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A suggested solution to the problem set at the exam in Advanced Macroeconomics January 15, 2014

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggregate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

For convenience the decision problem is restated here: Choose a plan $(L_t, I_t)_{t=0}^{\infty}$ to maximize

$$V_0 = \int_0^{\infty} \left(F(K_t, L_t) - G(I_t, K_t) - w_t L_t - (1-\sigma)I_t\right) e^{-rt} dt \quad \text{s.t.} \tag{1.1}$$

$$L_t \geq 0, I_t \text{ free (i.e., no restriction on } I_t\text{)}, \tag{1.2}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{1.3}$$

$$K_t \geq 0 \text{ for all } t. \tag{1.4}$$

Here F is a CRS neoclassical production function with inputs of capital and labor denoted K_t and L_t, respectively, G is a capital installation cost function, I_t is gross investment, w_t is a given market real wage, $r > 0$ is a given constant real interest rate, and $\delta > 0$ is a given constant capital depreciation rate. There is a constant investment subsidy $\sigma \in (0, 1)$. The installation cost function G satisfies, for all (I, K),

$$G(0, K) = 0, \; G_I(0, K) = 0, \; G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0. \tag{1.5}$$

a) The current-value Hamiltonian is

$$H(K, L, I, q, t) \equiv F(K, L) - G(I, K) - wL - (1-\sigma)I + q(I - \delta K), \tag{1.6}$$

[1] The solution below contains more details and more precision than can be expected at a three hours exam.

where q (to be interpreted economically below) is the adjoint variable associated with the dynamic constraint (1.3). For each $t \geq 0$ we maximize H w.r.t. the control variables. Thus, $\partial H/\partial L = F_L(K, L) - w = 0$, i.e.,

$$F_L(K, L) = w; \tag{1.7}$$

and $\partial H/\partial I = -G_I(I, K) - (1 - \sigma) + q = 0$, i.e.,

$$1 - \sigma + G_I(I, K) = q. \tag{1.8}$$

Next, we partially differentiate H w.r.t. the state variable and set the result equal to $rq - \dot{q}$, where r is the discount rate in (1.1):

$$\frac{\partial H}{\partial K} = F_K(K, L) - G_K(I, K) - q\delta = rq - \dot{q}. \tag{1.9}$$

Then, the Maximum Principle says that for an interior optimal path (K_t, L_t, I_t) there exists an adjoint variable q, which is a continuous function of t, written q_t, such that for all $t \geq 0$ the conditions (1.7), (1.8), and (1.9) hold and the transversality condition

$$\lim_{t \to \infty} K_t q_t e^{-rt} = 0 \tag{1.10}$$

is satisfied.

b) q_t is the shadow price of installed capital, i.e., the value to the firm (in optimum) of having one extra unit of installed capital. In view of $G_{II} \neq 0$, (1.8) defines optimal investment, I_t, as an implicit function of q_t, σ, and K_t :

$$I_t = \mathcal{M}(q_t, \sigma, K_t) \equiv \tilde{\mathcal{M}}(q_t - (1 - \sigma), K_t). \tag{1.11}$$

Although not necessary, given the question, we find the partial derivatives of $\mathcal{M}$ by taking the total differential on both sides in (1.8):

$$-d\sigma + G_{II}(I_t, K_t)dI_t + G_{IK}(I_t, K_t)dK_t = dq_t.$$

Thus

$$\frac{\partial I_t}{\partial q_t} = \frac{\partial I_t}{\partial \sigma} = \frac{1}{G_{II}(I_t, K_t)} > 0, \quad \text{and} \quad \frac{\partial I_t}{\partial K_t} = -\frac{G_{IK}(I_t, K_t)}{G_{II}(I_t, K_t)},$$

where the latter cannot be signed without further specification.

c) From now $G(I, K) = \beta \frac{I^2}{2K}$, where $\beta > 0$. Consequently,

$$G_I(I, K) = \beta I/K, \tag{1.12}$$

$$G_K(I, K) = -\beta I^2 K^{-2}/2 = -\frac{\beta}{2}\left(\frac{I}{K}\right)^2. \tag{1.13}$$

Substituting (1.12) into (1.8) gives $1 - \sigma + \beta I/K = q$, from which follows

$$\frac{I_t}{K_t} = \frac{1}{\beta}(q_t - 1 + \sigma). \tag{1.14}$$

d) The first differential equation entering the (K, q) dynamics is obtained by inserting (1.14) into (1.3):

$$\dot{K}_t = \frac{1}{\beta}(q_t - 1 + \sigma - \beta\delta)K_t, \qquad K_0 > 0 \text{ given.} \tag{1.15}$$

