Chapter 8

Optimal capital accumulation

In Barro's dynasty model of the previous chapter, coordination across generations is brought about through a competitive market mechanism and bequests induced by parental altruism. We will now study resource allocation in a context where we imagine that the coordination across generations is brought about by a benevolent and omniscient *social planner* discounting the utility of future generations at a certain rate. The study of such problems was initiated already by the British mathematician and economist Frank P. Ramsey (1903-1930). The modeling framework is therefore sometimes referred to as Ramsey's optimal saving problem. While the original contribution by Ramsey was in continuous time, here we take a discrete time approach and leave the continuous-time formulation for Chapter 10.

The coordination across generations leads to the social planner's *modified golden rule.* Whether the planning horizon is finite or infinite, the associated time path of the economy features a distinctive stability attribute, known as the *turnpike property.*

In the first section below, to solve the social planner's dynamic optimization problem, we use the simple substitution method. In Section 8.2 the more general and advanced mathematical tool called *optimal control theory* is applied (in its discrete time version). The questions of existence, uniqueness, and convergence over time of the solution are examined in Section 8.3. Optimal control theory is also applicable to cases where other optimality criteria than maximization are needed. This is the topic of Section 8.4. The sections 8.2, 8.3, and 8.4 are relatively technical and can be skipped in a first reading. Hence their headings are marked by an asterisk.

8.1 Command optimum

As to demography, technology, and individual preferences the framework is as in the Diamond OLG model with Harrod-neutral technological progress and the notation is the same. Chapter 3 concluded that a competitive market economy in this framework may suffer from dynamic inefficiency, hence absence of Pareto optimality. In addition to this problem, however, one should be aware that even if resource allocation is Pareto optimal, it may not be satisfactory from a societal point of view. Pareto optimality is a weak optimality criterion. For example, if in a Diamond economy each young generation has very high impatience, they save very little for their old age and the economy may gradually shrink to the detriment of future generations. Nevertheless, this can easily be a Pareto optimal resource allocation. Similarly, if the queen of Denmark received almost all consumption goods (and satiation were impossible), while the rest of the population received just what is needed for subsistence, that would be a Pareto optimum.

Pareto optimality should be seen as only a minimum requirement of social organization. A more ample optimality criterion is based on a *social welfare function*, that is, an objective function which aggregates the welfare levels of the various members of society, possibly including the as yet unborn members, into an index of "social welfare". How can a social welfare function be constructed in a democratic society with conflicting interests? As is well-known from Arrow's Impossibility Theorem (see, e.g., Mas-Colell et al., 1995, Chapter 21), no definite answer satisfying a series of "natural" minimal requirements, including Pareto-optimality and independence of irrelevant alternatives, can be given. The theory of public economics can help clarify achievable and, according to well-defined criteria, desirable properties of a social welfare function. But in the last instance a social welfare function relies on ethics and political choice.

8.1.1 A social planner

Consider a hypothetical centrally planned economy with a *benevolent* and *omniscient* social planner who can dictate every aspect of production and distribution within the constraints given by technology and initial resources. The demography and individual preferences are as in the Diamond OLG model of Chapter 3. It is assumed that the social planner

- knows and respects the individual preferences as to the distribution of individual consumption over own lifetime;
- discounts the utility of future generations at a constant effective rate $\bar{R}$, which *may* deviate from the effective intergenerational discount rate of the individuals.

Technically feasible paths

The number of young is $L_t = L_0(1+n)^t$, where $L_0 > 0$ and $n > -1$. As in both the Diamond and the Barro model, the aggregate production function is

$$Y_t = F(K_t, \mathcal{T}_t L_t) \equiv \mathcal{T}_t L_t f(\tilde{k}_t), \tag{8.1}$$

where F has constant returns to scale and is neoclassical so that $f' > 0$ and $f'' < 0$. The technology level $\mathcal{T}_t$ grows at a constant exogenous rate $g \geq 0$. To save notation we chose measurement units such that $\mathcal{T}_0 = 1$, whereby $\mathcal{T}_t = (1+g)^t$.[1] Only the young work and they all supply one unit of labor per period (the social planner ensures full employment). The dynamic resource constraint is

$$K_{t+1} = K_t + Y_t - C_t - \delta K_t, \qquad K_0 > 0, \quad 0 \leq \delta \leq 1. \tag{8.2}$$

Aggregate consumption C_t satisfies

$$C_t = L_t c_{1t} + L_t(1+n)^{-1} c_{2t}. \tag{8.3}$$

Dividing through by L_t, isolating c_{1t} and using (8.2) and (8.1) yields

$$c_{1t} \equiv \tilde{c}_{1t}\mathcal{T}_t = \left[f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - (1+n)(1+g)\tilde{k}_{t+1} \right] \mathcal{T}_t - (1+n)^{-1} c_{2t}, \tag{8.4}$$

where $\tilde{k}_0 > 0$ is given. Essentially, this is just an aggregate book-keeping relation saying that consumption by each young equals what is available per young minus what is used for investment and consumption by the old.

Let the historically given initial effective capital-labor ratio be $\bar{\tilde{k}}_0 \geq 0$. Then a *technically feasible* path from time 0 to time T is a sequence $\left\{ (\tilde{k}_t, c_{1t}, c_{2t}) \right\}_{t=0}^{T-1}$ such that $\tilde{k}_0 = \bar{\tilde{k}}_0$ and for $t = 0, 1, 2, \ldots, T-1$, the non-negativity constraints $c_{1t} \geq 0$, $c_{2t} \geq 0$, and $\tilde{k}_{t+1} \geq 0$ hold and the equation (8.4) is satisfied. A *technically feasible* path with *infinite horizon* is defined similarly for $T \to \infty$.

To begin with our social planner is assumed to have a *finite* planning horizon. For many practical planning problems this is certainly the realistic case. A planner often has inadequate information about available resources and technology in the far future and may consequently refrain from very-long horizon planning, be it infinite or finite.

[1] In this chapter we use $\mathcal{T}$ to denote the technology level, whereas T will be an integer representing the planning horizon.

Finite planning horizon

The social planner's problem with a finite planning horizon, T, is to select from the set of technically feasible paths the best one according to a criterion function and possibly a specific *terminal constraint*, see below. If such a member of the set of technically feasible paths exists, it is called an *optimal path* or - in casual jargon - a *command optimum.*

In the present context the criterion function, also known as a *social welfare function*, is:

$$\begin{aligned} W_0 &= (1+\rho)^{-1}u(c_{20}) + \sum_{t=0}^{T-2}(1+\bar{R})^{-(t+1)}[u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1})] \\ &\quad +(1+\bar{R})^{-T}u(c_{1T-1}). \end{aligned} \tag{8.5}$$

The private rate of time preference is ρ (> -1), there is no utility from leisure, and $u' > 0$, $u'' < 0$. To avoid corner solutions (where, for some t, $c_{1t} = 0$ or $c_{2t} = 0$), we impose the usual No Fast Assumption

$$\lim_{c\to 0} u'(c) = \infty. \tag{A1}$$

The social planner's effective intergenerational discount rate, $\bar{R}$, is defined by

$$(1+\bar{R})^{-1} = (1+R)^{-1}(1+n), \tag{8.6}$$

where R is the pure intergenerational discount rate which enters the utility discount factor by which the social planner translates the lifetime utility of a member of a given generation into equivalent utility units for a member of the previous generation. Until further notice, nothing is assumed about R (except of course the conceptual restriction $R > -1$) and so, similarly, $\bar{R} > -1$. If $R = 0$, we have $(1+\bar{R})^{-1} = 1+n$, implying that the social planner weighs the per capita lifetime utility obtained by each generation according to the size of that generation. If $R > 0$, we have $(1+\bar{R})^{-1} < 1+n$, and so future generations get less weight, while $R < 0$ implies that future generations get more weight.

The dynasty criterion function as formulated in the Barro model, cf. Chapter 7, viewed the stream of present and future utilities from the perspective of the young parent who in a market economy takes the welfare of the descendents into account. The social welfare function (8.5) is slightly more inclusive in that it takes into account that also in the first period, period 0, there is a trade-off between utilities of two coexisting generations, the young and the old. Possibly the allocation preferred by the social planner involves a transfer from the currently young to the currently old. That kind of transfer would never happen in the

Barro model where parents' utility did not enter children's utility function and children were exempted from responsibility for parental debts (in accordance with a normal legal system of a market economy).

It is important that the social planner faces a historically given $\tilde{k}_0$. When studying the golden rule problem in Chapter 3, we just asked: what is the highest sustainable path of consumption? We did not ask: given we are at some arbitrary $\tilde{k}_0$ today, where should we go and how fast? But this is what a planning problem is about. Given a criterion function, which may involve discounting and diminishing marginal utility, the problem is to find an optimal route to follow, *starting from a historically given initial condition.*

In spite of the finite planning horizon, the planner may give some weight to what happens after period $T-1$. We therefore introduce the *terminal constraint,*

$$\tilde{k}_T \geq \bar{\tilde{k}}_T, \tag{8.7}$$

where $\bar{\tilde{k}}_T \geq 0$.

Solving the social planner's problem

The planner's problem is: choose a plan $\left(c_{20}, \{(c_{1t}, c_{2t+1})\}_{t=0}^{T-2}, c_{1T-1}\right)$ to maximize W_0 subject to the constraints (8.4), (8.7), and non-negativity of c_{1t}, c_{2t}, and $\tilde{k}_{t+1}$. To solve the problem we insert (8.4) (both as it looks and shifted one period forward) into (8.5) and maximize w.r.t. c_{2t} and $\tilde{k}_{t+1}$. An interior solution[2] will for $t = 0, 1, \ldots, T-1$ satisfy the first-order conditions:

$$\frac{\partial W_0}{\partial c_{2t}} = (1+\bar{R})^{-t}(1+\rho)^{-1}u'(c_{2t}) - (1+\bar{R})^{-(t+1)}u'(c_{1t})(1+n)^{-1} = 0,$$

and

$$\begin{aligned}\frac{\partial W_0}{\partial \tilde{k}_{t+1}} &= (1+\bar{R})^{-(t+1)}u'(c_{1t}) \cdot [-(1+n)(1+g)]\,\mathcal{T}_t \\ &\quad +(1+\bar{R})^{-(t+2)}u'(c_{1t+1})\left[f'(\tilde{k}_{t+1}) + 1 - \delta\right]\mathcal{T}_{t+1} = 0.\end{aligned}$$

These two conditions can be written

$$(1+\rho)^{-1}u'(c_{2t}) = (1+\bar{R})^{-1}u'(c_{1t})\frac{1}{1+n} \quad \text{and} \tag{8.8}$$

$$u'(c_{1t}) = (1+\bar{R})^{-1}u'(c_{1t+1})\frac{1+f'(\tilde{k}_{t+1})-\delta}{1+n}, \tag{8.9}$$

[2] An *interior* solution is a solution with the property that for no $t \in [0, T-1]$ is a boundary point of the set of technically feasible paths reached.

respectively, for $t = 0, 1, ..., T - 1$. Condition (8.8) is a $MC = MB$ condition (in terms of utility) referring to the distribution of consumption across generations in the same period. It states that, from the point of view of the social planner, the utility loss by transferring one unit of consumption from the old in period t to the young in the same period must equal the utility gain obtained by the young (who can now consume more), discounted by the intergenerational discount rate $\bar{R}$. The rate of transformation is $1/(1 + n)$, since, for every old there are $1 + n$ young.

Condition (8.9) is a $MC = MB$ condition referring to the distribution of consumption across time *and* generations. It states that the utility loss by decreasing the consumption of the young in period t by one unit must equal the utility gain obtained by the young in the next period discounted by the intergenerational discount rate $\bar{R}$. The rate of transformation is $\left[f'(\tilde{k}_{t+1}) + 1 - \delta\right]/(1 + n)$, since the saved unit is invested and gives a gross return of $f'(\tilde{k}_{t+1}) + 1 - \delta$ in period $t + 1$, but at the same time, for every young in period t there are $1 + n$ young in period $t + 1$.

Replacing $u'(c_{1t+1})$ in (8.9) by its value from (8.8), shifted one period ahead, we end up with

$$u'(c_{1t}) = (1 + \rho)^{-1}u'(c_{2t+1})(1 + f'(\tilde{k}_{t+1}) - \delta). \tag{8.10}$$

This relation is identical to the familiar condition, the Euler equation, for individual intertemporal utility optimization in the Diamond model, if we insert the equilibrium relation $r_{t+1} = f'(\tilde{k}_{t+1}) - \delta$. The central planner thus holds the individual's intertemporal first-order condition in the market economy in respect.

But (8.10) only ensures that the *relative* consumption across time and generations is "right" (optimal). For the general "level" of the consumption path to be "right", we need that the terminal constraint (8.7) holds with strict equality:

$$\tilde{k}_T = \bar{\tilde{k}}_T. \tag{8.11}$$

Such a terminal optimality condition is called a *transversality condition.* The intuition behind it is that the alternative, $\tilde{k}_T > \bar{\tilde{k}}_T$, would reflect overaccumulation, since higher discounted utility, W_0, could be obtained by consuming more in period $T - 1$ (or an earlier period) without violating the terminal constraint (8.7).

As formally shown in Section 8.2, the conditions above are not only necessary but also sufficient for an optimal solution. So the optimal allocation over generations and time is characterized by (8.4), (8.8), (8.9), and (8.11). Apart from the presence of technological progress, the first three of these equations are the same as in the competitive market economy when the bequest motive is operative. The equations (7.12), (7.7), and (7.10) (where $r_{t+1} = f'(k_{t+1}) - \delta$) from Barro's dynasty model in Chapter 7 confirm this. So the resource allocation in a perfectly-competitive market economy with altruistic parents looks similar to that brought about by a social planner with an effective intergenerational discount rate equal to the private one. We have not yet established full equivalence, however, since the family dynasty in the Barro model has an *infinite* horizon. To get a comparable situation we now consider an infinite planning horizon in the social planner's problem.

