Advanced Macroeconomics 2. Exercises

26.09.2008. Christian Groth

Problem set IV: Intertemporal optimization and OLG models in continuous time

IV.1 *Short questions*

a) State with your own words what the standard No-Ponzi-Game condition says.

b) Is the standard No-Ponzi-Game condition a constraint in the maximization problem or does it express a property of the solution?

c) The No-Ponzi-Game condition belongs to problems with an infinite horizon. What is the analogue condition for a problem with finite horizon?

d) State with your own words what the perfect foresight transversality condition of a household with infinite horizon says.

e) In a standard consumption/saving problem, is the household's transversality condition a constraint in the maximization problem or does it express a property of the solution?

IV.2 *On implications of the NPG condition.* Consider the following budget constraint for an agent with infinite horizon:

$$\dot{a}(t) = ra(t) + w(t) - c(t), \qquad a(0) \text{ historically given}, \tag{*}$$

$$\lim_{t\to\infty} a(t)e^{-rt} \geq 0, \tag{**}$$

where the notation is as usual ($a(t)$ is financial wealth and $w(t)$ is labor income, the supply of labor being one unit per time unit). To make things easy we have assumed that the rate of return on savings is a positive constant r (if you want to interpret this problem as a problem within the Blanchard model of overlapping generations in continuous time, expand r with the mortality rate p).

a) Can a consumption plan implying a negative $a(t)$ for all $t > t_1$ (t_1 is some fixed positive number), i.e., a positive debt $d(t) \equiv -a(t)$ for all $t > t_1$, satisfy the No-Ponzi-Game condition condition (from now just denoted NPG)?

b) Assume $d(0) \equiv -a(0) > 0$. Can a consumption plan implying that, for all $t \geq 0$, current labor income $w(t)$ covers current consumption plus some constant fraction of current interest payments $rd(t)$, that is, $w(t) = c(t) + \beta rd(t)$, where $0 < \beta < 1$, be consistent with the NPG condition?

c) Assume again $d(0) \equiv -a(0) > 0$. Can a consumption plan implying that, for all $t \geq 0$, current labor income covers only current consumption or less than that, be consistent with the NPG condition?

d) Suppose the economy is in a steady state with *no technical progress* and that the agent not only lives forever, but has also the same effective labor supply forever. Then we can put $w(t)$ equal to a given constant, $w > 0$. "In this case it is not possible to follow the consumption plan described in question b), if $d(0) > 0$; what is violated is the described plan, not the consistency of this plan with the NPG condition." Do you agree or not agree in these statements? Why?

Consider the infinite horizon consumer problem (with $\rho > 0$): choose a consumption plan $c(t)_{t=0}^{\infty}$, to maximize $\int_0^\infty \ln c(t)\, e^{-\rho t} dt$ s.t. $c(t) > 0$ and (*) and (**).

e) Assuming $\rho = r$ and $0 < rd(0) < w$, determine the highest initial level of consumption which is consistent with both the Keynes-Ramsey rule and the NPG condition. *Hint*: the differential equation $\dot{x} - ax = -b$, where a and b are constants, has the solution $x(t) = [x(0) - b/a]\, e^{at} + b/a$. Comment on the result.

f) Suppose that due to technical progress, the growth rate of the real wage is a positive constant $g < g + \rho = r$. Under this condition, solve the consumer problem above. *Hint*: it is convenient at some step in the analysis to solve for the time path of $d(t)$, using the fact that a differential equation $\dot{x} - ax = -be^{ct}$, where a, b, and c are constants, $a \neq c$, has the solution $x(t) = \left[x(0) - \frac{b}{a-c}\right] e^{at} + \frac{b}{a-c} e^{ct}$.

g) Assuming $d(0) > 0$, find that specific time path of the debt/income ratio, $d(t)/w(t)$, which is implied by the solution to f).

h) Show that given $g < g + \rho = r$ (as described under f)) a consumption plan as described in question b) is followed. Find the implied value of β.

IV.3 *A positive technology shock* An economy is described by the two differential equations

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \frac{\omega + n + p}{n + p}\tilde{c}_t - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{1}$$

$$\dot{\tilde{c}}_t = \left[f'(\tilde{k}_t) - \delta - \rho + \omega - g\right]\tilde{c}_t - (n + p)(\rho + p)\tilde{k}_t, \tag{2}$$

and the condition that for any fixed pair (v, t_0), where $t_0 \geq 0$ and $v \leq t_0$,

$$\lim_{t\to\infty} a_{v,t}e^{-\int_{t_0}^{t}(f'(\tilde{k}(s))-\delta+p)ds} = 0. \tag{3}$$

