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A suggested solution to the problem set at the re-exam in Advanced Macroeconomics February 15, 2016

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggregate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

The model is:

$$\dot{C}_t = (F_K(K_t, L) - \delta - \rho)C_t - m(\rho + m)(K_t + B_t), \tag{1}$$

$$\dot{K}_t = F(K_t, L) - \delta K_t - C_t - G, \qquad K_0 > 0 \text{ given}, \tag{2}$$

$$\dot{B}_t = [F_K(K_t, L) - \delta]\, B_t + G - T_t, \qquad B_0 > 0 \text{ given}, \tag{3}$$

the condition

$$\lim_{t\to\infty} B_t e^{-\int_0^t [F_K(K_s,L)-\delta]ds} \leq 0, \tag{4}$$

and a requirement that households satisfy their transversality conditions. Here, C_t is aggregate private consumption, K_t is physical capital, L is population = labor supply, B_t is public debt, G is government spending on goods and services, T_t is net tax revenue (= gross tax revenue−transfer payments), and F is an aggregate neoclassical production function with constant returns to scale and satisfying the Inada conditions. The other symbols stand for parameters and all these are positive; L and G are positive constants. A dot over a variable denotes the derivative w.r.t. time t.

[1] The solution below contains more details and more precision than can be expected at a three hours exam.

a) Parameters: δ = capital depreciation rate, m = mortality rate (= birth rate, no population growth), ρ = pure rate of time preference (utility discount rate, a measure of impatience). The model relies on the simplifying assumption that for a given individual the probability of having a remaining lifetime, X, longer than some arbitrary number x is $P(X > x) = e^{-mx}$, the same for all (i.e., independent of age).

Eq. (1) shows how the increase per time unit in aggregate private consumption is determined. The first term on the right-hand side reflects the individual Keynes-Ramsey rule at time t (instantaneous utility is assumed to be logarithmic). In general equilibrium with perfect competition, $r_t = F_K(K_t, N) - \delta$. The second term on the right-hand side reflects the generation replacement. The arrival of newborns is Nm per time unit. The newborns enter the economy with *less financial* wealth than the "average citizen". This lowers aggregate consumption by $m(\rho+m)A_t$ per time unit, where A_t is aggregate private financial wealth. In general equilibrium in the closed economy we have $A_t = K_t + B_t$.

Eq. (2) is essentially just national income accounting for a closed economy with public sector. There is no population growth and no technology growth.

Eq. (3) says that the increase per time unit in real public debt equals the real budget deficit, that is, total government expenditure (interest payments plus spending on goods and services) minus net tax revenue. This tells us that the budget deficit is entirely debt-financed (i.e., no money financing).

Finally, the condition (4) is the No-Ponzi-Game condition for the government in general equilibrium (recall $r_s = F_K(K_s, N) - \delta$).

b) Given a balanced budget for all $t \geq 0$, we have $\dot{B}_t = 0$ in (3) so that B_t in (1) and (2) is a constant, B_0. Then these two differential equations constitute a self-contained dynamic system for which we can draw a phase diagram. We introduce two benchmark values of K, namely the golden rule value, K_{GR}, and a "critical" value, $\bar{K}$. These are defined by,

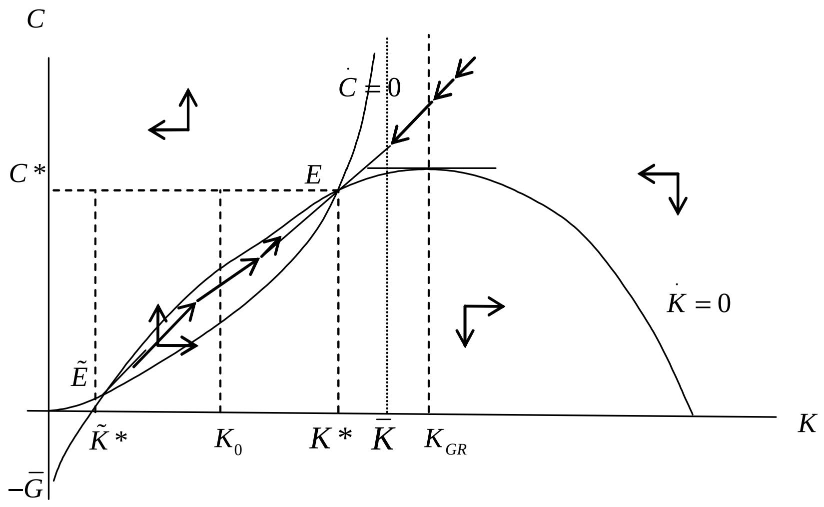
$$F_K\left(K_{GR}, N\right) - \delta = 0, \qquad \text{and} \qquad F_K\left(\bar{K}, N\right) - \delta = \rho > 0, \tag{1.1}$$

respectively. In view of the Inada conditions and $\delta > 0$, both values exist and are unique (since $F_{KK} < 0$). We have $\bar{K} < K_{GR}$, since $\rho > 0$ and $F_{KK} < 0$.

