Advanced Macroeconomics
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A suggested solution to the problem set at the exam January 19, 2004

Four hours. No auxiliary material

1. Solution to Problem 1

For convenience we repeat the basic equations. The model is:

$$\dot{C}_t = (F_K(K_t, L) - \delta - \rho)C_t - p(\rho + p)(K_t + B_t), \tag{1.1}$$

$$\dot{K}_t = F(K_t, L) - \delta K_t - C_t - G, \tag{1.2}$$

$$\dot{B}_t = [F_K(K_t, L) - \delta] B_t + G - T_t, \tag{1.3}$$

together with the condition

$$\lim_{t\to\infty}(K_t + B_t)e^{-\int_0^t [F_K(K_s,L)-\delta+p]ds} = 0, \tag{1.4}$$

and a requirement that the government remains solvent.

a) Evidently, the model is a Blanchard OLG model for a closed economy with public debt and lump-sum taxation. In this version there is a constant population, and technical progress is ignored. Individuals have finite, but uncertain remaining lifetime. The parameter p is the death rate, i.e., p is the expected number of deaths per time unit, say per year, relatively to the size of population. The model relies on the simplifying assumption that for a given individual the probability of having a remaining lifetime, X, longer than some arbitrary number τ is $P[X > \tau] = e^{-p\tau}$, the same for all independently of age. It follows that for any person the probability of dying within one year from now is approximately equal to p. Since a constant population is assumed, the birth rate is equal to p. At the aggregate level the model appeals to the law of large numbers and considers the actual number of deaths (births) per year to be indistinguishable from the expected number.

Individuals can buy life annuity contracts from life insurance companies. These companies have negligible administrative costs so that in equilibrium with free entry (zero profits), the rate of return on these contracts is $r + p$ until death, where r is the safe (real) rate of interest (actuarial fairness).

Subject to their intertemporal budget constraint, individuals maximize expected lifetime utility

$$E_t\left[U_t\right] = E_t\left[\int_t^{t+X} e^{-\rho(v-t)} \log c_{j,v} dv\right] = \int_t^{\infty} e^{-(\rho+p)(v-t)} \log c_{j,v} dv,$$

where $c_{j,v}$ is consumption (per time unit) at time v for an individual born at time j, and ρ is the "pure" rate of time preference (a measure of impatience). Labour supply is inelastic and equal to 1 unit per time unit. This leads to the Keynes-Ramsey rule (with elasticity of marginal utility equal to one)

$$\dot{c}_{j,t} \equiv \frac{\partial c_{j,t}}{\partial t} = \left[r_t + p - (\rho + p)\right] c_{j,t} = (r_t - \rho) c_{j,t}, \tag{1.5}$$

which, combined with the transversality condition, (1.4), implies the consumption function

$$c_{j,t} = (\rho + p)(a_{j,t} + \tilde{w}_{j,t}), \tag{1.6}$$

where $a_{j,t}$ is financial wealth, and $\tilde{w}_{j,t}$ is "human capital" (PDV of future net earnings),

$$\tilde{w}_{j,t} \equiv \int_t^{\infty} (w_v - \frac{T_v}{L}) e^{-\int_t^v (r_s + p) ds} dv. \tag{1.7}$$

Here, w_v is the real wage at time v, and T_v/L is the per capita lump sum tax at time v.

Aggregating over all times of birth and relying on perfect competition and clearing on factor markets, (1.6) leads to (1.1), using the fact that aggregate financial wealth, A_t, must be equal to $K_t + B_t$.

An interpretation of (1.1) runs as follows. The first term in (1.1) reflects the Keynes-Ramsey rule (1.5) and the fact that $r_t = F_K(K_t, L) - \delta$ in equilibrium. The subtraction of the term $p(\rho + p)(K_t + B_t)$ in (1.1) is due to a *generation replacement effect*. Indeed, in every short instant some people die and some people are born. The first group has financial wealth, but the last group has not. The arrival of newborns is Lp per time unit, and since they have no financial wealth the inflow of these people lowers aggregate consumption by $p(\rho + p)A_t$ per time unit. Indeed, the average financial wealth in the population is A_t/L, and the consumption effect of this is $(\rho + p)A_t/L$, cf. (1.6). This implies that, *ceteris paribus*, aggregate consumption is reduced by

$$Lp(\rho + p)\frac{A_t}{L} = p(\rho + p)A_t$$

per time unit. Since $A_t = K_t + B_t$, this explains the last term in (1.1).

The second differential equation, (1.2), is easier. Since δ is the rate of physical capital depreciation, (1.2) is just a way of writing the national income identity for a closed economy: Gross investment, $\dot{K}_t + \delta K_t$, equals gross national income, $F(K_t, L)$, minus the sum of private and public consumption, $C_t + G$.

The differential equation (1.3) gives the increase per time unit in real public debt as equal to the budget deficit, that is, total government expenditure (interest payments plus spending on goods and services) minus net tax revenue. This tells us that the budget deficit is completely debt-financed (no money financing).

(This answer is more detailed than necessary.)

b) Given B_0 and a balanced budget for all $t \geq 0$, we have, from (1.3),

$$T_t = (F_K(K_t, L) - \delta)B_0 + G. \tag{1.8}$$

In order to be able to draw the phase diagram we introduce two baseline values of K, namely the golden rule value, K_g, and the "critical" value, $\bar{K}$, defined by,

$$F_K\left(K_g, L\right) - \delta = 0, \qquad \text{and} \qquad F_K\left(\bar{K}, L\right) - \delta = \rho,$$

