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A suggested solution to the problem set at the exam in Advanced Macroeconomics January 15, 2015

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggregate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

The given equations from the Blanchard OLG model are:

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{\lambda + b}{b}\tilde{c}_t - (\delta + g + b - m)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{1.1}$$

$$\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho + \lambda - g\right]\tilde{c}_t - b(\rho + m)\tilde{k}_t, \tag{1.2}$$

together with the condition that for any fixed pair (t_0, v_0), where $t_0 \geq 0$ and $v \leq t_0$,

$$\lim_{t\to\infty} a_{t,v} e^{-\int_{t_0}^{t}(f'(\tilde{k}(s))-\delta+m)ds} = 0. \tag{1.3}$$

Notation: $\tilde{k}_t \equiv K_t/(T_t L_t)$ and $\tilde{c}_t \equiv C_t/(T_t N_t) \equiv c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively; N_t is population, L_t is labor supply, and T_t is the technology level, all at time t; f is a production function in intensive form, satisfying $f(0) = 0$, $f' > 0$, $f'' < 0$, and the Inada conditions. Finally, $a_{t,v}$ is financial wealth at time t of an individual born at time v. The remaining symbols stand for parameters and it is assumed all these are strictly positive. Furthermore, $\rho \geq b - m \geq 0$ and $\lambda < \delta + \rho + g$.

[1] The solution below contains more details and more precision than can be expected at a three hours exam.

a) Parameters: $\lambda =$ retirement rate, $b =$ birth rate, $\delta =$ capital depreciation rate, $g =$ growth rate of technology, $m =$ mortality rate, $\rho =$ pure rate of time preference (utility discount rate, a measure of impatience). The model relies on the simplifying assumption that for a given individual the probability of having a remaining lifetime, X, longer than some arbitrary number x is $P(X > x) = e^{-mx}$, the same for all (i.e., independent of age). Individual labor supply is assumed to decline exponentially at rate λ with increasing age. Both population and labor force grow at the rate $n \equiv b - m$.

The equation (1.1) is essentially just national income accounting for a closed economy. Isolating $f(\tilde{k}_t)$ on one side we have aggregate gross income per unit of effective labor on this side and consumption plus gross investment per unit of effective labor on the other side. Because the technology-corrected capital-labor ratio, $\tilde{k}_t$, has employment, L_t, in the denominator while the technology-corrected per-capita consumption, $\tilde{c}_t$, has population, N_t, in the denominator, $\tilde{c}_t$ is multiplied by the inverse of the participation rate, $L_t/N_t = b/(\lambda + b)$.

As to the first term on the right-hand side of (1.2), notice that instantaneous utility in the Blanchard OLG model is logarithmic, so that the individual Keynes-Ramsey rule at time t for a person born at time v is simply

$$\frac{\partial c_{t,v}}{\partial t} = [r_t + m - (\rho + m)]\, c_{t,v} = (r_t - \rho) c_{t,v}. \tag{1.4}$$

In general equilibrium with perfect competition, $r_t = f'(\tilde{k}_t) - \delta$. The corresponding Keynes-Ramsey rule for growth-corrected per-capita consumption is therefore

$\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho - g\right] \tilde{c}_t$. But due to the gradual replacement of dying elder individuals with low labor supply by younger individuals supplying more labor, the first term on the right-hand side of (1.2) also includes $+\lambda\tilde{c}_t$.

The second term on the right-hand side of (1.2) represents another aspect of generation replacement. The arrival of newborns is Nb per time unit. The fact that they have *more human* wealth than those who they replace has already been taken into account by the mentioned $+\lambda\tilde{c}_t$. But the newborns enter the economy with *less financial* wealth than the "average citizen". This lowers aggregate consumption by $b(\rho+m)A_t$ per time unit, where A_t is aggregate private financial wealth. In general equilibrium in the closed economy (without government debt) we have $A_t = K_t$. Correcting for population and technology growth, we end up with a lowering of $\dot{\tilde{c}}_t$ equal to $b(\rho + m)\tilde{k}_t$. This explains the second term in (1.2).