Given $B_t = B_{0,}$ equation (2) shows that $\dot{K} = 0$ for

$$C = F(K, N) - \delta K - G,$$

cf. the strictly concave $\dot{K} = 0$ locus in Fig. 1.1.



Equation (1) shows that $\dot{C} = 0$ for

$$C = \frac{m(\rho + m)(K + B_0)}{F_K(K, N) - \delta - \rho}. \tag{1.2}$$

Thus, along the $\dot{C} = 0$ locus,

$$K \nearrow \bar{K} \Rightarrow C \to \infty$$

and

$$K \searrow 0 \Rightarrow C \to 0,$$

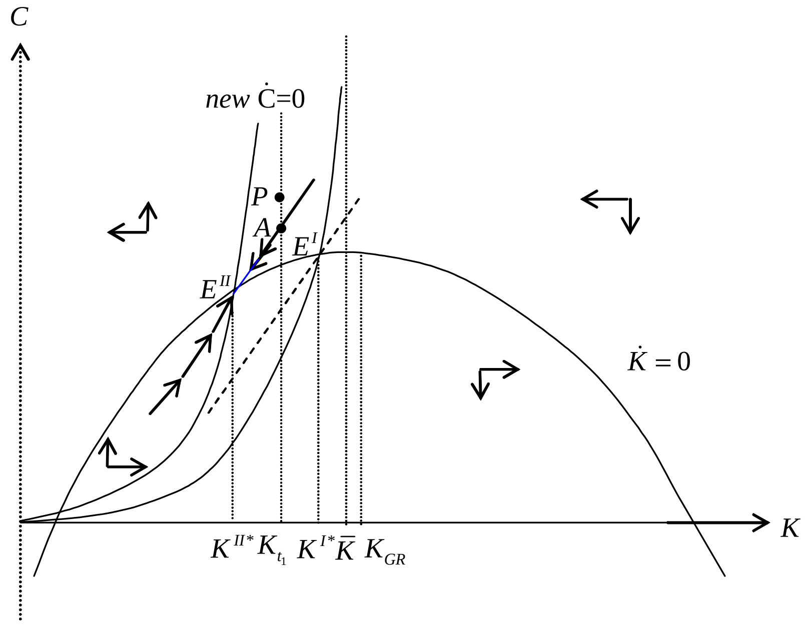
the latter result following from the lower Inada condition ($\lim_{K\to 0} F_K(K, N) = \infty$). The $\dot{C} = 0$ locus is shown as the strictly convex curve in Fig. 1.1.

We are told that G and B_0 are “modest” relative to the production possibilities of the economy for the given K_0. This means that the $\dot{C} = 0$ curve crosses the $\dot{K} = 0$ curve for *two* positive values of K. Fig. 1.1 shows these steady states as the points E and Ẽ with coordinates (K^*, C^*) and $(\tilde{K}^*, \tilde{C}^*)$, respectively. Obviously, $\tilde{K}^* < K^* < \bar{K}$.

The direction of movement in the different regions of Fig. 1.1 are determined by the differential equations, (1) and (2), and shown by arrows. It is seen that E is a saddle point, whereas Ẽ is a totally unstable steady state. Since G and B_0 are “modest", we have that the lower steady-state value of K, $\tilde{K}^*$, is smaller than K_0, as shown in the figure.

The capital stock is predetermined, whereas consumption is a jump variable. The slope of the saddle path is not parallel with the C axis. The divergent paths can be ruled out as equilibrium paths since they violate either the transversality conditions of the households (paths that in the long run point South-East in Fig. 1.1) or the NPG condition[2] of the

[2] And therefore also the transversality condition.



households (paths that in the long run point North-West in the diagram). It follows that the system is saddle-point stable. The saddle path is the only trajectory satisfying *all* the conditions of general equilibrium (individual utility maximization for given expectations, profit maximizing firms, continuous market clearing, and fulfilled expectations). Hence, initial consumption, C_0, is determined as the ordinate to the point where the vertical line $K = K_0$ crosses the saddle path, and over time the economy moves along the saddle path, approaching the steady state point E with coordinates (K^*, C^*).

c) Let $B_0 = B_0^I < B_0^{II}$ and $K_0 = K_0^I = K_0^{II}$. Based on (1.2), Fig. 1.2 illustrates. In the long run Country II has less capital and a lower consumption level, due to the crowding-out effect of government debt in a full-employment economy. (So far, ignore the vertical line at $K = K_{t_1}$ and the point A.)

d) A given fiscal policy is called *sustainable* if by applying its spending and tax rules forever, the government stays solvent. An operational criterion for sustainability is whether the fiscal policy can be deemed compatible with upward boundedness of the public debt-to-income ratio.

From now on we just consider a single country, Country I. We are told that from t_0,

$$T_t = \bar{T} < T^*, \tag{1.3}$$

where T^* is the tax revenue in the old steady state.