Infinite planning horizon

In the social welfare function (8.5) we let $T \to \infty$. As before, we consider a historically given initial effective capital-labor ratio $\tilde{k}_0 > 0$. When $T \to \infty$, the function (8.5) becomes a criterion which in itself assigns a proper weight to what happens "ultimately". There is no need – in fact it makes no sense – to "translate" the terminal constraint (8.7) into a terminal constraint like $\lim_{T\to\infty} \tilde{k}_T \geq \bar{\tilde{k}}$ except if $\bar{\tilde{k}} = 0$. And in that case, the condition is redundant since we in any case, by definition of capital, have the technical feasibility condition

$$\tilde{k}_t \geq 0 \text{ for all } t. \tag{8.12}$$

The first-order conditions (8.8) and (8.9) are still conditions which an interior optimal solution has to satisfy. It remains to set up the necessary transversality condition for the infinite horizon case. A first problem is that with $T \to \infty$, W_0 may no longer be bounded from above. In that situation we cannot maximize W_0, and as long as maximization is our optimality criterion, no optimal solution exists. While in Section 8.4 we consider other optimality criteria, we here maintain maximization as criterion for optimality. And provided an optimal solution exists, it must satisfy the transversality condition

$$\lim_{T\to\infty} \left(\frac{1+g}{1+\bar{R}}\right)^T u'(c_{1T-1})(1+n)\tilde{k}_T = 0. \tag{8.13}$$

A substantiation of the necessity of this condition for the problem at hand is contained in Appendix C. Here we attempt an intuitive understanding. On the one hand, high $\tilde{k}$ is good for future production. On the other hand, overaccumulation such that some consumption possibilities are postponed forever should be avoided. That is, the "ultimate" $\tilde{k}$ should not be too high. For a moment, imagine as before there is a finite horizon, T, with terminal constraint $\tilde{k}_T \geq \bar{\tilde{k}}_T$. Consider a technically feasible plan with $\tilde{k}_T > \bar{\tilde{k}}_T$, i.e., the terminal constraint is not binding. This plan will not be optimal if the extra consumption in period $T-1$, made feasible by a small decrease of $\tilde{k}_T$, creates extra discounted utility. Indeed, by a small decrease in k_T for the benefit of the young in period $T-1$, we get $\Delta c_{1T-1} = -(1+n)\Delta k_T > 0$, since for every young in period $T-1$ there are $1+n$ young in period T. A change in $\tilde{k}_T$ of size $\Delta\tilde{k}_T = -1$ amounts to a change in k_T of size $\Delta k_T = -(1+g)^T$, since $k_T = \tilde{k}_T(1+g)^T$ (in view of $\mathcal{T}_0 = 1$). Hence, with $\Delta\tilde{k}_T = -1$, the utility gain for generation $T-1$ resulting from this reduction of $\tilde{k}_T$ is $u'(c_{1T-1})(1+n)(1+g)^T$. The present value of this gain, evaluated from the social planner's point of view, which is the point of view of generation -1 (the old generation in period 0), is $(1+\bar{R})^{-T}u'(c_{1T-1})(1+n)(1+g)^T$. If this present value is positive, it cannot be optimal to end up with $\tilde{k}_T > \bar{\tilde{k}}_T$. This explains the transversality condition (8.11). But if the present value were zero (say because c_{1T-1} and c_{2T-1} were already above a certain level of saturation), then $\tilde{k}_T > \bar{\tilde{k}}_T$ need not be inconsistent with optimality.

With finite horizon, T, the necessary transversality condition can thus be written

$$\left(\frac{1+g}{1+\bar{R}}\right)^T u'(c_{1T-1})(1+n)(\tilde{k}_T - \bar{\tilde{k}}_T) = 0. \tag{8.14}$$

This is a manifestation of the "complementary slackness condition" from the general theory of static maximization subject to inequality constraints. Here it says that if more capital than required is leftover, its discounted shadow price as seen from time 0 must be nil.[3]

In the infinite horizon case we have $\bar{\tilde{k}}_T = 0$. Then, taking the limit in (8.14) for $T \to \infty$ gives the condition (8.13). This "natural" extension of (8.14), with $\bar{\tilde{k}}_T = 0$, to the limit for $T \to \infty$ is valid in the present problem (in less proto-type economic problems such an extension need not be valid).

To obtain compatibility with a balanced growth path when $g > 0$, we have to specify $u(c)$ to be a CRRA function, $c^{1-\theta}/(1-\theta)$, with elasticity of marginal utility equal to $\theta > 0$. Then, in (8.13), we can substitute $u'(c_{1T-1}) = c_{1T-1}^{-\theta}$. In order that maximization of the social welfare function is possible, we need that the sum of discounted utilities, W_0, converges for $T \to \infty$ (an thereby remains bounded). To ensure this condition satisfied, we have to assume a positive intergenerational discount rate, $\bar{R}$, satisfying

$$1+\bar{R} > (1+g)^{1-\theta}. \tag{A2}$$

[3] A *discounted shadow price* (measured in some unit of account) of a good is, from the point of view of the buyer, the maximum number of units of account (here consumption utilities) that the optimizing buyer is willing to offer at *time 0* to obtain one extra unit of the good, here $\tilde{k}_{t+1}$, at *time* $t+1$.

This discounting might seem unjust and unethical towards future generations. Yet, we shall see that to the extent the future generations are favoured by better technology, they in fact tend to end up with higher lifetime utility than their predecessors in spite of the discounting in the social welfare function.

The transversality condition (7.8) from the Barro model for a market economy with positive bequests in Chapter 7 can be shown to be equivalent to (8.13). Thus, in view of the equilibrium condition $r_t = f'(\tilde{k}_t) - \delta$, the equations describing the equilibrium path in the Barro model for a competitive market economy with positive bequests are the same as those describing the social planner's solution. A necessary and sufficient condition for positive bequests in a steady state of that model was given by the condition (7.36). In terms of $\bar{R}$ we may restate this condition as

$$1 + \bar{R} < \frac{1 + r_D}{(1+n)(1+g)}(1+g)^{1-\theta}, \tag{8.15}$$

where R_D is the steady-state interest rate in the associated well-behaved Diamond economy. This inequality is compatible with (A2) whenever $1+r_D > (1+n)(1+g)$, that is, whenever the associated Diamond economy has an effective capital-labor ratio in steady state below the golden-rule value.

To summarize:

PROPOSITION 1 *(equivalence theorem)* Consider a perfectly competitive market economy with technology, demography, labor force, and preferences as described above. Assume $u(c) = c^{1-\theta}/(1-\theta)$ and, if $g > 0$, (A2). Suppose (8.15) holds and that households have perfect foresight and perfect computation ability. Let $\tilde{k}_0 > 0$ be given. If initial conditions are such that the bequest motive is operative for all $t \geq 0$, then the resulting resource allocation is the same as that brought about by a social planner facing the same technology and initial resources and having a positive effective intergenerational discount rate, $\bar{R}$, equal to the private one.

Proof. As the text above indicate, the laws of movement of the two models are the same, except that we have not yet shown that the transversality conditions of the two models are equivalent. This is done in Appendix C. Since the resource allocation brought about by the social planner is unique (the argument is given in Section 8.2.2), so is that of Barro's market economy. □

Social discounting: different views

To get perspective on Proposition 1, note that there may be good reasons that a social planner should have a *lower* intergenerational discount rate $\bar{R}$ than the private sector. One reason is put forward by Nobel laureate Amartya Sen who refers to what he calls the *isolation paradox* (Sen 1961). Suppose each old has an altruistic concern for *all* members of the next generation. Then a transfer from any member of the old generation to the heir entails an externality that benefits all other members of the old generation. A nation-wide coordination (political agreement) that internalizes these externalities would raise bequests, which corresponds to a lowering of the intergenerational utility discount rate, $\bar{R}$.

More generally, members of the present generations may be willing to join in a collective contract of more saving and investment by all, though unwilling to save more in isolation. Other reasons for a relatively low social discount rate have been suggested. One is based on the *super-responsibility argument:* the government has responsibility over a longer time horizon than those currently alive. Another is based on the *dual-role argument*: the members of the currently alive generations may in their political or public role be more concerned about the welfare of the future generations than they are in their private economic decisions.

Varieties of utilitarianism An elementary principle may be that the utility of all individuals should enter with the same weight in the social welfare function. Adherence to this principle implies that the per capita lifetime utility obtained by each *generation* from own lifetime consumption should be weighed by the size of that generation. This size is here growing at the rate n. Thus, from a social point of view the effective discount factor used to compare the per capita utility of the next generation with that of the present generation should be $1+n$. This principle may be called *utilitarianism* and implies a pure intergenerational discount rate, R, equal to zero, see (8.6). If we ignore technological progress, this "generation indifference" principle agrees with the principle that when setting up a social welfare criterion we should imagine that members of current and future generations agree on the value of R before knowing which generation they belong to (the "veil of ignorance" principle). Then, as long as we ignore technological progress, it seems likely that $R = 0$ would be agreed upon.

Taking into account that sustained technological progress is prevalent, many economists argue, however, that because future generations are likely to benefit from better technology, a counterbalancing positive value of R is ethically acceptable. This principle, sometimes called *discounted utilitarianism,* means that less weight is placed on the utility of members of future generations than on the utility of members of the present generations.[4] In spite of an $R > 0$, future generations may be better off than present generations if there is sustained growth in some measure of per capita welfare. The steady state considered in the next sub-section has this property when $g > 0$.

An alternative – or supplementary – argument for a positive R, though small,

[4]In contrast, the principle of *discounted average utilitarianism* uses R as effective intergenerational discount rate, thereby ignoring that generations generally are of different size.

is that there is a positive but small probability of a meteorite or an atomic war obliterating the human race in the future. Some argue that this is the *only* ethically acceptable reason for an $R > 0$. This is for instance the position taken by the "Stern Review" (the popular name for the voluminous report *The Economics of Climate Change,* published 2007 by the British economist Nicholas Stern and his associates and dealing with the huge negative externalities on future generations caused by the greenhouse gas emissions to the atmosphere). Because of the positive but small probability of extinction, the Stern Review contends that R on an annual basis should be considerably less than 0.001.[5]

8.1.2 The modified golden rule of the command optimum

In view of the equivalence theorem, it can be no surprise that also the social planner's solution in steady state satisfies the *modified golden rule.*

In steady state $\tilde{k}_t = \tilde{k}^*$, $\tilde{c}_{1t} = \tilde{c}_1^*$, and $\tilde{c}_{2t} = \tilde{c}_2^*$ for all $t = 0, 1, 2, ...,$ where $\tilde{k}^*$, $\tilde{c}_1^*$, and $\tilde{c}_2^*$ are positive constants. Hence, $c_{1t} = \tilde{c}_1^* \mathcal{T}_t = \tilde{c}_1^*(1+g)^t$. Owing to the CRRA utility function we have $u'(c_{1t}) = c_{1t}^{-\theta}$. Consequently, in steady state the first-order condition (8.9) reads

$$c_1^{*-\theta} = (1+\bar{R})^{-1} \left[c_1^*(1+g)\right]^{-\theta} \frac{f'(\tilde{k}^*) + 1 - \delta}{1+n}.$$

Solving for the net marginal productivity of capital, we get $f'(\tilde{k}^*) - \delta = (1 + \bar{R})(1+n)(1+g)^\theta - 1$. The capital intensity satisfying this condition is called the *modified-golden-rule* capital intensity, $\tilde{k}_{MGR}$, i.e.,

$$f'(\tilde{k}_{MGR}) - \delta = (1+\bar{R})(1+n)(1+g)^\theta - 1. \tag{8.16}$$

The conclusion is that optimality of a steady state requires that the capital intensity in this steady-state results in a net marginal productivity of capital equal to the right-hand side of (8.16). In the previous chapter we considered Barro's model of a competitive market economy with an intergenerational discount rate $\bar{R}$. We saw that as long as the bequest motive is operative, the steady state interest rate, $r^* = f'(\tilde{k}^*) - \delta$, in that model takes exactly the *modified-golden-rule* value given by the right-hand side of (8.16).

To help ensuring that a positive $\tilde{k}^*$ satisfying the modified golden rule *exists*, we assume the following combined technology and parameter condition

$$\begin{aligned} \lim_{\tilde{k}\to 0} f'(\tilde{k}) &> (1+\bar{R})(1+n)(1+g)^\theta - (1-\delta) \text{ and} \qquad \text{(A3)} \\ \lim_{\tilde{k}\to \infty} f'(\tilde{k}) &< (1+n)(1+g) - (1-\delta). \end{aligned}$$

[5] This is lower than the calibrated private own-generation preference $\tilde{R}$ equal to 0.0092, see Section 7.4 of Chapter 7.

Together with (A2), these two inequalities imply $\lim_{\tilde{k}\to 0} f'(\tilde{k}) > (1+\bar{R})(1+n)(1+g)^{\theta} - (1-\delta) > (1+n)(1+g) - (1-\delta) > \lim_{\tilde{k}\to\infty} f'(\tilde{k})$. By continuity of f', hereby, existence of both the modified-golden-rule capital intensity, $\tilde{k}_{MGR}$, and the "simple" golden-rule capital intensity, $\tilde{k}_{GR}$, is ensured, the latter being defined by $f'(\tilde{k}_{GR}) = (1+g)(1+n) - (1-\delta)$. With $\delta > 0, \bar{R} \geq 0, n \geq 0$, and $g \geq 0$, the Inada conditions are of course sufficient but not necessary for (A3) to be satisfied.

When the effective intergenerational discount rate is at its lower bound, $\bar{R} = (1+g)^{1-\theta} - 1$, given by (A2), the rule (8.16) simplifies to the simple *golden rule* where $\tilde{k}^* = \tilde{k}_{GR}$. This is the rule saying that to obtain the highest possible sustainable consumption path, the net marginal product of capital should in steady state equal the "natural growth rate" of GDP, which in turn equals $g + n + gn$;[6] when g and n are "small", this sum can be approximated by $g+n$. In this limiting case maximization of W_0 does not make sense (when $T \to \infty$), since W_0 will not be bounded. One can in such a case sometimes use another optimality criterion, be it the *overtaking* criterion or the more robust *catching-up* criterion. Indeed, our social planning problem turns out to be well-defined in terms of both these criteria when $1 + \bar{R} = (1+g)^{1-\theta}$; and the first-order conditions (8.4), (8.8), and (8.10) are still necessary for an optimal solution (see Section 8.4 below).