Finally, (1.3) is a transversality condition as seen from time t_0 for a person born at time v. The condition says that the No-Ponzi-Game condition is not over-satisfied (a

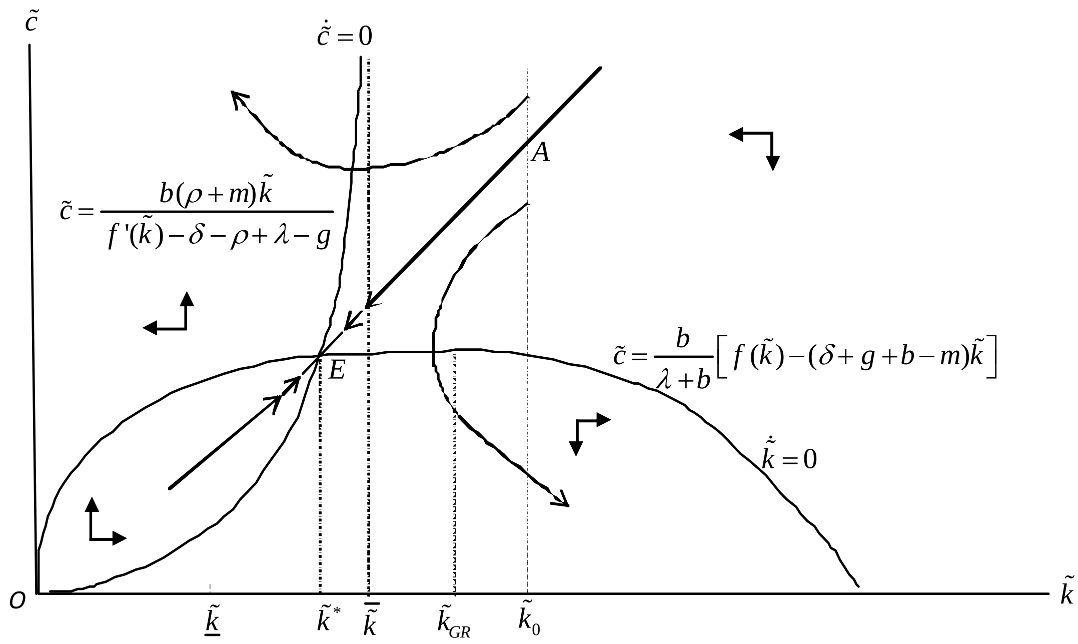

Figure 1.1:

necessary condition for individual optimality).

b) The equation describing the $\dot{\tilde{k}} = 0$ locus is

$$\tilde{c} = \frac{b}{\lambda + b}\left[f(\tilde{k}) - (\delta + g + b - m)\tilde{k}\right]. \tag{1.5}$$

The equation describing the $\dot{\tilde{c}} = 0$ locus is

$$\tilde{c} = \frac{b\,(\rho + m)\,\tilde{k}}{f'(\tilde{k}) - \delta - \rho + \lambda - g}. \tag{1.6}$$

Let $\bar{\tilde{k}}$ be defined by

$$f'(\bar{\tilde{k}}) - \delta = \rho - \lambda + g. \tag{1.7}$$

That is, $\bar{\tilde{k}}$ is defined as the value of $\tilde{k}$ at which the denominator of (1.6) vanishes. Such a value exists since, in addition to the Inada conditions, the inequality

$$\lambda < \delta + \rho + g$$

is assumed to hold. Another key value of $\tilde{k}$ is the golden-rule value, $\tilde{k}_{GR}$, determined by the requirement

$$f'\left(\tilde{k}_{GR}\right) - \delta = n + g, \qquad \text{where } n = b - m.$$

The phase diagram and the $\dot{\tilde{k}} = 0$ and $\dot{\tilde{c}} = 0$ loci are shown in Fig. 1.1. The $\dot{\tilde{c}} = 0$ locus is everywhere to the left of the line $\tilde{k} = \bar{\bar{\tilde{k}}}$ and is asymptotic to this line for $\tilde{k} \to \bar{\bar{\tilde{k}}}$. The figure also shows the steady-state point, E, where the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus. The corresponding capital intensity is $\tilde{k}^*$, to which corresponds the (growth-corrected) per-capita consumption level $\tilde{c}^*$. Fig. 1.1 depicts a case where $\bar{\bar{\tilde{k}}} \leq \tilde{k}_{GR}$ so that $\tilde{k}^* < \tilde{k}_{GR}$, that is, the economy is dynamically efficient. Yet, since we may have $b - m \leq \rho < \lambda + b - m$, so that $\bar{\bar{\tilde{k}}} > \tilde{k}_{GR}$, dynamic inefficiency cannot be ruled out theoretically (a typical feature of an OLG model).