Reordering (1.9) gives $\dot{q}_t = (r + \delta)q_t - F_K(K_t, \bar{L}) + G_K(I_t, K_t)$. Inserting (1.13) gives the second differential equation

$$\begin{aligned} \dot{q}_t &= (r + \delta)q_t - F_K(K_t, \bar{L}) - \frac{\beta}{2}\left(\frac{q_t + \sigma - 1}{\beta}\right)^2 \\ &= (r + \delta)q_t - F_K(K_t, \bar{L}) - \frac{(q_t + \sigma - 1)^2}{2\beta}. \end{aligned} \tag{1.16}$$

e) From (1.15) follows that

$$\dot{K}_t \gtreqless 0 \text{ for } q_t \gtreqless 1 - \sigma + \beta\delta \equiv q^*, \quad \text{respectively,} \tag{1.17}$$

where $q^* > \beta\delta$ since $\sigma < 1$. As indicated in Fig. 1.1, the $\dot{K} = 0$ locus is horizontal in the (K, q) plane. From (1.15) follows that

$$\dot{q} = 0 \text{ for } 0 = (r + \delta)q - F_K(K, \bar{L}) - \frac{(q + \sigma - 1)^2}{2\beta}. \tag{1.18}$$

In view of $F_{KK}(K, \bar{L}) < 0$, the $\dot{q} = 0$ locus is downward-sloping in the (K, q) plane and, by (1.16),

$$\dot{q} \gtrless 0 \text{ for points above and below the } \dot{q} = 0 \text{ locus, respectively,} \tag{1.19}$$

as indicated in Fig. 1.1.

Since F satisfies the Inada conditions, there exists $K^* > 0$ such that the second equation in (1.18) holds with $q = q^*$; since $F_{KK} < 0$, K^* is a unique. Hence there exists a unique steady state, denoted E in Fig. 1.1, with coordinates K^* and q^*, respectively. Net investment is zero in the steady state. The directions of movement in the different regions of the phase diagram follow from (1.17) and (1.19) and are indicated by arrows in Fig. 1.1. The arrows taken together show that the steady state is a saddle point. We have one predetermined variable, K, and one jump variable, q, the saddle path is not parallel to the jump-variable axis, and the diverging paths can be ruled out (they can be shown to violate the transversality condition). Hence the steady state is saddle-point stable.

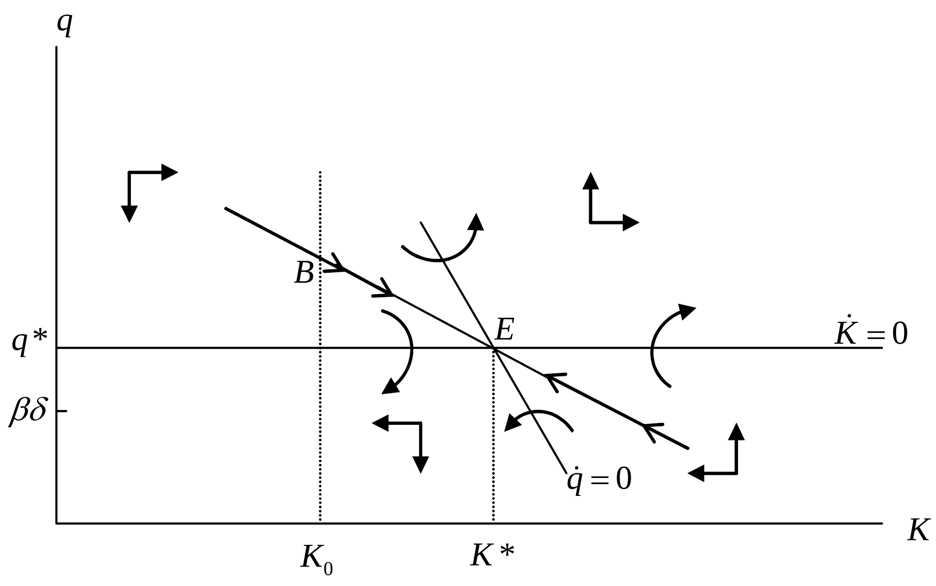

Figure 1.1:

For a given $K_0 > 0$, the path starting at the point B in Fig. 1.1 and moving along the saddle path towards the steady state is the unique solution, (K_t, q_t), to the firm's investment problem. This path *is* a solution because: (a) as the steady state satisfies the transversality condition (1.10), so does any path that converges towards the steady state; and (b) the Hamiltonian is jointly concave in (L, I, K). Moreover, the converging path is the *only* solution because the divergent paths can be shown to violate the transversality condition.

f) The shift at time t_0 to $\sigma' > \sigma$ reduces q^* and implies a downward shift in the $\dot{K} = 0$ locus as indicated in Fig. 1.2. The $\dot{q} = 0$ locus is shifted upwards. The new steady-state value of K, $K^{*\prime}$, is larger than the old, K^*. In the present case a graphical argument for this is sufficient. An algebraic argument may use that in steady state, $q^* + \sigma - 1 = \beta\delta$ so that the second equation in (1.18) reads $0 = (r + \delta)q^* - F_K(K^*, \bar{L}) - \beta\delta^2/2$; since $F_{KK} < 0$, a lower q^* implies a higher K^*.[2]

At time t_0 the shadow price, q, jumps down to a level corresponding to the point A in Fig. 1.2, thereby remaining larger than the new steady-state value, $q^{*\prime}$. Then net investment is positive and the pair (K_t, q_t) moves gradually along the new saddle path towards the new steady state, E'.