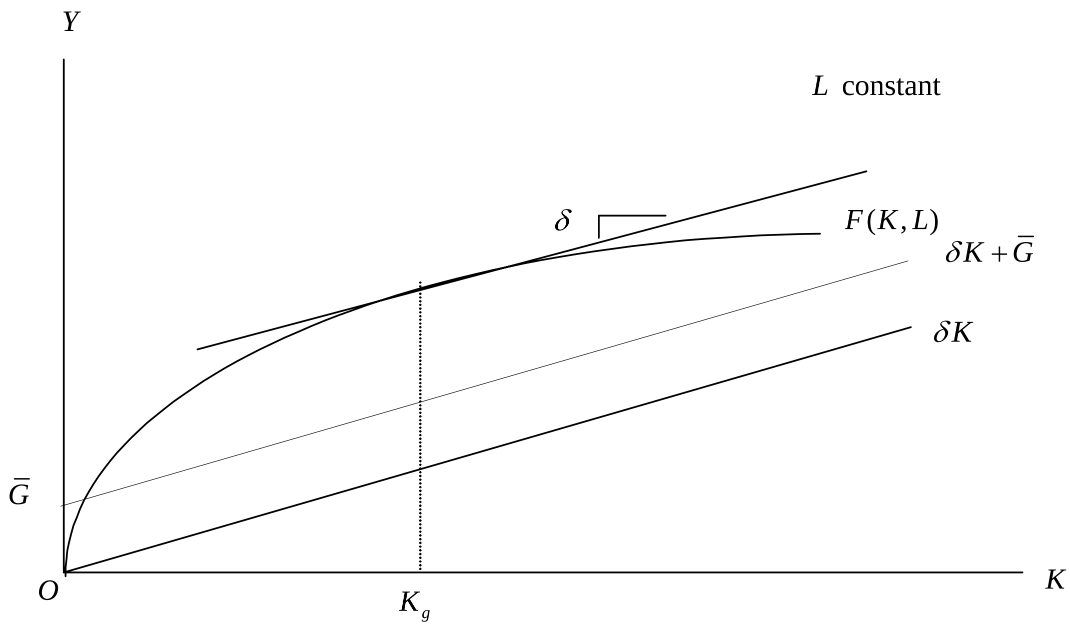


Figure 1.1:

respectively. In view of the assumption that the production function F satisfies the Inada conditions (and $\delta > 0$), both values exist and are unique (since $F_{KK} < 0$). We have $\bar{K} < K_g$, in view of $\rho > 0$ and $F_{KK} < 0$.

Equation (1.2) shows that $\dot{K} = 0$ for

$$C = F(K, L) - \delta K - G.$$

Fig. 1.1 illustrates how (in principle) the $\dot{K} = 0$ locus in Fig. 1.2 is constructed.

Equation (1.1) shows that $\dot{C} = 0$ for

$$C = \frac{p\,(\rho + p)\,(K + B_0)}{F_K(K, L) - \delta - \rho}.$$

Hence, along the $\dot{C} = 0$ locus

$$\frac{dC}{dK}\Big|_{\dot{C}=0} = p\,(\rho + p)\,\frac{F_K(K, L) - \delta - \rho - (K + B_0)\,F_{KK}\,(K, L)}{(F_K(K, L) - \delta - \rho)^2},$$

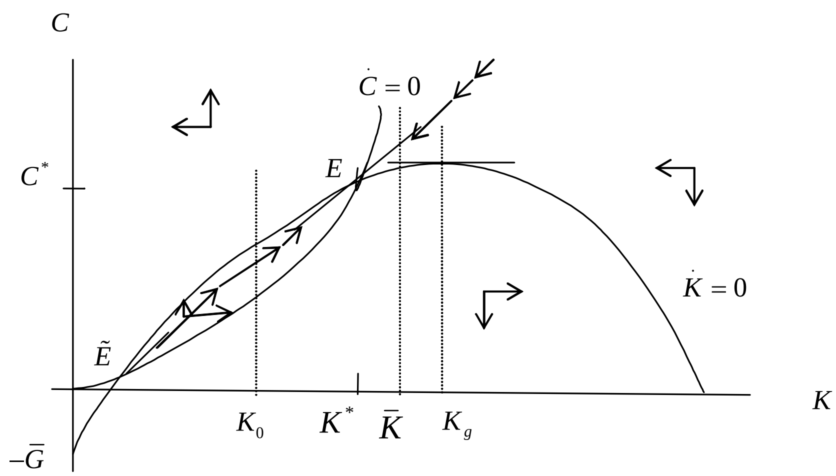


Figure 1.2:

which is positive, when $F_K(K, L) - \delta > \rho$, that is, when $K < \bar{K}$. Further, along the $\dot{C} = 0$ locus

$$K \nearrow \bar{K} \Rightarrow C \to \infty.$$

In view of the lower Inada condition, along the $\dot{C} = 0$ locus we have, in addition,

$$K \searrow 0 \Rightarrow C \to 0.$$

The $\dot{C} = 0$ locus is shown in Fig. 1.2. It is assumed that, given K_0, G and B_0 are "modest" relative to the production possibilities of the economy. Then the $\dot{C} = 0$ curve crosses the $\dot{K} = 0$ curve for *two* positive values of K. Fig. 1.2 shows these steady states as the points E and Ẽ with coordinates (K^*, C^*) and $(\tilde{K}^*, \tilde{C}^*)$, respectively. Obviously, $\tilde{K}^* < K^* < \bar{K}$.

The direction of movement in the different regions of Fig. 1.2, as determined by the differential equations, (1.1) and (1.2), are shown by arrows. It is seen that E is a saddle point, whereas Ẽ is totally unstable. Since G and B_0 are “modest", we have that initial K, K_0, is larger than $\tilde{K}^*$ as shown in the figure.