With the simple golden rule as the benchmark case, the predicate "modified" in the term "modified golden rule" should be interpreted in the following way. If the intergenerational discount rate is higher than needed to "compensate" for technological progress (i.e., $\bar{R} > (1+g)^{1-\theta} - 1$), then the social planner prefers a permanent effective capital-labor ratio $\tilde{k}_{MGR,}$ lower than $\tilde{k}_{GR}$. Though society could attain a higher consumption path in the long run, if $\tilde{k}_{GR}$ were strived for, this would not compensate for the cost in the form of reduced current consumption required to arrive at $\tilde{k} = \tilde{k}_{GR}$ (or stay there instead of moving to $\tilde{k}_{MGR}$). The long-run benefit is lower relative to this cost, the higher is the discount rate $\bar{R}$.[7]

8.1.3 The turnpike property

In Section 8.3 it is shown that when $1+\bar{R} > (1+g)^{1-\theta}$, the effective capital-labor ratio, $\tilde{k}_t$, along the optimal path converges for $t \to \infty$ to the unique steady-state value $\tilde{k}^* = \tilde{k}_{MGR}$. Even in the golden rule case, $1 + \bar{R} = (1+g)^{1-\theta}$, with

[6] In our discrete time setting the growth rate of $Y = TLf(\tilde{k}^*)$ in steady state is $(1+g)(1+n) - 1 = g + n + gn$.

[7] In the absence of technical progress, (8.16) simplifies to $f'(k_{MGR}) - \delta = (1+\bar{R})(1+n) - 1 = \bar{R} + n + \bar{R}n \approx \bar{R} + n$. This and similar approximations should be taken with caution because the period length is not the usual one year, but around 30 years; therefore, the term $\bar{R}n$ might not be negligible. Yet, the example calibrated at the end of Section 7.4 of the previous chapter gives $\bar{R}n \simeq 0.037 \cdot 0.270 \simeq 0.010$, which is relatively small compared to $\bar{R} + n \simeq 0.307$.

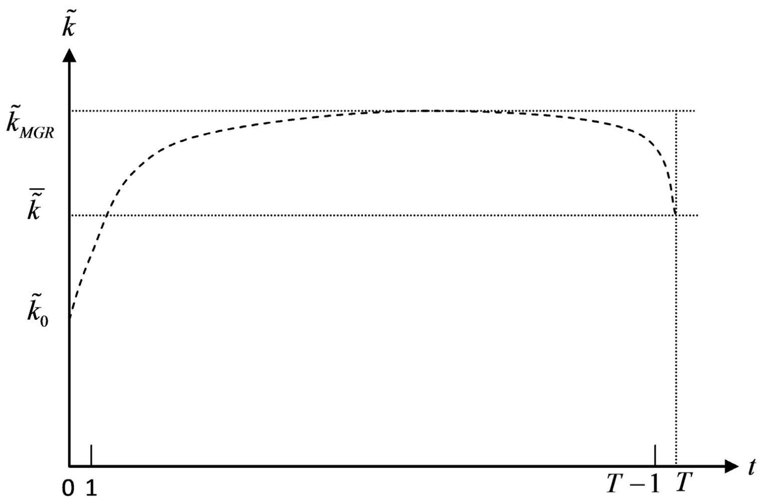

Figure 8.1: Finite horizon. Most of the time the optimal path has $\tilde{k}$ close to $\tilde{k}_{MGR}$.

overtaking or catching-up as optimality criterion, cf. Section 8.4, convergence towards $\tilde{k}^* = \tilde{k}_{GR}$ for $t \to \infty$ holds. These stability properties of the golden rule and modified-golden-rule paths are sometimes called "turnpike" properties. It is more common, however, to reserve the term "turnpike" specifically to a situation with a *finite* planning horizon.

We will here state, without proof, one such turnpike theorem for a finite-horizon planning problem like the one considered above:

> If the planning horizon T is "large", then the best, i.e., optimal, way to go from any initial (effective) capital-labor ratio $\tilde{k}_0 > 0$ to a specified terminal capital-labor ratio $\bar{\tilde{k}} \geq 0$ is to stay arbitrarily close to the modified-golden-rule capital-labor ratio, $\tilde{k}_{MGR}$, for most of the time.

Fig. 8.1 illustrates this turnpike property of the modified-golden-rule path ($\tilde{k} = \tilde{k}_{MGR}$) in a situation where at first glance moving close to $\tilde{k}_{MGR}$ seems a detour. Compare with a highway (turnpike): the shortest way to go from Hellerup to Helsinore is to take Strandvejen. Yet, it is faster to go by the highway.[8] This is similar to the economic planning problem. The intuition is that, first, the desire for consumption smoothing favors steady consumption compared to oscillating consumption. Second, the $\tilde{k} = \tilde{k}_{MGR}$ path entails the proper balance between consumption now and consumption later, given the intergenerational

[8] Hellerup is a suburb of Copenhagen. Strandvejen ("Beach Road") is the name of the narrow and winding road along the coast to Helsinore.

discount rate and the growth rates of the labor force and technology. Hence, if the economy is initially above $\tilde{k}_{MGR}$, some initial dissaving pays, both because dissaving directly means more consumption now and because maintaining a lower k in the future requires less saving in the future. If alternatively, as in Fig. 8.1, the economy is initially below $\tilde{k}_{MGR}$, the cost in terms of forgone early consumption of moving up to $\tilde{k}_{MGR}$ is compensated when finally we move down to $\bar{\tilde{k}}_{T+1}$.

8.2 Optimal control theory and the social planner's problem*

Heretofore we have solved intertemporal optimization problems by using the substitution method. The advantage of this method is that it is simple and straightforward. There are cases, however, where the method does not work. Another drawback is that the method is not immediately supported by a general mathematical machinery providing necessary and sufficient conditions for optimality and for existence of solutions to a broad class of optimization problems. Fortunately, two alternative mathematical methods are available which are backed up by such general machinery: *optimal control theory* (where the Hamiltonian function and shadow prices are key concepts) and *dynamic programming* (where the value function and the Bellman equation are key concepts). Here we will apply optimal control theory to the social planner's problem. Moreover, the questions of existence, uniqueness, and precise characterization of the solution are examined. A useful feature of the method of optimal control theory is that it delivers dynamics of the shadow prices.

The main result within optimal control theory is Pontryagin's Maximum Principle, in brief the *Maximum Principle*. In its continuous time version this principle was developed in the 1950s by the soviet mathematician L. S. Pontryagin and his associates with a view to engineering applications, control of rockets, satellites, etc.[9] Since then, the method has been applied also in medicine, biology, ecology, and economics. In economics the method is applied to a wide range of topics including the study of consumption versus saving, optimal taxation, firms' fixed capital investment, inventory control, pollution problems, and extraction of natural resources. Based on Pontryagin's principles a solution technique for *discrete time* dynamic optimization problems has been developed and it is a special case of this technique we will now use for solving the problem of a society's optimal capital accumulation. We first consider the finite horizon case and next the infinite horizon case.

[9] Pontryagin et al. (1962).

8.2.1 Decomposing the social planner's problem

To prepare the ground we first have to convert the social planner's problem into a form convenient for the application of the discrete time Maximum Principle. This conversion is of interest also in its own right. With $\beta \equiv (1+\bar{R})^{-1} \equiv (1+R)^{-1}(1+n) \in (0,1)$ and $\gamma \equiv (1+\rho)^{-1} > 0$, the social welfare function with a finite horizon is

$$W_0 = \gamma u(c_{20}) + \sum_{t=0}^{T-2} \beta^{t+1}[u(c_{1t}) + \gamma u(c_{2t+1})] + \beta^T u(c_{1T-1}). \tag{8.17}$$

We order terms after periods instead of generations:

$$\begin{aligned} W_0 &= \gamma u(c_{20}) + \sum_{t=0}^{T-2} \beta^{t+1} u(c_{1t}) + \gamma \sum_{t=0}^{T-2} \beta^{t+1} u(c_{2t+1}) + \beta^T u(c_{1T-1}) \\ &= \beta u(c_{10}) + \gamma u(c_{20}) + \sum_{t=1}^{T-1} \beta^{t+1} u(c_{1t}) + \gamma \sum_{t=1}^{T-1} \beta^t u(c_{2t}) \\ &= \beta u(c_{10}) + \gamma u(c_{20}) + \sum_{t=1}^{T-1} \beta^t \left(\beta u(c_{1t}) + \gamma u(c_{2t})\right) \\ &= \sum_{t=0}^{T-1} \beta^t \left(\beta u(c_{1t}) + \gamma u(c_{2t})\right). \end{aligned}$$

We name the function $\tilde{u}(c_{1t}, c_{2t}) \equiv \beta u(c_{1t}) + \gamma u(c_{2t})$ the *social planner's period utility function*. The arguments of this function are the per capita consumption in the young and the old generation, respectively, alive in period t.

With the social welfare function written this way, the optimization problem can be decomposed into *two separate problems*. One is the intertemporal problem: how to choose between less aggregate consumption in period t and more aggregate consumption later. The other is a static one: given aggregate consumption per unit of labor, $c_t \equiv C_t/L_t$, in period t, how should this consumption be shared among old and young?

The social planner's optimized period utility function

Let us take the second problem first. Since the problem is a static one, to save notation, we suppress the time index. The problem is: given $c > 0$,

$$\max_{c_1, c_2} \tilde{u}(c_1, c_2) = \beta u(c_1) + \gamma u(c_2) \quad \text{s.t.} \tag{8.18}$$

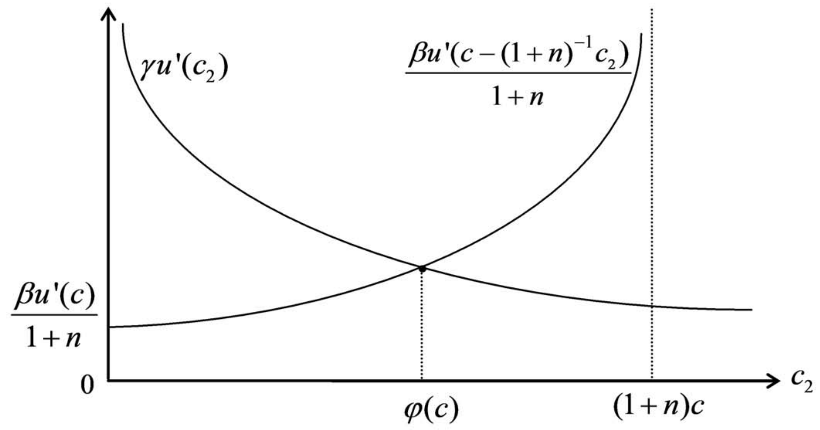

Figure 8.2: Finding c_2 as a solution of (8.20) for given $c > 0$.

$$\begin{aligned} c_1 + (1+n)^{-1}c_2 &= c, \\ c_1 &\geq 0, c_2 \geq 0. \end{aligned} \tag{8.19}$$

After substituting the constraint, $c_1 = c - (1+n)^{-1}c_2$, into the $\tilde{u}$ function, we find the first-order condition

$$\gamma u'(c_2) = \beta u'(c_1)(1+n)^{-1} = \beta u'\left(c - (1+n)^{-1}c_2\right)(1+n)^{-1}. \tag{8.20}$$

This equation defines c_2 as an implicit function of c, $c_2 = \varphi(c)$, where $c_2 > 0$ in view of (A1), cf. Fig. 8.2. By implicit differentiation, we find

$$\varphi'(c) = (1+n)\frac{\beta u''(c_1)}{(1+n)^2\gamma u''(c_2) + \beta u''(c_1)} \in (0, 1+n). \tag{8.21}$$

On this basis it is convenient to introduce a new function, $v(c)$, defined by

$$v(c) \equiv \tilde{u}(c - (1+n)^{-1}\varphi(c), \varphi(c)) = \beta u(c - (1+n)^{-1}\varphi(c)) + \gamma u(\varphi(c)). \tag{8.22}$$

This function, named the *social planner's optimized period utility function*, inherits its key properties from u. In optimum we have

$$\begin{aligned} v'(c) &= \beta u'(c_1)\left[1 - (1+n)^{-1}\varphi'(c)\right] + \gamma u'(c_2)\varphi'(c) \\ &= \beta u'(c_1) - \left[\beta u'(c_1)(1+n)^{-1} - \gamma u'(c_2)\right]\varphi'(c) = \beta u'(c_1), \end{aligned} \tag{8.23}$$

in view of (8.20). This property is a manifestation of the *envelope theorem.* Indeed, $v(c)$ is a composite function of c. Considering $\beta u(c-(1+n)^{-1}c_2)+\gamma u(c_2)$ $\equiv \tilde{\tilde{u}}(c, c_2)$ as the exterior function and $c_2 = \varphi(c)$ as the interior function, the envelope theorem says that the total derivative of $\tilde{\tilde{u}}$ w.r.t. c equals the partial derivative w.r.t. c.[10] That is, $\tilde{\tilde{u}}_1(c, c_2) + \tilde{\tilde{u}}_2(c, c_2)\varphi'(c) = \tilde{\tilde{u}}_1(c, c_2) = \beta u'(c_1)$, where the second but last equality comes from the fact that $\tilde{\tilde{u}}_2(c, c_2) = 0$ in optimum and the last equality from the definition of $\tilde{\tilde{u}}(c, c_2)$. Moreover,

$$v''(c) = \beta u''(c_1)\left[1 - (1+n)^{-1}\varphi'(c)\right] < 0, \tag{8.24}$$

in view of the fact that $u''(c_1) < 0$ and $1 - (1+n)^{-1}\varphi'(c) > 0$, by (8.21). Finally, by (8.19), we have that $c \to 0$ implies $c_1 \to 0$, from which follows $\lim_{c\to 0} v'(c) = \lim_{c_1\to 0} \beta u'(c_1) = \infty$, in view of the No Fast Assumption, (A1), stated in Section 8.1.

The intertemporal optimization problem

The intertemporal optimization problem is to choose a plan $\{c_t\}_{t=0}^{T-1}$ so as to

$$\max W_0 = \sum_{t=0}^{T-1} \beta^t v(c_t) \text{ s.t.} \tag{8.25}$$

$$c_t \geq 0, \tag{8.26}$$

$$\tilde{k}_{t+1} = \frac{f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - c_t/(1+g)^t}{(1+g)(1+n)}, \quad \tilde{k}_0 = \bar{\tilde{k}}_0 > 0, \tag{8.27}$$

$$\tilde{k}_t \geq 0, \text{ for } t = 0, 1, 2, ..., T-1, \text{ and } \tilde{k}_T \geq \bar{\tilde{k}} \geq 0, \tag{8.28}$$

where $\bar{\tilde{k}}_0$and $\bar{\tilde{k}}$ are given numbers. The constraint (8.27) comes from (8.19) in combination with the dynamic resource constraint (8.4), using that $\tilde{c}_t = c_t/(1+g)^t$, given $\mathcal{T}_0 = 1$.

