, a certain employment level is needed. Find this employment level as a function of K_t and y^d. Let your result be denoted $L(K_t, y^d)$.

The real interest rate faced by the firm is denoted r and is, until further notice, a positive constant. As seen from time 0, the firm solves the following decision problem:

$$\max_{(I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} \left[py^d - wL(K_t, y^d) - I_t - G(I_t)\right] e^{-rt} dt \quad \text{s.t.}$$

$$I_t \text{ free (i.e., no restriction on } I_t),$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given},$$

$$K_t \geq 0 \text{ for all } t.$$

b) Briefly interpret this decision problem.

c) Denoting the adjoint variable q, derive the first-order conditions and state the necessary transversality condition (TVC) for a solution. *Hint:* the TVC has the standard form for an infinite horizon optimal control problem with discounting.

d) The optimal investment level, I_t, can be written as an implicit function of q_t. Show this.

e) Construct a phase diagram for the (K, q) dynamics, assuming that a steady state with $K > 0$ exists. Let the steady state value of K be denoted K^*. For an arbitrary $K_0 > 0$, indicate in the diagram the movement of the pair (K_t, q_t) along the optimal path.

Assume that until time t_1, the economy has been in steady state. Then, unexpectedly, the aggregate demand level, and thereby y^d, shifts to a new constant level $y^{d\prime} < y^d$ and is rightly expected to remain at that level for a long time.

f) Illustrate by the same or a new phase diagram what happens on impact and gradually over time. Comment on the implied effect on investment on impact and in the long run.

Assume instead that it is the interest rate which at time t_1 shifts to a new constant level $r' > r$ and is rightly expected to remain at that level for a long time.

g) Illustrate by the same or a new phase diagram what happens on impact and gradually over time. Comment on the implied effect on investment on impact and in the long run.

h) As a modified scenario, imagine that the fall in demand at time t_1 considered under f) was in fact due to a rise in the interest rate at time t_1. Compare the implied combined effect on investment on impact and in the long run with the isolated effects under f) and g), respectively.

i) Relate the results in f) and g) to the signs of the partial derivatives of the investment function in a standard IS-LM model. Comment.

VI.5 Consider a single firm with production function

$$Y_t = K_t^{\alpha}(T_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where Y_t, K_t, and L_t are output, capital input, and labor input per time unit at time t, respectively. Time is continuous and T_t is the technology level, growing over time at a constant rate $\gamma > 0$. The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0, \qquad \delta > 0,$$

where I_t is gross investment per time unit at time t and δ is the capital depreciation rate. Cash flow (in real terms) at time t is

$$R_t = K_t^{\alpha}(T_t L_t)^{1-\alpha} - J_t - w_t L_t - I_t,$$

where w_t is the real wage and J_t represents capital installation costs given by

$$J_t = \beta \frac{I_t^2}{2K_t}.$$

There is perfect competition in all markets and no uncertainty. The real interest rate faced by the firm is a constant $r > 0$.

a) Set up the firm's intertemporal production and investment problem as a standard optimal control problem, given that the firm wants to maximize its market value.

Let the adjoint variable be denoted q_t.

b) Derive the first-order conditions and state the necessary transversality condition (TVC). *Hint:* along the optimal plan the partial derivative of the current-value Hamiltonian w.r.t. the state variable equals the difference between the discount rate multiplied by the adjoint variable and the time derivative of the adjoint variable; the TVC has the standard form for an infinite horizon optimal control problem with discounting.

c) What is the economic interpretation of q_t? On the basis of one of the first-order conditions, express the optimal investment level at time t as a function of q_t and K_t.

From now on, assume that the firm is a representative firm in a small open economy.

d) Suppose the government wants to stimulate firms' investment and from time t_0 on implements a subsidy σ, $0 < \sigma < 1$, so that to attain an investment level I, purchasing the investment goods involves a cost of $(1 - \sigma)I$. Assuming the subsidy is financed by some tax not affecting firms' behavior (for example a tax on households' consumption), will the government attain its goal? Make sure that you substantiate your answer by a formal proof.

Define $\tilde{k}_t \equiv K_t/(T_t L_t)$ and assume that labor supply grows at the constant rate $n \geq 0$.

e) Show that $\dot{\tilde{k}}_t = (aq_t - b)\tilde{k}_t$, where a and b are constants.

f) On the basis of another of the first-order conditions from question b), derive an equation for $\dot{q}_t$ in terms of q_t and $\tilde{k}_t$.

g) Suppose $r > \gamma + n$. Draw a phase diagram in the $(\tilde{k}, q)$ plane and illustrate the evolution of the economy for $t \geq 0$, assuming that $0 < \tilde{k}_0 < \tilde{k}^*$, where $\tilde{k}^*$ is the steady-state value of $\tilde{k}$. *Hint:* it can be shown that in a neighborhood of the steady state, the $\dot{q} = 0$ locus is negatively sloped.

VI.6 Consider a small open economy facing a constant real interest rate, given from the world market. Markets are competitive. Labor supply is inelastic and constant over time and there is no technical progress. The government contemplates introduction of an 'investment subsidy', σ, such that to buy I machines, each with a price equal to one unit of account, firms have to pay $(1-\sigma)I$ units of account, where σ is a constant, $0 < \sigma < 1$. The private sector is in a steady state and is not aware of these governmental considerations. "In this setting, Tobin's q-theory of investment predicts that by introducing and maintaining the investment subsidy σ, the government will be able to stimulate aggregate net investment temporarily, but not permanently." True or false? Why?

VI.7 *Short question.* In a standard Ramsey model and a model based on the q-theory of investment the circumstances under which firms optimize are different. Give a brief characterization of this difference and its implications.

VI.8 *Short question.* In many simple macroeconomic models a firm's acquisition of its capital input is described as if the firm solves a sequence of static profit maximization problems. One can imagine circumstances where this description of firms' behavior is not adequate, however. Give a brief account of what such circumstances might be and what alternative approach might be relevant.

VI.9 Consider a single firm that chooses a plan $(L_t, I_t)_{t=0}^{\infty}$ to maximize

$$V_0 = \int_0^{\infty} \left(F(K_t, L_t) - G(I_t, K_t) - w_t L_t - (1-\sigma)I_t\right) e^{-rt} dt \quad \text{s.t.} \qquad (1)$$

$$L_t \geq 0, I_t \text{ free (i.e., no restriction on } I_t), \qquad (2)$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{3}$$
$$K_t \geq 0 \text{ for all } t \geq 0. \tag{4}$$

Here F is a neoclassical production function with CRS and satisfying the Inada conditions. Inputs of capital and labor are denoted K_t and L_t, respectively; G is a function representing the capital installation costs, I_t is gross investment, w_t is a given real wage, $r > 0$ a given constant real interest rate, and $\delta > 0$ a given constant capital depreciation rate. There is a constant investment subsidy $\sigma \in (0, 1)$. The installation cost function satisfies

$$G(0, K) = 0, \; G_I(0, K) = 0, \; G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0,$$

for all (I, K).

a) Set up the current-value Hamiltonian. Let the adjoint variable be denoted q_t. Derive the first-order conditions and state the necessary transversality condition (TVC). *Hint:* In this problem the necessary TVC has the standard form for an infinite horizon optimal control problem with discounting.

b) Interpret q_t. Show that the optimal level of investment at time t is a function of q_t, K_t and σ.

From now we consider the special case $G(I, K) = \beta \frac{I^2}{2K}$, where $\beta > 0$.

c) Express the optimal I_t/K_t as a function of q_t and σ.

Assume that the firm is a representative firm in a small open economy (SOE) with perfect competition in all markets and free mobility of financial capital, but no mobility of labor across borders. The labor force in SOE is a constant, $\bar{L}$.

d) Show that in this setup the firm's first-order conditions results into two coupled differential equations in K_t and q_t.

e) Construct the corresponding phase diagram and show in the diagram the evolution of (K_t, q_t) for an arbitrary initial value $K_0 > 0$. Comment!

f) Assume that until time $t_0 > 0$ the system has been in a steady state with $(K, q) = (K^*, q^*)$. The the government, unexpectedly, raises the investment subsidy to the level $\sigma' > \sigma$. Assume that the investment subsidy is rightly expected to stay at the new level forever. Illustrate by the phase diagram (possibly a new one) the path followed by (K_t, q_t) for $t > t_0$. Comment!

g) Consider the statement: "The increase in the investment subsidy has a temporary effect om gross investment, not a permanent." True or false? Why?

VI.10 *Short question.* Give a brief account of the main points in Tobin's q-theory of investment.

Chapter 7

Uncertainty, expectations, and speculative bubbles

VII.1 Let $\{X_t\}$ be a stochastic process in discrete time. Suppose $Y_t = X_t + e_t$, where $X_t = X_{t-1} + \varepsilon_t$ and e_t and ε_t are white noise.

a) Is $\{X_t\}$ a random walk? Why or why not?

b) Is $\{Y_t\}$ a random walk? Why or why not?

c) Calculate the rational expectation of X_t conditional on all relevant information up to and including period $t-1$.

d) What is the rational expectation of Y_t conditional on all relevant information up to and including period $t-1$?

e) Compare with the subjective expectation of Y_t based om the adaptive expectations formula with adjustment speed equal to one.

VII.2 Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), abundant capacity, and output determined by demand:

$$\begin{aligned} Y_t &= D_t = C_t + \bar{I} + G_t, && (1) \\ C_t &= \alpha + \beta Y^e_{t-1,t}, \qquad \alpha > 0,\ 0 < \beta < 1, && (2) \\ G_t &= (1-\rho)\bar{G} + \rho G_{t-1} + \varepsilon_t, \qquad \bar{G} > 0,\ 0 < \rho < 1, && (3) \end{aligned}$$

where the endogenous variables are Y_t = output (= income), D_t = aggregate demand, C_t = consumption, and $Y^e_{t-1,t}$ = expected output (income) in period t as seen from period $t-1$, while G_t, which stands for government spending on goods and services, is considered exogenous as is ε_t, which is white noise. Finally, investment, $\bar{I}$, and the parameters α, β, ρ, and $\bar{G}$ are given positive constants.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in period $t-1$.

a) Solve for Y_t.

b) Find the income multiplier (partial derivative of Y_t) with respect to a change in G_{t-1} and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in G_{t-1} and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.

VII.3 Consider arbitrage between equity shares and a riskless asset paying the constant rate of return $r > 0$. Let p_t denote the price at the beginning of period t of a share that at the end of period t yields the dividend d_t. As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, \ldots$. Suppose agents have rational expectations and care only about expected return (risk neutrality).

a) Write down the no-arbitrage condition.