As we shall see, the new fiscal policy, $(G, \bar{T})$, is *not* sustainable. In the explanation of this, the phase diagram in Figure 1 is of no use because it is no longer valid. Indeed, after time t_0, *three* differential equations, determining changes in C, K, and B, are active. Moreover, while (2) and (3) still hold, (1) need not. This is because of the uncertainty about *when* and *how* a fiscal tightening will take place. Hence, for the time being we can not use the phase diagram.

Nevertheless, we have sufficient information to settle the question about fiscal sustainability. First, as a result of the tax cut at time t_0, a budget deficit arises, hence $\dot{B}_t > 0$ at least for a while. Moreover, the tax cut makes current generations feel wealthier, hence they increase their consumption. They do so in spite of being forward-looking and anticipating that the current fiscal policy sooner or later must come to an end (because it is not sustainable, as we have claimed and shall see in a moment). The prospect of higher taxes in the future dampens the increase in consumption, but does not prevent it, since part of the future taxes will fall on new generations entering the economy.

The rise in C, combined with unchanged $\bar{G}$, implies negative net investment so that K begins to fall. Hence, for $t > t_0$,

$$K_t < K^* < \bar{K}. \tag{1.4}$$

So r remains positive, and at least for a while rises due to the negative net investment in capital. From this follows that for $t > t_0$,

$$r_t = F_K\left(K_t, N\right) - \delta > F_K\left(K^*, N\right) - \delta > F_K\left(\bar{K}, N\right) - \delta = \rho > 0. \tag{1.5}$$

On this basis our syllabus describes three different approaches for showing that the fiscal policy $(G, \bar{T})$ is not sustainable.

Approach 1: Sustained rise in the debt-income ratio. The negative net investment continues. And along with the falling K, we have a falling aggregate income, $Y_t = F\left(K_t, N\right)$. So we are in a situation where the interest rate remains larger than the long-run output growth rate which in the absence of growth in technology or labor force is clearly non-positive. The falling K implies falling Y.

The combination of a rising B and falling Y implies a forever rising debt-income ratio, B/Y. The private sector will understand that bankruptcy is threatening and nobody will buy government bonds except at a low price, which means a higher interest rate. The high interest rate only aggravates the problem. That is, the fiscal policy $(G, \bar{T})$ breaks down.

This reasoning, based on the evolution of the debt-income ratio, is probably the easiest. Two other approaches are described in our syllabus. Both build on the observations that after t_1, (1.4) and (1.5) hold. Together, these observations imply an interest rate forever larger than the long-run output growth rate which (in the assumed case of no growth in technology and labor force) is zero or even negative. In this situation, to be sustainable, fiscal policy has to satisfy the NPG condition.

Approach 2 shows that the current fiscal policy violates NPG.

Approach 3 shows that the current fiscal policy violates the government's intertemporal budget constraint (GIBC). The hint alludes to this approach. Starting from the intertemporal government budget constraint we check whether the primary budget surplus, $\bar{T} - G$, which rules after time t_0, satisfies

$$\int_{t_0}^{\infty} (\bar{T} - G) e^{-\int_{t_0}^{t} r_s ds} dt \geq B_{t_0}, \tag{1.6}$$

where $B_{t_0} = B_0 > 0$. Obviously, if $\bar{T} - G \leq 0$, (1.6) is not satisfied. Suppose $\bar{T} - G > 0$. Then

$$\int_{t_0}^{\infty} (\bar{T} - G) e^{-\int_{t_0}^{t} r_s ds} dt < \int_{t_0}^{\infty} (\bar{T} - G) e^{-r^*(t-t_0)} dt = \frac{\bar{T} - G}{r^*} < B_0 = B_{t_0},$$

where the first inequality comes from $r_s > r^*$ for $s > t_0$ (cf. Approach 1), the first equality from carrying out the integration $\int_{t_0}^{\infty} e^{-r^*(t-t_0)} dt$, and, finally, the second inequality from the facts that $r^* B_0 + \bar{G} - T^* = 0$ (from the old steady state) and $\bar{T} < T^*$. So the intertemporal government budget constraint is not satisfied from which follows that the current fiscal policy is not sustainable.

e) We use Figure 1.2 as our the phase diagram for $t > t_1$. Now K^{II*} should not be interpreted as referring to Country II, but to Country I after time t_t in the sense of representing the long-run value of Country I's capital after time t_1. The debt level from time t_1 and onwards, B_{t_1}, exceeds B_0. Hence, the $\dot{C} = 0$ locus has turned counterclockwise, cf. (1.2). So the new steady-state value of K, here denoted K^{II*}, must be smaller than the original one, K^*.

Immediately after t_1, we must have $K_{t_1} < K^*$, as indicated in the figure. This follows from the negative net investment in the time interval (t_0, t_1). As the figure is drawn, K_{t_1} is larger than the new steady-state level, K^{II*}. The equilibrium conditions in the model imply that consumption at time t_1, when people become aware of the fiscal tightening and also immediately feel it, must jump to a level equal to the ordinate of the point of intersection between the vertical line $K = K_{t_1}$ and the new saddle path, i.e., the point A in Fig. 1-2. Right before t_1 the economy was at some point like P in Fig. 1-2.

