Advanced macroeconomics 2 Short Note 3.

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Solution formulas for linear differential equations of first order

In the lecture notes and other places we sometimes consider explicit solutions of linear differential equations. Apart from my general recommendation that you provide yourself with a handy mathematics manual (e.g., Berck and Sydsæter, Economists' Mathematical Manual, Springer-Verlag), I list here the most important formulas.[1]

I. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = (x_0 - x^*)e^{-at} + x^*, \text{ where } x^* = \frac{b}{a}. \tag{1}$$

II. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} \int_0^t b(s) e^{as} ds. \tag{2}$$

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-at} + e^{-at} c \int_0^t e^{(a+h)s} ds = (x_0 - \frac{c}{a+h})e^{-at} + \frac{c}{a+h} e^{ht}.$$

III. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(0) = x_0$. Solution:

$$x(t) = x_0 e^{-\int_0^t a(\tau)d\tau} + e^{-\int_0^t a(\tau)d\tau} \int_0^t b(s) e^{\int_0^s a(\tau)d\tau} ds. \tag{3}$$

A simple application of Rule I is the following. Suppose we know that a variable x grows at the constant rate g. Thus x satisfies the differential equation $\dot{x}(t)/x(t) = g$, i.e., $\dot{x}(t) - gx(t) = 0$. To apply rule 1 we set $a = -g$ and $b = 0$. Then, by (1),

$$x(t) = x_0 e^{gt}.$$

[1] The Math appendix in Barro and Sala-i-Martin, *Economic Growth*, second ed., 2004, may also be useful.

Suppose instead that x is known to grow at the time dependant rate $g(t)$. Thus x satisfies the differential equation $\dot{x}(t)/x(t) = g(t)$, i.e., $\dot{x}(t) - g(t)x(t) = 0$. To apply Rule III we set $a(t) = -g(t)$ and $b(t) = 0$ for all t. Then, by (3),

$$x(t) = x_0 e^{\int_0^t g(\tau)d\tau}.$$

This is the "basic growth formula". By defining the average growth rate as $\bar{g} = \frac{1}{t}\int_0^t g(\tau)d\tau$, the result can also be written

$$x(t) = x_0 e^{\bar{g}t}.$$

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