Suppose dividends follow the process $d_t = \bar{d} + \varepsilon_t$, where $\bar{d}$ is a positive constant and ε_t is white noise, observable in period t, but not known in advance.

b) Find the fundamental solution for p_t and let it be denoted p_t^*. *Hint:* given $y_t = aE_t y_{t+1} + c\ x_t$, the fundamental solution is $y_t = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Suppose someone claims that the share price follows the process

$$p_t = p_t^* + b_t,$$

with a given $b_0 > 0$ and, for $t = 0, 1, 2, \ldots$,

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases}$$

where $q_t = f(b_t)$, $f' < 0$.

c) What is an asset price bubble and what is a rational asset price bubble?

d) Can the described b_t process be a rational asset price bubble? *Hint:* a bubble component associated with the inhomogenous equation $y_t = aE_t y_{t+1} + c\, x_t$ is a solution, different from zero, to the homogeneous equation, $y_t = aE_t y_{t+1}$.

VII.4 *The housing market in an old city quarter (partial equilibrium analysis, discrete time)* Consider the housing market in an old city quarter with unique amenity value (for convenience we will speak of "houses" although perhaps "apartments" would fit real world situations better). Let H be the aggregate stock of houses (apartments), measured in terms of some basic unit (a house of "normal size", somehow adjusted for quality) existing at a given point in time. No new construction is allowed, but repair and maintenance is required by law and so H is constant through time. Notation:

p_t = the real price of a house (stock) at the start of period t,
m = real maintenance costs of a house (assumed constant over time),
$\tilde{R}_t$ = the real rental rate, i.e., the price of housing services (flow),
R_t = $\tilde{R}_t - m =$ the *net* rental rate = net revenue to the owner per unit of housing services.

Let the housing services in period t be called S_t. Note that S_t is a *flow:* so and so many square meter-months are at the disposal for utilization (accommodation) for the owner or tenant during period t. We assume the rate of utilization of the house stock is constant over time. By choosing appropriate measurement units the rate of utilization is normalized to 1, and so $S_t = 1 \cdot H$. The prices p_t, m, and R_t are measured in *real* terms, that is, deflated by the consumer price index. We assume perfect competition in both the market for houses and the market for housing services.

Suppose the aggregate demand for housing services in period t is

$$D(\tilde{R}_t, Z_t), \qquad D_1 < 0, D_2 > 0, \tag{*}$$

where the stochastic variable Z_t reflects a factor that in our partial equilibrium framework is exogenous, say the present value of expected future labor income in the region.

a) Set up an equation expressing equilibrium in the market for housing services. In the $(H, \tilde{R})$ plane, for given Z_t, illustrate how $\tilde{R}_t$ is determined.

b) Show that the equilibrium *net* rental rate at time t can be expressed as an implicit function of H, Z_t, and m, written $R_t = \tilde{\mathcal{R}}(H, Z_t, m) \equiv R(H_t)$. Sign the partial derivatives of $\tilde{\mathcal{R}}$ w.r.t. H, Z_t, and m. Comment.

Suppose a constant tax rate $\tau_R \in [0, 1)$ is applied to rental income, after allowance for maintenance costs. In case of an owner-occupied house the owner still has to pay the tax $\tau_R R_t$ out of the imputed income, R_t, per house per year. Assume further there is a constant property tax rate $\tau_p \geq 0$ applied to the market value of houses. Finally, suppose a constant tax rate $\tau_r \in [0, 1)$ applies to interest income, whether positive or negative. We assume capital gains are not taxed and we ignore all complications arising from the fact that most countries have tax systems based on nominal income rather than real income. In a low-inflation world this limitation may not be serious.

We assume housing services are valued independently of whether the occupant owns or rents. We further assume that the market participants are risk-neutral and that transaction costs can be ignored. Then in equilibrium,

$$\frac{(1-\tau_R)R_t - \tau_p p_t + p_{t+1}^e - p_t}{p_t} = (1-\tau_r)r, \qquad (**)$$

where p_{t+1}^e denotes the expected house price next period as seen from period t, and r is the real interest rate in the loan market. We assume that r is constant over time, $r > 0$.

c) Interpret (**).

Assume from now that the market participants have rational expectations (and that they know the stochastic process which R_t follows as a consequence of the process of Z_t).

d) Derive the expectational difference equation in p_t implied by (**).

e) Find the fundamental value of a house, assuming R_t does not grow "too fast". *Hint:* write (**) on the standard form for an expectational difference equation and use that the fundamental solution of the standard equation $y_t = aE_t y_{t+1} + c\, x_t$ is $y_t = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Denote the fundamental value p_t^*. Assume R_t follows the process

$$R_t = \bar{R} + \varepsilon_t, \qquad (***)$$

where $\bar{R}$ is a positive constant and ε_t is white noise with variance σ^2.

f) Find p_t^* under these conditions.

g) How does $E_{t-1}p_t^*$ (the conditional expectation one period beforehand of p_t^*) depend on each of the three tax rates? Comment.

h) How does $Var_{t-1}(p_t^*)$ (the conditional variance one period beforehand of p_t^*) depend on each of the three tax rates? Comment.

VII.5 *A housing market with bubbles (partial equilibrium analysis)* We consider the same setup as in Exercise VII.4, including the equations (*), (**), and (***).

Suppose that until period 0 the houses were owned by the municipality. But in period 0 the houses are sold to the public at market prices. Suppose that by coincidence a large positive realization of ε_0 occurs and that this triggers a stochastic bubble of the form

$$\begin{aligned} b_{t+1} &= [1 + \tau_p + (1 - \tau_r)r]\, b_t + \varepsilon_{t+1}, \qquad t = 0, 1, 2, ..., \quad (\hat{\ }) \\ \text{where} \quad E_t\varepsilon_{t+1} &= 0 \quad \text{and} \quad b_0 = \varepsilon_0 > 0. \end{aligned}$$

Until further notice we assume b_0 is large enough relative to the stochastic process $\{\varepsilon_t\}$ to make the probability that b_{t+1} becomes non-positive negligible.

a) Can (^) be a rational bubble? You should answer this in two ways: 1) by using a short argument based on theoretical knowledge, and 2) by directly testing whether the price path $p_t = p_t^* + b_t$ is arbitrage free. Comment.

b) Determine the value of the bubble in period t, assuming ε_{t-i} known for $i = 0, 1, ..., t$.

c) Determine the market price, p_t, and the conditional expectation $E_t p_{t+1}$. Both results will reflect a kind of "overreaction" of the market price to the shock ε_t. In what sense?

d) It may be argued that a stochastic bubble of the described ever-lasting kind does not seem plausible. What kind of arguments could be used to support this view?

e) Still assuming $b_0 > 0$, construct a rational bubble which has a constant probability of bursting in each period $t = 1, 2, ...$

f) What is the expected further duration of the bubble as seen from any period $t = 0, 1, 2, ...,$ given $b_t > 0$? *Hint:* $\sum_{i=0}^{\infty} iq^i(1-q) = q/(1-q)$.[1]

g) If the bubble is alive in period t, what is the probability that the bubble is still alive in period $t+s$, where $s = 1, 2, ...$? What is the limit of this probability for $s \to \infty$?

h) Assess this last bubble model.

i) Housing prices are generally considered to be a good indicator of the turning points in business cycles in the sense that house prices tend to move in advance of aggregate economic activity, in the same direction. In the language of business cycle analysts housing prices are a *procyclical leading indicator*. Do you think this last bubble model fits this observation? *Hint:* consider how a rise in p affects residential investment and how this is likely to affect the economy as a whole.

VII.6 *Short question*

"Under the hypothesis of rational expectations, speculative bubbles cannot arise in general equilibrium." True or false? Why?

[1] Here is a proof of this formula. $\sum_{i=0}^{\infty} iq^i(1-q) = (1-q)q\sum_{i=0}^{\infty} iq^{i-1} = (1-q)q\sum_{i=0}^{\infty} dq^i/dq = (1-q)qd\left(\sum_{i=0}^{\infty} q^i\right)/dq = (1-q)qd\left(1-q\right)^{-1}/dq = (1-q)q\left(1-q\right)^{-2} = q\left(1-q\right)^{-1}$. □

Borrowed from Jeppe Druedahl:

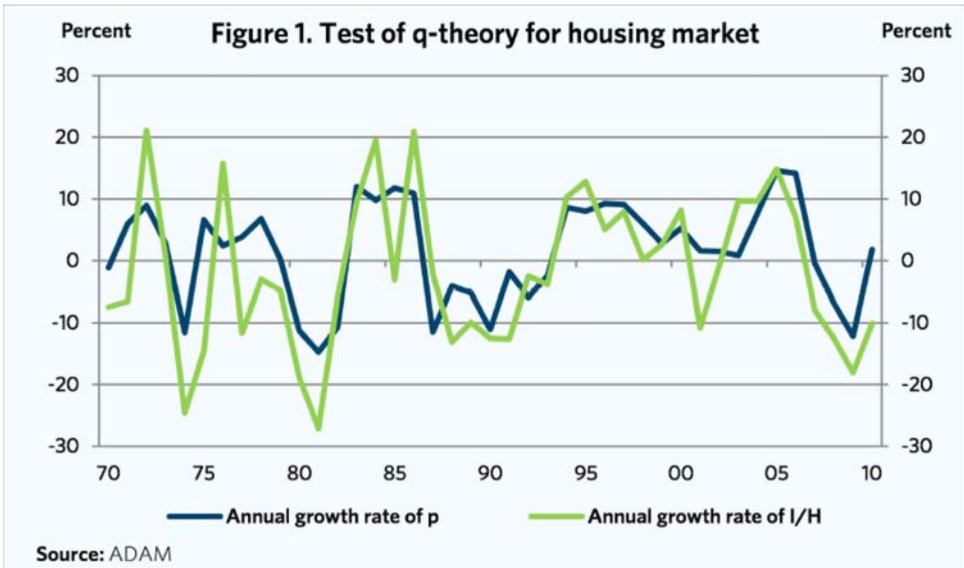


Figure 7.1

VII.7 *The housing sector.* We consider the housing sector in a small open economy facing a constant real risk-free interest rate, $r > 0$, given from the world financial market. Time is continuous. Let:

H_t = aggregate stock of houses,
S_t = aggregate housing services,
p_t = (real) price of houses,
R_t = (real) price of housing services,
$\dot{p}_t^e$ = expected increase per time unit in p_t,
I_t^H = aggregate gross housing investment (residential construction),
δ = rate of physical depreciation on houses,
τ_R = tax on (imputed) rental income after allowance for depreciation,
τ_p = tax on housing property,
τ_r ≡ tax on interest income.

We assume that S_t is proportional to H_t, and we normalize the factor of proportionality to be one:

$$S_t = H_t, \tag{1}$$
$$R_t = R(H_t), \quad R' < 0. \tag{2}$$

a) Briefly explain (1) and (2), including why $R' < 0$ is plausible.

Suppose housing services are valued independently of whether you own or rent. Assume further there is no uncertainty and that there are no transaction costs. Then an equilibrium condition is

$$\frac{(1-\tau_R)(R(H_t)-\delta p_t)-\tau_p p_t+\dot{p}_t^e}{p_t}=(1-\tau_r)r. \tag{3}$$

b) Explain why (3) is an equilibrium condition.