The directions of movement in the different regions of the phase diagram are determined by the differential equations (1.1) and (1.2); the directions are shown by arrows. The arrows taken together show that the steady state, E, is a saddle point. Moreover, we have one predetermined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$, the saddle path is not parallel to the jump-variable axis, and the diverging paths can be ruled out as equilibrium paths (see below). Hence the steady state is saddle-point stable.

The saddle path is the only path that satisfies *all* the conditions of general equilibrium (individual utility maximization for given expectations, profit maximization by firms, continuous market clearing, and perfect foresight). The other paths in the diagram are diverging and either violate the transversality conditions of the individuals (paths that in the long run point South-East) or the NPG conditions of the individuals[2] (paths that in the long run point North-West). Hence, equilibrium initial consumption is determined as the ordinate, $\tilde{c}_0$, to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path. Over time the economy moves from this point, along the saddle path, towards the steady state.

c) In view of stability we have for $t \to \infty$,

$$r_t = f'(\tilde{k}_t) - \delta \to f'(\tilde{k}^*) - \delta \equiv r^*, \tag{1.8}$$

where r^* is the long-run rate of return. From the definition of $\bar{\bar{\tilde{k}}}$ and the fact that $\bar{\bar{\tilde{k}}} > \tilde{k}^*$, follows

$$f'(\bar{\bar{\tilde{k}}}) - \delta = \rho + g - \lambda < f'(\tilde{k}^*) - \delta = r^*,$$

in view of $f'' < 0$. So the lower end point of the interval to which r^* belongs is $\rho + g - \lambda$.

d) To answer how a shift in b affects r^* we have to find out, how it affects $\tilde{k}^*$. An unambiguous conclusion can be obtained in the following way. In steady state we have

[2] And therefore also the transversality conditions.

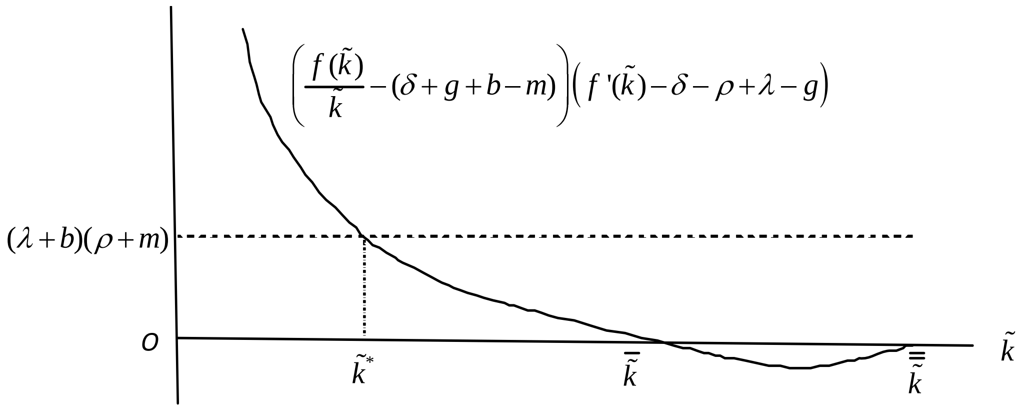

Figure 1.2:

$\dot{\tilde{k}} = 0$ and $\dot{\tilde{c}} = 0$ at the same time, implying that the right-hand sides of (1.5) and (1.6) must equal each other. By ordering we get

$$\left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + g + b - m)\right)\left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right] = (\lambda + b)(\rho + m). \qquad (1.9)$$

Fig. 1.2 illustrates. A lower b makes the left-hand side of (1.9) larger for fixed $\tilde{k}$ and the right-hand side smaller. Thereby $\tilde{k}^*$ is raised and so $r^* = f'(\tilde{k}^*) - \delta$ becomes lower ($f''(\tilde{k}^*) < 0$).

e) As already noted, a downward shift in b raises $\tilde{k}^*$ which makes r^* lower. A simultaneous reduction in m, so that also $n = b - m$ is lowered, makes the left-hand side of (1.9) larger for fixed $\tilde{k}$ and the right-hand side smaller. So $\tilde{k}^*$ is raised further, and this contributes further to a lowering of r^*.[3]

f) A lower g makes the left-hand side of (1.9) larger and does not affect the right-hand side. This implies a further rise in $\tilde{k}^*$ and thus a further reduction in r^*.

g) In the new interpretation output and employment are primarily demand-determined. A downward shift in aggregate demand may for instance be triggered by a bursting housing bubble leading to a financial crisis and defaults. A credit crunch arises in the banking sector, thus causing a drop in spending on especially durable consumption and investment by households and firms in need of credit. A fall in output and employment sets in, which may prompt precautionary saving and thus, in "the second round", cause a further fall in aggregate demand and employment (a vicious spiral).

