Advanced Macroeconomics.   Exercise solutions

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An answer to Problem II.11

II.11

a) The parent cares about the descendants' utility:

$$U_t = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) + (1+R)^{-1}(1+n)U_{t+1}.$$

By forward substitution of U_{t+1}, U_{t+2}, etc., we arrive at (*), since $\lim_{j\to\infty}(\frac{1+n}{1+R})^{j+1}U_{t+j+1} = 0$, in view of $R > n$ and no technical progress.

b) We insert into U_t the effective intergenerational discount rate $\bar{R}$, defined by $1+\bar{R} \equiv (1+R)/(1+n)$, and the two period budget constraints in order to consider the objective of the parent as a function, $\tilde{U}_t$, of the decision variables, s_t and b_{t+1}. Then, wrt. s_t we get

$$\begin{aligned}\frac{\partial \tilde{U}_t}{\partial s_t} &= -u'(c_{1t}) + (1+\rho)^{-1}u'(c_{2t+1})(1+r) = 0, \text{ i.e.,} \\ u'(c_{1t}) &= (1+\rho)^{-1}u'(c_{2t+1})(1+r). \qquad \text{(FOC1)}\end{aligned}$$

Wrt. b_{t+1} we get, when the constraint $b_{t+1} \geq 0$ is not binding (Case a),

$$\begin{aligned}\frac{\partial \tilde{U}_t}{\partial b_{t+1}} &= (1+\rho)^{-1}u'(c_{2t+1})\left[-(1+n)\right] + (1+\bar{R})^{-1}u'(c_{1t+1}) \cdot 1 = 0, \text{ i.e.,} \\ (1+\rho)^{-1}u'(c_{2t+1}) &= (1+\bar{R})^{-1}u'(c_{1t+1})\frac{1}{1+n}. \qquad \text{(FOC2a)}\end{aligned}$$

When the constraint $b_{t+1} \geq 0$ is binding (Case b), we get

$$\begin{aligned}\frac{\partial \tilde{U}_t}{\partial b_{t+1}} &= (1+\rho)^{-1}u'(c_{2t+1})\left[-(1+n)\right] + (1+\bar{R})^{-1}u'(c_{1t+1}) \cdot 1 \leq 0, \text{ i.e.,} \\ (1+\rho)^{-1}u'(c_{2t+1}) &\geq (1+\bar{R})^{-1}u'(c_{1t+1})\frac{1}{1+n}. \qquad \text{(FOC2b)}\end{aligned}$$

Comment: in Case a the optimal b_{t+1} satisfies (FOC2a) which says that the parent's utility cost of increasing the bequest by one unit in an interior optimum must equal the discounted utility benefit derived from the next generation having $1/(1+n)$ more units per member for consumption in the same period. In Case b, however, the optimal $b_{t+1} = 0$, i.e., we have a corner solution, which has the property that the parent's utility cost of increasing the bequest by one unit either exceeds or equals the discounted utility benefit derived from the next generation having $1/(1+n)$ more units per member for consumption in the same period.

c) Since the agents are at an interior solution and $r = R$ (the modified golden rule property), the economy must be in a steady state. A detailed proof goes as follows. Inserting (FOC2a) on the right-hand side of (FOC1) gives

$$u'(c_{1t}) = (1 + \bar{R})^{-1} u'(c_{1t+1}) \frac{1+r}{1+n}. \qquad (**)$$

Given $r = R$, we have $1 + \bar{R} \equiv (1+r)/(1+n)$ so that (**) gives

$$u'(c_{1t}) = u'(c_{1t+1}), \qquad (1)$$

which implies $c_{1t} = c_{1t+1}$, since $u'' < 0$. By (FOC1), forwarded one period, we have

$$u'(c_{2t+2}) = (1+\rho)(1+r)^{-1} u'(c_{1t+1}) = (1+\rho)(1+r)^{-1} u'(c_{1t}) = u'(c_{2t+1}),$$

by (1) and (FOC1), respectively. It follows that $c_{2t+1} = c_{2t+2}$. Thus, the economy is in a steady state.