Figure 7.1 shows the year-to-year percentage change in p_t and I_t^H/H_t in Denmark since 1970. Data are from ADAM (Annual Danish Aggregate Model).

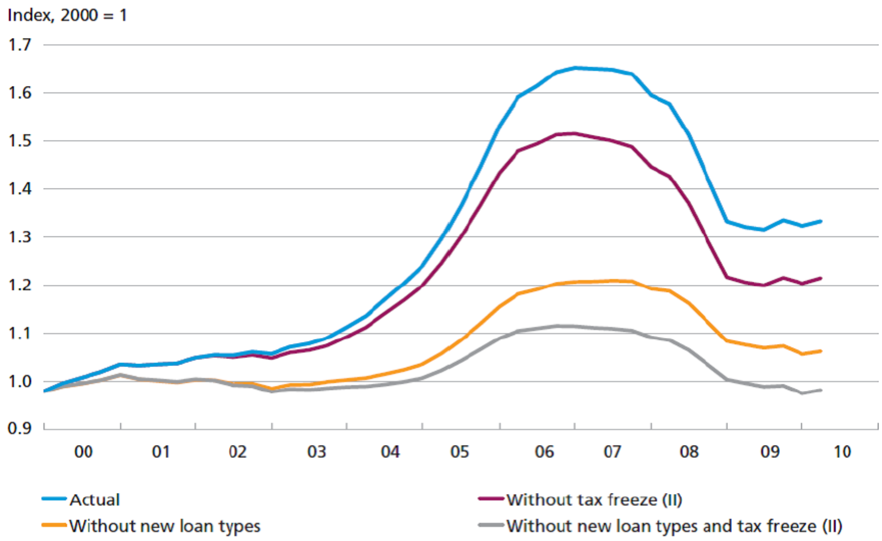
c) Does the housing sector model of Chapter 15.2 in Lecture Notes fit with Figure 7.1?

In Denmark, as in a lot of other countries, the housing market has changed somewhat in the last 15-20 years. In Denmark new loan types in terms of both adjustable-rate mortgage loans (from 1996?) and deferred amortization loans (from last quarter 2003) were introduced for the first time and from 2002 on the so-called 'tax freeze' implied that the property tax was frozen measured in *nominal* kroner (so it was falling in real terms). Figure 7.2 is from a simulation study by the Danish central bank (Nationalbanken) and indicates the estimated effects.

d) For a given $\dot{p}_t^e$, what should the effects of the tax freeze be according to our model? Do the predictions fit qualitatively with the empirical study?

e) For a given $\dot{p}_t^e$, what are the effects of the new loan types likely to be in a more realistic model? Do the predictions from the model fit qualitatively with the empirical study?

REAL HOUSE PRICE WITH AND WITHOUT NEW LOAN TYPES AND FREEZE, WITH PROPORTIONAL ADJUSTMENT OF UNEXPLAINED PART — Chart 3.4



Note: Counterfactual scenario based on estimated demand relation. Adjustable-rate loans are not assumed to be reflected until the 1st quarter of 2000. In the absence of the freeze on property value tax, the imputed rate of property value tax of the MONA databank is assumed to be kept constant as a ratio of the housing stock calculated at market value. The freeze on property value tax is modelled in accordance with method (II) as a permanent reduction in property value tax by 63 per cent in the 1st quarter of 2002, cf. Box 3.3. The red line depicts the total effect of new loan types (blue line) and the freeze on property value tax (yellow line).

Source: MONA databank and own calculations.

Figure 7.2

Chapter 8

Money and prices

VIII.1 *Some quotations.*

a) "When the growth rate of money supply is high, liquidity is low." Can you make sense of this?

b) *Oscar Wilde* once said: "When I was young, I used to think that money is the most important thing in life; now that I am old, I know it is." What might an economist's answer to this be?

VIII.2 *Short questions.*

a) Define the concept of "money".

b) What are the *essential* functions of money?

c) Is it possible to establish a connection between "money in the utility function" and the cash-in-advance constraint (= Clower constraint)? *Hint:* If c_t is desired consumption, and $\bar{c}_t$ is actual consumption, then the cash-in-advance constraint implies $\bar{c}_t = \min(c_t, M_t/P_t)$, where M_t is money holding, and P_t is the nominal price.

d) "Classical representative agent monetary macroeconomics suggests that money is neutral and superneutral. Hence, according to this kind of theory, the rate of money growth is irrelevant." Discuss whether these statements are valid.

e) "The Sidrauski model contains a satisfactory micro-based theory of money demand". Do you agree? Why or why not?

f) "According to the Sidrauski model, hyperinflation driven by expectations cannot occur in general equilibrium." Evaluate this statement.

g) List some cases where money is not superneutral.

h) "According to Keynesian Economics, money is neutral, but not superneutral." Evaluate this statement.

i) What is meant by the "Friedman zero interest rule" or the "Friedman money satiation result"?

j) A controversial question in monetary theory is whether the Sidrauski model and similar classical (or new classical) models are a suitable or not suitable framework for the design of monetary policy. Briefly discuss this question!

VIII.3 *Money in the long and short run (comparing Keynesian and classical views)* Consider the Keynesian money demand hypothesis,

$$M_t^d = P_t \cdot L(Y_t, i_t), \qquad L_Y > 0, L_i < 0, \tag{*}$$

where $P_t > 0$ is the general price level (in terms of money), $L(\cdot)$ is a real money demand function ("L" for liquidity), Y_t is aggregate production per time unit, and $i > 0$ is the nominal interest rate on short-term bonds. Let time be continuous and define the short-term real interest rate by

$$r_t \equiv i_t - \pi_t,$$

where $\pi_t \equiv \dot{P}_t/P_t$ (the inflation rate). Since asset markets move fast, it is natural to assume that the money market clears continuously, i.e.,

$$M_t = P_t \cdot L(Y_t, i_t), \qquad \text{for all } t \geq 0,$$

where M_t is the money supply.

Let the growth rate of any positive variable x be denoted g_x. Abstracting from fluctuations around the trend, suppose that $g_M = \mu$ and $g_Y = \gamma$, both constant, and that the real interest rate is a constant, r.

a) Briefly, interpret the signs of the partial derivatives of the money demand function in (*).

b) Show that if the money demand function is specified as $L(Y_t, i_t) = Y_t/V(i_t)$, where $V'(i) > 0$, then a constant long-run inflation rate, π, is consistent with the model and that this π satisfies a simple equation where also μ and γ enter. Relate to your empirical knowledge.

Recall that the (income) velocity of money is defined as P_tY_t/M_t. Classical monetary theory (the Quantity Theory of Money) claims that for given monetary institutions the velocity of money is a constant and thus independent of the nominal interest rate.

c) What is the prediction implied by the classical theory concerning the long-run inflation rate, given μ and γ? Comment in relation to the above Keynesian result.

We now consider a short time interval, say a month, where the level of the money supply is practically constant, apart from level shifts implemented by the central bank, through open market operations as part of its monetary policy. Suppose the money supply in this way shifts from M to $M' > M$.

d) According to the classical monetary theory (which is more or less shared by the "monetarists", like Milton Friedman, and also to some extent by the "new classical" theorists like Robert Lucas), which variable will respond in the short run and how?

e) What is the likely further effect on aggregate demand, output, and employment as long as the rate of inflation does not respond very much?

f) Now answer d) and e) from the point of view of Keynesian theory where P_t and even g_P are "sticky" in the short run?

Figure 8.1 is taken from Stock and Watson: "Business Cycle Fluctuations in US Macroeconomic Time series", *Handbook of Macroeconomics, vol. 1A*, 1999, p. 51. Based on US data 1921-1996, the authors estimate the regression $\log(M_t/P_t) = \alpha + \beta_y \log Y_t + \beta_r i_t + \varepsilon_t$ (where i_t is the "commercial paper rate", an unsecured promissory note with a fixed maturity of no more than 270 days) and find the following estimates and standard errors:

$$\beta_y = 0.868 \quad (0.070) \qquad \beta_r = -0.094 \quad (0.018)\,.$$

g) Discuss the relationship between the data and the mentioned theories.

VIII.4 *Short questions*

a) Briefly describe different monetary transmission mechanisms. Make sure that both classical and Keynesian-style monetary transmission channels are included in the list. Briefly discuss.

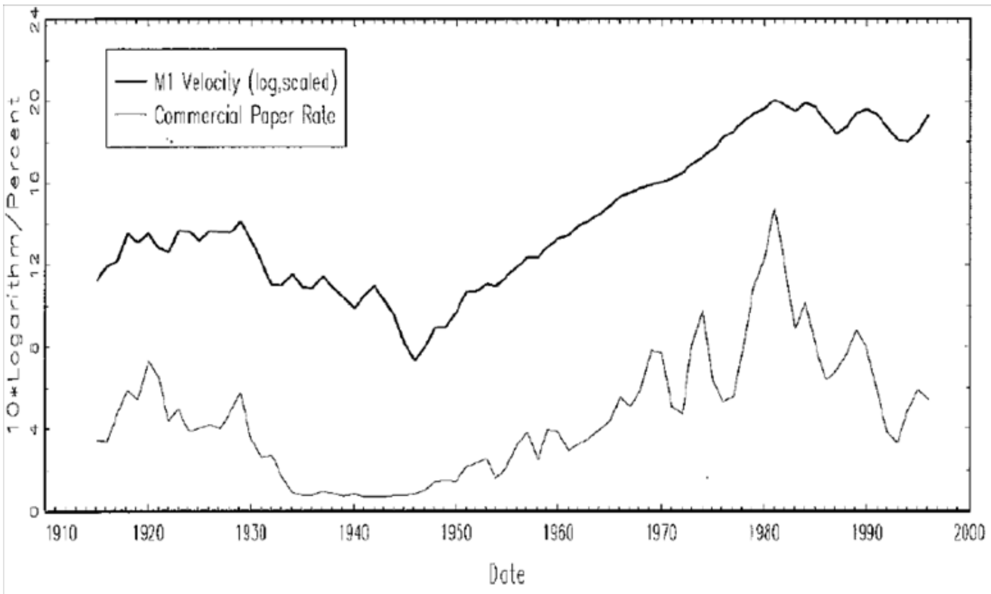


Fig. 4.4. M1 and the commercial paper rate.

Figure 8.1

b) To combat the sharply rising unemployment in the U.S., the Obama government decided to increase government spending substantially and the U.S. now has a huge government budget deficit. In response to this policy, economist Robert Barro from Harvard University declared that the policy was likely to raise expected future taxation considerably and as a result of this there would not be any stimulating effect on current aggregate demand and production. Briefly discuss.

c) "According to Sidrauski's monetary Ramsey model, the long-run growth rate in the money supply is irrelevant for welfare." True or false? Why?

