Advanced Macroeconomics 2

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A suggested solution to Problem X.2

X.2 The decision problem, as seen from period 0, is:

$$\max E_0(U_0) = E_0[\sum_{t=0}^{T-1}(\log c_t - \gamma\frac{\sigma}{1+\sigma}\ell_t^{(1+\sigma)/\sigma})(1+\rho)^{-t}] \qquad \text{s.t.}$$

$$c_t \geq 0, 0 \leq \ell_t \leq 1, \tag{1}$$

$$a_{t+1} = (1+r_t)a_t + w_t\ell_t - c_t, \qquad a_0 \text{ given}, \tag{2}$$

$$a_T \geq 0. \tag{3}$$

a) Defining $\tilde{U}_t \equiv (1+\rho)^t U_t$, the remainder of the problem as seen from period t (t $= 0, 1, ...$) is:

$$\begin{aligned} \max \ E_t\tilde{U}_t &= (1+\rho)^t E_t U_t \\ &= \log c_t - \gamma\frac{\sigma}{1+\sigma}\ell_t^{(1+\sigma)/\sigma} + (1+\rho)^{-1}E_t[\log c_{t+1} - \gamma\frac{\sigma}{1+\sigma}\ell_{t+1}^{(1+\sigma)/\sigma} + ...] \qquad (4) \\ &\text{s.t. (1) - (3)}, \quad a_t \text{ given.} \end{aligned}$$

To solve the problem we will use the substitution method. First, from (2) we have

$$c_t = (1+r_t)a_t + w_t\ell_t - a_{t+1}, \qquad \text{and} \tag{5}$$

$$c_{t+1} = (1+r_{t+1})a_{t+1} + w_{t+1}\ell_{t+1} - a_{t+2}.$$

Substituting this into (4), the problem is reduced to one of maximizing the function $E_t\tilde{U}_t$ wrt. $(\ell_t, a_{t+1}), (\ell_{t+1}, a_{t+2}), ..., (\ell_{T-1}, a_T)$. We get

$$\frac{\partial E_t\tilde{U}_t}{\partial \ell_t} = \frac{1}{c_t}w_t - \gamma\ell_t^{1/\sigma} = 0,$$

that is,

$$\gamma\ell_t^{1/\sigma} = \frac{1}{c_t}w_t \qquad t = 0, 1, 2, ..., T-1, \tag{*}$$

and

$$\frac{\partial E_t\tilde{U}_t}{\partial a_{t+1}} = \frac{1}{c_t}\cdot(-1) + (1+\rho)^{-1}E_t[\frac{1}{c_t+1}(1+r_{t+1})] = 0,$$

that is,

$$\frac{1}{c_t} = (1+\rho)^{-1} E_t[\frac{1}{c_t+1}(1+r_{t+1})], \qquad t = 0, 1, 2, ..., T-2. \qquad (**)$$

In view of the solvency condition (3), in the last period consumption must be

$$c_{T-1} = (1+r_{T-1})a_{T-1} + w_{T-1}\ell_{T-1},$$

since it is not optimal to end up with $a_T > 0$ (indeed, the transversality condition is $a_T = 0$).

b) The first-order condition (*) describes the trade-off between leisure in period t and consumption in the same period. The condition says that in the optimal plan, the cost (in terms of current utility) of increasing labor supply by one unit is equal to the benefit of obtaining an increased labor income and using this increase for extra consumption (i.e., marginal cost = marginal benefit).

The other first-order condition, (**), describes the trade-off between consumption in period t and consumption in period $t+1$, as seen from period t. The optimal plan must satisfy that the current utility loss by decreasing consumption c_t by one unit is equal to the discounted expected utility gain next period by having $1 + r_t$ extra units available for consumption, namely the gross return on saving one more unit (i.e., marginal cost = marginal benefit).

