Advanced Macroeconomics 2 Short Note 1

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Answers to questions about timing convention and intertemporal budget constraints

I have been asked three questions. The first is about the timing convention used in the course and the two others are about intertemporal budget constraints appearing on p. 116-117 in Section 4.1.2 (Bequests and Ricardian equivalence).

Question 1 *What is the timing convention in the lecture notes?*

The capital stock accumulated by the end of period $t-1$, and available for production in period t, is denoted K_t. Thus, $K_t = (1-\delta)K_{t-1} + I_{t-1}$ and $Y_t = F(K_t, L_t)$, where $F(\cdot)$ is the aggregate production function. Correspondingly, $\tilde{r}_t$ is the rental rate for renting a unit of capital in period t. And r_t is the rate of interest on a loan from the end of period $t-1$ to the end of period t. In this context it is useful to think of "period t" as running from date t to date $t+1$, i.e., the time interval $[t, t+1)$ on a continuous time axis (with time unit equal to the period length). Still, all decisions are taken at discrete points in time, $t = 0, 1, 2, ...$ ("dates"). We imagine that receipt of payments for work and lending and payment for consumption in period t occur at the end of period t. These timing conventions are common in discrete-time growth and business cycle theory.[1] They are convenient because they make switching between discrete and continuous time analysis easy.

Question 2 *What is the connection between the flow budget constraint and the intertemporal budget constraint of the govenrment (p. 116)?*

To see the connection, note that, by forward substitution, the flow budget constraint

[1] In contrast, in the standard finance literature, K_t would denote the *end-of-period* t stock that begins to yield its services *next* period.

gives

$$
\begin{aligned}
B_t &= (1+r)^t(T_t - G_t) + (1+r)^{-1}B_{t+1} \\
&= \sum_{i=0}^{j}(1+r)^{-(i+1)}(T_{t+i} - G_{t+i}) + (1+r)^{-(j+1)}B_{t+j+1} \\
&= \sum_{i=0}^{\infty}(1+r)^{-(i+1)}(T_{t+i} - G_{t+i}),
\end{aligned}
$$

assuming that government debt at most grows at a rate less than r, so that

$$\lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} = 0.$$

This is needed for government solvency.[2]

Question 3 *How is the intertemporal budget constraint of the dynasty, (4.18), related to the period budget constraints of a member of generation* t*?*

This relates to p. 117. With lump-sum taxation and constant interest rate, r, the period budget constraints of a member of generation t are

$$c_{1t} + s_t = w_t - \sigma_t + b_t, \qquad \text{and} \tag{1}$$

$$c_{2t+1} + (1+n)b_{t+1} = (1+r)s_t. \tag{2}$$

We isolate s_t in (2), substitute into (1), and reorder to get, by forward substitution,

$$
\begin{aligned}
b_t &= c_{1t} + \frac{c_{2t+1}}{1+r} - (w_t - \sigma_t) + \frac{1+n}{1+r}b_{t+1} \\
&= \sum_{i=0}^{j}(\frac{1+n}{1+r})^i\left[c_{1t+i} + \frac{c_{2t+i+1}}{1+r} - (w_{t+i} - \sigma_{t+i})\right] + (\frac{1+n}{1+r})^{j+1}b_{t+j+1} \\
&= \sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i\left[c_{1t+i} + \frac{c_{2t+i+1}}{1+r} - (w_{t+i} - \sigma_{t+i})\right], \qquad (3)
\end{aligned}
$$

assuming $\lim_{j\to\infty}(\frac{1+n}{1+r})^{j+1}b_{t+j+1} = 0$, in view of $r > n$. For every old in any given period there are $1+n$ young. We therefore multiply through in (3) by $1+n$ and reorder:

$$(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(c_{1t+i} + \frac{c_{2t+i+1}}{1+r}) = (1+n)b_t + (1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(w_{t+i} - \sigma_{t+i}).$$

To this we add the period budget constraint of an old member of the dynasty,

$$c_{2t} + (1+n)b_t = (1+r)s_{t-1},$$

[2] A detailed account of government solvency issues is postponed to Chapter 9.

and get the consolidated intertemporal budget constraint of the dynasty in period t:

$$c_{2t} + (1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(c_{1t+i} + \frac{c_{2t+i+1}}{1+r}) = (1+r)s_{t-1} + (1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(w_{t+i} - \sigma_{t+i}),$$

where $(1+n)b_t$ has been cancelled out on both sides. Dividing through by $1+r$ and reordering gives

$$\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i} + (1+n)c_{1t+i}\right] = s_{t-1} + (1+n)\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i} - \sigma_{t+i}).$$

This is the intertemporal budget constraint, as seen from the beginning of period t, of a dynasty with one old member in period t. With L_{t-1} old members, this becomes

$$\begin{aligned} & L_{t-1}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i} + (1+n)c_{1t+i}\right] \\ = \; & L_{t-1}s_{t-1} + L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i} - \sigma_{t+i}) = A_t + H_t, \end{aligned}$$

where A_t is the financial wealth in the beginning of period t and H_t is the human wealth, as defined in (4.19). Dividing through by N gives (4.18).