We have here written the optimization problem on the standard form for an optimal control problem in discrete time. In the language of optimal control theory, c_t is a *control variable* in the sense of an instrument which the optimizing agent is able to directly control. Sometimes the alternative term *decision variable* is used. The set of admissible values of the control variable is called the *control region,* here the set of non-negative numbers. The variable $\tilde{k}_t$ entering the first-order difference equation (8.27) is called a *state variable.* It is in each period a predetermined variable and its value in the next period is not directly chosen, but is implied by the change caused by the chosen value of the control variable in the current period. When the first-order difference equation for the state variable is ordered such that it has the value of the state variable "next period" isolated on the left-hand side of the equation, as in (8.27), it is known as a *transition*

[10] Appendix A of Chapter 7.

function. There is given an *initial* value, $\bar{\tilde{k}}_0$, of the state variable, interpreted as historically determined. In many cases, including the present one, there will be a a *terminal constraint,* that is, a restriction on what values the state variable is allowed to take at the terminal date. Here this is represented by the constraint $\tilde{k}_T \geq \bar{\bar{k}}$ in (8.28). Owing to the nature of the state variable in the present problem, in (8.28) is added the non-negativity constraint on $\tilde{k}_t$ for $t = 0, 1, 2, ..., T-1$.[11]

To choose the "best" control, we need of course a criterion from which to choose. This is provided by the objective function (8.25), also known as the *criterion function.* There are problems where, contrary to the present case, both the control and the state variable (or only the latter) enter the criterion function. The model could, for instance, include environmental quality as a state variable. This state variable would then naturally enter the period utility function as a separate argument besides consumption. The solution procedure described below is also applicable in such cases. In economics we are sometimes interested in *minimizing* a criterion function. The function could for instance represent costs of a given project. In that case we can simply multiply the criterion function by the factor minus 1 and then maximize.

A path $\left\{\left(\tilde{k}_t, c_t\right)\right\}_{t=0}^{T-1}$ that satisfies the constraints (8.26), (8.27), and (8.28) is, in the terminology of optimal control theory, an *admissible path.* If we have $\bar{\bar{k}} = 0$, an admissible path in the present problem is the same as what in our general terminology in Chapter 3 is called a *technically feasible path.* If the terminal constraint has $\bar{\bar{k}} > 0$, the set of admissible paths is the subset of technically feasible paths satisfying $\tilde{k}_T \geq \bar{\bar{k}} > 0$. Anyway, an admissible path $\left\{\left(\tilde{k}_t, c_t\right)\right\}_{t=0}^{T-1}$ that solves the problem is an *optimal path* or simply a *solution.* A solution, $\left\{\left(\tilde{k}_t, c_t\right)\right\}_{t=0}^{T-1}$, is an *interior* solution if for all t, $\tilde{k}_t > 0$ and $c_t > 0$.

8.2.2 Applying Pontryagin's Maximum Principle

After making sure that the dynamic constraint, here (8.27), is written as a *transition function,* i.e., in the form $\tilde{k}_{t+1} = \psi(\tilde{k}_t, \tilde{c}_t, t)$, we are ready to solve the

[11] As we have stated the problem, here as well as in Section 8.1, we have implicitly assumed that the capital good can instantaneously (without cost) be converted into a consumption good and thus be consumed. Otherwise, the control region should be replaced by $0 \leq c_t \leq (1+g)^t f(\tilde{k}_t)$ (in which case a "complementary slackness" condition for all t must be added). In our formulation there is only an implicit – and less strict – upper constraint on c_t, namely $c_t \leq (1+g)^t \left[f(\tilde{k}_t) + (1-\delta)\tilde{k}_t\right]$. This weak inequality is implied by (8.27) combined with $\tilde{k}_{t+1} \geq 0$ for all t. It reflects that the consumed amount can never exceed the available amount of goods.

optimization problem. The four-step solution procedure described below applies to a large class of discrete-time intertemporal optimization problems in macroeconomics:

1. Set up the *current-value Hamiltonian function* associated with the problem:

$$\begin{aligned} H(\tilde{k}_t, c_t, \lambda_t, t) &\equiv v(c_t) + \lambda_t \psi(\tilde{k}_t, \tilde{c}_t, t) \\ &= v(c_t) + \lambda_t \frac{f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - c_t/(1+g)^t}{(1+g)(1+n)}, \quad (8.29) \end{aligned}$$

   for $t = 0, 1, ..., T-1$, where λ_t is an *adjoint variable* (also called a *co-state variable*) associated with the dynamic constraint in the problem. That is, λ_t is an auxiliary variable which is analogous to the Lagrange multiplier in static optimization.

2. For $t = 0, 1, ..., T-1$, differentiate H partially w.r.t. to the control variable. If looking for an *interior* solution, put the partial derivative equal to zero:

$$\frac{\partial H}{\partial c_t} = v'(c_t) - \lambda_t (1+g)^{-(t+1)}(1+n)^{-1} = 0, \quad (8.30)$$

   that is,

$$v'(c_t) = \lambda_t (1+g)^{-(t+1)}(1+n)^{-1}, \quad (8.31)$$

   for $t = 0, 1, ..., T-1$.[12]

3. Differentiate H partially w.r.t. the state variable. Then put the result for period t equal to the *adjoint variable* dated $t-1$, multiplied by the inverse of the discount factor in the objective function, that is,

$$\frac{\partial H}{\partial \tilde{k}_t} = \lambda_t \frac{f'(\tilde{k}_t) + 1 - \delta}{(1+g)(1+n)} = \lambda_{t-1}\beta^{-1}, \quad (8.32)$$

   for $t = 1, 2, ..., T-1$. In this way a first-order difference equation in the adjoint variable is obtained.

4. Now apply the *Maximum Principle* which (for this problem type) states: an interior optimal path $\left\{ \left(\tilde{k}_t, c_t \right) \right\}_{t=0}^{T-1}$ will satisfy that there exists an adjoint variable, λ_t, such that (8.30) and (8.32), for $t = 0, 1, ..., T-1$, and

[12] If we wish to allow for boundary solutions, (8.30) is replaced by a more general condition, which (surprisingly) only if H is concave in c is equivalent to requiring that c_t should maximize H. For details, see Feichtinger and Hartl (1986, p. 504 ff.).

$t = 1, ..., T-1$, respectively, hold along the path, and the *transversality condition*,

$$\beta^{T-1}\lambda_{T-1}(\tilde{k}_T - \bar{\tilde{k}}) = 0, \tag{8.33}$$

is satisfied.

The optimality condition (8.31) can be seen as a $MC = MB$ condition. The current utility *cost* by decreasing c_t by one unit equals the left-hand side of (8.31). The associated *gain* comes from capital next period being increased by the amount $\Delta K_{t+1} = -\Delta C_t = -L_t \Delta c_t = L_t \cdot 1$ for $\Delta c = -1$. Thereby, in view of (8.27), $\tilde{k}_{t+1}$ is approximately increased by the amount

$$\Delta\tilde{k}_{t+1} \approx \frac{\partial \tilde{k}_{t+1}}{\partial c_t}\Delta c_t = -(1+g)^{-(t+1)}(1+n)^{-1}\Delta c_t = (1+g)^{-(t+1)}(1+n)^{-1}. \tag{8.34}$$

So, when $\Delta c = -1$, the optimality condition (8.31) can be rewritten $v'(c_t) = \lambda_t \partial\tilde{k}_{t+1}/\partial c_t \approx \lambda_t \Delta\tilde{k}_{t+1}$, where the right-hand side is the gain obtained through the increase in $\tilde{k}_{t+1}$, measured in period$-t$ utility units. The adjoint variable λ_t can thus be interpretated as the *shadow price,* measured in current utility units, of the marginal unit of capital (per unit of effective labor) next period along the optimal path.

This interpretation of λ_t is confirmed if we rewrite (8.31) as

$$\lambda_t = v'(c_t)(1+g)^{t+1}(1+n). \tag{8.35}$$

Imagine a *decrease* of next period's effective capital-labor ratio, $\tilde{k}_{t+1}$, by one unit. The right-hand side of (8.35) then directly expresses the current utility gain obtained by this. The gain derives from the unit-decrease in $\tilde{k}_{t+1}$ allowing an increase in consumption per worker in period t by $\Delta c_t \approx (1+g)^{t+1}(1+n)$ units (invert (8.34) for $\Delta\tilde{k}_{t+1} = -1$). When we multiply this by the marginal utility of consumption in period t, $v'(c_t)$, we get the total utility gain in period t from this reallocation. Since in the optimal plan such a marginal reallocation leaves total welfare unchanged, the left-hand side of (8.35) must measure the current utility worth of the marginal unit of capital (per unit of effective labor) next period along the optimal path.

The key importance of the condition (8.32) lies in its message that the shadow price λ_t satisfies a certain difference equation. Substituting (8.35), as it stands and for t replaced by $t-1$, into (8.32) and reordering gives

$$v'(c_{t-1}) = \beta v'(c_t)\frac{f'(\tilde{k}_t) + 1 - \delta}{1+n}. \tag{8.36}$$

This is of the same form as the consumption Euler equation in (8.9). : on the margin one unit of account (here the output good) must in the optimal plan be equally valuable in its two alternative uses: consumption in period $t-1$ or saving, resulting in per capita net return $(f'(\tilde{k}_t)+1-\delta)/(1+n)$, which gives rise to extra consumption in period t, the discounted utility value of which is measured by the right-hand side of (8.36).

Finally, the transversality condition, (8.33), has the form of a complementary slackness condition. Here it implies $\tilde{k}_T = \bar{\tilde{k}}$. Indeed, in view of (8.35), an optimal plan has $\lambda_{T-1} = v'(c_{T-1})(1+g)^T(1+n) > 0$ (no saturation). Thus, if $\tilde{k}_T > \bar{\tilde{k}}$ ("over-satisfaction" of the terminal constraint in (8.28)), then higher welfare could be obtained by decreasing $\tilde{k}_T$ and increasing c_{T-1} correspondingly. There would be scope for this change without violating the terminal constraint (8.28).

Technical remark. A Hamiltonian function is often just called a *Hamiltonian.* More importantly, the prefix "current-value" is used to distinguish it from the *present-value Hamiltonian.* The latter is defined as $\hat{H} \equiv \beta^t H$ with $\beta^t \lambda_t$ in the second term substituted by μ_t, which is the associated (discounted) adjoint variable. Applying the present-value Hamiltonian involves a similar solution procedure except that step 3 is replaced by $\partial \hat{H}/\partial \tilde{k}_t = \mu_{t-1}$, and in the transversality condition, $\beta^{T-1}\lambda_{T-1}$ is replaced by μ_{T-1}. The two solution procedures are equivalent. For many economic problems the *current-value* Hamiltonian has the advantage that it makes the interpretation simpler. In the current-value Hamiltonian the adjoint variable, λ_t, which acts as a shadow price of the state variable, is a *current* price rather than a *discounted* price as μ_t. □

The Maximum Principle gives *necessary* conditions for an optimal plan. That is, from the above analysis we know that any interior optimal solution to the social planner's problem *must* satisfy the above conditions. These conditions are helpful for finding a *candidate* for an optimal solution, but they do not guarantee that this candidate *is* an optimal solution or that there at all *exists* an optimal solution (as the problem is phrased).[13] For these concerns we must appeal to *sufficient* conditions for an optimal plan and to circumstances which verify that an optimal solution *exists.* Regarding the sufficiency issue, note that our Hamiltonian (8.29) is for every t jointly concave in $\tilde{k}_t$ and c_t; indeed, the first term in (8.29) is (strictly) concave in c_t and the second term is (strictly) concave in $\tilde{k}_t$ and concave in c_t. Then, for problems like the present one, it can be shown that a plan $\left\{\left(\tilde{k}_t, c_t\right)\right\}_{t=0}^{T-1}$ which satisfies the first-order conditions and the transversality

[13] Sydsæter et al. (2008, p. 373) provide the following example of non-existence of a solution. A person wants to keep a pan of boiling water as close as possible to the constant temperature of 100°C for one hour when it is being heated on an electric burner whose only control is an on/off switch. Assuming there is no cost of switching, there is no limit to the number of times the burner should be turned on and off.

condition *is* optimal.[14] Consequently, these conditions are both necessary and sufficient for optimality.

Regarding the issue of *existence* of an optimal solution to the social planner's problem, as far as the present finite horizon case is concerned, existence is easily proved on the basis of the *extreme value theorem* (see Appendix A).

Infinite horizon

In (8.25) and (8.28) we let $T \to \infty$ and set $\bar{\tilde{k}} = 0$. Then the social planner's dynamic problem is a standard "neoclassical optimal growth problem" with exogenous technological progress. The problem is also called "Ramsey's optimal saving problem in discrete time" because it has a close connection to Frank Ramsey's classical analysis of a society's optimal saving (Ramsey, 1928).

With infinite horizon the first-order conditions (8.30) and (8.32) are still necessary conditions for an interior solution. The "natural" extension of the necessary transversality condition (8.33), with $\bar{\tilde{k}} = 0$, to an infinite horizon is

$$\lim_{T\to\infty} \beta^{T-1}\lambda_{T-1}\tilde{k}_T = 0. \tag{8.37}$$

As touched upon in Section 8.1, in prototype-economic problems, like the present one, such a direct extension of a necessary transversality condition from a finite to an infinite horizon *is* valid. The condition (8.37) says that the present value (in utility units) of the capital stock "left over" at infinity must be zero. Otherwise there is over-accumulation (or no optimal solution exists because W_0 is unbounded from above when $T \to \infty$).

Since the first-order condition (8.35) holds for any interior solution, (8.37) can be written

$$\lim_{T\to\infty} \beta^{T-1}v'(c_{T-1})(1+g)^T(1+n)\tilde{k}_T = 0. \tag{8.38}$$

For later use this is a more convenient form for the necessary transversality condition. In Appendix C a proof of the necessity of (8.38) is sketched.

Sufficiency of the first-order and transversality conditions In the social planner's optimization problem with infinite horizon, since the Hamiltonian is concave w.r.t. $(\tilde{k}_t, c_t)$ and both the state variable and its shadow price λ_t are nonnegative for every t, the first-order conditions together with the transversality condition are both necessary and sufficient for an interior optimal solution.[15] These conditions may be called the *Mangasarian conditions* because they are

[14] Sydsæter et al. (2008, p. 445)

[15] Sydsæter et al. (2008, p. 447).

analogue to the sufficient conditions established by the American mathematician, Olvi Mangasarian, for continuous-time optimal control problems (Mangasarian, 1966). Moreover, it turns out that the Hamiltonian (8.29) is for every t *strictly concave* in $(\tilde{k}_t, c_t)$.[16] It then follows that a solution to the social planner's optimal control problem is *unique*.