[2]In the hint to this question there was a typo: β in the equation $q^* + \sigma - 1 = \beta\delta$ was omitted. Although this deficiency does not affect the logic of the argument, it may have caused some delay for those (rather few) examinees trying to apply an algebraic argument. This has been taken into account in the grading.

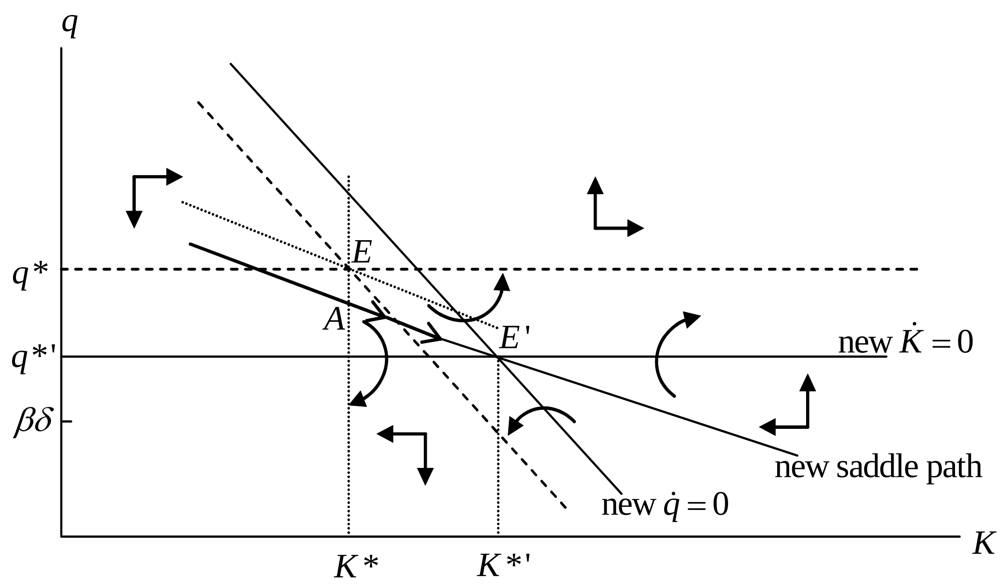

Figure 1.2:

The economic intuition is that the larger investment subsidy stimulates investment. As the capital stock is thereby gradually increased, however, the marginal productivity of capital decreases and so does therefore q_t until the new steady state is reached. The steady-state capital stock is larger than before because maintaining it is less costly than before due to the investment subsidy.

2. Solution to Problem 2

For convenience the equations of the model are restated here:

$$\dot{Y}_t = \lambda(D(Y_t, R_t, \tau) + G - Y_t), \quad \lambda > 0, 0 < D_Y < 1, D_R < 0, -1 < D_\tau < 0 \tag{2.1}$$

$$\frac{M_t}{P_t} = L(Y_t, i_t), \qquad L_Y > 0,\ L_i < 0 \tag{2.2}$$

$$R_t = \frac{1}{q_t}, \tag{2.3}$$

$$\frac{1 + \dot{q}_t^e}{q_t} = r_t^e, \tag{2.4}$$

$$r_t^e \equiv i_t - \pi_t^e, \tag{2.5}$$

$$\pi_t = \pi, \tag{2.6}$$

where the superscript e denotes subjective expectation. Further, Y_t = output, q_t = real price of a consol paying one unit of output per time unit forever, G = government spending on goods and services, M_t = money supply, P_t = output price, R_t = real long-term interest rate, i_t = nominal short-term interest rate, r_t = real short-term interest rate, and $\pi_t \equiv \dot{P}_t/P_t$ = rate of inflation. The variables λ, τ, G, and π are exogenous constants. The initial values Y_0 and P_0 are historically given.

In questions a) - f) it is assumed that the central bank (from now CB) maintains the real money supply, $m_t \equiv M_t/P_t$, at a given constant level, m, by letting the (nominal) money supply grow at a rate equal to the rate of inflation.

a) The model is essentially Blanchard's dynamic IS/LM model, which describes the adjustment in the "very short run" towards a "short-run equilibrium" with respect to output and interest rates. The adjustment of output to demand takes time and during the adjustment process also demand changes (since the output level and asset prices are among its determinants). There are three financial assets: money, a long-term inflation-indexed bond, and a short-term bond.

Eq. (2.1) tells how output adjusts to demand; the parameter λ is the speed of adjustment. Output demand depends positively on current income (to reflect, e.g., that a fraction of the consumers are credit constrained) and negatively on the long-term real interest rate. In particular investment is likely to depend negatively on this rate. Also consumption tends to be negatively dependent on the long-term rate due to the substitution and wealth effects. Regarding τ an obvious interpretation is that it measures fiscal tightness.

Eq. (2.2) expresses equilibrium in the money market. Real money demand depends positively on Y, because Y is a proxy for the number of transactions per time unit, and negatively on the short-term nominal interest rate, the opportunity cost of holding money. The money market and other asset markets are assumed to clear instantaneously.