The dynamics of the economy after t_1 implies gradual movement along the new saddle path towards the new steady state, E^{II}.

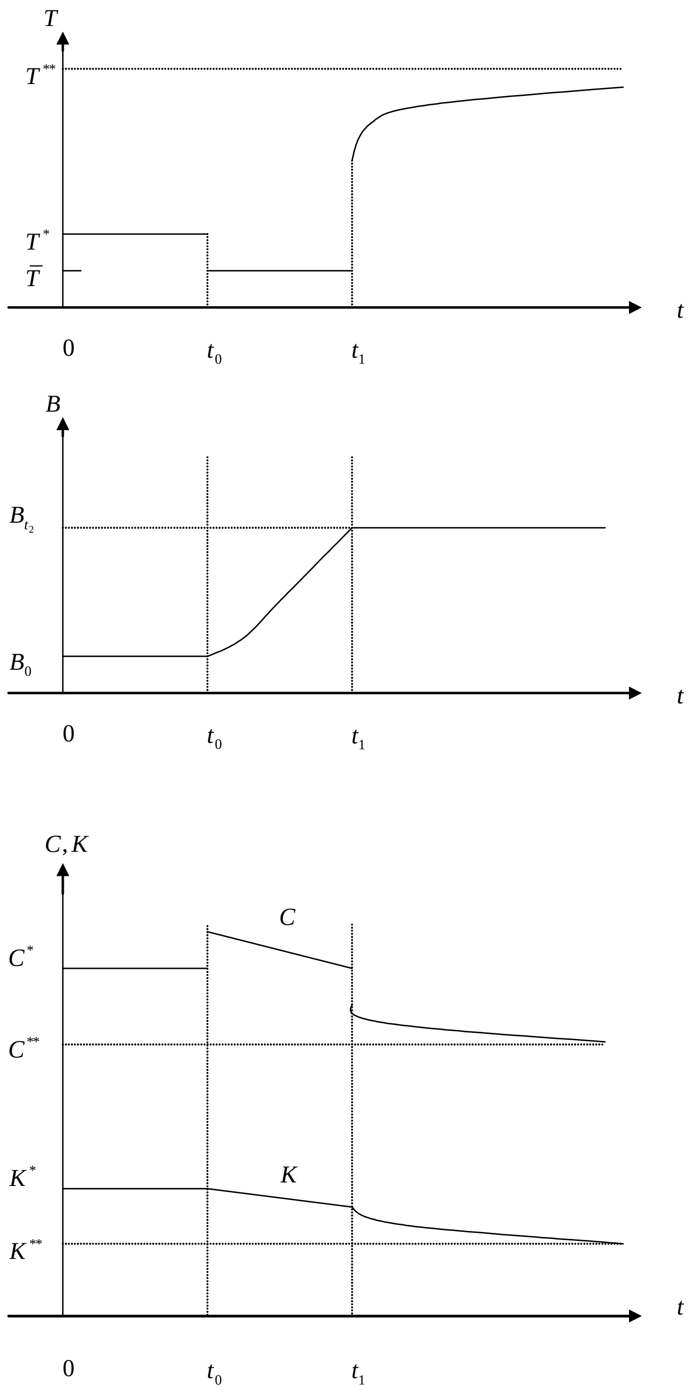
Alternatively, it seems possible that K_{t_1} could be smaller than K^{II*} so that the new initial point, A, is to the left of the new steady state E^{II}. This case is not illustrated in Figure 2. Which of the two cases will come about is difficult to tell. It is a question of an intricate balance between the size of the tax cut and its duration.

f) *Figure 1.3 shows the case* $K_{t_1} > K^{II*}$. The two upper panels show the time profiles of T and B, respectively. The first visualizes that the increase in taxation at time t_1 is larger than the decrease at time t_0. This is due to public expenses being larger after t_1 because both the government debt B_{t_1} and the interest rate, $F_K(K_t, N_t) - \delta$, are higher. The further gradual rise in T_t towards its new steady-state level is due to the rising interest service along with a rising interest rate, caused by the falling K.

The middle panel of Fig. 1.3 is self-explanatory.

The lower panel of Fig. 1.3 visualizes that the tax cut at time t_0 results in an upward jump in consumption. This implies negative net investment, so that K begins to fall. The size of the upward jump in consumption at time t_0 and the subsequent time path of consumption in the time interval $[t_0, t_1)$ can not be precisely pinned down. We can not even be sure that C will be gradually falling within this time interval. Therefore the downward-sloping time path of C in the lower panel of Fig. 1.3 in this time interval illustrates just one of the possibilities. At time t_1, the uncertain moment of the expected fiscal tightening becomes a matter of fact, and consumption drops to a lower level due to the lower level of after-tax human wealth.