c) We rewrite (*) as

$$\gamma \ell_t^{1/\sigma} c_t = w_t. \qquad (*')$$

Apart from the finite horizon (which is not important in this context), the intertemporal utility function above could easily be a specification of the preferences of a representative household in a RBC model. Further, the RBC theory maintains that factor prices are always such that there is no unemployment. Hence, the prediction from the RBC theory is the same as that from condition (*), namely that, since employment is procyclical and fluctuates almost as much as GDP, and consumption and employment are positively correlated, real wages will also be procyclical and fluctuate almost as much as output. But according to the stylized fact (iii), real wages are only weakly procyclical and do not fluctuate much. This is one of the often mentioned difficulties faced by RBC theory.

d) By replacing t by $t+1$ in (*) we get

$$\gamma \ell_{t+1}^{1/\sigma} = \frac{1}{c_{t+1}} w_{t+1}$$

so that

$$(\frac{\ell_t}{\ell_{t+1}})^{1/\sigma} = \frac{c_{t+1}}{c_t}\frac{w_t}{w_{t+1}}. \tag{6}$$

Ignoring uncertainty, (**) gives

$$\frac{c_{t+1}}{c_t} = (1+\rho)^{-1}(1+r_{t+1}).$$

Substituting this into (6) and solving gives

$$\frac{\ell_t}{\ell_{t+1}} = (1+\rho)^{-\sigma}(\frac{w_t}{w_{t+1}/(1+r_{t+1})})^{\sigma}. \tag{7}$$

We see from this expression that σ is the elasticity of ℓ_t/ℓ_{t+1} wrt. w_t/w_{t+1}. Hence, σ measures what is called the *elasticity of intertemporal substitution* (in labor supply). From microeconometric studies we have estimates of this parameter. These estimates are quite small, at least for men (in the range 0 to 1.5, in many studies considerably below 1). And since fluctuations in w_t/w_{t+1} in the data are also small, it is difficult to reconcile the theory with the stylized fact (i) saying that employment fluctuates almost as much as GDP.

e) If fluctuations in the real wage are negligible, is it then likely that fluctuations in r_{t+1} could be a driving force behind fluctuations in employment? According to equation (7) one might be tempted to answer "yes". At least (7) indicates a positive relation between ℓ_t/ℓ_{t+1} and r_{t+1}. The interpretation of this relation is that a high interest rate has a negative substitution effect on leisure in the current period, hence positive substitution effect on current labor supply.

But if the real wage doesn't fluctuate, and an attempt is made to explain fluctuations in employment by fluctuations in the real interest rate, then, by (*), one would expect a *negative* correlation between employment and consumption. But the stylized fact (ii) tells the opposite.

f) We now reintroduce uncertainty. Indeed, there is now also uncertainty as to the prospect of employment in the future. The decision problem, as seen from period 0, can now be written:

$$\max E_0(U_0) = E_0[\sum_{t=0}^{T-1}(\log c_t - \gamma\frac{\sigma}{1+\sigma}\ell_t^{(1+\sigma)/\sigma})(1+\rho)^{-t}] \qquad \text{s.t.}$$

$$c_t \geq 0, 0 \leq \ell_t \leq \min(z_t, 1),$$

$$a_{t+1} = (1+r_t)a_t + w_t\ell_t - c_t, \qquad a_0 \text{ given,}$$

$$a_T \geq 0.$$

where $z_t \geq 0$ is the exogenous maximum employment offered the household in period t (this constraint coming from the demand side at the labor market).

When the employment constraint $\ell_t \leq z_t$ is binding, (*) is replaced by

$$\gamma \ell_t^{1/\sigma} \leq \frac{1}{c_t} w_t. \tag{8}$$

The interpretation of the possibility that $<$ obtains is: although in the optimal plan, the cost (in terms of current utility) of increasing labor by one unit is *less* than the benefit of obtaining an increased labor income and using this increase for extra consumption, this desired increase in employment cannot be realized, due to the exogenous employment constraint (involuntary unemployment).

g) Yes, within this extended framework it is possible to reconcile theory with the stylized facts. Indeed, rewriting (8) as

$$\gamma \ell_t^{1/\sigma} c_t \leq w_t,$$

we see that when $<$ is in force, there is scope for employment to be procyclical and fluctuate almost as much as GDP (fact (i)) and for consumption and employment to be positively correlated (fact (ii)), whereas real wages do not fluctuate much (fact (iii)).

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