When the "long-term real interest rate", R_t, is identified with the internal rate of return on the consol, we have

$$q_t = \int_t^\infty 1 \cdot e^{-R_t(s-t)} ds = \frac{1}{R_t}.$$

Reordering gives (2.3).

Eq. (2.4) is a no-arbitrage condition saying that the expected real rate of return on holding the consol one time unit equals the expected real short-term interest rate, r_t^e. So there is no risk premium (which is explained below). Eq. (2.5) defines the expected real interest rate, r_t^e. Finally, eq. (2.6) states the simplifying assumption that the actual rate of inflation is a constant π. So both the initial price level and the inflation rate are "sticky", i.e., not affected by short-run fluctuations in aggregate demand.

We now assume expectations are rational (model consistent). As there are no stochastic

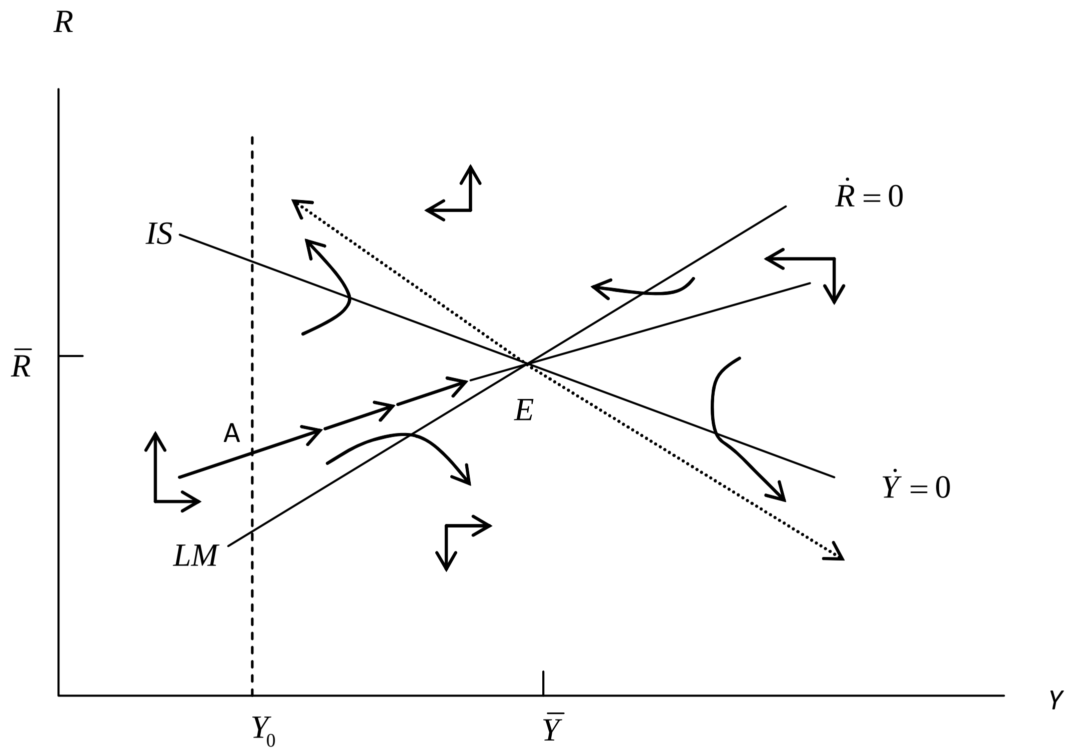

Figure 2.1:

elements in the model, this amounts to perfect foresight. In addition we also assume that speculative bubbles never arise.

b) In view of perfect foresight, $\pi_t^e = E_t\pi_t = \pi_t = \pi > 0$ for all t. This implies that $r_t^e = i_t - \pi \equiv r_t$. Similarly, $\dot{q}_t^e = E_t\dot{q}_t = \dot{q}_t$. Substituting into (2.4), we get

$$\frac{1}{q_t} + \frac{\dot{q}_t}{q_t} = R_t - \frac{\dot{R}_t}{R_t} = i_t - \pi. \tag{7}$$

As the CB maintains the real money supply at the constant level, m, by (2.2) we have $m = L(Y_t, i_t)$. This equation defines i_t as an implicit function of Y_t and m, i.e.,

$$i_t = i(Y_t, m), \quad \text{with } i_Y = -L_Y/L_i > 0, \quad i_m = 1/L_i < 0. \tag{2.1}$$

Inserting this function into (7) and reordering, we have

$$\dot{R}_t = [R_t - (i(Y_t, m) - \pi)]\, R_t. \tag{8}$$

Combining this with (2.1), we thus have two coupled differential equations in R and Y, that is, a dynamic system in Y and R.

Given $R > 0$, (8) implies

$$\dot{R} \gtreqless 0 \qquad \text{for} \qquad R \gtreqless i(Y,m) - \pi, \qquad \text{respectively.} \tag{2.2}$$

We have $\frac{\partial R}{\partial Y}|_{\dot{R}=0} = i_Y = -L_Y/L_i > 0$. The $\dot{R} = 0$ locus is thus an upward-sloping curve, named the "LM curve" in Fig. 2.1.