Chapter 9

IS-LM dynamics in closed and open economies

IX.1 Consider the following model in continuous time for a closed economy:

$$
\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, R_t, \tau) + G - Y_t), && (1)\\
\lambda &> 0, \quad 0 < D_Y < 1, \quad D_R < 0, \quad -1 < D_\tau < 0, \\
\frac{M_t}{P_t} &= L(Y_t, i_t), \qquad L_Y > 0, \; L_i < 0, && (2)\\
R_t &= \frac{1}{q_t}, && (3)\\
\frac{1 + \dot{q}_t^e}{q_t} &= r_t^e, && (4)\\
r_t^e &\equiv i_t - \pi_t^e, && (5)\\
\pi_t &= \pi, && (6)
\end{aligned}
$$

where the superscript e denotes subjective expectation. Further, Y_t = output, q_t = real price of a consol paying one unit of output per time unit forever, G = government spending on goods and services, M_t = money supply, P_t = output price, R_t = real long-term interest rate, i_t = nominal short-term interest rate, r_t = real short-term interest rate, and $\pi_t \equiv \dot{P}_t/P_t$ = rate of inflation. The variables λ, τ, G, and π are exogenous positive constants. The initial values Y_0 and P_0 are historically given.

In questions a) - f) we assume that the central bank maintains the real money supply, $m_t \equiv M_t/P_t$, at a given constant level, m, by letting the (nominal) money supply grow at a rate equal to the rate of inflation.

a) Briefly interpret the model including the parameters.

Suppose expectations are rational and that speculative bubbles never arise.

b) To characterize the movement over time of the economy, derive from the model a dynamic system in Y and R. Draw the corresponding phase diagram and illustrate the path that the economy follows for an arbitrary $Y_0 > 0$. Comment.

Now we will consider effects of shifts in policy. Suppose that the economy has been in its steady state until time $t_0 > 0$.

c) Then, at time t_0, an unanticipated downward shift in G occurs. But after this shift everybody rightly expects G to remain unchanged forever. Graphically illustrate by means of a phase diagram. Comment. *Hint:* the following formula may be helpful for intuition:

$$R_t = \frac{1}{q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_u du} ds}.$$

d) Suggest a "free" interpretation of G and the downward shift in G such that the result under c) can be seen as a "rough" picture of events in the wake of the financial crisis 2008-2009.

e) What is the sign of the slope of the yield curve immediately after the shock to G? Comment.

We now consider a different monetary policy. Suppose that the central bank applies the short-term nominal interest rate as the monetary policy instrument and does so in accordance with the following rule:

$$i_t = \max(0, \alpha + \beta Y_t), \qquad (*)$$

where α and β are constants, $\beta > -L_Y/L_i$.

f) Maintaining the interpretation from d), briefly answer question c) under these circumstances.

g) Compare the output stabilization capability of this policy rule to that of the original monetary policy above. *Hint:* compare the slope of the $\dot{R} = 0$ locus under the two alternative policies.

Suppose that the economy is close to a steady state and that the government finds the level of economic activity (output and employment) unsatisfactorily low.

h) In this situation the government decides to raise the level of spending to $G' > G$. So, at time $t_1 > t_0$ the government announces a shift to G' to be implemented at time $t_2 > t_1$ and maintained forever. After this announcement everybody rightly expect this fiscal policy to be carried out as announced. Graphically illustrate by means of a phase diagram what happens to R_t and Y_t for $t \geq 0$. Comment.

i) Given the intension of the government: 1) Is it a good or bad idea to let the time interval (t_1, t_2) be short? Why? 2) Is it a good or bad idea to have the monetary policy (*) replaced by the interest rate targeting policy $i_t = \bar{i}$ for a while? Why?

IX.2 In this problem we change the policy regime in the model of Problem IX.1 so that i_t becomes the instrument and equal to a positive constant, i, as long as the monetary authority does not decide to change it. Now M_t is endogenous.

a) To characterize the movement over time of the economy, derive from the model a dynamic system in Y and R. Draw the corresponding phase diagram and illustrate the path that the economy follows. Comment.

Suppose that the economy has been in its steady state until time t_0.

b) Then an unanticipated downward shift in i occurs, but after this shift everybody rightly expect i to remain unchanged forever. Graphically illustrate by means of a phase diagram and time profiles what happens to Y_t, M_t, R_t, and r_t. Comment.

c) Assume instead that at time t_0, the monetary authority credibly announces a downward shift in the instrument variable to take place at time $t_1 > t_0$. After this shift everybody rightly expect i to remain unchanged forever. Graphically illustrate by means of a phase diagram and time profiles what happens to Y_t, R_t, r_t and M_t for $t \geq t_0$. Comment.

d) Briefly discuss the model.

IX.3 Consider the model and policy regime from Problem IX.2.

a) Determine the short-term and long-term real interest rates in steady state and find an implicit solution for Y in steady state. How does Y in steady state depend on i?

Now, suppose that the economy has been in its steady state until time t_0.

b) Answer b) from problem IX.2 under the assumption that the unanticipated shift in the short-term rate is *upward*. Comment.

c) Answer c) from problem IX.2 under the assumption that the anticipated shift in the short-term rate is *upward*. Comment.

IX.4 Consider the following continuous time model which focuses on "very-short run" dynamics of a closed economy:

$$\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, R_t, \alpha) + G - Y_t), && (1)\\
\lambda &> 0, \quad 0 < D_Y < 1, \quad D_R < 0, \quad D_\alpha > 0, &&\\
\frac{M_t}{P_t} &= L(Y_t, i_t), \qquad L_Y > 0, \; L_i < 0, && (2)\\
\pi_t &= \pi, && (3)\\
R_t &= \frac{1}{q_t}, && (4)\\
\frac{1 + \dot{q}_t^e}{q_t} &= r_t, && (5)\\
r_t &= i_t - \pi_t^e, && (6)
\end{aligned}$$

where a dot over a variable denotes the derivative w.r.t. time t, and the superscript e denotes subjective expectations. Further, Y_t = output, q_t = real price of a consol paying one unit of output per time unit forever, α = index of "level of confidence", G = government spending on goods and services, M_t = money supply, P_t = output price, i_t = nominal short-term interest rate, and $\pi_t \equiv \dot{P}_t/P_t$ = rate of inflation. The tax revenue function is implicit in the demand function D. The variables α, G, and π are exogenous positive constants. The initial values Y_0 and P_0 are historically given.

In questions a) - f) we assume that the central bank maintains the real money supply, $m_t \equiv M_t/P_t$, at a given constant level, m, by letting the (nominal) money supply grow at a rate equal to the rate of inflation.

a) Briefly interpret the equations (1) - (6) as well as R_t and r_t.

From now on, suppose that expectations are rational and that speculative bubbles do not arise.

b) To characterize the movement of the economy over time, derive from the model a dynamic system in Y_t and R_t. Draw the corresponding phase diagram, assuming that parameters are such that there exists a steady state with a nominal short-term interest rate $\bar{i} > \pi$. *Hint:* it may be useful to find an expression for i_t in terms of output and the real money supply.

c) Illustrate the path that the economy follows for an arbitrary $Y_0 > 0$. Comment.

Suppose that the economy has been in its steady state ("short-run equilibrium") until time $t_0 > 0$.

d) Then an unanticipated shift in the level of confidence to a value $\alpha' < \alpha$ occurs. But after this shift everybody rightly expects the new level of confidence to be maintained for a long time. Illustrate by a phase diagram what happens to Y_t and R_t over time. Illustrate in another figure the time profiles of i, r, R, and Y for $t \geq 0$. Briefly explain in words. *Hint:* the following formula may be helpful:

$$R_t = \frac{1}{q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds}.$$

e) What is the sign of the slope of the yield curve immediately after the shock? Comment.

Imagine that at time $t_1 > t_0$ the economy has virtually settled down in the new steady state.

f) For simplicity inflation is not endogenous in the model. As a crude representation of the operation ("behind the scene") of a Phillips curve, however, we imagine that at time $t_2 > t_1$ there is a jump down in the inflation rate to a negative level π'. The consolidated government/central bank instantly lowers $\dot{M}_t/M_t$ to the same level so that the real money supply does not change. Suppose that the market participants rightly expect these new circumstances to last for a long time. Illustrate by a phase diagram what happens to Y_t and R_t over time. Illustrate in another figure the time profiles of i, r, R, and Y for $t \geq t_1$. Whether the new steady state level of R exceeds or does not exceed the steady state level before time t_0 is ambiguous. Why?

Imagine that at time $t_3 > t_2$ the economy has virtually settled down in the new steady state. Let the nominal interest rate in this steady state be denoted $\bar{i}''$. Suppose $\bar{i}''$ is practically equal to zero. The fiscal and monetary authorities find the situation unsatisfactory and decide a coordinated fiscal and monetary policy involving a shift of G to $G' > G$ and a continuous adjustment of the money supply through open market operations so as to maintain the short-term nominal interest rate at the target level $i = \bar{i}''$ until the recession is over.

g) Rewrite one of the basic equations of the model so as to indicate how the nominal money supply must move according to the new monetary policy rule.

h) Within the model (with inflation staying at π' for a relatively long time), is the coordinated fiscal and monetary policy more potent than a conventional expansionary fiscal policy in isolation (that is, with a monetary policy that, as above, maintains the real money supply)? Is it more potent than a conventional expansionary monetary policy in isolation (that is, with a "passive" fiscal policy as above)? Explain your answers.

i) To the extent the stimulation of aggregate demand succeeds in raising economic activity, it is likely that (outside the model) inflation also gradually rises towards a more "normal level". Is this feature likely to counteract or support the effectiveness of the coordinated fiscal and monetary policy? Explain.

IX.5 We consider a small open economy (SOE) satisfying (approximately):

1. Perfect mobility across borders of financial capital, but no mobility of labor.
2. Domestic and foreign financial claims are perfect substitutes (no uncertainty).
3. Domestic and foreign output goods are imperfect substitutes.

Suppose the short-term behavior of the economy can be described by the following model in continuous time. Given the function $D(Y_t, R_t, \frac{XP^*}{P})$, where $0 < D_Y < 1, D_R < 0$, and $D_{\frac{XP^*}{P}} > 0$, the model is:

$$\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, R_t, \frac{XP^*}{P}) + G - Y_t), && \lambda > 0, \\
i_t &= i^*, \\
\frac{M_t}{P} &= L(Y_t, i_t), && L_Y > 0,\ L_i < 0, \\
R_t &= \frac{1}{q_t}, \\
\frac{1 + \dot{q}_t^e}{q_t} &= r_t, \\
r_t &\equiv i_t - \pi_t^e.
\end{aligned}$$

The endogenous variables are: $Y_t \equiv$ output, $R_t \equiv$ long-term real interest rate, $i_t \equiv$ short-term nominal interest rate, $M_t \equiv$ money supply, $q_t \equiv$ real price of a long-term bond (a consol), $r_t \equiv$ short-term real interest rate and $\pi_t^e \equiv$ expected (forward-looking) rate of inflation, where t is time. The superscript e signifies an expected value. A dot over a variable denotes the time derivative. The variables X, P^*, P, G, τ, and i^* are exogenous and constant; their interpretation is as follows: $X \equiv$ nominal exchange rate, $P^* \equiv$ foreign price level, $P \equiv$ domestic price level, $G \equiv$ government spending on goods and services, and $i^* \equiv$ foreign short-term nominal interest rate. The parameter λ is constant. The initial value, Y_0, of Y is predetermined. We assume that expectations are rational and speculative bubbles never occur.

a) Briefly interpret the model.

b) To characterize the movement over time of the economy, derive two coupled differential equations in Y and R.

c) Construct the corresponding phase diagram and illustrate the path the economy follows. Comment.

d) Determine the long-term real interest rate in steady state.

e) How does Y in steady state depend on i^* and G?