It remains to show that there *exists* an admissible path satisfying the Mangasarian conditions and to characterize the properties of such a path. Are for instance oscillations possible or does the solution display a monotonic pattern over time? Here, dynamic analysis and a phase diagram are useful.

8.3 The transitional dynamics*

To allow for balanced growth, thereby making the dynamic analysis easily tractable, we specialize the investigation to the case where the household's period utility function is of CRRA form:[17]

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0. \tag{8.39}$$

In this case the partial similarity between social planner's optimized period utility function, v, and the household's period utility function, u, commented on in connection with the "envelope condition" (8.23) above, becomes complete:

LEMMA 1 Suppose $u(c)$ in (8.17) is a CRRA function with parameter θ. Then so is the social planner's optimized period utility function, $v(c)$, defined in (8.22). We can thus write

$$v(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0. \tag{8.40}$$

Proof. Substituting $u'(c_i) = c_i{}^{-\theta}$ and $u''(c_i) = -\theta c_i{}^{-\theta-1}$, for $i = 1, 2$, into (8.20) and (8.21) gives $\phi'(c) = c_2/c$ in view of (8.19). Then, by (8.24), $v''(c)c = \beta u''(c_1)\left[c - (1+n)^{-1}c_2\right] = \beta u''(c_1)c_1$ in optimum. Combining this with (8.23), we have

$$-\frac{cv''(c)}{v'(c)} = -\frac{c_1 u''(c_1)}{u'(c_1)} = \theta, \tag{8.41}$$

where the last equality is implied by (8.39). The result (8.41) holds for any $c > 0$. From Appendix A of Chapter 3 we know that up to a positive linear transformation v must then be of the form (8.40). □

[16] This is seen by applying the same method as used in Appendix E of Chapter 10 for the continuous time case.

[17] As mentioned in the previous chapter, in problems with infinite horizon it is an advantage not to have to bother with additive constants in the instantaneous utilities. Otherwise, convergence of the sum (8.25) for $T \to \infty$ may go by the board. Hence we write the CRRA function without subtraction of the constant $1/1-\theta)$.

Remark. The key in the proof is the observation that when the household's period utility function, u, is CRRA, the planner's chosen c_2 is a function, $\phi(c)$, of the current per capita consumption level, c, with derivative $\phi'(c) = c_2/c$. This result means that the elasticity of c_2 w.r.t. the "budget", c, is 1. Consequently, the chosen c_1 and c_2 are proportional to c and to each other, implying that the social planner's period utility function $\tilde{u}(c_1, c_2)$ in (8.18) is homothetic.[18] This implication of u being CRRA should be no surprise. From Chapter 4 we know this implication for an additive utility function with sub-utility functions that are CRRA with the same parameter θ. Also, recall that in case $\theta = 1$, the expression on the right-hand side of (8.39) and (8.40) should be interpreted as $\ln c$. □

The model can be reduced to two coupled first-order difference equations in $\tilde{k}_t$ and $\tilde{c}_t$. In (8.36) we replace t by $t+1$, apply (8.40), and reorder to get

$$c_{t+1}^{-\theta} = \beta^{-1} \frac{1+n}{f'(\tilde{k}_{t+1}) + 1 - \delta} c_t^{-\theta}. \tag{8.42}$$

In view of $c_t = \tilde{c}_t(1+g)^t$, we have $c_t^{-\theta} = \tilde{c}_t^{-\theta}(1+g)^{-\theta t}$. Substituting into (8.42) and rearranging yields

$$\tilde{c}_{t+1}^{-\theta} = \beta^{-1} \frac{(1+g)^{\theta}(1+n)}{f'(\tilde{k}_{t+1}) + 1 - \delta} \tilde{c}_t^{-\theta}. \tag{8.43}$$

The transition function (8.27) can be written

$$\tilde{k}_{t+1} = \frac{f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - \tilde{c}_t}{(1+g)(1+n)} \equiv h(\tilde{k}_t, \tilde{c}_t), \tag{8.44}$$

where

$$h_1 = \frac{f'(\tilde{k}_t) + 1 - \delta}{(1+g)(1+n)} > 0, \tag{8.45}$$

$$h_2 = -\frac{1}{(1+g)(1+n)} < 0. \tag{8.46}$$

Substituting (8.44) into (8.43) gives

$$\tilde{c}_{t+1} = \left(\beta \frac{f'(h(\tilde{k}_t, \tilde{c}_t)) + 1 - \delta}{(1+g)^{\theta}(1+n)} \right)^{1/\theta} \tilde{c}_t. \tag{8.47}$$

The equations (8.44) and (8.47) constitute a system of two coupled first-order difference equations in $\tilde{k}_t$ and $\tilde{c}_t$ and these equations are autonomous, i.e.,

[18] See Appendix C of Chapter 4.

they do not depend on t separately. The initial $\tilde{k}$ is historically given by the value $\bar{\tilde{k}}_0$. From now, to save notation, we let the symbol $\tilde{k}_0$ itself directly indicate a historically given initial value of $\tilde{k}$. But the initial $\tilde{c}$, $\tilde{c}_0$, is up to the social planner's choice, hence endogenous. As a substitute for knowing $\tilde{c}_0$ in advance, we have, fortunately, the transversality condition (8.38).

We may restate the parameter inequality (A2) from Section 8.1 as an upper bound on the utility discount factor, $\beta \equiv (1+\bar{R})^{-1}$, this way:

$$0 < \beta < (1+g)^{\theta-1}, \quad \text{where } g \geq 0. \tag{A2'}$$

This inequality will ensure boundedness from above of the social welfare function. Similarly, the condition (A3) on the range of the marginal productivity of capital can be restated as

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \beta^{-1}(1+n)(1+g)^{\theta} - (1-\delta) \text{ and } \lim_{\tilde{k}\to\infty} f'(\tilde{k}) < (1+n)(1+g) - (1-\delta), \tag{A3'}$$

Together with (A2'), these two inequalities imply $\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \beta^{-1}(1+n)(1+g)^{\theta} - (1-\delta) > (1+n)(1+g) - (1-\delta) > \lim_{\tilde{k}\to\infty} f'(\tilde{k})$. By continuity of f', hereby, existence of both the golden-rule capital intensity, $\tilde{k}_{GR}$, and the modified-golden-rule capital intensity, $\tilde{k}_{MGR}$, is ensured.[19]

Phase diagram

By a phase diagram for the dynamic system (8.44) – (8.47) is meant a graph in the $(\tilde{k}, \tilde{c})$ plane showing projections of the time paths, $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$, that are consistent with the system for alternative arbitrary initial points, $(\tilde{k}_0, \tilde{c}_0)$. The phase diagram is shown in the lower panel of Fig. 8.3.

In the phase diagram the curve marked by "$\tilde{k}_{t+1} = \tilde{k}_t$" shows the points $(\tilde{k}, \tilde{c})$ with the property that if $\tilde{k}_t = \tilde{k}$, then $\tilde{c}$ is the value of $\tilde{c}_t$ such that $\tilde{k}_{t+1}$ in the dynamic equation (8.44) takes the value $\tilde{k}$, that is, the same value as $\tilde{k}_t$ has. In brief, the locus for $\tilde{k}_{t+1} = \tilde{k}_t$ is made up by the pairs $(\tilde{k}, \tilde{c})$ at which (8.44) generates no change from t to $t+1$ in the effective capital-labor ratio. The pairs $(\tilde{k}, \tilde{c})$ with this property satisfy the equation

$$\tilde{c} = f(\tilde{k}) - [(1+g)(1+n) - (1-\delta)]\,\tilde{k} \equiv \tilde{c}(\tilde{k}),$$

by (8.44). The graph representing this equation in the phase diagram is called the *nullcline for* $\tilde{k}$. The example shown Fig. 8.3 has the graph going through the origin, i.e., $f(0) = 0$ is presumed. Here capital is thus essential. But all the

[19] The often presumed Inada conditions, $\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0$, are stricter than (A3') and not necessary.

conclusions we shall consider go through also when capital is not essential. So we only impose the condition $f(0) \geq 0$.

The upper panel of Fig. 8.3 illustrates how the graph of $\tilde{c}(\tilde{k})$ can be constructed as the vertical distance between the curve $\tilde{y} = f(\tilde{k})$ and the line $\tilde{y} = [(1+g)(1+n) - (1-\delta)]\,\tilde{k}$ (to save space, the proportions are distorted). Both the upper and lower panel indicate the position of the golden rule capital intensity, $\tilde{k}_{GR}$, defined by $f'(\tilde{k}_{GR}) = (1+g)(1+n) - (1-\delta)$. In the upper panel the tangent to the f curve having this slope tells where $\tilde{k}_{GR}$ is. In the lower panel, $\tilde{k}_{GR}$ is at the point where the tangent to the *OEB* curve is horizontal, namely where $\tilde{c}'(\tilde{k}) = f'(\tilde{k}) - [(1+g)(1+n) - (1-\delta)] = 0$. In view of (A3'), the technology guarantees existence of such a value of $\tilde{k}$.

The *horizontal arrows* in the lower panel indicate the direction in which $\tilde{k}$ moves if the economy is not at the $\tilde{k}_{t+1} = \tilde{k}_t$ locus. These directions are determined by (8.44). Above the $\tilde{k}_{t+1} = \tilde{k}_t$ locus, consumption is so high and saving so low that the capital intensity shrinks ($\tilde{k}_{t+1} < \tilde{k}_t$). Below the $\tilde{k}_{t+1} = \tilde{k}_t$ locus, consumption is so low and saving so high that the capital intensity grows ($\tilde{k}_{t+1} > \tilde{k}_t$).

Now, consider the *nullcline for* $\tilde{c}$, i.e., the $\tilde{c}_{t+1} = \tilde{c}_t$ locus. This is the collection of points $(\tilde{k}, \tilde{c})$ with the property that no change in $\tilde{c}$ is generated by the dynamic equation (8.47). These points are such that the pair $(\tilde{k}, \tilde{c})$ satisfies the equation

$$h(\tilde{k}, \tilde{c}) = \tilde{k}_{MGR}, \tag{8.48}$$

where h is the function defined in (8.44), and $\tilde{k}_{MGR}$ is the modified-golden-rule capital intensity. This follows from (8.47), since $\tilde{k}_{MGR}$, as defined in (8.16), is given by

$$f'(\tilde{k}_{MGR}) + 1 - \delta = \beta^{-1}(1+g)^{\theta}(1+n), \tag{8.49}$$

in view of $\beta \equiv (1+\bar{R})^{-1}$.

Equation (8.48) defines $\tilde{c}$ as an implicit function of $\tilde{k}$, i.e., $\tilde{c} = \eta(\tilde{k})$, where $\eta'(\tilde{k}) = -h_1/h_2 > 0$, by (8.45) and (8.46). Consequently, the $\tilde{c}_{t+1} = \tilde{c}_t$ locus has positive slope. In Fig. 8.3 it is represented by the curve *DEF*. This curve must cross the $\tilde{k}_{t+1} = \tilde{k}_t$ locus exactly where $\tilde{k} = \tilde{k}_{MGR}$. Indeed, in view of the definition of the function h in (8.44), the $\tilde{c}_{t+1} = \tilde{c}_t$ locus is such that $\tilde{k}_{t+1} = \tilde{k}_{MGR}$. Hence, at the point where the $\tilde{c}_{t+1} = \tilde{c}_t$ locus crosses the $\tilde{k}_{t+1} = \tilde{k}_t$ locus, we have $\tilde{k}_t = \tilde{k}_{t+1} = \tilde{k}_{MGR}$. Therefore, the (non-trivial) steady-state value of $\tilde{k}$ is $\tilde{k}^* = \tilde{k}_{MGR}$. The corresponding steady-state value of $\tilde{c}$ is called $\tilde{c}^*$, cf. the point E in Fig. 8.3.[20]

The *vertical arrows* in the figure indicate the direction in which $\tilde{c}$ moves if the economy is not at the $\tilde{c}_{t+1} = \tilde{c}_t$ locus. These directions are determined by

[20] The point D in Fig. 8.3 is located where the $\tilde{c}_{t+1} = \tilde{c}_t$ locus crosses the $\tilde{k}$-axis. This happens to be at a $\tilde{k} > 0$ if capital is essential (see Appendix D), but this is not crucial for any of our conclusions.

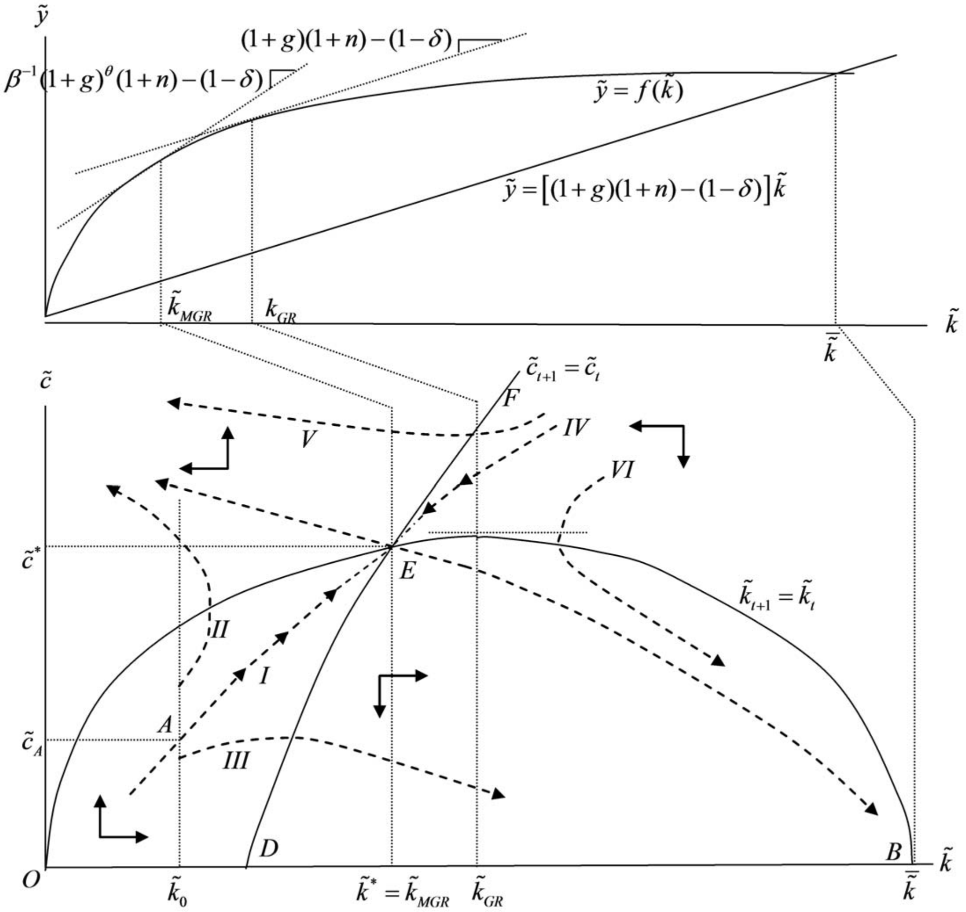

Figure 8.3: Assembly of the phase diagram.