From (2.1) we have

$$\dot{Y} \gtreqless 0 \qquad \text{for} \qquad D(Y,R,\tau) + G \gtreqless Y, \qquad \text{respectively.} \tag{2.3}$$

Hence, $\frac{\partial R}{\partial Y}|_{\dot{Y}=0} = (1 - D_Y)/D_R < 0$. The $\dot{Y} = 0$ locus is thus a downward-sloping curve, named the "IS curve" in Fig. 2.1.

The figure shows the direction of movement in the different regions, as described by (2.2) and (2.3). We see that the steady state point, E, with coordinates $(\bar{Y}, \bar{R})$, is a saddle point. We have one predetermined variable, Y, and one jump variable, R, the saddle path is not parallel to the jump-variable axis, and the diverging paths can be ruled out (they can be shown to violate the no-bubbles assumption). Hence the steady state is saddle-point stable. For a given $Y_0 < \bar{Y}$, the path followed by the economy for $t \geq 0$ is indicated by arrows. It is a movement along the saddle path from the point A to the steady state point E.

Both the stable arm (with positive slope) and the unstable arm (the negatively sloped stippled curve) are displayed in the figure.

c) The effect of the downward shift in G is shown in Fig. 2.2. When G shifts downward, the LM curve is not affected, but the IS curve is shifted downward. Hence, the long-term interest rate immediately jumps down to R_A, reflecting that the market value of the consol jumps up in the expectation of lower short-term interest rates as a result of the recession. This is where the given formula,

$$R_t = \frac{1}{q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_u du} ds}, \tag{2.4}$$

is useful. This formula, valid in the absence of bubbles, indicates that the long-term interest rate is a weighted average of the expected future short-term rates.

The mechanism behind the jump is as follows. The lower level of government spending implies lower output demand. This triggers an expectation of decreasing Y and therefore also an expectation of decreasing i and r in view of the gradually reduced money demand. The implication is, by (2.4), a lower R already immediately after time t_0, as illustrated in Fig. 2.2. As time proceeds and the economy gets closer to the expected low future values

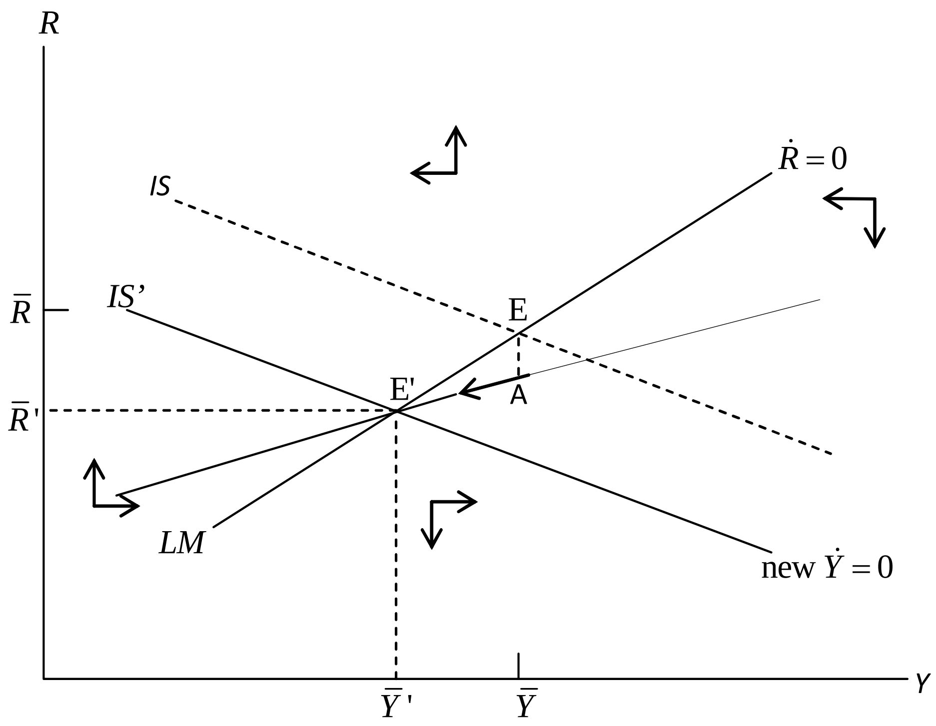

Figure 2.2:

of r, these lower values gradually become dominating in the determination of R. Hence, R gradually decreases towards its new steady-state value, the same as that for r.

d) The result under c) can be seen as a "rough" picture of events in the wake of the financial crisis 2008-2009 if we interpret G as the level of "confidence" and the downward shift in G as representing the huge adverse shock to aggregate demand triggered by the financial meltdown 2008-09.

e) The yield curve immediately after the shock will be negatively sloped. Indeed, immediately after t_0 we have $R_t < r_t \approx \bar{r} \equiv \bar{R} = i(\bar{Y}, m) - \pi$. This is because it takes time for Y and r to fall while the long-term rate jumps down anticipating the future downward movement of the short-term rate.