Figure 1.4 shows the case $K_{t_1} < K^{II*}$. In this case the tax revenue after t_1 has to exceed what is required in the new steady state. During the subsequent adjustment, the taxation level will be gradually falling which reflects the gradual fall in the interest rate generated by the rising K, cf. Fig. 1.3. Private consumption will at time t_1 jump to a level *below* the new (in itself lower) steady-state level, $C^{*\prime}$.



T
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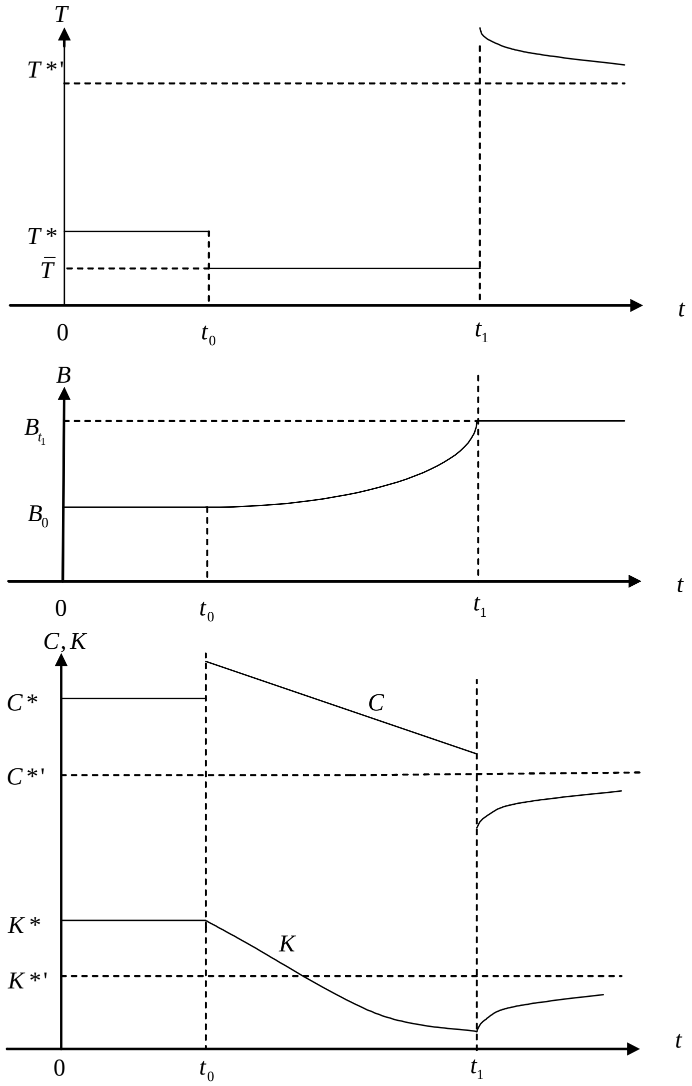
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Solution to
Problem 2 (15/2-2016) (in Danish)

$$\max V_0 = \int_0^\infty [F(K_t, L_t) - G(I_t, K_t) - w_t L_t - p I_t] e^{-rt} dt$$

ub. (1)

$L_t \geq 0$, I_t „fri" (2)

$\dot{K}_t = I_t - \delta K_t$, $K_0 > 0$ givne, (3)

$K_t \geq 0 \quad \forall t$ (4)

$G(0,K)=0$, $G_I(0,K)=0$, $G_{II}(I,K)>0$ og $G_K(I,K) \leq 0$

a) $H = F(K,L) - G(I,K) - wL - pI + q(I - \delta K)$

$\frac{\partial H}{\partial L} = F_L(K,L) - w = 0 \Rightarrow F_L(K,L) = w$ (5)

$\frac{\partial H}{\partial I} = -G_I(I,K) - p + q = 0 \Rightarrow p + G_I(I,K) = q$ (6)

$\frac{\partial H}{\partial K} = F_K(K,L) - G_K(I,K) - q\delta = -\dot{q} + rq$ (7)

$\lim_{t \to \infty} K_t q_t e^{-rt} = 0$ (TVC)

b) (6) kan fortolkes som $MC = MB$.

q_t kan derfor fortolkes som skyggeprisen på installeret kapital på tidspunkt t (dvs. q_t er værdien for virksomheden af den marginale enhed installeret K).

b) forts. $(2) \Rightarrow p + G_I(I,k) = q$

$\Rightarrow G_I(I,k) = q - p \underset{G_{II} \neq 0}{\Rightarrow} I = \tilde{M}(q,p,k) \equiv M(q-p,k)$ (9)

$M(0,k) = 0$

$G_{II}(I,k) M_{q-p}(q-p,k) = 1 \Rightarrow M_{q-p}(q-p,k) = \frac{1}{G_{II}} > 0$

c) $G(I,k) = \frac{1}{2} I^2/k \Rightarrow G_I = I/k$

$G_k = \frac{1}{2} I^2 (-k^{-2}) = -\frac{1}{2}\left(\frac{I}{k}\right)^2$

$(2) \Rightarrow p + \frac{I}{k} = q$

Altså $\underline{\frac{I_t}{k_t} = q_t - p}$ (10)

d) $\underline{w_t = F_L(K_t, \bar{L})}$

e) (3) og (10) $\Rightarrow$

$\dot{k} = I - \delta k = (q-p)k - \delta k = (q - p - \delta)k$ (*)