(8.47). To the left of the $\tilde{c}_{t+1} = \tilde{c}_t$ locus, we have $\tilde{k} < \tilde{k}_{MGR}$, hence the marginal productivity of capital is above the *MGR* level. Given this high rate of return, it is optimal to postpone consumption in order to enjoy higher consumption later and so $\tilde{c}_{t+1} > \tilde{c}_t$ in (8.47). To the right of the $\tilde{c}_{t+1} = \tilde{c}_t$ locus, we have $\tilde{k}_{t+1} > \tilde{k}_{MGR}$, hence the marginal productivity of capital is below the *MGR* level. Given this low rate of return, impatience "wins" and encourages consumption now to the detriment of consumption in the future. So $\tilde{c}_{t+1} < \tilde{c}_t$ in (8.47).

The unique convergent path is the unique solution By construction the first-order conditions are satisfied along the trajectories in the phase diagram. In particular they are satisfied at the steady-state point E. We claim that also the necessary transversality condition (8.38), with $v'(c_{t-1}) = c_{t-1}^{-\theta}$, is satisfied at E.

Indeed, since $c_{t-1}^{-\theta} = (\tilde{c}_{t-1}(1+g)^{t-1})^{-\theta}$, the transversality condition can be written

$$\lim_{t\to\infty} \left[\beta(1+g)^{1-\theta}\right]^{t-1} \tilde{c}_{t-1}^{-\theta}(1+g)(1+n)\tilde{k}_t = 0. \tag{8.50}$$

In the steady state $\tilde{c}_{t-1}$ and $\tilde{k}_t$ can be replaced by the constants $\tilde{c}^*$ and $\tilde{k}^*$, respectively. So (8.50) is satisfied in view of (A2'). The transversality condition will then hold also along any trajectory converging towards the steady state E. Since $\tilde{k}_0$ is predetermined, the economy must at the initial date be at some point on the vertical line $\tilde{k} = \tilde{k}_0$ in Fig. 8.3. Among the infinitely many admissible values of c_0, the social planner looks for an optimal one, knowing that over time optimal consumption must move according to (8.47).

We claim that the solution is to choose $\tilde{c}_0$ such that the economy converges to the steady state, E, for $t \to \infty$. As we shall see, this requires the unique choice $c_0 = \tilde{c}_A$ in Fig. 8.3. The argument goes as follows. Consider the arrows in the phase diagram. They suggest that the steady state, E, is a *saddle point.* Graphically, by this is meant a steady-state point with the following property: there exists exactly two paths (one from each side of $\tilde{k}^*$) which tends towards the steady state E, cf. the two stippled curves, I and IV, through E pointing North-East and South-West, respectively. All other paths move away from the steady state and asymptotically approach one of the diverging paths represented by the two stippled curves through E pointing North-West and South-East, respectively. In Appendix D, where a formal definition of a saddle point in terms of eigenvalues is given, we also give an algebraic proof that the steady state *is* a saddle point and thereby that the arrows along the paths I and IV can rightly be interpreted as indicating convergence over time towards the steady state E; indeed, monotonous (non-oscillating) convergence can be shown. The reason that an algebraic proof is needed is that a phase diagram for a dynamic system in *discrete time* can only *suggest* the convergence property. Contrary to a system in continuous time, the system in discrete time will not move continuously along one of the trajectories. Only a countable number of points on the trajectory will be observed (this is why the solution curves in the diagram are stippled). Jumps forth and back across the steady state can not *apriori* be ruled out.

The two converging paths, I and IV, in Fig. 8.3 are called *saddle paths.* In combination they make up what is known as the *stable branch* (or *stable arm).* The two diverging paths going through E in combination make up the *unstable branch* (or *unstable arm*).

It follows that choosing $\tilde{c}_0 = \tilde{c}_A$ in Fig. 8.3 implies choosing a path which converges to the steady state E. Since the Mangasarian conditions hold, we conclude that this path, path I as the figure is drawn, *is* an optimal solution. If instead $\tilde{k}_0$ were above $\tilde{k}^*$, the optimal path would be on the upper stable branch, path IV in Fig. 8.3. Knowing from the end of Section 8.2.2 that an optimal solution will be unique, we have thus reached the conclusion that a solution to the social planner's problem *exists*, is *unique*, and *converges* to the *modified golden rule.*

Let us try to get some more direct understanding why all the other dynamic paths in the diagram can be ruled out as solutions. Paths starting *below* the saddle path (such as path III in the diagram) entail so low consumption, given the value of the state variable $\tilde{k}$, and so high investment that the economy in finite time ends up in the regime with $\tilde{k} > \tilde{k}_{GR}$ and $\tilde{k}$ still growing. This implies that the transversality condition (8.50) is violated, thus signifying "overaccumulation". To see this, suppose, without loss of generality, that already at time 0, we have $\tilde{k}_0 > \tilde{k}_{GR}$ along such a path. Then $\tilde{k}_t > \tilde{k}_{GR}$ for all $t \geq 0$ so that

$$f'(\tilde{k}_{t+1}) + 1 - \delta < f'(\tilde{k}_{GR}) + 1 - \delta = (1+g)(1+n)$$

for all $t \geq 0$. By (8.43), lagged two periods, this implies, for $t \geq 2$,

$$\tilde{c}_{t-1}^{-\theta} > \beta^{-1}(1+g)^{\theta-1}\tilde{c}_{t-2}^{-\theta} \geq \left[\beta^{-1}(1+g)^{\theta-1}\right]^{t-1} \tilde{c}_0^{-\theta},$$

by backward iteration. Consequently,

$$\lim_{t\to\infty} \left[\beta(1+g)^{1-\theta}\right]^{t-1} \tilde{c}_{t-1}^{-\theta} \geq \tilde{c}_0^{-\theta} > 0.$$

Since in addition $\lim_{t\to\infty} \tilde{k}_t = \bar{\tilde{k}} > 0$ along the path, the transversality condition (8.50) is thus violated. If $\tilde{k}_0$ were above $\tilde{k}^*$, it is paths like VI that are relevant. Also these paths signify "overaccumulation" in the long run.

All paths starting *above* the saddle path (such as path II in the diagram) reach $\tilde{k} = 0$ in finite time. This follows from the fact that before this possibly happens, there is a sequence of periods where not only is $\tilde{c}_t$ so large that $\tilde{k}_t$ is decreasing, but $\tilde{c}_t$ is at the same time increasing over time. Then, sooner or later, all capital is used up, and consumption, $\tilde{c}$, drops sharply to at most $f(0) \geq 0$. That this road to a foreseeable disaster in finite time will be avoided by the optimizing planner seems intuitive. If $\tilde{k}_0$ were above $\tilde{k}^*$, it is paths like V that are relevant, and they have the same doomsday implication.

Taking stock

Let us sum up:

PROPOSITION 2 *(existence, uniqueness, and convergence)* Assume (A2') and (A3'). Let $\tilde{k}_0 > 0$. Then there exists a unique optimal path, $\left\{(\tilde{k}_t, \tilde{c}_t)\right\}_{t=0}^{\infty}$. It starts at the point on the saddle path which corresponds to the given $\tilde{k}_0$. The optimal path then follows the saddle path and converges toward the steady state,

E. The steady state has $\tilde{k}^* = \tilde{k}_{MGR}$. The modified-golden-rule condition, (8.49), thus holds in the long run.

As the steady state is a saddle point, the convergence property of the optimal solution is known as *saddle-point stability.* For details see Appendix D, which also contains an algebraic formula for the optimal time path, based on a linear approximation of the dynamic system around the steady state.

Given the optimal path for $\tilde{k}_t$ and $\tilde{c}_t$, the optimal paths for other variables are easily found. For instance, given the optimal $\tilde{c}_t$, the optimal level of consumption per unit of labor in the economy will be $c_t = \tilde{c}_t(1+g)^t$ where $\tilde{c}_t \to \tilde{c}^*$ for $t \to \infty$. To find the optimal distribution of consumption between generations we insert this c_t into the first-order condition (8.20) from the static optimization problem. In view of the CRRA utility specification, this first-order condition amounts to

$$\gamma c_{2t}^{-\theta} = \beta \left(c_t - (1+n)^{-1} c_{2t} \right)^{-\theta} (1+n)^{-1}. \tag{8.51}$$

Given c_t, this equation has a unique solution in c_{2t}, from which we finally find $c_{1t} = c_t - (1+n)^{-1}c_{2t}$, by (8.19).

Note the chain of causality concerning the long-run properties of the optimal solution. First, the preference parameters (β and θ) and growth parameters (g and n) determine the net marginal productivity of capital in steady state according to the modified golden rule. For instance, a lower intergenerational discount factor β (less weight on future generations) and, when $g > 0$, a higher θ (more weight on individual consumption smoothing) reduces the steady-state capital intensity $\tilde{k}_{MGR}$, thereby raising the net marginal productivity of capital. A higher rate of technical progress, g, has a similar effect. In the next step the technology factors (the production function f and capital depreciation rate δ) determine the capital intensity and consumption per unit of effective labor in steady state. Finally, the optimal distribution of consumption between young and old is in every period determined by the lifetime preference parameters, $\gamma \equiv (1+\rho)^{-1}$ and θ, and the social planner's effective intergenerational discount factor, $\beta \equiv (1+\bar{R})^{-1}$.

Combining Proposition 1 and Proposition 2, we can infer that assuming (A2') and (A3'), also the Barro model of a market economy for which (8.15) holds (so that positive bequests obtain) will feature a unique and stable steady state which satisfies the modified golden rule.

Technical remark. ....(hent fra Ch8-2016-1, s. 329) Then the criterion function (8.25) should be replaced by (8.x) and c_t in (8.27) by $c_{1t} + (1+n)^{-1}c_{2t}$. The Hamiltonian should be rewritten correspondingly. The solution procedure would be to maximize the rewritten Hamiltonian with respect to the *two* control variables, c_{1t} and c_{2t}. In a different context, a case with two control variables (and continuous time) is considered in Chapter 17. □

8.4 The overtaking and catching-up criteria*

Here we will consider the case where the utility discount factor is at its upper bound given in (A2'), that is

$$\beta = (1+g)^{\theta-1}. \tag{A2"}$$

Then the $\tilde{k}_{MGR}$, defined by (8.49), coincides with the *golden-rule* capital intensity, $\tilde{k}_{GR}$, given by the requirement

$$f'(\tilde{k}_{GR}) + 1 - \delta = (1+g)(1+n).$$

In this case, maximization of W_0 in (8.25) with $T = \infty$ (infinite horizon) does not make sense, since the social welfare function, W_0, is now unbounded from above. Because of the relatively high discount factor (and thereby low discount rate), we are in a situation where the distant future contributes sufficiently much to the value of the social welfare function to imply unboundedness from above of this function.

What can be done in this situation? Just leaving the problem for good because a maximum of W_0 does not exist seems unsatisfactory. According to our intuition, when the discount rate is low, gradually approaching the golden rule from the historically given $\tilde{k}_0$ should have *some* kind of optimality attribute in this situation. Failing to come close to $\tilde{k}_{GR}$ in finite time would imply a forgone "opportunity of infinite gain".

Fortunately, there are other ways to meaningfully rank (at least partially) alternative technically feasible paths with infinite horizon.[21] The simplest alternative optimality criterion is named the *overtaking criterion*. The idea is to replace infinite sums ($T = \infty$) with finite sums ($T < \infty$) and then consider how a certain *difference* behaves as $T \to \infty$. As above, let the period utility function be denoted $v(c)$. Let the sequence $\{c_t\}$ be an arbitrary technically feasible per capita consumption path (by this is meant that it is the per capita consumption part of a technically feasible path $\left\{(\tilde{k}_t, c_t)\right\}$). Let the sequence $\{\hat{c}_t\}$ be a particular technically feasible per capita consumption path which we wish to test for optimality. So $\{\hat{c}_t\}$ is our "candidate" for an optimal path.

Define

$$D_T \equiv \sum_{t=0}^{T-1} \beta^t v(\hat{c}_t) - \sum_{t=0}^{T-1} \beta^t v(c_t), \tag{8.52}$$

[21] This is only relevant in the context of a social planner. In a market economy, as described by the Barro model of Chapter 7, an equilibrium with positive bequests can only exist if β is *lower* than the right-hand side of (A2").