[3] A fall in λ has an ambiguous effect on $\tilde{k}^*$.

An upward shift in aggregate demand may for instance be triggered by a wave of optimism due to invention of new consumer goods whereby households' spending may be profoundly stimulated (historical examples: automobiles, the radio, the ICT revolution). Similarly, news about new technology may suddenly stimulate firms' investment spending (think of the economic expansion leading to the dot.com bubble). For an open economy a rise in exports of the kind of goods the economy produces may spark a virtual upward demand spiral in the economy.

Shifts in monetary or fiscal policy are also examples of circumstances that may suddenly change aggregate demand.

It may be added (although the notion of "demand shock" is not appropriate here) that some economists fear that a gradual reduction in aggregate demand may be a consequence of reduced population growth leading to less demand for residential construction and less capital investment to furnish new workers with production equipment.

h) In accordance with monetary policy in recent decades we consider the nominal interest rate on short-term government bonds to be the "instrument" used by the CB to affect aggregate demand. Let this interest rate be denoted i_t. Let the inflation rate, $\dot{P}_t/P_t$, be denoted π_t. Then the real interest rate of relevance for households' consumption and firms' investment is $r_t = i_t + \omega - \pi_t$, where $\omega \geq 0$ is the spread between the "official" nominal interest rate i_t and the nominal interest rate faced by the non-bank general public. For simplicity, let us ignore ω. Then aggregate demand depends negatively on the expected real interest rate $r_t^e = i_t - \pi_t^e$. To ease the discussion, let $\pi_t^e = \pi_t$, so that we do not have to distinguish between r_t^e and r_t. We further assume, realistically, that π_t is relatively sticky within the time horizon relevant for policy intervention during business-cycles.

The term "zero lower bound", ZLB, refers to the fact that the nominal interest rate even on short-term bonds can (essentially) not go below zero. This is because agents would prefer holding cash at zero interest rather than bonds at negative interest. So,

$$i_t = \max(0, i_t^p),$$

where i_t^p is the level of the interest rate *desired* by the CB, the "policy rate". The ZLB becomes a binding constraint when, in a recession, the interest rate needed to obtain the desired stimulus of aggregate demand is negative. In this situation conventional monetary policy can thus only bring i_t down to zero, which is not sufficient for recovery of the economy. The economy is in a "liquidity trap".

So, the situation calls for additional policies. See answer to j) below.

i) To answer this, let us consider the "normal case" where i_t^p is given by a Taylor rule which in the present context could have the form:

$$i_t^p = \hat{\imath} + \alpha_1 \frac{Y_t - Y_t^*}{Y_t^*} + \alpha_2 \frac{\pi_t - \hat{\pi}}{\hat{\pi}}, \qquad \alpha_1 > 0, \alpha_2 > 1,$$

where $\hat{\imath} \equiv r^* + \hat{\pi}$, Y_t is actual output, Y_t^* is NAIRU output, and $\hat{\pi}$ the inflation target.

For a given $\hat{\pi}$, a reduced r^* implies a lower i_t^p everything else equal. So the distance to ZLB becomes smaller and therefore the likelihood that the ZLB becomes binding is raised.

j) Conceivable alternatives or supplements to conventional monetary policy that may lessen a possible tendency to "secular stagnation" due a binding ZLB include:

a) *Raising the inflation target* is a possible non-conventional monetary policy to deal with the problem. A difficulty is that the announcement of a higher inflation target may not be taken as credible.

b) *Quantitative easing (QE).* This stands for an unconventional monetary policy where the CB increases the monetary base by buying government bonds (short- as well as long-term) and/or private-sector financial assets. Also collateralized lending by the CB to the banks is a form QE.

QE may reduce the maturity premium on long-term government bonds and thereby the spread between i_t and the nominal interest rate on corporate bonds and bank loans. This stimulates aggregate demand.