f) We are now asked to consider a different monetary policy, namely one where the CB directly applies the short-term nominal interest rate as the monetary policy instrument and does so in accordance with the following "truncated" Taylor rule:

$$i_t = \max(0, \alpha + \beta Y_t), \qquad (*)$$

where α and β are constants, $\beta > -L_Y/L_i$. Also under this policy are the qualitative

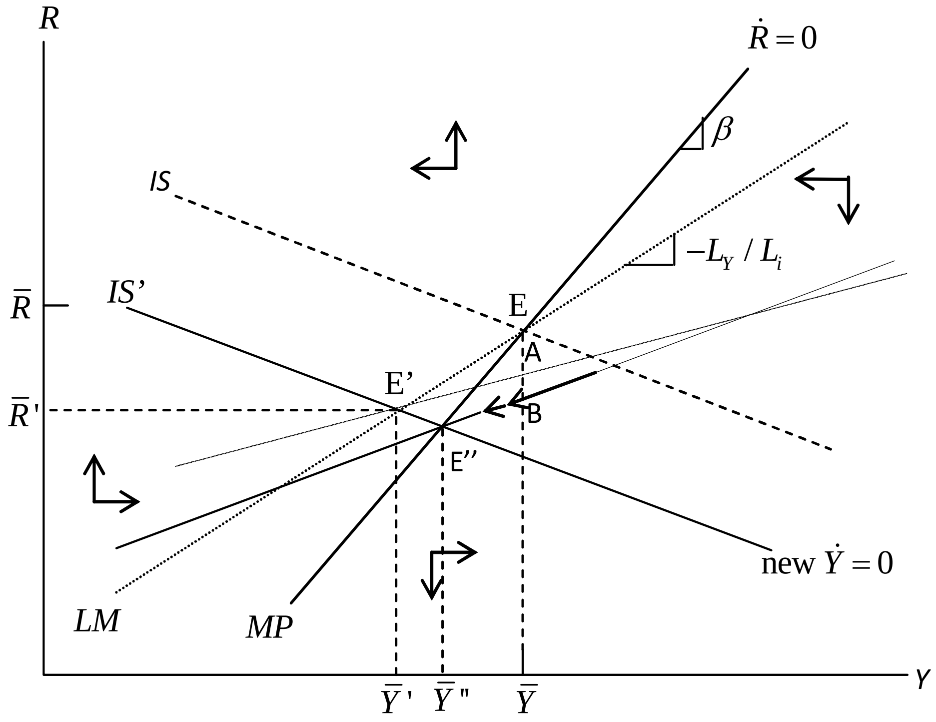

Figure 2.3:

features of Fig. 2.2 valid, but as $\beta > -L_Y/L_i$ the monetary policy curve (MP in Fig. 2.3) is now steeper than the old (LM in Fig. 2.2). The short-term interest rate is gradually reduced in response to the gradual fall in output. This is now the direct result of the decision by the CB to buy short-term bonds from the market to the extend needed to comply with (*).

g) In view of $\beta > -L_Y/L_i$, this active monetary policy implies that in response to the adverse demand shock, the short-term interest rate is reduced more than before along the $\dot{R} = 0$ curve (for a given reduction in Y). The output stabilization capability of the new policy rule is thus greater than that of the old more passive (monetarist) policy rule, cf. Fig. 2.3, where $\bar{Y}'' > \bar{Y}'$.

h) If the expansionary fiscal policy were implemented immediately at time t_1, anticipating the future higher short-term rates, the long-term rate would immediately jump up to point C in Fig. 2.4 and then the economy would gradually move along the new saddle path from C to the new steady state E"'. Owing to the time lag between announcement and implementation, however, the upward jump in the long-term rate will be smaller, to point D in Fig. 2.4, say. Then in the time interval (t_1, t_2) the dynamics are determined by the old phase diagram in Fig. 2.3. The economy will follow that path (DF in Fig.