Suppose that the economy has been in its steady state until time t_0.

f) At time t_0 an unanticipated upward shift in the foreign short-term nominal interest rate occurs. But after this shift everybody rightly expects the foreign short-term nominal interest rate to remain unchanged for a very long time. Illustrate by a phase diagram and by graphical time profiles what happens to R_t, Y_t, r_t, and M_t for $t \geq t_0$. Comment.

g) Assume instead that at time t_0, people in the SOE become aware that a monetary tightening in the leading countries in the world economy is underway. As a crude representation of this, suppose the agents rightly expect an upward shift in the foreign short-term nominal interest rate to take place at time $t_1 > t_0$. After this shift everybody rightly expects the foreign short-term nominal interest rate to remain unchanged for a very long time. Illustrate by a phase diagram and by graphical time profiles what happens to R_t, Y_t, r_t, and M_t for $t \geq t_0$. Comment.

h) Now imagine the scenario is somewhat different from that described in g). Until time $t_2 > t_1$ everything is as described in g). But at time t_2, due to now foreseeable unemployment problems, the government of the SOE credibly announces an upward shift in G to take place at time $t_3 > t_2$. After this shift everybody rightly expects G to remain unchanged for a very long time. The size of the shift in G is such as to reestablish, in the long run, an output level equal to that attained at time t_1. Illustrate by a phase diagram and by graphical time profiles what happens to R_t, Y_t, r_t, and M_t for $t \geq t_2$. Comment.

IX.6 We consider a small open economy (SOE) satisfying (approximately) the conditions 1-3 in Problem IX.5. Suppose the short-term behavior of the economy can be described by the following model in continuous time. Given the function $D(Y_t, R_t, \frac{XP^*}{P}, \tau)$, where $0 < D_Y < 1$, $D_R < 0, D_{\frac{XP^*}{P}} > 0$, and $-1 < D_\tau < 0$, the model is:

$$\begin{aligned}
Y_t^d &= D(Y_t, R_t, \frac{XP^*}{P}, \tau) + G, \\
\dot{Y}_t &= \lambda(Y_t^d - Y_t), \qquad \lambda > 0, \\
i_t &= i^*, \\
\frac{M_t}{P} &= L(Y_t, i_t), \qquad L_Y > 0, \ L_i < 0, \\
R_t &= 1/q_t, \\
\frac{1 + E_t \dot{q}_t}{q_t} &= r_t, \\
r_t &\equiv i_t - E_t \pi_t.
\end{aligned}$$

Notation: $Y_t^d \equiv$ output demand, $Y_t \equiv$ output, $R_t \equiv$ long-term real interest rate, $i_t \equiv$ short-term nominal interest rate, $M_t \equiv$ money supply, $q_t \equiv$ real price of a long-term bond (a consol), $r_t \equiv$ short-term real interest rate, and $\pi_t \equiv$ (forward-looking) rate of inflation. The variables X, P^*, P, G, τ, and i^*

are exogenous and constant; their interpretation is as follows: $X \equiv$ nominal exchange rate, $P^* \equiv$ foreign price level, $P \equiv$ domestic price level, $G \equiv$ government spending on goods and services, $\tau \equiv$ tax parameter and $i^* \equiv$ foreign short-term nominal interest rate. The parameter λ is constant. The initial value, Y_0, of Y is given. The symbol E_t denotes expectation conditional on information available at time t. Expectations are rational and there are never speculative bubbles.

a) Briefly interpret the model.

b) To characterize the movement over time of the economy, derive two differential equations in Y and R.

c) Construct the corresponding phase diagram and illustrate the path the economy follows. Comment.

d) Determine the long-term real interest rate in steady state.

e) How does Y in steady state depend on G and τ, respectively? Considering alternative fiscal tightening policies, compare the effect (on Y in steady state) of $dG = -1$ to that of increasing τ by an amount, $d\tau$, so that net tax revenue, $\mathcal{T}$, is increased by 1, i.e., $d\mathcal{T} = 1$, assuming $\mathcal{T} = \tau + T(Y)$, where $0 < T'(Y) < 1$. Comment.

Suppose that the economy has been in its steady state until time t_0.

f) At time t_0 an unanticipated upward shift in τ occurs, but apart from this shift, everybody expects τ to remain unchanged forever. Illustrate by a phase diagram and by graphical time profiles what happens to Y_t, M_t, r_t, and R_t for $t \geq t_0$. Comment.

g) Assume instead that at time t_0, due to foreseeable problems of fiscal sustainability arising in the "aging society", the government credibly announces an upward shift in τ to take place at time $t_1 > t_0$. Illustrate by a phase diagram and by graphical time profiles what happens to Y_t, M_t, r_t, and R_t for $t \geq t_0$. Comment.

Hint: the answer to this question may be easier than one might immediately think.

IX.7 Consider a small open economy satisfying (approximately) the conditions 1-3 in Problem IX.5. Suppose the short-term behavior of the economy can be described by the following model in continuous time. Given the function $D(Y_t, r_t, x_t)$, where $0 < D_Y < 1, D_r < 0$ and $D_x > 0$, the model is:

$$
\begin{aligned}
Y_t^d &= D(Y_t, r_t, x_t), \\
\dot{Y}_t &= \lambda(Y_t^d - Y_t), \qquad\qquad \lambda > 0, \\
\frac{M}{P} &= L(Y_t, i_t), \qquad\qquad L_Y > 0,\ L_i < 0, \\
i_t &= i^* + \frac{\dot{X}_t^e}{X_t}, \\
r_t &\equiv i_t - \pi_t^e, \\
x_t &\equiv \frac{X_t P^*}{P}.
\end{aligned}
$$

The endogenous variables are: Y_t^d = output demand, Y_t = output, r_t = real interest rate, x_t = real exchange rate, i_t = nominal interest rate, X_t = nominal exchange rate, π_t^e = expected (forward-looking) rate of inflation, all at time t; the superscript e denotes expectation. The variables M, P, P^*, and i^* are exogenous and constant; their interpretation is as follows: M = money supply, P = domestic price level, P^* = foreign price level, and i^* = foreign nominal interest rate. The parameter λ is constant. The initial value, Y_0, of Y is given. Expectations are rational and speculative bubbles never occur.

a) Briefly interpret the model.

b) To characterize the movement over time of the economy, derive two differential equations in Y and X.

c) Construct the corresponding phase diagram and illustrate the path that the economy follows. Comment.

Suppose that the economy has been in steady state until time t_0.

d) At time t_0 an unanticipated tightening of monetary policy (downward shift in M) occurs. After t_0 everybody rightly expects the money supply to remain at the new lower level, M', forever. Illustrate by a phase diagram and a separate figure with time profiles what happens to Y_t, X_t, and r_t for $t \geq t_0$. Comment.

e) Assume instead that at time t_0, due to foreseeable overheating problems everybody become aware that the monetary authority will at time $t_1 > t_0$ carry into effect a shift in money supply to the level $M' < M$. Illustrate by a phase diagram and a separate figure with time profiles what happens to Y_t, X_t, and r_t for $t \geq t_0$. Comment.

f) Briefly discuss the model.

IX.8 Consider a small open economy satisfying (approximately) the conditions 1-3 in Problem IX.5. Suppose the short-term behavior of the economy can be described by the following model in continuous time:

$$\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, r_t, x_t) - Y_t), \quad \lambda > 0, 0 < D_Y < 1, D_r < 0, D_x > 0, \\
\frac{M}{P} &= L(Y_t, i_t), \qquad\qquad L_Y > 0,\ L_i < 0, \\
i_t &= i^* + \frac{\dot{X}_t^e}{X_t}, \\
r_t &\equiv i_t - \pi_t^e, \\
x_t &\equiv \frac{X_t P^*}{P}.
\end{aligned}$$

The endogenous variables are: Y_t = output, r_t = real interest rate, x_t = real exchange rate, i_t = nominal interest rate, X_t = nominal exchange rate, π_t^e = expected (forward-looking) rate of inflation, all at time t; the superscript e denotes expectation. The variables M, P, P^*, and i^* are exogenous and constant; their interpretation is as follows: M = money supply, P = domestic price level, P^* = foreign price level and i^* = foreign nominal interest rate. The parameter λ is constant. The initial value, Y_0, of Y is given. Expectations are rational and speculative bubbles never occur.

a) Briefly interpret the first three equations of the model.

b) Derive two differential equations in Y and X that characterize the movement over time of the economy.

c) Construct the corresponding phase diagram and illustrate the path that the economy follows for $t \geq 0$. Comment.

Suppose that the economy has been in steady state until time t_0.

d) At time t_0 an unanticipated upward shift in M occurs. After t_0 everybody rightly expects the money supply to remain at the new higher level, M', forever. Illustrate by a phase diagram and a separate figure with time profiles what happens to Y_t, X_t and r_t for $t \geq t_0$. Explain in detail by words the economic intuition behind what happens.

e) Assume instead that at time t_0 everybody become aware that the monetary authority will at time $t_1 > t_0$ carry into effect a shift in money supply to the level $M' > M$. Illustrate by a phase diagram and a separate figure with time profiles what happens to Y_t, X_t and r_t for $t \geq t_0$. Comment.

IX.9 In this problem we consider short-run aspects of a small open economy (SOE) satisfying:

(i) Perfect mobility across borders of financial capital, but no mobility of labor.

(ii) Domestic and foreign financial claims are perfect substitutes.

(iii) Domestic and foreign output goods are imperfect substitutes.