Notation is: $\tilde{k}_t \equiv K_t/(T_tN_t)$ and $\tilde{c}_t \equiv C_t/(T_tN_t) \equiv c_t/T_t$, where K_t and C_t are aggregate capital and aggregate consumption, respectively, and N_t is population = labor supply, all at time t. Further, T_t is a measure of the technology level, $T_t = T_0e^{gt}$, where $T_0 > 0$ is given and f is a production function on intensive form, satisfying $f(0) = 0$, $f' > 0$, $f'' < 0$ and the Inada conditions. Finally, $a_{v,t}$ denotes planned financial wealth at time $t \geq t_0$ of an individual born at time $v \leq t_0$. The remaining symbols stand for parameters and all these are positive. Furthermore, $\rho \geq n$.

a) Briefly interpret the three above equations, including the parameters.

b) Assume $\omega < \delta + \rho + g$. Draw a phase diagram and illustrate the path the economy follows, given some arbitrary positive initial value of $\tilde{k}$. Can the divergent paths be excluded? Why or why not?

c) Is dynamic inefficiency theoretically possible in this economy? Why or why not?

Assume the economy has been in steady state until time 0. Then an unanticipated technology shock occurs so that T_0 is replaced by $T_0' > T_0$. After this shock everybody rightly expects T to grow forever at the same rate, g, as before.

d) Illustrate by a new phase diagram what happens to $\tilde{k}$ and $\tilde{c}$ on impact, i.e., immediately after the shock, and in the long run.

e) What happens to the real interest rate on impact and in the long run? Comment.

f) Why is the sign of the impact effect on the real wage ambiguous (at the theoretical level) as long as f is not specified further?[1]

g) What happens to the real wage in the long run?

h) Why is the sign of the impact effect on c ambiguous? *Hint:* $cN \equiv C = (\rho + p)(K + H)$.

i) What happens to $c \equiv C/N$ in the long run?

IV.4 *Productivity speed up* The basic model for this problem is the same as in problem IV.3. Assume the economy has been in steady state until time 0. Then an unanticipated shift in g to a higher positive level occurs. Hereafter everybody rightly expects g to remain at this new level forever.

a) What happens to the real interest rate on impact and in the long run? Comment.

b) Illustrate by a new phase diagram what happens. There might be different possibilities to consider. Comment.

c) Compare two closed economies, A and B, that can be described by this model and have the same production function, the same g and ρ, the same initial conditions, K_0, T_0, and N_0, and the same n. The only difference is that country B for some reason has a higher health level and therefore lower p than country A (and lower b, since n is the same). "Country B will in the long run experience a higher level of labor productivity, Y/N, than country A". True or false? Why?

IV.5 *Theory of the rate of return in the long run.* Consider a Blanchard OLG model for a closed economy as described in Problem IV.3.

a) The model entails a simple theory of the rate of return, r^*, in the long run. For example, the model implies that r^* must belong to an open interval, defined by the parameters ρ, g, ω, δ, n, and p. Show this. (*Hint:* as to one of the end points of this interval you may use the fact that the steady-state value of $\tilde{k}$ can be shown to be larger than some $\underline{\tilde{k}}$ satisfying the requirement $f'(\underline{\tilde{k}}) - \delta \leq \rho + g + n + p$.)

[1] Remark: for "empirically realistic" production functions (having elasticity of factor substitution larger than elasticity of production wrt. capital), the impact effect *is* positive, however. See Exercise Problem II.12, c).

b) We are interested not only in knowing this interval, but also in a formula displaying the elasticity of r^* wrt. life expectancy. Derive such a formula. What is the sign of this elasticity?

c) Compare the theory of the rate of return implied by this model to other simple theories.

Although absent from our models so far, uncertainty and risk of bankruptcy are significant parts of reality. They explain firms' unwillingness to finance all their investment by debt in spite of the lower rate of return on debt than on equity. In this way the excess of the rate of return on equity over that on debt, the equity premium, is sustained.

A simple, behavioral theory of the equity premium goes as follows. Firms prefer to finance a fraction b_f by bonds and the remaining fraction, $1 - b_f$, by equity. Households prefer a portfolio with a fraction b_h of financial wealth in bonds and the fraction $1 - b_h$ in equities. In view of risk aversion and memory of historical stock market crashes, the theory assumes $b_h > b_f$.

Thus, out of a long-run payout equal to $1 + r^*$ per real unit invested in firms' physical capital, $(1 + r^*)b_f$ constitutes the return to the bond owners and $(1 + r^*)(1 - b_f)$ the return to the equity owners. Let r_b^* denote the rate of return on bonds and r_e^* the rate of return on equity.

d) Find $1 + r_b^*$, $1 + r_e^*$, and the equity premium, $(1 + r_e^*)/(1 + r_b^*)$.

e) It is generally believed that b_h is nowadays lower than in the long aftermath of the Great Depression. How has this affected the equity premium, according to this simple theory?

f) How do you think the recent global financial crisis will affect the equity premium? Why?

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