QE may also work through relaxing the intertemporal budget constraint of the government by effectively financing a budget deficit by money instead of new government bonds. In this context QE is a kind of coordinated fiscal and monetary policy.

c) *Expansionary fiscal policy* as such is also an option. Most macroeconomists agree that when the economy is in a liquidity trap, fiscal policy multipliers tend to be large. This is so for several reasons. One is that there will be no financial crowding out as long as the aim of the CB is to maintain i_t as low as possible. Another reason is that the economic situation which has triggered the liquidity trap is also a situation where involuntary unemployment tends to be considerable.

Moreover, times with low interest rates are the right time for public investment of which a part is normally financed by borrowing.

More structural policies are also conceivable, such as subsidizing fertility by child benefits, invest more in public health and support pharmaceutical research to improve life expectancy.

2. Solution to Problem 2

We consider a "monopolistic competition setup". Firm i has the production function

$$y_i = n_i^{\alpha}, \qquad 0 < \alpha < 1,$$

where y_i is output and n_i is labor input, $i = 1, 2, \ldots, m$. Aggregate output demand in the period considered is

$$Y^d = \frac{M}{\beta P},$$

where M is base money (there is no private banking sector), P is the "general price level", and $\beta \in (0, 1)$ is a parameter reflecting consumers' patience.

The nominal wage rate is denoted W and η is a parameter, $\eta > 1$. Given W, P, and M, firm i sets a price P_i so as to maximize

$$\begin{aligned} \Pi_i &= P_i y_i - W y_i^{1/\alpha} = P_i \left(\frac{P_i}{P}\right)^{-\eta} \frac{M}{m\beta P} - W \left(\left(\frac{P_i}{P}\right)^{-\eta} \frac{M}{m\beta P}\right)^{1/\alpha} \qquad (*) \\ &\equiv \Pi(P_i, P, W, M). \end{aligned}$$

a) Π_i is the nominal profit of firm i in that $P_i y_i$ is the value of the firm's output (= sales), and $W y_i^{1/\alpha}$ is the associated total cost. As displayed by the production function, labor is the only input, and $y_i^{1/\alpha}$ indicates the needed labor to produce y_i. The term $\left(\frac{P_i}{P}\right)^{-\eta} \frac{M}{m\beta P}$ in (*) indicates the demand (sales) constraint faced by firm i :

$$y_i \leq \left(\frac{P_i}{P}\right)^{-\eta} \frac{M}{m\beta P}. \qquad (**)$$

The right-hand side of (**) shows that this demand depends negatively on the price P_i charged by the firm relative to the general price level, P. The absolute value of the price elasticity of this demand is $\eta > 1$ (with $\eta \leq 1$, there would be no finite profit maximizing price). The demand faced by the firm also depends on the firm's "share" of the aggregate output demand as indicated by $M/(\beta P)$ being divided by m.

In most cases of interest, it pays the firm to produce up to the level where the demand constraint becomes binding. Hence, in (*) there is "=" rather than "≤".

We are now told that $\bar{P}_i$ is the profit-maximizing price set in advance, given the expectation $M^e = M$, the same for all firms. But the actual money stock turns out to

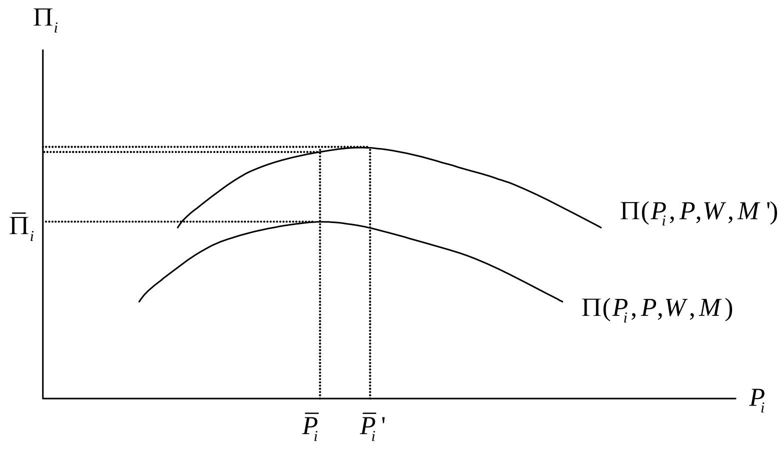

Figure 2.1:

be $M' > M$. Moreover, the firm expects the general price and wage levels, P and W, respectively, to remain unchanged. Finally, there is a given "menu cost" $c > 0$.