$\dot{k} \gtreqless 0 \Rightarrow q \gtreqless p + \delta \equiv q^*$, jf. de horisontale pile i Figur 2.1

(4) $\Rightarrow$

$\dot{q} = (r+\delta)q - F_k(k, \bar{L}) - \frac{1}{2}(q-p)^2$ (**)

f) $\dot{q} = 0 \Rightarrow F_k(k, \bar{L}) = (r+\delta)q - \frac{1}{2}(q-p)^2$

$\dot{q} = 0$ og $\dot{k} = 0 \Rightarrow \overbrace{F_k(k^*, \bar{L})}^{k = k^*, \text{ hvor}} = (r+\delta)(p+\delta) - \frac{1}{2}\delta^2$ (***)

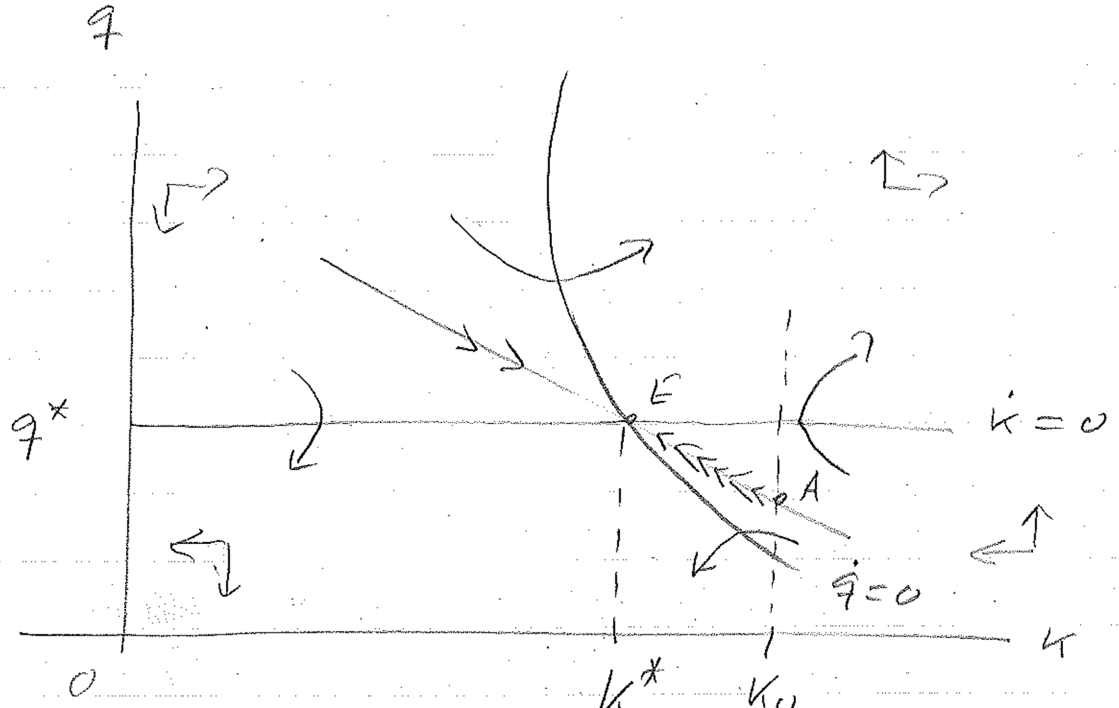
$= (r+\delta)p + r\delta + \delta^2 - \frac{1}{2}\delta^2$

$= (r+\delta)p + r\delta + \frac{1}{2}\delta^2 > 0 \Rightarrow k^* > 0$ entydig.

Check vedr. hældn. på $\dot{q} = 0$ kurven:

$F_{kk} \cdot dk = (r+\delta)dq - (q-p)dq = (r + \delta - q + p)dq$

$\Rightarrow \left.\frac{dq}{dk}\right|_{\dot{q}=0} = \frac{F_{kk}}{r + \delta - q + p}$, hvor $\frac{F_{kk}}{r + \delta - q^* + p} = \frac{F_{kk}}{r} < 0$



Figur 2.1

Af (**) ses, at for fast q, men øget K, bliver $\dot{q}$ større, idet $F_K(K, \bar{I})$ bliver mindre, da $F_{KK} < 0$.

Til højre for $\dot{q} = 0$-kurven er altså $\dot{q} > 0$ og til venstre er $\dot{q} < 0$.

Heraf følger de vertikale pile i Figur 2.1.