More specifically, it is assumed that at least outside the zero lower bound on the interest rate:

$$\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, r_t, x_t, \alpha) + G - Y_t), && (1)\\
\lambda &> 0, \quad 0 < D_Y < 1, \quad D_r < 0, \quad D_x > 0, \quad D_\alpha > 0, &&\\
\frac{M}{P} &= L(Y_t, i_t), \qquad\qquad L_Y > 0,\ L_i < 0, && (2)\\
i_t &= i^* + \frac{\dot{X}_t^e}{X_t}, && (3)\\
r_t &\equiv i_t - \pi_t^e, && (4)\\
x_t &\equiv \frac{X_t P^*}{P}. && (5)
\end{aligned}$$

Time is continuous. The endogenous variables are: $Y_t =$ output, $i_t =$ nominal interest rate, $X_t =$ nominal exchange rate, $\pi_t^e =$ expected (forward-looking) rate of inflation, all at time t; the superscript e denotes expectation. The variables α, M, P, P^*, and i^* are exogenous and constant; their interpretation is as follows: $\alpha =$ a demand shift parameter, $G =$ government spending on goods and services, $M =$ money supply, $P =$ domestic price level, $P^* =$ foreign price level, and $i^* =$ nominal world market interest rate. The parameter λ is constant. The initial value, Y_0, of Y is given. Expectations are rational. We assume speculative bubbles never occur.

a) Briefly interpret the model.

b) Derive two key differential equations and construct a phase diagram portraying the dynamics of the economy in (Y, X) space. Indicate the path that the economy follows for $t \geq 0$. Comment.

Suppose that the SOE is initially in steady state. Then, unexpectedly, a recession in the leading economies in the world comes about and gives rise to an offsetting monetary policy in these countries. As a crude representation of these events vis-a-vis our SOE we "translate" them into two unanticipated parameter shifts occurring at time $t_0 > 0$: a shift in the demand shift parameter to $\alpha' < \alpha$ and a shift in the world interest rate to $i^{*\prime} < i^*$, where $i^{*\prime}$ is close to zero.

c) Suggest an interpretation of the fall in the demand shift parameter.

d) Assume that after time t_0 the public in the SOE rightly expects that the mentioned two new parameter values will remain in force for a long time and that no policy change in the SOE will occur. Under these circumstances illustrate how the SOE evolves for $t \geq t_0$, using a phase diagram as well as a figure with time profiles of Y_t, X_t, and r_t, presupposing that the sign of the steady-state effect on X is dominated by the influence from the fall in the world interest rate. Explain the economic intuition.

e) As an alternative scenario imagine that at time $t_1 > t_0$ the monetary and fiscal authorities in the SOE find the situation unsatisfactory and contemplate monetary and fiscal measures to stimulate economic activity. It is soon realized, however, that neither conventional monetary policy (upward shift in M) nor conventional fiscal policy (upward shift in G) will be very effective. Give plausible reasons for this unfavorable outlook.

f) May a combination of conventional monetary and fiscal policy work? Why or why not?

g) If international coordination of fiscal policy is possible, can this then improve the outlook? Why or why not?

IX.10 In this problem we consider a model of a closed economy in the "very short run". Time is continuous. The assumptions are:

$$\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, R_t, \rho) + G - Y_t), & (1)\\
\lambda &> 0, \quad 0 < D_Y < 1, \quad D_R < 0, \quad D_\rho > 0, &\\
\frac{M_t}{P_t} &= L(Y_t, i_t), \qquad L_Y > 0,\ L_i < 0, & (2)\\
R_t &= \frac{1}{q_t}, & (3)\\
\frac{1 + \dot{q}_t^e}{q_t} &= r_t^e, & (4)\\
r_t^e &\equiv i_t - \pi_t^e, & (5)\\
\pi_t &= \pi, & (6)\\
i_t &= \alpha_0 + \alpha_1 Y_t, \qquad \alpha_1 > 0. & (7)
\end{aligned}$$

where a dot over a variable denotes the derivative w.r.t. time t, and the superscript e indicates expected value (until further notice a subjective expectation). Further, Y_t = output, R_t = real long-term interest rate, q_t = real price of a consol paying one unit of output per time unit forever, i_t = nominal short-term interest rate, ρ = level of "confidence", G = government spending on goods and services, M_t = money supply, P_t = output price, and $\pi_t \equiv \dot{P}_t/P_t$ = rate of inflation. The tax revenue function is implicit in the demand function $D(\cdot)$. The variables ρ, G, and π are exogenous positive constants. The initial values Y_0 and P_0 are historically given. In questions a) - e) it is assumed that exogenous variables and initial conditions are such that $i_t > 0$ for all $t \geq 0$.

a) Briefly interpret the model. Make sure you explain why the real long-term interest rate can be written as in (3).

From now on we assume perfect foresight and that speculative bubbles do not arise.

b) To characterize the movement of the economy over time, derive from the model a dynamic system in Y_t and R_t. Comment on what the role of equation (2) is in the model.

c) Draw the corresponding phase diagram, assuming that parameters are such that there exists a steady state with a nominal short-term interest rate $\bar{i} > \pi$. Illustrate the path that the economy follows for an arbitrary $Y_0 > 0$. Comment.

Now suppose that the economy is already in its steady state ("short-run equilibrium"). Let the steady state values of Y and R be denoted $\bar{Y}$ and $\bar{R}$, respectively.

d) Find an analytical expression for the effect on $\bar{Y}$ of a unit increase in G (the "spending multiplier").

Unexpectedly an adverse demand shock occurs at time $t_1 > 0$ (that is, a shift of confidence level from ρ to $\rho' < \rho$). We assume that after this shift everybody rightly expect the new level of confidence to be maintained for a long time.

e) Illustrate by a phase diagram what happens to Y_t and R_t over time, presupposing that a steady state with $R > 0$ still exists. Illustrate in another figure the time profiles of i, r, R, and Y for $t \geq t_1$. Explain the intuition in words. *Hint:* in the absence of bubbles the following formula holds:

$$R_t = \frac{1}{q_t} = \frac{1}{\int_t^{\infty} e^{-\int_t^s r_\tau d\tau} ds}.$$

f) Briefly evaluate the model.

IX.11 *Uncovered interest parity in discrete time.* Starting with discrete time (period analysis), let Japan be the domestic country and define:

$$\begin{aligned} X_t &= \text{nominal exchange rate (¥ per \$)}, \\ P_t &= \text{domestic price level}, \\ P^* &= \text{foreign price level}, \\ i_t &= \text{nominal (short-term) interest rate}, \\ i^* &= \text{foreign (short-term) nominal interest rate.} \end{aligned}$$

The uncovered interest parity (UIP) condition is

$$1 + i_t = \frac{1}{X_t}(1 + i_t^*) E_t X_{t+1}.$$

a) Interpret.

b) Show that for both i_t^* and $\frac{E_t X_{t+1} - X_t}{X_t}$ "small",

$$i_t \approx i_t^* + \frac{E_t X_{t+1} - X_t}{X_t}.$$

Hint: You may use that $\frac{E_t X_{t+1}}{X_t} = \frac{E_t X_{t+1} - X_t}{X_t} + 1$ and that the product of two small numbers is "*very* small".

IX.12 *Short questions*

a) The expectations theory of the term structure predicts that the long-term interest rate tends to be higher than the short-term interest rate." True or false? Comment.

b) We consider a Blanchard-and-Fischer-style dynamic short-run model of a small open economy with fixed prices (zero inflation), floating exchange rate, and no risk premium. It takes time for output to adjust to changes in output demand. The central bank pursues the policy $i_t = i^0 + i^1 Y_t$, where i_t is the short-term nominal interest rate, Y_t is aggregate output, i^0 and i^1 are constants and $i^1 > 0$. Suppose the economy has been in steady state until time t_0. Then the central bank unexpectedly changes i^0 to a lower constant, $i^{0\prime}$. After t_0 everybody rightly expects the monetary policy to remain $i_t = i^{0\prime} + i^1 Y_t$. "There will be no exchange rate overshooting phenomenon generated by this policy shift." True or false? Comment.

Chapter 10

Financial intermediation, business cycles

X.1 *Short questions*

In the wake of the Great Recession 2008- there has been much debate about expansionary fiscal policy and the size of the government spending multiplier. One view is that deficit-financed government spending even in a recession is not capable of expanding aggregate demand.

Economist A, sharing this view, says: "Given the government intertemporal budget constraint, the present value of future taxes will have to rise exactly as much as the current budget deficit is increased by the spending."

a) Evaluate this statement.

Economist B, says:

"For the sake of argument, let us assume that Barro's infinitely-lived family dynasties make up an acceptable approximative description of the household sector, that taxes are lump-sum, and that credit markets are perfect."

Economist A says:

"Fine. My argument is the following. If the government introduces a new program that will spend \$100 billion a year forever, then taxes must ultimately go up by the present-value equivalent of \$100 billion forever. In view of consumption smoothing it is reasonable that consumers want to reduce consumption by the same amount every year to offset this tax burden; then consumer spending will fall by \$100 billion per year to compensate, wiping out any expansionary effect of the government spending."

b) Can you make sense of this. If yes, how? If no, why?

Now Economist B says:

"That is besides the point. Expansionary fiscal policy with a view to help the economy out of recession is temporary, not permanent. Suppose the government will increase spending by \$100 billion per year for only 1 or 2 years, not forever. Then the spending program will expand aggregate demand, even if you have Barro-style family dynasties and perfect credit markets."

c) Can you make sense of this. If yes, how? If no, why?

d) State what the proposition of Ricardian Equivalence exactly asserts. In what sense is the above debate not directly a debate about this proposition?

X.2 *The IS-BL model* We consider the static IS-BL model of Bernanke and Blinder. The key symbols are here:

$$\begin{aligned}
D &= \text{ demand deposits (earn no interest)}, \\
\rho &= \text{ required reserve-deposit ratio, } \rho \in [0,1), \\
M_0 &= \text{ monetary base}, \\
E &\equiv M_0 - \rho D = \text{ excess reserves (earn no interest)}, \\
L^s &= \text{ supply of bank loans}, \\
\sigma &= \text{ shift parameter measuring perceived riskiness of supplying bank loans}, \\
B &= B_b + B_n = \text{ nominal stock of government bonds held by the private sector.}
\end{aligned}$$

The consolidated commercial banks face the constraint

$$E + L^s + B_b = (1-\rho)D = \text{ disposable deposits.} \tag{*}$$

Currency held by the non-bank public is ignored. The price level is predetermined and set equal to 1. The inflation rate is exogenous. Additional exogenous variables are: ρ, σ, M_0, and G.