b) Here we consider the case where $\Delta M \equiv M' - M$ is "infinitesimal". Consider the marginal effect of a higher M, i.e.,

$$\begin{aligned} \frac{d\Pi}{dM}(\bar{P}_i, P, W, M) &= \frac{\partial \Pi}{\partial P_i}(\bar{P}_i, P, W, M)\frac{\partial P_i}{\partial M} + \frac{\partial \Pi}{\partial M}(\bar{P}_i, P, W, M) \qquad (***) \\ &= 0 + \frac{\partial \Pi}{\partial M}(\bar{P}_i, P, W, M). \end{aligned}$$

The first term on the right-hand side of (***) vanishes at the profit maximum because $\frac{\partial \Pi}{\partial P_i}(\bar{P}_i, P, W, M) = 0$, i.e., the profit curve is flat at the profit-maximizing price $\bar{P}_i$, cf. Fig. 2.1. Moreover, since our thought experiment is one where P and W remain unchanged, there is no indirect effect of the rise in M via P or W. Thus, only the direct effect through the fourth argument of the profit function is left. And this effect is the same whether or not the price is changed. This result reflects the *envelope theorem*: in an interior optimum, the total derivative of a maximized function w.r.t. a parameter (here M) equals the partial derivative w.r.t. that parameter.

It follows that in this case there is no incentive to change the price.

c) Yes, the logic is somewhat different for a finite $\Delta M = M' - M$, because in this case, there *is* an opportunity cost of *not* changing price. It can be shown that the new top of the profit curve is north-east of the old, as indicated in Fig. 2.1. It follows that by not changing price a discernible profit gain is left unexploited. Still, if the rise in M in not "too large", the slope of the profit curve at the old price $\bar{P}_i$ may still be small enough to be dominated by the menu cost. The key point is that for a "small" ΔM, the opportunity cost by not changing price will be "very small", as also illustrated in Fig. 2.1. (By taking a second-order Taylor approximation, the opportunity cost can be shown to be proportional to $(\Delta M/M)^2$, i.e., of "second order".)

Given the menu cost $c > 0$, the net gain of *changing* price may thus easily be a negative number, in which case there is again no incentive to change price. We say that in this case the menu cost is *operative.*

For a larger ΔM, the opportunity cost may exceed c, and then there *is* an incentive to change price.

d) The *rule of the minimum* is the principle that, given a predetermined price $\bar{P}_i$, the desired level of production satisfies

$$y_i = \min \left[D(\frac{\bar{P}_i}{P}, Y^d), y^c(W, \bar{P}_i) \right], \tag{2.1}$$

saying that desired output is determined as the minimum of demand and the *classical* supply. Given rising marginal costs, MC, the classical supply is the output level at which marginal cost equals the predetermined price. Up to the level $y^c(W, \bar{P}_i)$, for fixed $P_i = \bar{P}_i$ it is beneficial for the firm to satisfy demand since MC is below the price.

As long as menu costs are operative, shifts in the position of the downward-sloping demand curve due to changes in aggregate demand, Y^d, does not induce a price change but instead a *quantity* change. This may be the situation also for the other firms as well, and so the general price level, P, will remain the same, *unless* W is increased. This "unless" is important because the condition might not hold. Indeed, the increased production requires increased employment, and workers or workers' unions might thus increase their wage demands.

Nonetheless, within certain limits the assumption of an unchanged W can be defended by one of the following two alternative arguments. The first argument is that also craft unions may face "menu costs" if they, outside the ordinary recurrent negotiations with the employers, try to obtain wage increases. Hence, the same envelope logic as above may apply for wages changes.

The second argument is that there may be *involuntary unemployment* in the labor market. This means that there are people around without a job although they are as qualified as the employed workers and are ready and willing to take a job at the going wage or even a somewhat lower wage.

e) The three stylized short-run facts are:

1. Shifts in aggregate demand are largely accommodated by changes in quantities rather than changes in nominal prices.

2. Large movements in quantities are often associated with little or no movement in relative prices (including the real wage).