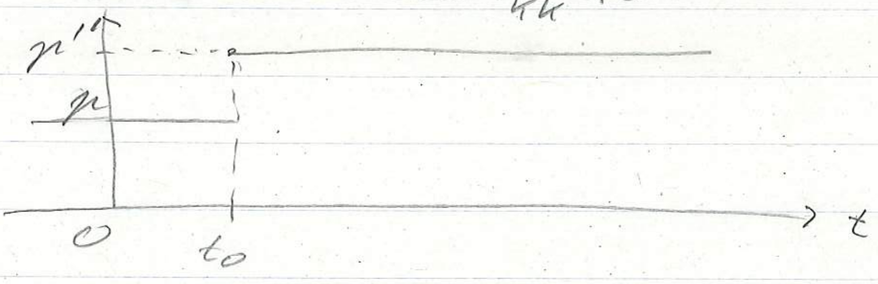
Den entydige st.st. E ses at være et saddelpunkt.

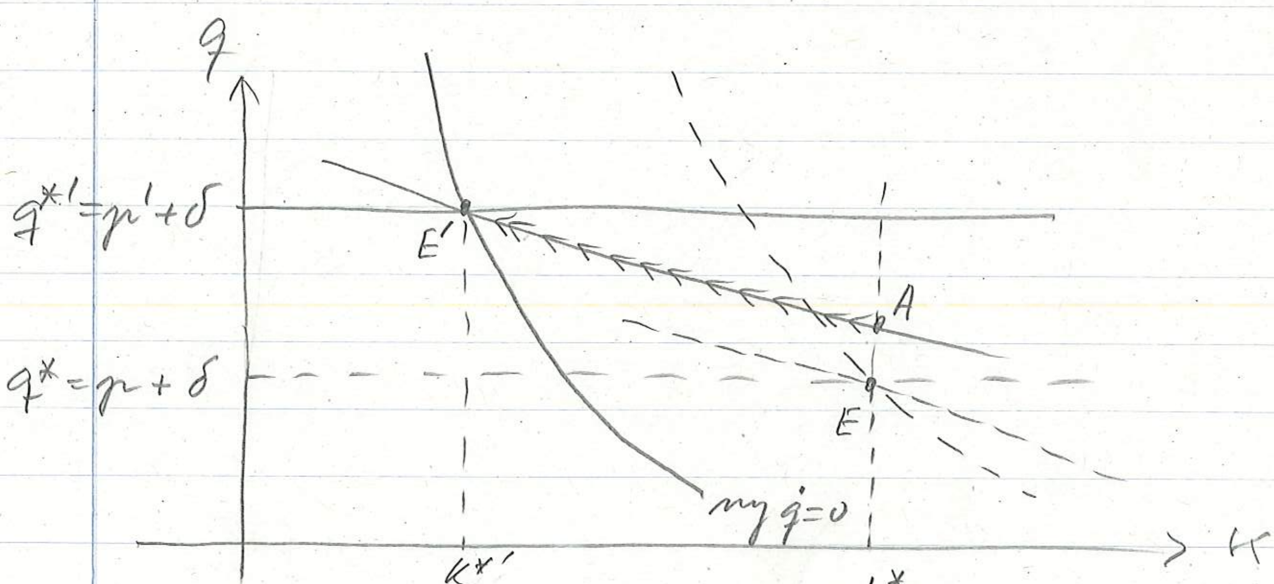
g) $q^* = \rho + \delta$

h) Lad $K_0 > K^*$ som på Figur 2.1. Da de divergerende baner ikke opfylder (TVC), kan de udelukkes. På tidspunkt 0 må økonomien være på saddelbanen og dermed i punktet A på Figur 2.1

q_c er altså lav og bruttoinvesteringerne derfor så lave, at der er negative nettoinvesteringer. Herved vil K være aftagende og økonomien vil bevæge sig mod st.st. i E. At q er stigende i denne proces skyldes, at F_K er voksende.

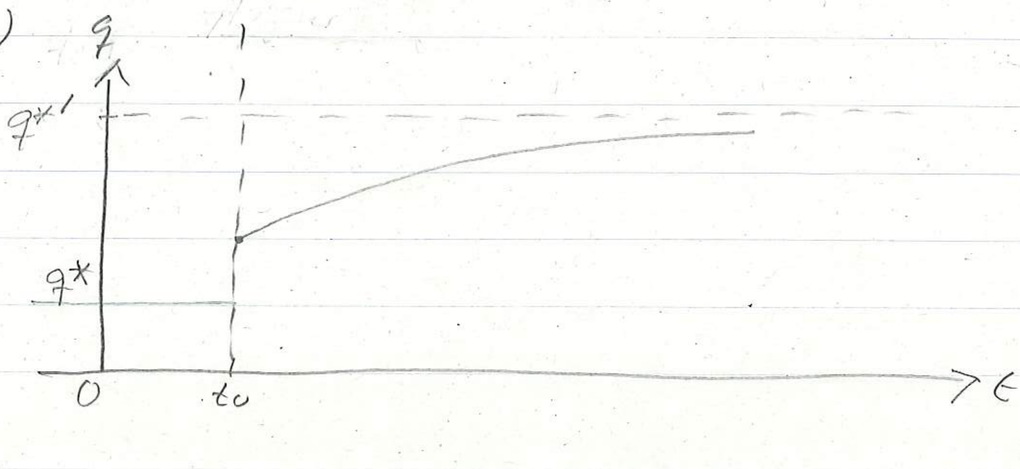
i) Da $\rho' > \rho$, er $K^{*'} < K^*$. Det følger af (***) på s.12 kombineret med $F_{KK} < 0$.