The model leads to the following equilibrium conditions,

$$D = mm(i_B)M_0 = M(Y, i_B), \tag{MM}$$

where

$$mm'_{i_B} > 0, \quad M'_Y > 0, \quad M'_{i_B} < 0,$$

$$\ell(i_B, i_L, \sigma)(1-\rho)mm(i_B)M_0 = L(Y, i_B, i_L), \tag{BL}$$

where

$$\ell'_{i_B} < 0,\ \ell'_{i_L} > 0,\ \ell'_{\sigma} < 0,\ L'_Y > 0,\ L'_{i_B} > 0,\ L'_{i_L} < 0,$$

and

$$Y = Y^d(Y, i_B, i_L) + G, \tag{YY}$$

where

$$0 < Y^{d\prime}_Y < 1,\ Y^{d\prime}_{i_B} < 0,\ Y^{d\prime}_{i_L} < 0.$$

a) Briefly interpret these three equations. Why do we not have to write down an equilibrium condition for the third asset market, the bond market?

b) Equation (BL) gives i_L as an implicit function of Y, i_B, σ, and M_0 :

$$i_L = f(Y, i_B; \sigma, M_0), \tag{**}$$

with partial derivatives

$$\begin{aligned} f'_Y &> 0, \\ f'_{i_B} &> 0, \quad \text{(if } mm'_{i_B} \text{ is not “too large”, which we assume)} \\ f'_\sigma &> 0, \\ f'_{M_0} &< 0. \end{aligned}$$

Briefly, interpret these signs.

c) In (Y, i_B) space give a graphical illustration of the general equilibrium of the model. Briefly explain why the MP curve has positive slope and the IS curve negative slope.

d) Suppose M_0 is increased by an open market operation ($\Delta M_0 = -\Delta B > 0$). Illustrate in (Y, i_B) space how the MP curve and the IS curve are affected. Sign the effects on Y and i_B, respectively. Are the signs unambiguous? Why or why not? *Hint:* to answer this and the subsequent questions, curve shifting - together with a comment - is sufficient.

e) Suppose G is increased. Illustrate in (Y, i_B) space what happens. Sign the effects on Y and i_B. Explain.

f) Suppose a financial crisis is on the way and that an increased riskiness of making loans is perceived. Sign the effects on Y, i_B, and i_L. Explain.

g) Relate to your knowledge about what in 2008 happened to many countries that were affected by the financial crisis.

h) In continuation of f), if the monetary base remains unchanged, what is likely to happen to the money supply in response to a rise in σ? Why? Relate to your knowledge about what actually happened to M_1^s in the US in the early part of the Great Depression.

X.3 *Short questions*

a) In new-Keynesian theory, what does "nominal rigidities" mean and what does "real rigidities" mean (warning: a better term for the latter is "real price rigidities" and still better is "real price insensitiveness").

b) Both nominal and real rigidities are important for persistence of real effects of changes in the money supply. Give a brief intuitive explanation.

c) Briefly describe the difference between the concepts "wage curve" and "Phillips curve". Is it possible to unify them theoretically? Empirically? Comment.

X.4 *Precautionary saving* Consider a given household facing uncertainty about future labor income. For simplicity, assume the household supplies one unit of labor inelastically. The household never knows for sure whether it will be able to sell that amount of labor in the next period. Given the time horizon $T \geq 2$, the decision problem is:

$$\max E_0(U_0) = E_0[\sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t}] \quad \text{s.t.} \tag{1}$$

$$c_t \geq 0, \tag{2}$$

$$a_{t+1} = (1+r_t)a_t + w_t\ell_t - c_t, \qquad a_0 \text{ given}, \tag{3}$$

$$a_T \geq 0. \tag{4}$$

where $u' > 0$ and $u'' < 0$. Think of "period t" as the time interval $[t, t+1)$; the last period within the planning horizon T is thus period $T-1$. Real financial wealth is denoted a_t, and w_t (> 0) is the real wage, whereas ℓ_t is the exogenous amount of employment offered by the market in period t, $0 \leq \ell_t \leq 1$. The real rate of return on financial wealth is called r_t, and E_0 is the expectation operator, conditional on the information available in period 0. This information includes knowledge of all variables up to period 0, including that period. There is uncertainty about future values of r_t, w_t, and ℓ_t, but the household knows the stochastic processes that these variables follow.

a) Interpret (1) - (4).

b) Derive the Euler equation. *Hint:* consider maximization of $E_t\tilde{U}_t$ for $t = 0, 1, 2, ...$, where $\tilde{U}_t \equiv (1+\rho)^t U_t$. Comment.

c) Determine the consumption in period $T-1$, given the financial wealth a_{T-1}. Comment.

d) With a CRRA utility function, what is the sign of u'''?

From now, assume our $u(c)$ satisfies $u''' > 0$, that is, marginal utility is strictly convex ("prudence").

e) Draw a graph in (c, u') space illustrating how marginal utility of consumption depends on the consumption level c.

Suppose from now that there is no uncertainty about the future value of r_t, only about future employment and therefore labor income.

f) Consider the decision problem as seen from period 1 and assume period 2 is the last period (i.e., $T = 3$). The consumption level chosen in period 1 will determine a_2. Let there be two possible outcomes for labor income in period 2, say y_L and y_H, each with probability $\frac{1}{2}$. Write down c_2 as a function of a_2 for each of the possible labor income outcomes.

g) Let the diagram from e) represent the situation and enter the two possible values, c_L and c_H, of c_2 on the c_2 axis and indicate how the expected marginal utility, $E_1u'(c_2)$, conditional on a_2, can be found graphically.

h) To find out the effect of an increased uncertainty, consider a mean-preserving spread in y_L and y_H. Let the two new possible values of c_2, conditional on the same a_2, be called c_L^* and c_H^*, respectively and indicate their position on the c_2 axis. Further, indicate how the new expected marginal utility, $E_1u'(c_2^*)$, conditional on a_2, can be found graphically.

i) Use the Euler equation relating c_1 and c_2 to establish how the increased uncertainty affects saving in period 1.

j) How is *precautionary saving* defined? Is precautionary saving present here? Why or why not?

k) Let $u(c)$ be a quadratic utility function:

$$u(c) = \eta c - \frac{1}{2}c^2, \qquad \eta > 0, \eta \text{ "large"}.$$

Will increased uncertainty about future labor income result in precautionary saving in this case? Why or why not?

X.5 *Keynesian concepts*

a) State and compare the "technical" definitions of the phenomena "underemployment" and "involuntary unemployment".

b) On the basis of your general macroeconomic knowledge, mention some models where the phenomenon of "involuntary unemployment" can occur and some models where it cannot occur.

c) Both nominal and real price rigidities are important for persistence of real effects of changes in the money supply. Give a brief intuitive explanation.

d) Give a brief account of the "minimum rule" and the Keynesian concept of effective demand.

X.6 *Short questions about business cycles*

a) What is the conceptual difference between "white noise fluctuations" and "business cycle fluctuations"?

b) In the theory of business cycle fluctuations a lot of terms fly around. What is meant by the following terms?

Impulse. Response. Propagation. Amplification. Persistence. Comovement.

c) Say in a few words, what are the shocks and the mechanisms that drive business cycle fluctuations,

1. according to the RBC theory?
2. according to Keynesian thinking?

d) How do real wages behave over the business cycle?

e) How has, at least in the US, productivity behaved over the business cycle?

f) Briefly compare and discuss the two theories mentioned under c).

X.7 *Business cycles: facts and theories* Consider a decision problem in discrete time for a given household facing uncertainty. To begin with we assume, in accordance with classical and new classical theory, that the household never expects having to face the problem of getting less employment than desired at the going wage. As seen from period 0, the decision problem is:

$$\begin{aligned} \max E_0 U_0 &= E_0 \left(\sum_{t=0}^{T-1} (\log c_t - \gamma \frac{\sigma}{1+\sigma} n_t^{(1+\sigma)/\sigma})(1+\rho)^{-t} \right) && \text{s.t.} \quad (1) \\ c_t &\geq 0, 0 \leq n_t \leq 1, && (2) \\ a_{t+1} &= (1+r_t)a_t + w_t n_t - c_t, \qquad a_0 \text{ given}, && (3) \\ a_T &\geq 0, && (4) \end{aligned}$$

where c = consumption, n = labor supply, a = financial wealth, r = real rate of return on financial wealth, and w = real wage. The parameters γ, σ, and ρ are all positive. We assume the upper boundary, 1, to labor supply is large enough so as to be never binding, given the environment in which the household acts. The symbol E_0 (generally E_t) denotes the mathematical expectation conditional on the information available in period 0 (generally t). This information includes knowledge of all realizations of the variables up to period 0, including that period. There is uncertainty about future values of r_t and w_t, but the household knows the stochastic processes which these variables follow.

a) Derive two first-order conditions, the first of which (call it (*)) describes the trade-off between consumption and labor supply in, say, period t, and the second of which (call it (**)) describes the trade-off between consumption in period t and consumption in period $t+1$, both conditions as seen from period t ($t = 0, 1, ...$). *Hint:* consider maximization of $E_t \tilde{U}_t$ for $t = 0, 1, 2, ...$, where $\tilde{U}_t \equiv (1+\rho)^t U_t$.

b) Interpret the two first-order conditions.

Among the "stylized facts" of business cycle fluctuations (based on time series data after detrending) are the following:

(i) Employment (aggregate labor hours) is strongly procyclical and fluctuates almost as much as GDP.

(ii) Aggregate consumption and employment are markedly positively correlated.

(iii) Real wages are weakly procyclical and do not fluctuate much.

c) Are these facts supportive or the opposite for the RBC theory in the light of the condition (*)? Discuss. *Hint:* it will prove convenient to rewrite (*) such that w_t is isolated on one side of the equation; ignoring the finite horizon, the decision problem above can be seen as that of the representative agent in an RBC model.

d) In order to simplify the discussion, suppose for a moment there is no uncertainty. Then find n_t/n_{t+1} as a function of w_t/w_{t+1}. From this expression, give an interpretation of the parameter σ. Relate this to the discussion under c), taking into account the empirical evidence concerning the elasticity of intertemporal substitution in labor supply ($0 < \sigma < 1.5$).

e) Within the market-clearing framework of the RBC approach, if fluctuations in the real wage are almost negligible, is it then likely that fluctuations in r_{t+1} could be the driving force behind fluctuations in employment? Relate your answer to your result under d), the condition (*), and the stylized facts above. *Hint:* given that fluctuations in the real wage are almost negligible, we can on the basis of (*) sign the expected correlation between consumption and employment and compare with fact (ii).

We now reintroduce the existence of uncertainty and reconsider the household's decision problem under the Keynesian hypothesis that the household may have to face rationing in the labor market and uncertainty concerning the prospect of employment in the future. That is, for $t = 0, 1, 2, ...$, we replace (2) by the constraint $c_t \geq 0$, $0 \leq n_t \leq \min(\bar{n}_t, 1)$, where $\bar{n}_t \geq 0$ is the exogenous maximum employment offered to the household in period t. The current value of $\bar{n}_t$ is known by the household, but not the future values.

f) Show that when $\bar{n}_t$ is binding, the equality sign in (*) is replaced by a weak inequality sign. Write down the new (*) and interpret.

g) Is it possible within this framework to reconcile theory with the stylized facts? Why or why not?

h) Suppose an adverse demand shock occurs so that $\bar{n}_t$ several periods ahead is expected to be binding and lower than otherwise. What is the likely effect on current consumption, c_t, of the household?

X.8 *Short questions*

a) Define what is meant by a *Beveridge curve*.