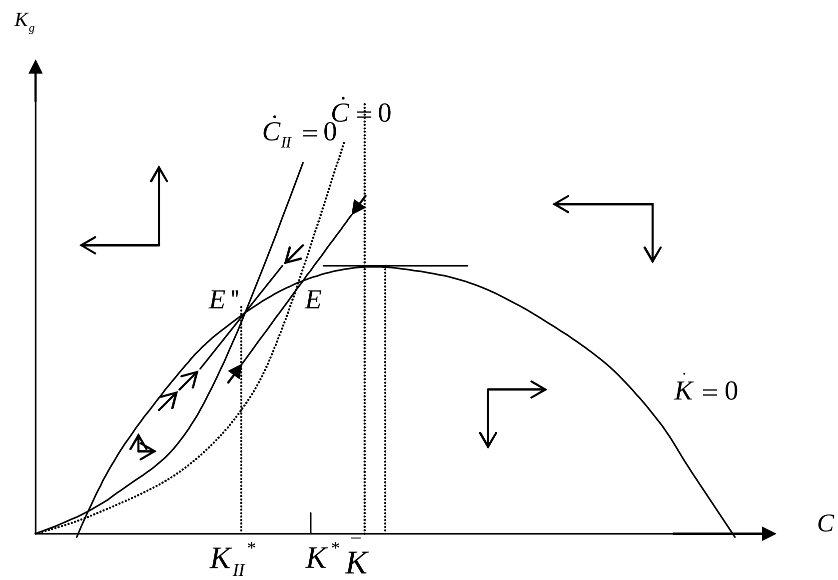


Figure 1.3:

The capital stock is predetermined, whereas consumption is a jump variable. It follows that the system is saddle-point stable. The only dynamic path satisfying *all* the conditions of general equilibrium (individual utility maximization for given expectations, continuous market clearing, perfect foresight) is the saddle path. The other dynamic paths in the diagram either violate the transversality condition of the individual household (paths that in the long run point South-East in Fig. 2) or the NPG condition[1] of the household (paths that in the long run point North-West in the diagram). Hence, initial consumption, C_0, is determined as the ordinate to the point where the vertical line $K = K_0$ crosses the saddle path, and over time the economy moves along the saddle path, approaching the steady state point E with coordinates (K^*, C^*). If $\tilde{K}^* < K_0 < K^*$, as in Fig. 1.2, then K and C grow over time until the steady state E is "reached".

[1] And therefore also the transversality condition.

c) Let $B_0 = B_0^I < B_0^{II}$ and $K_0 = K_0^I = K_0^{II}$. Fig. 1.3 illustrates. In the long run country II has less capital and a lower consumption level, due to the crowding-out effect of government debt.

d) We assume that until time t_0 (> 0) country I has been in the saddle-point stable steady state E with a balanced government budget. The level of public debt in this steady state is $B_0 > 0$, and tax revenue is, by (1.8),

$$T = (F_K(K^*, L) - \delta)B_0 + G \equiv T^*,$$

a positive constant, in view of $F_K(K^*, L) - \delta > \rho > 0$.

At time t_0 the government cuts taxes to a lower level $\bar{T}$, holding public consumption unchanged. That is, at least for a while after time t_0 we have

$$T_t = \bar{T} < T^*. \tag{1.9}$$

As a result $\dot{B}_t > 0$. The tax cut make current generations feel more wealthy, hence they increase their consumption. The rise in C combined with unchanged G implies negative net investment, and K begins to fall, implying a rising rate of interest r. For a while all the three differential equations that determine changes in C, K, and B are active. These dynamics are complicated and cannot, of course, be illustrated in a two-dimensional phase diagram.

In view of $K^* < \bar{K} < K_g$, we have $r^* = F_K(K^*, L) - \delta > F_K(K_g, L) - \delta = 0$. Therefore, r_t ($\geq r^*$) is strictly positive, and in order to be sustainable, as seen from time t_0, where the level of debt is $B_0 > 0$, a fiscal policy must satisfy the NPG condition

$$\lim_{t\to\infty} B_t e^{-\int_{t_0}^{t} r_s ds} \leq 0. \tag{1.10}$$

This requires

$$\lim_{t\to\infty} \frac{\dot{B}_t}{B_t} < \lim_{t\to\infty} r_t. \tag{1.11}$$

Now, the fiscal policy $(G, \bar{T})$ implies increasing public debt B_t. Indeed, we have, for $t > t_0$,

$$\begin{aligned} \dot{B}_t &= r_t B_t + G - \bar{T} \\ &> r^* B_0 + G - \bar{T} > r^* B_0 + G - T^* = 0, \end{aligned} \tag{1.12}$$

where the first inequality comes from $B_t > B_0 > 0$ and $r_t = F_K(K_t, L) - \delta > r^* = F_K(K^*, L) - \delta$, in view of $K_t < K^*$. This implies $B_t \to \infty$ for $t \to \infty$, and therefore, dividing by B_t in (1.12), we get

$$\frac{\dot{B}_t}{B_t} = r_t + \frac{G - \bar{T}}{B_t} \to r_t \qquad \text{for} \qquad t \to \infty. \tag{1.13}$$

But this violates the NPG condition (1.11), and the fiscal policy $(G, \bar{T})$ is not sustainable. Hence, sooner or later the fiscal policy must change, either by lowering public consumption or by raising taxes.

e) Now we extend the model by assuming age-dependent labour supply. Let the (inelastic) labour supply (per time unit) at time t for an individual born at time j be

$$E_{t-j} = me^{-\omega(t-j)},$$

where $\omega > 0$ is the "retirement rate", and m is a positive constant. Normalizing aggregate labour supply, N, so that $N = L$, implies fixing m such that

$$N = \int_{-\infty}^{t} me^{-\omega(t-j)} Lpe^{-p(t-j)} dj = L,$$

which gives

$$m = \frac{\omega + p}{p}.$$

The implication is that, whereas (1.2) is not affected by this extention, (1.1) changes to

$$\dot{C}_t = (F_K(K_t, L) - \delta - \rho + \omega) C_t - (\omega + p)(\rho + p)(K_t + B_t). \tag{1.14}$$

This change comes about because the young that replace the old enter the economy with *more* human capital than the old. This implies that, *ceteris paribus,* $d\tilde{W}_t = \omega\tilde{W}_t$ per time unit, where $\tilde{W}_t$ is aggregate human capital at time t. Hence,

$$\begin{aligned} dC_t &= (\rho + p)d\tilde{W} = (\rho + p)\omega\tilde{W} = \omega(C - (\rho + p)A) \\ &= \omega C - \omega(\rho + p)A, \end{aligned}$$

where the third equality comes from the aggregate consumption function, $C_t = (\rho + p)(A_t + \tilde{W}_t)$, implied by (1.6). Since $A = K + B$, this explains the difference between (1.14) and (1.1).