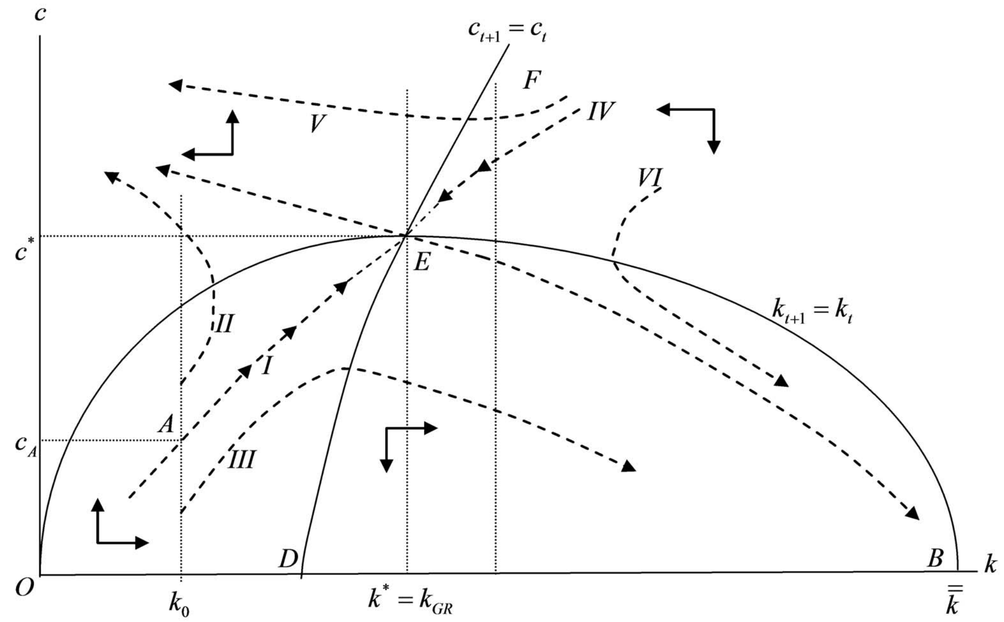

Figure 8.4: Phase diagram for the case $g = 0, \beta = 1$.

where $0 < \beta \leq 1$. Then the path $\{\hat{c}_t\}$ is *overtaking optimal*, if for any alternative technically feasible path, $\{c_t\}$, there exists an integer T' such that $D_T \geq 0$ for all $T \geq T'$. That is, if for every alternative feasible path, the candidate path, $\{\hat{c}_t\}$, has from some date on, cumulative utility up to *all* later dates at least as great as that of the alternative technically feasible path, then the candidate path is overtaking optimal. We say the candidate path is *weakly preferred* in case we just know that $D_T \geq 0$ for all $T \geq T'$. If $D_T \geq 0$ can be replaced by $D_T > 0$, we say it is *strictly preferred.*

Consider again a social planner facing the same objective function, demography, and technology as in the previous section, except that (A2') is replaced by (A2"). For simplicity, let us start with the case $g = 0$, i.e., there is no technological change. In this case, $\tilde{k}_t \equiv k_t \equiv K_t/L_t$ for all t, and $\beta = 1$ (i.e., $\bar{R} = 0$). To ensure existence of a steady state, assume

$$\lim_{k \to 0} f'(k) > n + \delta > \lim_{k \to \infty} f'(k). \qquad \text{(A3")}$$

Moreover, in the absence of technological change we do not require that the period utility function, $v(c)$, is of CRRA-type but only that it satisfies $v' > 0$, $v'' < 0$, as well as the No-Fast Assumption (A1). To avoid having to deal with unimportant technicalities, we also assume that capital is essential, i.e.,

$$f(0) = 0. \qquad \text{(A4)}$$

If the technically feasible path $\{\hat{c}_t\}$ is overtaking optimal, it must satisfy the first-order conditions (8.30) and (8.32) with (since local optimality remains necessary for global optimality). The new phase diagram, shown in Fig. 8.4, is similar to that in Fig. 8.3 except that now the steady state point E is placed at the top of the $k_{t+1} = k_t$ curve (since now $k^* = k_{MGR} = k_{GR}$). The steady state is still a saddle point and the associated saddle paths are trajectories I and IV in the figure. The point of intersection between the vertical line $k = k_0$ and the relevant saddle path is called A. The figure shows the case where $0 < k_0 < k_{GR}$, and trajectory I is the relevant saddle path. If instead $k_0 > k_{GR}$, trajectory IV is the relevant saddle path.

PROPOSITION 3 Assume $g = 0$, $\beta = 1$, the No-Fast Assumption (A1), (A3"), and (A4). In Fig. 8.4, the trajectory starting at point A and converging, along the saddle path, toward the steady-state point E is the unique overtaking-optimal trajectory. The steady state has $k^* = k_{GR}$ and thereby the golden-rule condition, $f'(k^*) - \delta = n$, holds in the long run.

We provide two substantiations of this proposition, an intuitive "proof" and a formal proof.

Intuitive "proof". Let $\{c_t^i\}$ be the sequence of consumption along a path of type $i =$ I, II,. . . ,VI in Fig. 8.4. We need only compare paths emanating from the vertical line $k = k_0$. Presupposing $k_0 < k_{GR}$, our optimality candidate is path $\left\{c_t^I\right\}$. We first compare this path with paths of type III. The phase diagram directly shows that for all $t = 0, 1, 2, ...$, we have $c_t^I > c_t^{III}$. Hence, $D_T > 0$ for all $T = 0, 1, 2, \ldots$. If instead $k_0 > k_{GR}$, the same argument makes clear that our optimality candidate $\left\{c_t^{IV}\right\}$ dominates paths of type VI.

Returning to the case $k_0 < k_{GR}$, we next compare $\left\{c_t^I\right\}$ with paths of type II. It can be shown that along a type II path at some point in time, $t_1 > 0$, all capital is used up (see Appendix D), so that $c_t^{II} = f(0) = 0$ for $t = t_1, t_1 + 1, \ldots$, in view of (A4). For every $T > t_1$ we now have

$$\begin{aligned} D_T &= \sum_{t=0}^{t_1-1} v(c_t^I) + \sum_{t=t_1}^{T-1} v(c_t^I) - \left(\sum_{t=0}^{t_1-1} v(c_t^{II}) + \sum_{t=t_1}^{T-1} v(0)\right) \\ &\geq \sum_{t=0}^{t_1-1} v(c_t^I) + \sum_{t=t_1}^{T-1} v(c_{t_1}^I) - \left(\sum_{t=0}^{t_1-1} v(c_t^{II}) + v(0)(T-t_1)\right) \qquad (8.53) \\ &= \sum_{t=0}^{t_1-1} \left(v(c_t^I) - v(c_t^{II})\right) + \left(v(c_{t_1}^I) - v(0)\right)(T-t_1) \end{aligned}$$

where the weak inequality is due to $c_t^I \geq c_{t_1}^I$, hence $v(c_t^I) \geq v(c_{t_1}^I)$, for $t = t_1, t_1 + 1, ...$ (here we use the monotonicity of c_t along path I shown in Appendix

D). The first term in the last row is a negative constant, whereas the last term is positive and grows linearly with T.[22] Hence, there exists a T' such that $D_T > 0$ for all $T \geq T'$. In the case $k_0 > k_{GR}$ it remains to compare $\{c_t^{IV}\}$ with paths of type V. In this case we replace $c_{t_1}^I$ in the second and third row of (8.53) by c_{GR} so that (8.53) is again valid. Again, the desired conclusion follows. □

Formal proof. The sequence $\{c_t^I\}$, described in the intuitive "proof" above, satisfies the Mangasarian sufficient conditions and is thereby a solution according to the overtaking-optimality criterion. Moreover, as the Hamiltonian is *strictly* concave in (k_t, c_t) for every t, the sequence $\{c_t^I\}$ is a *unique* solution according to the overtaking-optimality criterion. □

This result can be extended to the case of Harrod-neutral technological progress at constant rate $g > 0$ and CRRA utility, $v(c) = c^{1-\theta}/(1-\theta)$, $\theta > 0$. To also now end up at the golden rule, we let β satisfy (A2") for the given $g > 0$. We have $v(c_t) = (\tilde{c}_t(1+g)^t)^{1-\theta}/(1-\theta) = \tilde{c}_t^{1-\theta}(1+g)^{(1-\theta)t}/(1-\theta)$, so that, by (A2"), $\beta^t v(c_t) = \tilde{c}_t^{1-\theta}/(1-\theta) = v(\tilde{c}_t)$. Thus, with c_t replaced by $\tilde{c}_t$ and assumption (A3") by (A3') in Section 8.1, the logic in the proof of Proposition 3 goes through. Again, the trajectory along the saddle path from point A to point E, is the unique overtaking-optimal trajectory.

Generally, the overtaking criterion entails only a *partial* ordering of the alternative technically feasible paths. Hence there are cases where the overtaking criterion is not applicable. For example, technically feasible paths may oscillate with the implication that the role as the "better" path switches indefinitely between alternative technically feasible paths as $T \to \infty$. Then there is no path which is overtaking optimal.

A slightly more general optimality criterion is the *catching-up* criterion. Let again D_T be defined as in (8.52). Then the technically feasible path $\{\hat{c}_t\}$ is *catching-up optimal* if, when comparing with any alternative technically feasible path $\{c_t\}$, we have

$$\lim_{T\to\infty} D_T \geq 0. \tag{8.54}$$

Note that whenever a path is overtaking-optimal, it is also catching-up optimal, but not the other way round.[23]

Note also the welcome property, that whenever a path is optimal according to the traditional maximization criterion, it is also optimal according to both the overtaking and the catching-up criterion.

[22] In case $v(0)$ in (8.53) is not well-defined beforehand (for instance if $v(c) = \ln c$), we define $v(0) = -\infty$ and consider $c_t = 0$ as admissible.

[23] There are exceptional cases where a slightly more general definition of the *catching-up* criterion is relevant, see Appendix E. This is related to the distinction between $\lim_{T\to\infty}$ and $\liminf_{T\to\infty}$.

8.5 Concluding remarks

Chapter 7 described coordination across generations as brought about by competitive markets if a bequest motive due to parental altruism is and remains operative in a two-period OLG model. This is Barro's framework which is close to a representative agent model in discrete time.

In the present chapter we have looked at the intergenerational coordination problem from the perspective of a benevolent and omniscient social planner facing the same neoclassical CRS production function and initial resources as in the market economy. Whether the planning horizon is finite or infinite, the associated time path of the economy features a distinctive stability attribute, known as the *turnpike property*.

The fundamental result of the chapter, stated in Proposition 2, is that if the effective intergenerational discount rate is large enough to allow existence of dynamic general equilibrium in Barro's framework and existence of a maximum of social welfare in the social planner's infinite horizon problem, and if the range of the marginal productivity of capital as a function of the effective capital intensity is adequate, then the evolution of the economic system, whether governed by competitive markets or a social planner, is uniquely determined and implies convergence toward a steady state satisfying the modified golden rule. Moreover, the analysis leads to the *equivalence theorem* saying that when parental altruism lead to positive bequests, the resource allocation brought about by competitive markets is the same as that brought about by a social planner with a certain criterion function with infinite horizon. This result requires that (a) the social planner's criterion function respects individual preferences as to the distribution of own consumption across lifetime; (b) the social planner discounts the utility of future generations in the same way as the private families do.

In dealing with the dynamic optimization problems involved, we have described and applied two alternative methods, the simple *substitution method* and Pontryagin's *Maximum Principle* (in discrete time). If the effective intergenerational discount rate is *not* large enough to allow existence of a maximum of social welfare in the social planner's infinite horizon problem, other optimality criteria than maximization can sometimes be applied. To these belong the *overtaking criterion* and the *catching-up criterion*.

The period length in the models considered so far is half adult lifetime, that is, quite long. This is both an advantage and a weakness: an advantage because it simplifies a lot, but a weakness if we wish to study problems where, for example, year-by-year changes are of interest. Therefore, in the next chapter we will allow lifetime to consist of many periods. Indeed, we shall make a transition to continuous time analysis.

8.6 Literature notes

The "veil of ignorance" principle mentioned in Section 8.1 is one of the ethical ideas in the American philosopher John Rawls' *The theory of Justice* (Rawls 1971). This influential book proposed an alternative to utilitarianism, the *maximin criterion*. According to this criterion the social planner should maximize the utility of the worst-off individual. This principle is less applicable to evolutionary problems with technological progress than to static resource allocation problems. Other alternatives to utilitarianism include the (pure) *sustainability* principle according to which the social planner should maximize the level of per capita human welfare that can be sustained forever; the *human development* extension of the sustainability principle says that the social planner should maximize the per capita level of human welfare that can not only be sustained forever but is consistent with a given minimum growth rate in human welfare, see Roemer (2008).

Reasons for allowing disparity between the social planner's and the private intergenerational utility discount rate are discussed by Marglin (1963) and Sen (1967, 1982). Social discounting, when natural resources and environmental risks are taken into account, is treated in Lind et al. (1982), Heal (1998), Weitzman (2007), and Stern (2008).

Introductions to turnpike theory are provided by Chakravarty (1969), Burmeister (1980), Blanchard and Fischer (1989), and Bewley (2007). For comprehensive accounts, see McKenzie (1987) and Arkin and Evstigneev (1987).

The decomposition in Section 8.2.1 of the social planner's problem into two separate problems, a static and a dynamic one, is possible because of the assumed additive separability of the lifetime utility function. The case of non-separability gives more intricate results, see Michel and Venditti (1997).

The Russian mathematician Lev Pontryagin (1908-1988) developed, with his students, in 1956 the Maximum Principle in continuous time, the main theorem in optimal control theory. An English translation was published in 1962 (Pontryagin et al., 1962).[24] For non-specialist economists an accessible rigorous exposition of the Maximum Principle, in continuous as well as discrete time, is contained in Sydsæter et al. (2008). Other useful introductions to the Maximum Principle and related methods, in discrete time, can be found in Feichtinger and Hartl (1986) and Dixit (1990). A mathematically advanced account, including generalized optimality criteria, stochastic optimal control, and economic applications, is given in Arkin and Evstigneev (1987).

The argument in Appendix C for the necessity of the infinite horizon transver-

[24] Pontryagin lost his eyesight in the age of 14, but his mother read mathematical books and journal articles to him.

sality condition relies on the boundedness of the state variable, $\tilde{k}$, and is based on identification of increments to a function with its differential when the changes in the independent variables become "infinitely small" and thus constitute "infinitisimals". This kind of somewhat imprecise reasoning is common in economics, but largely abandoned by mathematicians. For a rigorous account, accessible for non-mathematicians, of the necessity of the transversality condition in a class of economic optimization problems in discrete time, see Kamihigashi (2002). Extended results can be found in for instance Becker and Boyd (1997) and Kamihigashi (2005).

8.7 Appendix

(in need of abbreviation and polishing)

A. Boundedness

In this appendix we will show *existence* of a solution to the social planner's problem with finite horizon. As a by-product appears some background material to be used in subsequent appendices. We start with the case without technological progress.

A stationary model Assuming no technological progress, the dynamic resource constraint in (8.27) reads:

$$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n}, \tag{8.55}$$

where $k_t \equiv K_t/L_t$, $c_t \equiv C_t/L_t$, $0 \leq \delta \leq 1$, $n > -1$, $f(0) \geq 0$, $f' > 0$, and $f'' < 0$. The path $\{(k_t, c_t)\}_{t=0}^{\infty}$ is *technically feasible* if it satisfies (8.55) for all $t \geq 0$, with $c_t \geq 0$ and $k_t \geq 0$, where k_0 equals the historically given initial capital-labor ratio.