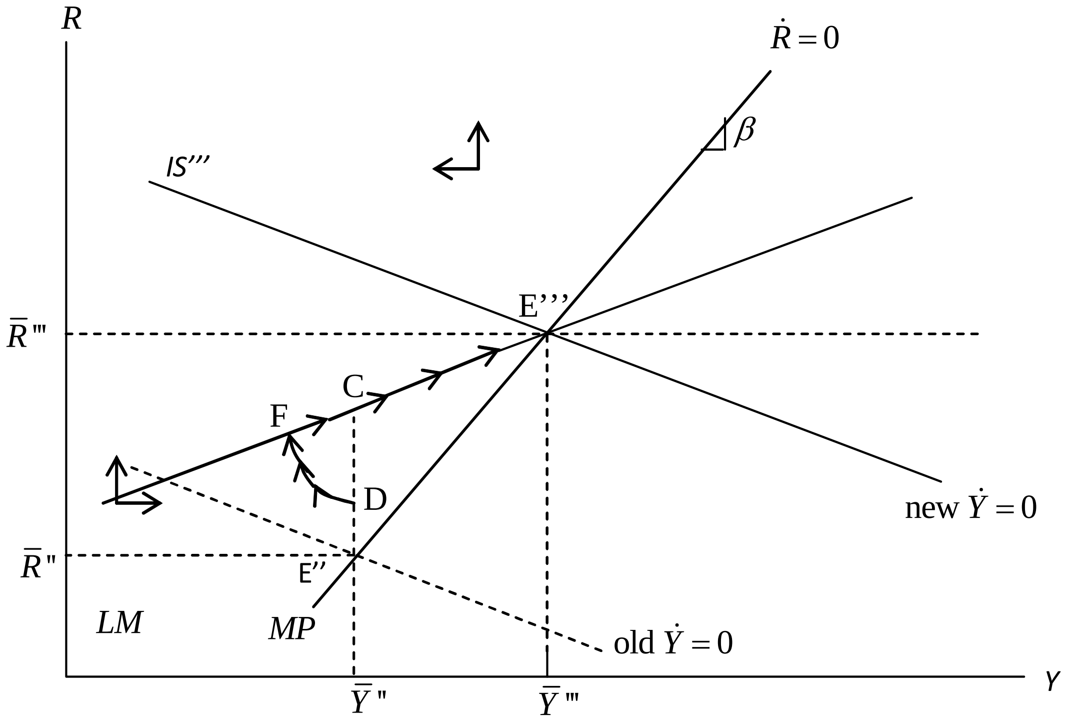

Figure 2.4:

2.4) for which, starting from a point on the vertical line $Y = \bar{Y}''$, it takes precisely $t_2 - t_1$ units of time to reach the new saddle path. The reason is that the market price, q, of the consol cannot have an *expected* discontinuity at time t_2, since such a jump would imply an infinite expected capital loss (or capital gain) per time unit immediately before $t = t_2$ by holding long-term bonds. Anticipating for example a capital loss, the market participants would want to sell long-term bonds in advance. The implied excess supply would generate an adjustment of q downwards until no longer a jump is expected to occur at time t_2. If instead a capital gain is anticipated, an excess demand would arise. This would generate in advance an upward adjustment of q, thus defeating the expected capital gain. This is the general principle that arbitrage prevents an expected jump in an asset price.

A credible announcement of future expansive fiscal policy thus has a temporary contractionary effect through the immediate upward jump in the long-term interest rate in the expectation of the higher future short-term rates. The reason is that the potentially counteracting force, the increase in G, has not yet taken place.

The qualitative features of the evolution of R and Y directly appears from the phase diagram in Fig. 2.4. Although not necessary (as question h) is stated), the time profiles of R and Y are portrayed directly in Fig. 2.5. With regard to r_t we have $r_t = i_t - \pi =$

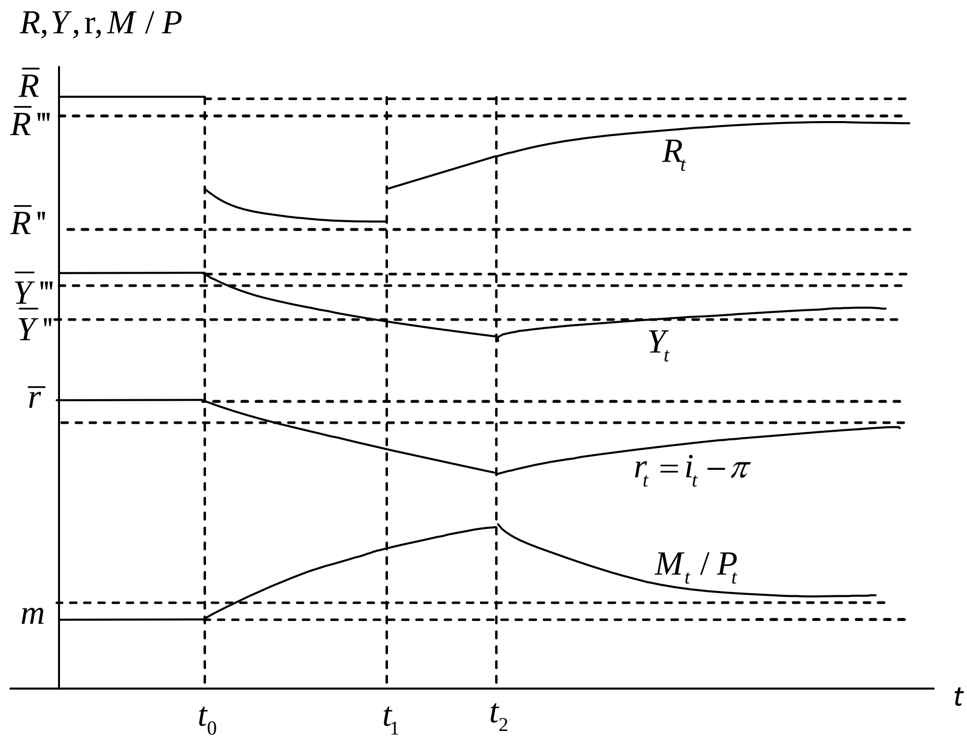

Figure 2.5:

$\alpha + \beta Y_t - \pi$. Therefore the evolution of r_t is simply a positive linear transformation of the evolution of Y_t. The fourth variable to be commented on is M_t. Admittedly, it would have been more straightforward if the question had been stated directly in terms of m_t instead of M_t. Anyway, we have $M_t \equiv P_t m_t$, where $m_t = L(Y_t, i_t)$, which is equivalent with $i_t = i(Y_t, m_t)$. Since the time path of P is exogenous and trivially given as $P_t = P_0 e^{\pi t}$, we may concentrate on the evolution of m_t (which exactly mirrors that of M_t in the case $\pi = 0$). The key equation here is $i_t = i(Y_t, m_t) = \alpha + \beta Y_t$. From $\beta > -L_Y/L_i = i_Y$ follows that when Y goes down, monetary policy becomes more lax than if m remained constant. So m must go up. Similarly, when Y goes up, monetary policy becomes more tight than if m remained constant. So m must go down, cf. Fig. 2.5. (As Fig. 2.5 is drawn, it is assumed that there is not a complete return to the pre-crisis output level.)

i) Given the intension of the government, it is a good idea to let the time interval (t_1, t_2) be short, possibly vanishing, thereby avoiding that the financial crowding out (rise in R) occurs before the demand stimulus through the higher G.

Given the intension of the government, it may also be a good idea to replace the monetary policy rule (*) with a fixed low policy rate $i_t = i$ for some time in order to postpone financial crowding until the recession/depression is over.

3. Solution to Problem 3

Although for convenience some formulas are inserted below, a purely verbal reasoning is enough.

a) The claim is true for the following reason. When the government as of period t_0 and onwards forever spends $\Delta\bar{G}$ more per period than otherwise, then the present value, from the perspective of period t_0, of future taxes will have to rise exactly by the present-value equivalent of $\Delta\bar{G}$ forever (which is $\sum_{i=0}^{\infty}\Delta\bar{G}(1+r)^{-(i+1)} = \Delta\bar{G}(1+r)^{-1}/(1-(1+r)^{-1})$ $= \Delta\bar{G}/r$) when the interest rate is $r > 0$). After-tax wealth is then reduced by the same amount. The representative household will thereby have to reduce the present value, from the perspective of period t_0, of its future consumption stream by the same amount. To smooth consumption, the representative household thus reduces its consumption level by $\Delta\bar{G}$ every period as of period t_0. So the sum $C_t + G_t$ is not raised by Policy B.

b) The claim is true for the following reason. When the government as of period t_0 temporarily, i.e., over $n+1$ periods, spends $\Delta\bar{G}$ more per period than otherwise, then the present value, from the perspective of period t_0, of future taxes will have to rise exactly by the present-value equivalent of $\Delta\bar{G}$ over these $n+1$ periods (which is $\sum_{i=0}^{n}\Delta\bar{G}(1+r)^{-(i+1)}$ $= \Delta\bar{G}(1+r)^{-1}(1-(1+r)^{-(n+1)})/(1-(1+r)^{-1}) = \Delta\bar{G}(1-(1+r)^{-(n+1)})/r$) when the interest rate is $r > 0$). After-tax wealth is then reduced by the same amount. The representative household will thus have to reduce the present value, from the perspective of period t_0, of its future consumption stream by the same amount. To smooth consumption, the representative household thus reduces its consumption level by x units per period as of period t_0 forever, where

$$\frac{x}{r} = \Delta\bar{G}\frac{(1-(1+r)^{-(n+1)})}{r} < \Delta\bar{G}\frac{1}{r}. \qquad (*)$$

It follows that $x < \Delta\bar{G}$. So the sum $C_t + G_t$ *is* raised temporarily by Policy B. There is thereby a temporary crowding out of investment.[3]

c) The claim is true for the following reason. In the first place, by a reasoning similar to that under b), Policy B raises the sum $C_t + G_t$ at least temporarily. Owing to unemployment and abundant production capacity, there need not be any crowding out of investment and so aggregate demand and employment may well increase in "the second

[3] A couple of the examinees interpret the term "Barro-style dynasties" to include the theoretically possible case where the intergenerational discount rate is large enough to *not* leave the bequest motive operative. Although the cited claims talk about the "representative household" and thereby in a sense invalidates this interpretation, it has been considered acceptable in the grading.

round". In this case, before-tax incomes increase and may through the "multiplier process" do so *more* (measured in present-value equivalents) than taxes are increased. So private consumption need not at all fall and may even rise.

d) The *Ricardian equivalence* proposition is the assertion that government debt is *neutral* in the sense that *for a given time path of government spending*, aggregate private consumption is unaffected by a temporary tax cut. The claim is that a temporary tax cut does not make the households feel richer because they are aware that the ensuing rise in government debt leads to a rise in future taxes the present value of which equals the current tax cut. In brief: the timing of (lump-sum) taxes does not matter.

The above debate is not directly about this since the debate is about possible effects of a *change* in the time path of government spending. More precisely, the debate is about whether Policy B can be expansionary in a setting where Ricardian equivalence *holds*.

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