d) The first row of the IBC of the government says that the PDV of spending = initial total wealth. The second row says that the PDV of taxes = PDV of spending plus initial debt. The third row says that the PDV of primary surpluses = initial debt.

e) The first equation reflects that the PDV of the consumption plan of the representative dynasty cannot exceed its initial total wealth; and the optimal consumption plan will imply equality. The second equation defines the human wealth of the dynasty as the PDV of after tax income to the young plus the pension to the old.

f) The consumption path $(c_{2t+i}, c_{1t+i})_{i=0}^{\infty}$ is not affected because the dynasty will not feel reacher. Although the pensions are now higher, sooner or later taxes will have to be increased in order that the IBC of the government can be satisfied. The PDV of these extra taxes will exactly match the PDV of the extra pensions. As the question is framed, this answer is sufficient. But if you were asked to "show" or "derive" the result, then you would have to deliver an argument like that in Chapter 4, p. 117.

In the present case, where $\gamma = 0$ (indicating that there is no government spending on goods and services), we get

$$\begin{aligned} B_t + H_t &= L_t \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}} \left[\sigma_{t+i} - \frac{\pi}{1+n} + w_{t+i} - \sigma_{t+i} + \frac{\pi}{1+n} \right] \\ &= L_t \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}} w_{t+i}. \end{aligned}$$

We see that the time profiles of π and σ have vanished and cannot affect $B_t + H_t$, hence, cannot affect the total wealth of the dynasty, $A_t + H_t = K_t + A_t^f + B_t + H_t$, where A_t^f is net foreign assets.

g) The consumption path $(c_{2t+i}, c_{1t+i})_{i=0}^{\infty}$ is not affected because the dynasty will not feel reacher. Although taxes on the young are now lower, sooner or later taxes will have to be increased in order that the IBC of the government can be satisfied. The PDV of the tax cuts will exactly match the PDV of the extra taxes later.

h) No, if the economy is in steady state, then $c_{1t} = c_{1t+1}$ so that $u'(c_{1t}) = u'(c_{1t+1})$, implying that (**) no longer holds. Indeed, in view of $R > r$, i.e., $1 + R = (1 + \bar{R})(1 + n) > 1 + r$, so that

$$u'(c_{1t}) > (1 + \bar{R})^{-1} u'(c_{1t+1}) \frac{1+r}{1+n}.$$

Substituting this into (FOC1) gives

$$(1+\rho)^{-1} u'(c_{2t+1}) > (1 + \bar{R})^{-1} u'(c_{1t+1}) \frac{1}{1+n}. \tag{2}$$

This shows that the bequest motive cannot be operative (we are in Case b). Indeed, the parent would prefer to leave negative bequests, but that is forbidden. Hence, $b_{t+1} = 0$.

i) Yes, resource allocation is affected because the old parent now has the opportunity to partly realize the preference for more consumption, still leaving no bequests.

j) Yes, resource allocation is affected because the young parent now has the opportunity to partly realize the preference for more consumption, both in the first and the second period of life, still leaving no bequests in the second period.

k) If $r > R$, a model like this is likely to run into trouble as a model for a small open economy. This is because, when $1 + r > 1 + R$, the rate of return is higher than what is required for inducing the saving (by the young) needed to maintain a steady state. Consequently, saving (by the young) will be higher, and the economy will accumulate wealth and capital forever and grow at a rate above n. In the long run the economy is no longer likely to be small relatively to the rest of the world. On average the rest of the world is likely to have an intergenerational discount rate equal to r and therefore will not be growing at a rate higher than n.

ℓ) Positive features of the Barro model include: it takes inheritance into account, constitutes a theoretically interesting benchmark case, provides a nice interpretation of the Ramsey model. More problematic features include: the model ignores that a large fraction of a population typically does not leave bequests and that an even smaller fraction does it for the reason assumed by Barro. Empirically, the model is rejected.