3. Nominal prices are sensitive to general changes in input costs.

The "story" about menu cost theory and the rule of the minimum offers one kind of explanation of primarily stylized fact no. 1. (Thereby it also to some extent offers an approach to explanation of stylized fact no. 2.)

f) In the new scenario we first consider the case where $\Delta W \equiv W' - W$ is "infinitesimal". Then

$$\begin{aligned} d\Pi &= \frac{d\Pi}{dW}(\bar{P}_i, P, W, M)dW = \frac{\partial \Pi}{\partial P_i}(\bar{P}_i, P, W, M)\frac{\partial P_i}{\partial W}dW + \frac{\partial \Pi}{\partial W}(\bar{P}_i, P, W, M)dW \\ &= 0 + \frac{\partial \Pi}{\partial W}(\bar{P}_i, P, W, M)dW. \end{aligned}$$

As at b), the envelope theorem thus applies and there is no incentive to change price.

For a finite $\Delta W = W' - W$, the answer is analogue to that at c) (we just have to replace M by W).

g) A rise in W means a rise in costs, everything else equal.

Since stylized fact no. 3 is about price *sensitiveness,* which is the opposite of price stickiness, it is unlikely that the menu cost theory could be helpful for an explanation of that fact. Even worse, according to the answer at f), the menu cost theory could, if anything, help to explain the opposite of fact no. 3.

3. Solution to Problem 3

a) The No-Ponzi-Game condition, NPG, of the government is the constraint on government borrowing that says that the government debt is not allowed to grow in the long run at a rate as high as (or even higher than) the interest rate. In discrete time:

$$\lim_{t\to\infty} B_t(1+r)^{-t} \leq 0, \tag{NPG}$$

where B_t is real government debt and r is the real interest rate (for simplicity assumed constant). (This formula and those below presuppose that there is no effective tax on interest income).

b) The government intertemporal budget constraint, GIBC, is the constraint saying that the present value of the expected future stream of government spending should not exceed government net wealth. GIBC as seen from period t can be written

$$\sum_{i=0}^{\infty} G_{t+i}(1+r)^{-(i+1)} \leq \sum_{i=0}^{\infty} T_{t+i}(1+r)^{-(i+1)} - B_t, \qquad \text{(GIBC)}$$

where G is government spending on goods and services and T is net tax revenue (=gross tax revenue minus transfers). The NPG and GIBC are related this way:

(i) (NPG) is satisfied if and only if (GIBC) is satisfied;

(ii) there is strict equality in (NPG) if and only if there is strict equality in (GIBC).

c) False. By definition a given fiscal policy is *sustainable* if by applying its spending and tax rules forever, the government stays solvent (able to meet the financial commitments as they fall due). The operational criterion for sustainability is whether the fiscal policy can be deemed compatible with *boundedness* of the public debt-to-income ratio, $b_t \equiv B_t/Y_t$, in the long run.

First, the "if" part of the cited statement is not valid because we *can* have

$$1 + g_Y < \lim_{t\to\infty} B_{t+1}/B_t < 1 + r.$$

Here, by the left-hand inequality, $\lim_{t\to\infty} b_{t+1}/b_t > 1$ so that the debt-income ratio explodes. Yet, the right-hand inequality shows that (NPG) is satisfied.

Second, the "only if" part of the cited statement essentially only holds if r exceeds the long-run GDP growth rate g_Y. (This is on the other hand also considered the "normal case"). To see this, suppose that $B_0 > 0$ and that the government levies taxes equal to its non-interest spending:

$$T_t = G_t \text{ for all } t \geq 0.$$

So taxes cover only the primary expenses while interest payments (and debt repayments when necessary) are financed by issuing new debt. That is, the government attempts a permanent debt roll-over.

Given the debt accumulation equation

$$B_{t+1} = (1+r)B_t + G_t - T_t,$$

the implied law of motion of the public debt-to-income ratio is

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y}\frac{B_t}{Y_t} \equiv \frac{1+r}{1+g_Y} b_t, \qquad b_0 > 0.$$

The solution to this linear difference equation is

$$b_t = b_0(\frac{1+r}{1+g_Y})^t.$$

We see that the permanent debt roll-over fiscal policy implies an exploding debt-to-income ratio if and only if $r > g_Y$.

It may be added that taking into account uncertainty complicates the story somewhat (details not part of syllabus). And taking into account the possibility of self-fulfilling expectations, governments should also worry about a "too high" *level* of the debt-to-income ratio even if *currently* there are no signs in the direction of an exploding debt-to-income ratio.

Remark. Although for convenience some formulas have been inserted above, a purely verbal account is enough if done properly.

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