Now $\dot{C} = 0$ for

$$C = \frac{(\omega + p)(\rho + p)(K + B_0)}{F_K(K, L) - \delta - \rho + \omega}.$$

We redefine the "critical" value, $\bar{K}$, by

$$F_K\left(\bar{K}, L\right) = \delta + \rho - \omega,$$

To ensure a solution in $\bar{K}$, we assume

$$\omega < \delta + \rho.$$

There are two cases to consider.

Case 1: $\omega \leq \rho$. Here $\rho - \omega \geq 0$, so that $\bar{K} \leq K_g$, implying that the new steady state value of K is below K_g. Hence, the new long-run rate of interest, r^*, is positive. Therefore, solvency of fiscal policy still requires the NPG condition to be satisfied, and the analysis of question d) above goes through. The fiscal policy $(G, \bar{T})$ is *not* sustainable.

Case 2: $\rho < \omega < \delta + \rho$. We now get $\bar{K} > K_g$, and it is possible that $K^* > K_g$. Hence, it is possible that $r^* = F_K(K^*, L) - \delta < F_K(K_g, L) - \delta = 0$, i.e., the long-run real rate of interest may be negative. In that case, solvency (and thereby sustainability) of fiscal policy does *not* require the NPG condition to be satisfied. Indeed, with $K_K(K_t, L) - \delta = r_t = r^* < 0$, (1.3) gives

$$B_t = (B_0 - B^*)e^{r^* t} + B^*, \text{ where } B^* = -\frac{G - \bar{T}}{r^*} \gtreqless 0 \text{ for } G \gtreqless \bar{T}.$$

Anyway, public debt does not explode; instead, it converges to the constant B^*. Indeed, even if there is a primary deficit (i.e., $G - \bar{T} > 0$), the positive payments that the government *receives* from the public as long as $B_t > 0$, finance, in the long run, that part of G which is not covered by $\bar{T}$. The fiscal policy $(G, \bar{T})$ *is* sustainable.

Of course, instead of this formal answer, an informal answer, based on awareness of the critical importance of $r > 0$ versus $r < 0$ (or more generally, $r > (\dot{Y}/Y)^*$ versus $r < (\dot{Y}/Y)^*$), is enough.

2. Solution to Problem 2

For convenience, the model is repeated here:

$$\dot{Y}_t = \lambda(D(R_t, Y_t) - Y_t), \qquad D_R < 0,\ 0 < D_Y < 1, \tag{2.1}$$

$$\frac{M_t}{P} = L(Y_t, i_t), \qquad L_Y > 0,\ L_i < 0. \tag{2.2}$$

$$R_t = 1/Q_t, \tag{2.3}$$

$$\frac{1 + \dot{Q}_t^e}{Q_t} = r_t, \tag{2.4}$$

$$r_t \equiv i_t - \pi_t^e, \tag{2.5}$$

a) Evidently, the model is Blanchard's dynamic IS/LM model, where the adjustment of output to demand takes time, and where there is a distinction between a long-term bond (a consol) and a short-term bond. Equation (2.1) tells how output adjusts to demand, D; the parameter λ is the speed of adjustment. Naturally, output demand depends positively on income = Y, and negatively on the long-term interest rate, since both consumption and investment are likely to depend negatively on this rate.

Equation (2.2) expresses equilibrium at the "money market". Naturally, real money demand depends positively on Y (a proxy for the number of transactions per time unit) and negatively on the short-term nominal rate of interest, the opportunity cost of holding money.

The inverse relation between the long-term interest rate and the market value of a long-term bond in equation (2.3) comes from the definition of the long-term rate as the internal rate of return on the long-term bond, i.e., the solution in R_t to

$$Q_t = \int_t^{\infty} e^{-R_t(s-t)} ds = \frac{1}{R_t}.$$

Equation (2.4) is the no-arbitrage condition saying that, absent uncertainty, the rate of return on the long-term bond is equal to the rate of return on the short-term bond. Finally, equation (2.5) defines r_t as the short-term nominal rate of interest minus the expected rate of inflation, hence, r_t is the short-term real rate of interest.

The information about i_t being the instrument of the monetary authority (the central bank) corresponds with the general conception nowadays about what central banks actually do (and are able to do).

b) The assumption of perfect foresight implies $\dot{Q}_t^e = E_t\dot{Q}_t = \dot{Q}_t$, and since the price level P is an exogenous constant in the model, we have $\pi_t^e = E_t\pi_t = \pi_t = 0$ for all t. Therefore, equation (2.5) reduces to $r_t = \bar{i} > 0$, given the policy $i_t = \bar{i}$. Further, (2.3) gives $\dot{Q}_t = d(R_t^{-1})/dt = -R_t^{-2}\dot{R}_t$. Together with (2.4) this entails

$$R_t - \dot{R}_t/R_t = r_t = \bar{i}.$$

Ordering gives

$$\dot{R}_t = (R_t - \bar{i})R_t. \tag{2.6}$$

The other differential equation is directly given by (2.1), which can be written

$$\dot{Y} = \tilde{D}(Y, R), \qquad \tilde{D}_Y = \lambda(D_Y - 1) < 0, \tilde{D}_R = \lambda D_R < 0. \tag{2.7}$$