In the wake of the full-blown financial and economic crisis in late 2008 a large fall in employment occurred in many countries, not least in the U.S. Two different stories could in principle explain this sharp fall in employment. One is a "Schumpeterian story" emphasizing technological and structural change. The other is a "Keynesian story".

b) During the recession a believer of the Schumpeterian story would expect "total separations", "quits", and "hiring" to rise, while a believer of the Keynesian story would expect "layoffs and discharges" to rise and "hiring" and "quits" to fall (the terms in quotation marks are the terms used by the Bureau of Labor Statistics in the U.S.). Briefly explain why the two types of believers would expect so.

c) What does the data on labor market flows in the U.S. published by the Bureau of Labor Statistics tell us in relation to these two types of explanations?

X.9 We consider a small open economy (SOE) satisfying the following conditions:

1. Perfect mobility across borders of financial capital, but no mobility of labor.
2. Domestic and foreign bonds are perfect substitutes and command the same expected rate of return.
3. Domestic and foreign output goods are imperfect substitutes.
4. Nominal prices are sluggish and follow an exogenous constant inflation path.

Aggregate output demand is

$$\begin{aligned}
Y_t^d &= C(Y_t^p, R_t) + I(Y_t, R_t) + N(Y_t, x_t) + G \equiv D(Y_t, R_t, x_t, \tau) + G, \qquad (*)\\
&\quad \text{where } Y_t^p \equiv Y_t - \mathbb{T} \text{ and } \mathbb{T} = \tau + T(Y),\ 0 < T' < 1,\\
0 &< C_{Y^p}(1-T') + N_Y < C_{Y^p}(1-T') < C_{Y^p}(1-T') + I_Y + N_Y \equiv D_Y < 1,\\
0 &< D_Y < 1, C_R + I_R \equiv D_R < 0, D_x > 0, -1 < D_\tau = -C_{Y^p} < 0.
\end{aligned}$$

Notation: Y_t^d is output demand, Y_t^p is after-tax income, Y_t is output, R_t is the long-term real interest rate, and x_t is the real exchange rate, XP_t^*/P_t, where X is a given and constant nominal exchange rate, and P_t is the domestic price level while P_t^* is the foreign price level; τ measures "fiscal tightness", G is government spending on goods and services, $\mathbb{T}$ is net tax revenue, and $T(Y)$ is a tax function.

We assume that the domestic (forward-looking) inflation rate, π, is constant and equals the foreign (forward-looking) inflation rate, π^*. Hence, P_t^*/P_t is a constant and so is the real exchange rate, from now denoted x.

The dynamics of the economy is described by the following equations:

$$\dot{Y}_t = \lambda(D(Y_t, R_t, x, \tau) + G - Y_t), \qquad \lambda > 0, \quad Y_0 > 0 \text{ given}, \tag{1}$$

$$\frac{M_t}{P_t} = L(Y_t, i^*), \qquad L_Y > 0,\ L_i < 0. \tag{2}$$

$$R_t = \frac{1}{q_t}, \tag{3}$$

$$\frac{1 + \dot{q}_t^e}{q_t} = r_t^e, \tag{4}$$

$$r_t^e \equiv i^* - \pi_t^e, \qquad \pi_t \equiv \frac{\dot{P}_t}{P_t}, \tag{5}$$

$$P_t = P_0 e^{\pi t}, \tag{6}$$

where i_t is the domestic short-term nominal interest rate, i^* the foreign short-term nominal interest rate, r_t the domestic short-term real interest rate, M_t the money supply, and q_t the real price of a long-term bond (a consol). The superscript e indicates an subjective expectation.

The variables x, τ, G, i^*, and π are exogenous positive constants, $\pi < i^*$. The initial values, Y_0 and P_0, are given. Expectations are rational and there are never speculative bubbles. The parameters are such that the speed of adjustment towards steady state is high.

a) Briefly interpret (2), (3), and (4) of the model.

b) To characterize the movement over time of the economy, derive two differential equations in Y and R, respectively.

c) Construct the corresponding phase diagram and illustrate the path the economy follows for an arbitrary $Y_0 > 0$. Comment.

d) Let the steady-state values of the long-term real interest rate and output be denoted $\bar{R}$ and $\bar{Y}$, respectively. Find these values. Finally, derive the spending and tax multipliers, $\partial\bar{Y}/\partial G$ and $\partial\bar{Y}/\partial\tau$.

Suppose that the economy has been in its steady state until time t_0.

e) Suppose the government is dissatisfied with the level of employment and at time $t_0 > 0$ decides (unexpectedly) to increase G to a higher level, G'. Suppose further that, owing to the automatic budget reaction under high unemployment, people rightly expect this higher level of spending to be maintained for quite some time without a rise in τ. Under the simplifying assumption that the new spending level is permanent, illustrate by a new phase diagram what happens for $t \geq t_0$.

f) Assume instead that at time t_0, the government credibly announces an upward shift in the level of government spending from G to G' to take place at time $t_1 > t_0$. Illustrate by a phase diagram and by graphical time profiles what happens to R_t, r_t, Y_t, and $m_t \equiv M_t/P_t$ for $t \geq t_0$. Comment. *Hint:* the answer to this question may be easier than one might immediately think.

g) Briefly evaluate the model.

X.10 Consider the following dynamic model in continuous time for a closed economy:

$$\begin{aligned}
\dot{Y}_t &= \lambda(D(Y_t, R_t, \alpha) + G - Y_t), \quad \lambda > 0, 0 < D_Y < 1, D_R < 0, D_\alpha < 0, \\
\frac{M_t}{P} &= L(Y_t, i_t), \qquad L_Y > 0,\ L_i < 0. \\
R_t &= \frac{1}{q_t}, \\
\frac{1 + \dot{q}_t^e}{q_t} &= r_t, \\
r_t &= i_t - \pi_t^e,
\end{aligned}$$

where the superscript e denotes subjective expectation. Further, Y_t = output, R_t = real interest rate on a consol, α = the general "state of confidence",

G = government spending on goods and services, M_t = money supply, P = output price, i_t = nominal short-term interest rate, π_t = rate of inflation, q_t = real price of a consol. The variables G, α, and P are exogenous positive constants. The initial value of Y, Y_0, is pre-determined.

As to monetary policy we assume that the monetary authority is capable of continuously adjusting the monetary base so that the short-term nominal interest rate remains at the target level. Until further notice suppose this level equals a positive constant, i.

a) Briefly interpret the equations of the model and the parameter λ.

Suppose expectations are rational and that speculative bubbles never arise.

b) To characterize the movement over time of the economy, derive from the model a dynamic system in Y and R. Draw the corresponding phase diagram and illustrate the path that the economy follows. Comment.

c) Determine the short-term and long-term real interest rates in steady state and find an implicit solution for Y in steady state. How does Y in steady state depend on G and i, respectively?

Suppose that the economy has been in steady state until time t_0.

d) Then an unanticipated shift in the state of confidence to $\alpha' < \alpha$ occurs. After this shift everybody rightly expects the state of confidence to remain unchanged for a long time. Suppose further that the market participants rightly expect no change in monetary and fiscal policy. Illustrate by a phase diagram and by graphical time profiles the evolution of Y_t, R_t, and r_t. Comment.

e) Assume instead that immediately after time t_0, the monetary authority responds to the confidence shock by lowering the target interest rate to a new constant $i' < i$. After this shift everybody rightly expects both the state of confidence and the targeted interest rate to remain unchanged for a long time. Illustrate by a phase diagram and by graphical time profiles the evolution of Y_t, R_t, and r_t. Comment.

f) Suppose that i' is very close to zero and that the monetary and fiscal authorities find the evolving level of economic activity (output and employment) unsatisfactory. What can be done to effectively stimulate the economic activity? Briefly, compare some alternatives you might think of.

X.11 *Short questions*

Among the “stylized facts” concerning business cycle fluctuations (based on time series data after de-trending) are the following:

(i) Employment (aggregate labor hours) is procyclical and fluctuates almost as much as GDP.

(ii) Aggregate consumption and employment are markedly positively correlated.

(iii) Real wages are weakly procyclical and do not fluctuate much.

Briefly discuss how these “stylized facts” relate to alternative business cycle theories.

Appendix A. Solutions to linear differential equations

For a general differential equation of first order, $\dot{x}(t) = \varphi(x(t), t)$, with $x(t_0) = x_{t_0}$ and where φ is a continuous function, we have, at least for t in an interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$,

$$x(t) = x_{t_0} + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau. \tag{*}$$

To get a confirmation, calculate $\dot{x}(t)$ from (*).

For the special case of a linear differential equation of first order, $\dot{x}(t) + a(t)x(t) = b(t)$, we can specify the solution. Three sub-cases of rising complexity are:

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = (x_{t_0} - x^*)e^{-a(t-t_0)} + x^*, \text{ where } x^* = \frac{b}{a}.$$

If $a = 0$, we get, directly from (*), the solution $x(t) = x_{t_0} + bt$.[1]

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)} \int_{t_0}^{t} b(s)e^{a(s-t_0)} ds.$$

[1] Some non-linear differential equations can be transformed into this simple case. For simplicity let $t_0 = 0$. Consider the equation $\dot{y}(t) = \alpha y(t)^{\beta}$, $y_0 > 0$ given, $\alpha \neq 0, \beta \neq 1$ (a Bernoulli equation). To find the solution for $y(t)$, let $x(t) \equiv y(t)^{1-\beta}$. Then, $\dot{x}(t) = (1-\beta)y(t)^{-\beta}\dot{y}(t) = (1-\beta)y(t)^{-\beta}\alpha y(t)^{\beta} = (1-\beta)\alpha$. The solution for this is $x(t) = x_0 + (1-\beta)\alpha t$, where $x_0 = y_0^{1-\beta}$. Thereby the solution for $y(t)$ is $y(t) = x(t)^{1/(1-\beta)} = \left(y_0^{1-\beta} + (1-\beta)\alpha t\right)^{1/(1-\beta)}$, which is defined for $t > -y_0^{1-\beta}/((1-\beta)\alpha$.

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$\begin{aligned} x(t) &= x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)}c\int_{t_0}^{t} e^{(a+h)(s-t_0)}ds \\ &= (x_{t_0} - \frac{c}{a+h})e^{-a(t-t_0)} + \frac{c}{a+h}e^{h(t-t_0)}. \end{aligned}$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau} + e^{-\int_{t_0}^{t} a(\tau)d\tau}\int_{t_0}^{t} b(s)e^{\int_{t_0}^{s} a(\tau)d\tau}ds.$$

Special case: $b(t) = 0$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau}.$$

Even more special case: $b(t) = 0$ and $a(t) = a$, a constant. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)}.$$

Remark 1 For $t_0 = 0$, most of the formulas will look simpler.

Remark 2 To check whether a suggested solution *is* a solution, calculate the time derivative of the suggested solution and add an arbitrary constant. By appropriate adjustment of the constant, the final result should be a replication of the original differential equation together with its initial condition.