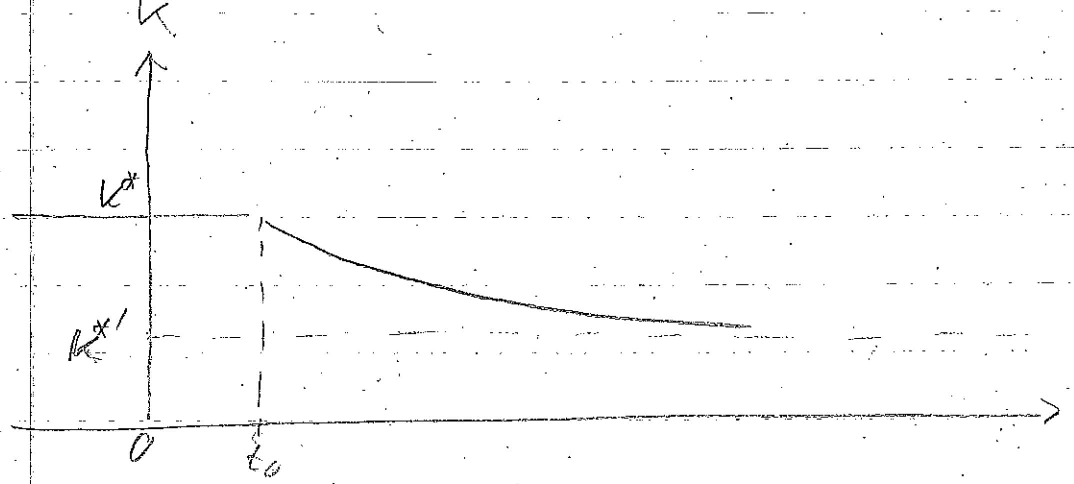




Figur 2.2

j)





Figur 2.3

Fortolkning: Den højere pris på investeringsgoder reducerer den på langt sigt ønskede kapitalbeholdning, fordi bruttoinvesteringer er blevet dyrere. Pga. det aftagende k er værdien, q, af det marginale k voksende, indtil den nye steady state med lavere k og højere q end i den gamle st.st. nås.

3. Solution to Problem 3

The no-arbitrage condition is

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r > 0, \qquad t = 0, 1, 2, \ldots. \tag{*}$$

where d_t is the dividend, p_t is the market price of the asset at time t, and E_t is the expectation operator conditional on the information available up to and including period t. We consider the stochastic process

$$b_t = \beta^{-t} x_t, \qquad t = 0, 1, 2, \ldots \qquad 0 < \beta < 1, \qquad b_0 > 0, \tag{**}$$

where $x_{t+1} = x_t + \varepsilon_{t+1}$, with $E_t \varepsilon_{t+1} = 0$, $t = 0, 1, 2, \ldots$.

To answer the stated question we use the hint. We forward (**) one period to get

$$b_{t+1} = \beta^{-(t+1)} x_{t+1}.$$

From this follows

$$E_t b_{t+1} = \beta^{-(t+1)} E_t x_{t+1} = \beta^{-(t+1)} x_t = \beta^{-1} \beta^{-t} x_t = \beta^{-1} b_t.$$

So b_t has the property $b_t = a E_t b_{t+1}$, $t = 0, 1, 2, \ldots$, where $a = (1+r)^{-1}$ if we set $\beta = a$ $= (1+r)^{-1}$. In this way, according to the hint, the process b_t satisfies the requirement for being a rational bubble component in the asset price.

It can thus *not* be ruled out that the stated claim may hold.

4. Solution to Problem 4

a) In an economy where taxes are lump sum, Ricardian equivalence is present if, for a given time path of future government spending, aggregate private consumption is unaffected by a temporary tax cut.

To put it differently: " ... unaffected by a change in the time profile of lump-sum taxes".

b) It takes some time for disequilibrium (a state of unrest, absence of balance) to result in equilibrium. *Arbitrage* is the activity aimed at exploiting risk-free profit opportunities made possible by disequilibrium (imbalance). Thereby, arbitrage is the mechanism through which *equilibrium* is brought about in the sense of no more risk-free profit opportunities being left.

The MIT economist seems aware that equilibrium need not be present beforehand, but requires the practice of arbitrage to be brought about. And since she might be the first person to spot the dollar note, she is ready to check whether the dollar note is false, that is, “to be the first to pick it up”.

The Chicago economist seems to forget that someone has to be the first - or more generally that temporary disequilibrium is part of reality.

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