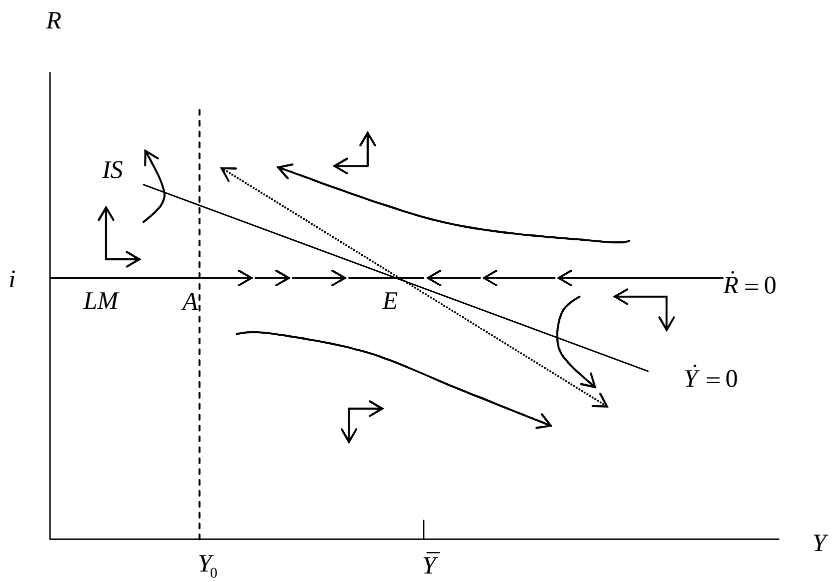


Figure 2.1:

The differential equations 2.6) and (2.7) constitute the dynamic system of the model.

To draw the corresponding phase diagram, note that (2.6) implies

$$\dot{R} = 0 \quad \text{for} \quad R = \bar{i}.$$

Hence, the $\dot{R} = 0$ locus (the "LM curve") is horizontal, cf. Fig. 2.1. Similarly, (2.7) implies

$$\dot{Y} = 0 \quad \text{for} \quad \tilde{D}(Y, R) = 0. \tag{2.8}$$

Totally differentiating this gives

$$\frac{dR}{dY}\,|_{\dot{Y}=0} = -\frac{\tilde{D}_Y}{\tilde{D}_R} < 0. \tag{2.9}$$

Hence, the $\dot{Y} = 0$ locus is downward-sloping as shown in Fig. 2.1. The figure also shows the direction of movement in the different regions, as determined by

(2.6) og (2.7). We see that the steady state point, E, is a saddle point.[2] This implies that two and only two solution paths – one from each side – converges towards E. These two saddle paths coincide with the $\dot{R} = 0$ locus. Since Y is (in this model) a predetermined variable, and R is a jump variable, the steady state is saddle-point stable.

At time $t = 0$, the economy must be somewhere on the vertical line $Y = Y_0$. In view of the absence of speculative bubbles, the explosive or implosive paths of Q in Fig. 2.1 cannot arise. Hence, we are left with the saddle path, the path AE in Fig. 2.1, as the unique solution to the model.

c) In steady state

$$r_t = \bar{r} = R_t = \bar{R} = \bar{i}, \tag{2.10}$$

and $Y = \bar{Y}$, where $(\bar{Y}, \bar{i})$ satisfies (2.8). Hence, inserting $(\bar{Y}, \bar{i})$ into (2.8), this equation defines $\bar{Y}$ as an implicit function of $\bar{i}$, $\bar{Y} = \phi(\bar{i})$. The derivative, $\phi'(\bar{i})$, is the inverse of (2.9), that is,

$$\frac{d\bar{Y}}{d\bar{i}} = -\frac{\tilde{D}_R}{\tilde{D}_Y} < 0.$$

A permanently higher short-term rate of interest implies a higher long-term rate of interest, hence lower output demand.

d) The unanticipated upward shift in the short-term interest rate, $\bar{i}$, to the new level, $\bar{i}'$ is depicted in Fig. 2.2. Fig. 2.3 shows that the $\dot{R} = 0$ locus (the LM curve) is shifted upward, whereas the $\dot{Y} = 0$ locus (the IS curve) is unaffected. Since the short-term interest rate is kept constant after t_0, the long-term rate immediately jumps to $\bar{i}'$ at time t_0 and stays constant thereafter. This is because the long-term rate is a kind of average of the future (constant) short-term rates.

[2] More formally, the determinant of the Jacobian matrix for the right hand sides of the two differential equations, evaluated in the steady state point $(\bar{Y}, \bar{R})$, is $\bar{R}\tilde{D}_Y < 0$.

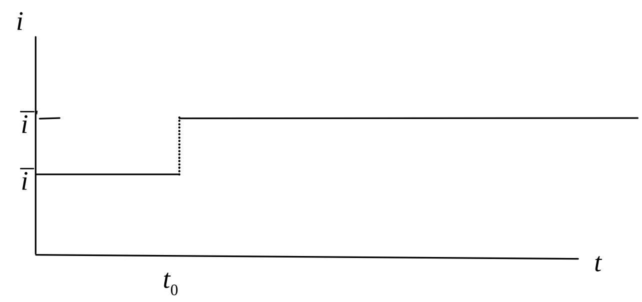


Figure 2.2:

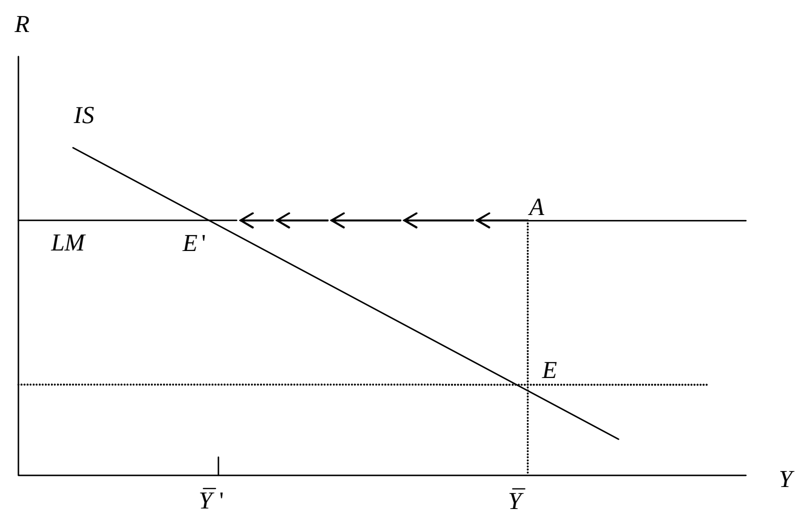
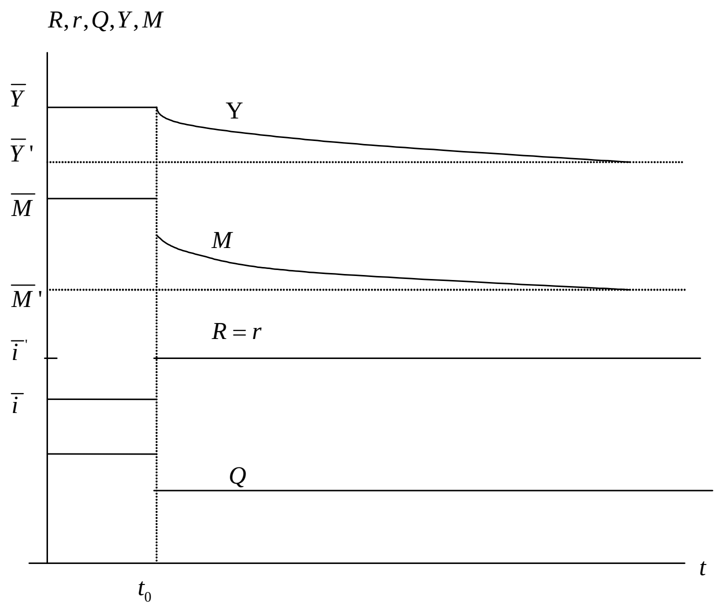


Figure 2.3:



R, r, Q, Y, M
$\overline{Y}$
$\overline{Y}'$
$\overline{M}$
$\overline{M}'$
$\overline{i}'$
$\overline{i}$
Y
M
$R = r$
Q
t
t_0

Figure 2.4:

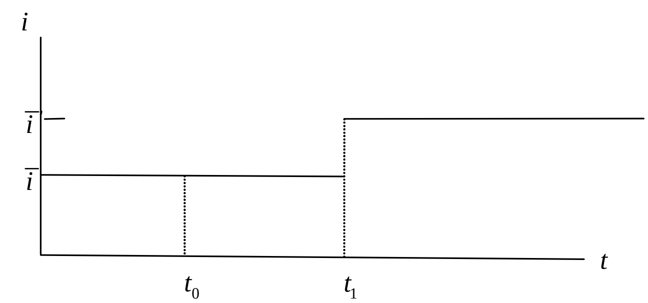


Figure 2.5:

Indeed, as indicated by the hint,

$$R_t = \frac{1}{Q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds},$$

so that for $r_t = \bar{i}'$ for all $t \geq t_0$,

$$R_t = \frac{1}{\int_t^\infty e^{-\bar{i}'(s-t)} ds} = \frac{1}{1/\bar{i}'} = \bar{i}'.$$

The higher R dampens output demand, and output gradually contracts during the approach to the new steady state E$'$. Figure 2.4 shows the time profiles of $Y, M, R, r,$ and Q.

d) Fig. 2.5 is an illustration. At time t_0 the monetary authority credibly announces an upward shift in the instrument variable to take place at time $t_1 > t_0$. The phase diagram is shown in Fig. 2.6, and the graphical time profiles are shown in Fig. 2.7. Already at time t_0, the time of the announcement, the long-term rate jumps upward, anticipating the later permanent increase in the short-term rate. This dampens demand, and output begins to contract already before t_1, that is, before the contractionary monetary policy is actually implemented. To prevent the gradual decline in transactions to lower the short-term interest rate,

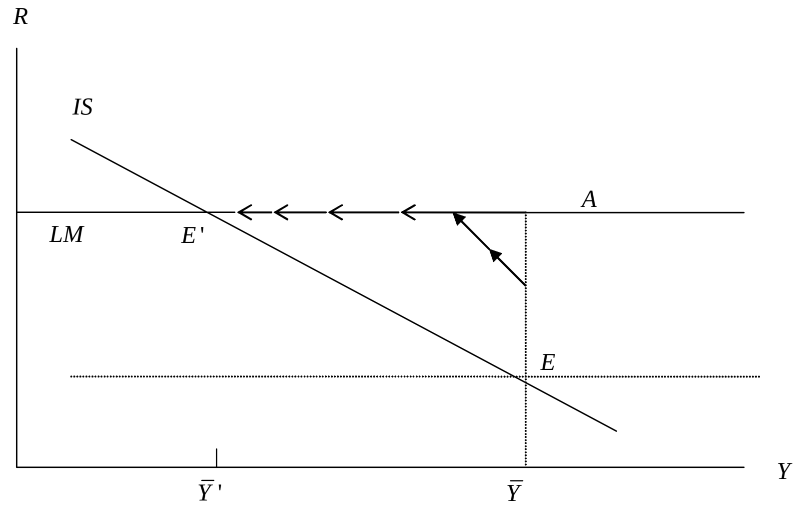


Figure 2.6:

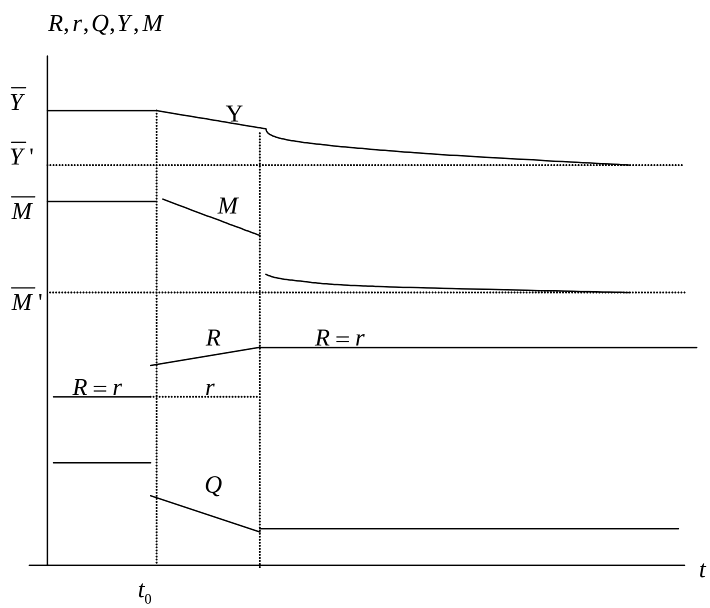


Figure 2.7:

money supply must be gradually reduced. At time t_1, to implement the announced increase in the short-term rate, money supply is reduced by a discrete amount and is, thereafter, gradually reduced further, along with the gradual reduction in output also after t_1 (which implies lower money demand).

3. Solution to Problem 3

The decision problem, as seen from period 0, is:

$$\max E_0(U_0) = E_0[\sum_{t=0}^{T-1}(\log c_t - \gamma\frac{\sigma}{1+\sigma}\ell_t^{(1+\sigma)/\sigma})(1+\rho)^{-t}] \qquad \text{s.t.}$$

$$c_t \geq 0, 0 \leq \ell_t \leq \bar{\ell}, \tag{3.1}$$

$$a_{t+1} = (1+r_t)a_t + w_t\ell_t - c_t, \qquad a_0 \text{ given}, \tag{3.2}$$

$$a_T \geq 0. \tag{3.3}$$

a) Defining $\tilde{U}_t \equiv (1+\rho)^t U_t$, the remainder of the problem as seen from period t $(t = 0, 1, ...)$ is:

$$\begin{aligned} \max\ E_t\tilde{U}_t &= (1+\rho)^t E_t U_t \\ &= \log c_t - \gamma\frac{\sigma}{1+\sigma}\ell_t^{(1+\sigma)/\sigma} + (1+\rho)^{-1}E_t[\log c_{t+1} - \gamma\frac{\sigma}{1+\sigma}\ell_{t+1}^{(1+\sigma)/\sigma} \\ &+ (\log c_{t+2} - \gamma\frac{\sigma}{1+\sigma}\ell_{t+2}^{(1+\sigma)/\sigma})(1+\rho)^{-1} + ...] \\ &\text{s.t. (3.1) - (3.3)}, \quad a_t \text{ given}. \end{aligned} \tag{3.4}$$

To solve the problem we will use the substitution method. First, from (3.2) we have

$$\begin{aligned} c_t &= (1+r_t)a_t + w_t\ell_t - a_{t+1}, \qquad \text{and} \\ c_{t+1} &= (1+r_{t+1})a_{t+1} + w_{t+1}\ell_{t+1} - a_{t+2}. \end{aligned} \tag{3.5}$$

Substituting this into (3.4), the problem is reduced to one of maximizing the function $E_t\tilde{U}_t$ w.r.t. $(\ell_t, a_{t+1}), (\ell_{t+1}, a_{t+2}), ..., (\ell_{T-1}, a_T)$. We get

$$\frac{\partial E_t\tilde{U}_t}{\partial \ell_t} = \frac{1}{c_t}w_t - \gamma \ell_t^{1/\sigma} = 0,$$

that is,

$$\gamma \ell_t^{1/\sigma} = \frac{1}{c_t}w_t \qquad t = 0, 1, 2, ..., T-1, \qquad (*)$$

and

$$\frac{\partial E_t\tilde{U}_t}{\partial a_{t+1}} = \frac{1}{c_t} \cdot (-1) + (1+\rho)^{-1}E_t[\frac{1}{c_t+1}(1+r_{t+1})] = 0,$$

that is,

$$\frac{1}{c_t} = (1+\rho)^{-1}E_t[\frac{1}{c_t+1}(1+r_{t+1})], \qquad t = 0, 1, 2, ..., T-2. \qquad (**)$$

In view of the solvency condition (3.3), in the last period consumption must be

$$c_{T-1} = (1+r_{T-1})a_{T-1} + w_{T-1}\ell_{T-1},$$

since it is not optimal to end up with $a_T > 0$ (indeed, the transversality condition is $a_T = 0$).

b) The first order condition (*) describes the trade-off between leisure in period t and consumption in the same period. The condition says that in the optimal plan, the cost (in terms of current utility) of increasing labour supply by one unit is equal to the benefit of obtaining an increased labour income and using this increase for extra consumption (i.e., marginal cost = marginal benefit).

The other first order condition, (**), describes the trade-off between consumption in period t and consumption in period $t+1$, as seen from period t. The optimal plan must satisfy that the current utility loss by decreasing consumption c_t by one unit is equal to the discounted expected utility gain next period by having $1 + r_t$ extra units available for consumption, namely the gross return on saving one more unit (i.e., marginal cost = marginal benefit).

c) We rewrite (*) as

$$\gamma \ell_t^{1/\sigma} c_t = w_t. \tag{*'}$$

Apart from the finite horizon (which is not important in this context), the intertemporal utility function above could easily be a specification of the preferences of a representative household in a RBC model. Further, the RBC theory maintains that factor prices are always such that there is no unemployment. Hence, the prediction from the RBC theory is the same as that from condition (*), namely that, since employment is procyclical and fluctuates almost as much as GDP, and consumption and employment are positively correlated, real wages will also be procyclical and fluctuate almost as much as output. But according to the stylized fact (iii), real wages are only weakly procyclical and do not fluctuate much. This is one of the often mentioned difficulties facing RBC theory.

d) Replacing t by $t+1$ in (*) gives

$$\gamma \ell_{t+1}^{1/\sigma} = \frac{1}{c_{t+1}} w_{t+1}$$

so that

$$(\frac{\ell_t}{\ell_{t+1}})^{1/\sigma} = \frac{c_{t+1}}{c_t} \frac{w_t}{w_{t+1}}. \tag{3.6}$$

Ignoring uncertainty, (**) gives

$$\frac{c_{t+1}}{c_t} = (1+\rho)^{-1}(1+r_{t+1}).$$

Substituting this into (3.6) and solving gives

$$\frac{\ell_t}{\ell_{t+1}} = (1+\rho)^{-\sigma}(\frac{w_t}{w_{t+1}/(1+r_{t+1})})^{\sigma}. \tag{3.7}$$

We see from this expression that σ is the elasticity of ℓ_t/ℓ_{t+1} w.r.t. w_t/w_{t+1}. Hence, σ measures what is called the intertemporal elasticity of substitution (in labour supply). From microeconometric studies we have estimates of this parameter. These estimates are quite small, at least for men (in the range 0 to 1.5, often considerably below 1). And since fluctuations in w_t/w_{t+1} in the data are also small, it is difficult to reconcile the theory with the stylized fact (i) saying that employment fluctuates almost as much as GDP.

e) If fluctuations in the real wage are negligible, is it then likely that fluctuations in r_{t+1} could be a driving force behind fluctuations in employment? According to equation (3.7) one might be tempted to answer "yes". At least (3.7) indicates a positive relation between ℓ_t/ℓ_{t+1} and r_{t+1}. The interpretation of this relation is that a high rate of interest has negative substitution and wealth effects on leisure in the current period, hence positive substitution and wealth effects on current labour supply.

But if the real wage doesn't fluctuate, and an attempt is made to explain fluctuations in employment by fluctuations in the real rate of interest, then, by (*'), one would expect a *negative* correlation between employment and consumption. But the stylized fact (ii) tells the opposite.

f) We now reintroduce uncertainty. Indeed, there is now also uncertainty as to the prospect of employment in the future. The decision problem, as seen from period 0, can now be written:

$$
\begin{aligned}
\max E_0(U_0) &= E_0[\sum_{t=0}^{T-1}(\log c_t - \gamma\frac{\sigma}{1+\sigma}\ell_t^{(1+\sigma)/\sigma})(1+\rho)^{-t}] \qquad \text{s.t.} \\
c_t &\geq 0, 0 \leq \ell_t \leq \min(x_t, \bar{\ell}), \\
a_{t+1} &= (1+r_t)a_t + w_t\ell_t - c_t, \qquad a_0 \text{ given}, \\
a_T &\geq 0.
\end{aligned}
$$

where $x_t \geq 0$ is the exogenous maximum employment offered the household in period t (this constraint coming from the demand side at the labour market).

When the employment constraint $\ell_t \leq x_t$ is binding, (*) is replaced by

$$\gamma\ell_t^{1/\sigma} < \frac{1}{c_t}w_t. \tag{3.8}$$

The interpretation is: Though in the optimal plan, the cost (in terms of current utility) of increasing labour by one unit is *less* than the benefit of obtaining an increased labour income and using this increase for extra consumption, this desired increase in employment cannot be realized, due to the exogenous employment constraint.

g) Yes, within this extended framework it is possible to reconcile theory with the stylized facts. Indeed, rewriting (3.8) as

$$\gamma \ell_t^{1/\sigma} c_t < w_t,$$

we see there is scope for employment to be procyclical and fluctuate almost as much as GDP (fact (i)) and for consumption and employment to be positively correlated (fact (ii)), whereas real wages do not fluctuate much (fact (iii)).

h) We now consider the effect of an increase in uncertainty, say a mean-preserving spread. Since the period utility function, $u(c)$, is logarithmic in the model, it implies strictly convex marginal utility ($(u')'' > 0$) so that increased uncertainty as to future labour income (whether it is uncertainty about future employment or the future real wage) results in "precautionary saving", that is, lower current consumption. Further, this conclusion cannot be overturned by the effect of increased uncertainty as to the rate of return on saving. This is because logarithmic utility implies a coefficient of relative risk aversion equal to one. And then increased uncertainty concerning the rate of return is neutral to saving.

4. Solution to Problem 4

a) "If leisure is a 'normal good', then the income effect as well as the wealth effect on leisure of an increase in the wage rate are capable at offsetting the substitution effect."

This is not true, since the income effect has the same sign as the substitution effect.

b) "Ignoring uncertainty, the difference between the short-term and the long-term interest rate is determined only by expectations."

This is true. Absent uncertainty, the no-arbitrage condition (2.4) above is valid. Then,

$$R_t = \frac{1}{Q_t} = r_t - \frac{Q_t^e}{Q_t} \gtreqless r_t \quad \text{for} \quad Q_t^e \lesseqgtr 0.$$

c) "In the Blanchard-Kiyotaki model with monopolistic competition and menu costs, there may be more than one equilibrium."

This is true, and the phenomenon is due to self-fulfilling expectations. For an individual firm, the minimum menu cost required for its price not to be increased in a situation with an increase in money supply (hence increase in output demand) depends positively on the aggregate proportion of firms that do change price. Hence, the following is possible. If every firm expects no other firm to change its price, the actual menu cost (the same for all firms) is large enough to prevent any price change, so that no firm changes its price. At the same time, if every firm expect all other firms to change their price, the actual menu cost is too small to prevent a price change, so that all firms change their price.

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