Assume

$$\delta + n > 0 \tag{8.56}$$

and

$$\lim_{k\to 0} f'(k) > \delta + n > \lim_{k\to\infty} f'(k). \tag{8.57}$$

Then, as indicated in Fig. 8.5, the graph of $f(k)$ is above the line $y = (\delta + n)k$ for k small and below for k large. In view of continuity of f and the fact that $f'' < 0$, there is a unique $\bar{k} > 0$ such that

$$f(k) \gtreqless (\delta + n)k \quad \text{for} \quad k \lesseqgtr \bar{k}, \tag{8.58}$$

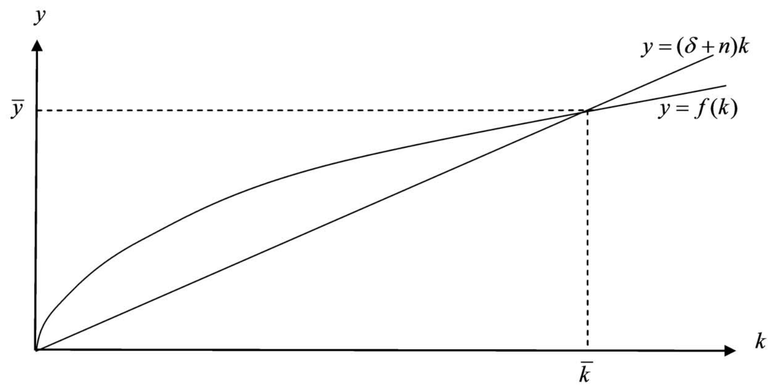

Figure 8.5: The maximum sustainable k is $bark$.

respectively. An implication of this is the following. Consider the investment per worker required to make up for capital depreciation per worker and to equip the net inflow of workers with capital in the same amount per worker which applies to the rest of the labor force. If the capital-labor ratio, k, is above $\bar{k}$, then this required investment per worker is *larger* than gross output per worker, y. Therefore, the required investment per worker can not be realized, hence it is technically impossible to maintain this high capital-labor ratio. To be precise:

LEMMA A1 (*boundedness of k and c*) Assume (8.56) and (8.57). Let $\bar{x}_0 \equiv \max\left\{k_0, \bar{k}\right\}$, where $\bar{k} > 0$ is defined in (8.58). Any technically feasible path, $\{(k_{t+1}, c_t)\}_{t=0}^{\infty}$, satisfies

$$k_t \leq \bar{x}_0, \tag{8.59}$$
$$c_t \leq f(\bar{x}_0) + (1-\delta)\bar{x}_0, \tag{8.60}$$

for $t = 0, 1, 2, ...$

Proof. By (8.55) and non-negativity of k_t for all t,

$$k_{t+1} \leq \frac{f(k_t) + (1-\delta)k_t}{1+n} \equiv \varphi(k_t), \tag{8.61}$$

where the inequality is due to $c_t \geq 0$. Note that for all $k > 0$, $\varphi(k) > 0$, $\varphi'(k) = (1+n)^{-1}\left[f'(k) + 1 - \delta\right] > 0$, and $\varphi''(k) = (1+n)^{-1}f'(k) < 0$; moreover, $\varphi(\bar{k}) = \bar{k}$. Hence,

$$\varphi(k) \gtreqqless k \text{ for } k \lesseqqgtr \bar{k}, \tag{8.62}$$

respectively.

We prove (8.59) by induction. Suppose that for a fixed $t \in \{0, 1, 2, ...\}$, $k_t \leq \bar{x}_0 = \max\left\{k_0, \bar{k}\right\}$. Then

$$k_{t+1} \leq \varphi(k_t) \leq \varphi(\bar{x}_0) \leq \bar{x}_0,$$

where the first (weak) inequality comes from (8.61), the second from the fact that $\varphi'(k) > 0$, and the last from $\bar{x}_0 \geq \bar{k}$ combined with (8.62). Obviously, $k_t \leq \bar{x}_0$ holds for $t = 0$. Hence (8.59) holds for all $t \geq 0$.

To prove (8.60), note that (8.55) combined with $k_{t+1} \geq 0$ implies

$$c_t \leq f(k_t) + (1-\delta)k_t \equiv \psi(k_t) \leq f(\bar{x}_0) + (1-\delta)\bar{x}_0,$$

where the last inequality follows from (8.59) combined with the fact that $\psi'(k) = f'(k) + 1 - \delta > 0$ for all $k > 0$. □

Technological progress. Reduction to a stationary model We now add Harrod-neutral technological progress. We show that this case can be reduced to a stationary case so that with an appropriate reinterpretation of the variables, the results in Lemma A1 apply.

With Harrod-neutral technological progress at the rate $g > 0$ we have, from (8.27), the dynamic resource constraint:

$$\tilde{k}_{t+1} = \frac{f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - \tilde{c}_t}{(1+g)(1+n)}, \tag{8.63}$$

where $\tilde{c}_t = c_t/(1+g)^t$. The assumptions (A2') and (A3') in Section 8.3 imply

$$(1+g)(1+n) - (1-\delta) > 0 \tag{8.64}$$

and

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > (1+g)(1+n) - (1-\delta) > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{8.65}$$

Hence, defining $1 + n' \equiv (1+g)(1+n)$, we see that (8.55), (8.56), and (8.57) are satisfied with n replaced by n' and k_t replaced by $\tilde{k}_t$, $t = 0, 1, 2, ...$ From Lemma A1 we now conclude that for all $t \geq 0$,

$$\tilde{k}_t \leq \bar{\bar{x}}_0, \tag{8.66}$$

$$\tilde{c}_t \leq f(\bar{\bar{x}}_0) + (1-\delta)\bar{\bar{x}}_0, \tag{8.67}$$

with $\bar{\bar{x}}_0 \equiv \max\left\{\tilde{k}_0, \bar{k}\right\}$, where $\bar{k} > 0$ is defined as in (8.58), but with n replaced by n' and k_t replaced by $\tilde{k}_t$.

This boundedness from above of $\tilde{k}_t$ and $\tilde{c}_t$ implies that k_t and c_t can not in the long run grow at a rate higher than g.

As to *existence* of a solution to the planner's optimization problem, (8.25) - (8.28), with $T < \infty$, we shall appeal to the *extreme value theorem.*[25] The period

[25]The *extreme value theorem* states that a continuous function defined on a closed and bounded set has both a maximum and a minimum.

utility function v and the production function f are continuous functions. By substitution of the constraint (8.27), for $t = 0, 1, ..., T-1$, into the objective function, this becomes a continuous function of $\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_T$. These $\tilde{k}$'s are non-negative and have an upper bound, given by (8.66). Hence, by the *extreme value theorem* the objective function has a maximum. Thereby a solution to the social planner's problem exists.

B. When $T \to \infty$, a high enough discount rate is needed for boundedness of the integral of discounted utilities

Here we shall substantiate the claim in Section 8.1 that with CRRA utility, $u(c) = c^{1-\theta}/(1-\theta)$, $\theta > 0$, the parameter restriction

$$1 + \bar{R} > (1+g)^{1-\theta} \tag{A2}$$

ensures that the social welfare function with an infinite horizon is bounded from above. As $\bar{R} > -1$ and $\beta \equiv (1+R)^{-1}$, this inequality is equivalent to

$$0 < \beta < (1+g)^{\theta-1}, \tag{A2'}$$

which is the simpler form applied in sections 8.2 and 8.3.

The social welfare function with infinite horizon introduced in Section 8.1 is:

$$W_0 = (1+\rho)^{-1}u(c_{20}) + \sum_{t=0}^{\infty} \beta^{t+1}[u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1})] \tag{8.68}$$

where $\rho > -1$, $u' > 0$, and $u'' < 0$. In Section 8.2 we showed that the problem of maximizing W_0 subject to technical feasibility could be decomposed into two problems, the static problem, (8.18) - (8.19), and the dynamic problem of maximizing

$$W_0 = \sum_{t=0}^{\infty} \beta^t v(c_t), \tag{8.69}$$

subject to technical feasibility. Here $c_t \equiv C_t/L_t$ and $v(c_t)$ is the social planner's optimized period utility function,

$$v(c_t) = \beta u(c_{1t}(c_t)) + (1+\rho)^{-1}u(c_{2t}(c_t)),$$

as defined in (8.22). Here we have inserted $\gamma \equiv (1+\rho)^{-1} > 0$. It always holds that $v' > 0$ and $v'' < 0$. In Lemma 1 of Section 8.3 it was shown that if $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$, then we can always choose the function v such that for the same θ, $v(c) = c^{1-\theta}/(1-\theta)$.

It is boundedness from above of the expression in (8.69) that is our concern. We begin with the case $g = 0$.

The stationary model Define $\bar{c} \equiv f(\bar{x}_0) + (1-\delta)\bar{x}_0$, where $\bar{x}_0$ is given in Lemma A1. In view of (8.60), we have for all $t \geq 0$, $c_t \leq \bar{c}$. Then, since $v' > 0$, $v(c_t) \leq v(\bar{c})$. Consequently,

$$W_0 \leq v(\bar{c}) \sum_{t=0}^{\infty} \beta^t = v(\bar{c}) \frac{1}{1-\beta},$$

for $0 < \beta < 1$, which is the form (A2') takes for $g = 0$. We also see that if $\beta \geq 1$, there always exists a $b > 0$ such that for $\tilde{v}(\bar{c}) \equiv v(\bar{c}) + b$, $\tilde{v}(\bar{c}) \sum_{t=0}^{\infty} \beta^t$ is not bounded from above. So, $0 < \beta < 1$ is sufficient as well as necessary (up to a positive constant added to the period utility function) for the social welfare function to be bounded from above.

The case of technological progress Consider the case $g > 0$ combined with $v(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$. Define $\bar{\tilde{c}} \equiv f(\bar{\tilde{x}}_0) + (1-\delta)\bar{\tilde{x}}_0$; here, $\bar{\tilde{x}}_0 \equiv \max\left\{\tilde{k}_0, \bar{k}\right\}$, where $\bar{k} > 0$ is defined as in (8.58), but with n replaced by n' and k_t replaced by $\tilde{k}_t$. We have $c_t \equiv \tilde{c}_t \mathcal{T}_t = \tilde{c}_t(1+g)^t \leq \bar{\tilde{c}}(1+g)^t$ in view of (8.67). There are two cases to consider.

Case 1: $\theta > 0$, $\theta \neq 1$. We get

$$v(c_t) = \frac{c_t^{1-\theta}}{1-\theta} \leq \frac{\bar{\tilde{c}}^{1-\theta}}{1-\theta}(1+g)^{(1-\theta)t}.$$

Consequently, with β satisfying (A2'),

$$W_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} \leq \sum_{t=0}^{\infty} \beta^t \frac{\bar{\tilde{c}}^{1-\theta}}{1-\theta}(1+g)^{(1-\theta)t} = \frac{\bar{\tilde{c}}^{1-\theta}}{1-\theta} \sum_{t=0}^{\infty} \left[\beta(1+g)^{1-\theta}\right]^t.$$

We see that this upper bound for W_0 is finite if $0 < \beta(1+g)^{1-\theta} < 1$, i.e., if (A2') holds. This inequality ensures that if $0 < \theta < 1$, then $0 < W_0 < \infty$, and if $\theta > 1$, then $-\infty < W_0 < 0$.

In a steady state we have $c_t = \tilde{c}^*(1+g)^t$, so that

$$W_0 = \frac{\tilde{c}^{*1-\theta}}{1-\theta} \sum_{t=0}^{\infty} \left[\beta(1+g)^{1-\theta}\right]^t,$$

which is finite if (A2') holds. If, on the other hand, $\beta \geq (1+g)^{\theta-1}$, then $W_0 = \infty$, if $0 < \theta < 1$, and $W_0 = -\infty$, if $\theta > 1$.

Case 2: $\theta = 1$, i.e., $u(c) = \ln c$. Here

$$W_0 = \sum_{t=0}^{\infty} \beta^t \ln c_t \leq \sum_{t=0}^{\infty} \beta^t \left[\ln \bar{\tilde{c}} + t\ln(1+g)\right] = (\ln \bar{\tilde{c}}) \sum_{t=0}^{\infty} \beta^t + \sum_{t=0}^{\infty} \beta^t t \ln(1+g).$$

This upper bound for W_0 is finite if $0 < \beta < 1$ which is the form taken by (A2'). The reason is that geometric decline (via β^t) outweighs arithmetic growth (via $t\ln(1+g)$).

In a steady state,

$$W_0 = \sum_{t=0}^{\infty} \beta^t \left[\ln \tilde{c}^* + t\ln(1+g)\right] = (\ln \tilde{c}^*) \sum_{t=0}^{\infty} \beta^t + \sum_{t=0}^{\infty} \beta^t t \ln(1+g),$$

which is bounded from above if $0 < \beta < 1$ (i.e., (A2') holds) and not bounded from above if $\beta \geq 1$, since in this case the second term will dominate and outweigh the first if $\ln \tilde{c}^* < 0$.

C. Transversality conditions with infinite horizon

In Section 8.1 as well as 8.2 we heralded a follow-up on the respective transversality conditions. It is convenient to begin with the one in Section 8.2.

Necessity of the transversality condition (8.38) In Section 8.2 we claimed that the transversality condition (8.38) must be satisfied by an interior solution to the optimization problem (8.25) − (8.28) with $T \to \infty$. Here we substantiate this claim by a heuristic argument known as the *unreversed arbitrage principle*.[26] For convenience we rewrite (8.38) as

$$\lim_{T\to\infty} \beta^{T-1} v'(c_{T-1})(1+g)^{T-1}(1+g)(1+n)\tilde{k}_T = 0. \tag{*}$$

Imposing the conditions (A2') and (A3') (Section 8.3), k_t cannot in the long run grow at a rate higher than the rate of technological progress. So the effective capital-labor ratio, $\tilde{k}_T$, will remain bounded from above for $T \to \infty$. Hence, $\lim_{T\to\infty}\left[(1+g)(1+n)\tilde{k}_T\right] < \infty$ in (*). It is therefore enough to show that

$$\lim_{T\to\infty} \beta^{T-1} v'(c_{T-1})(1+g)^{T-1} = 0 \tag{8.70}$$

along an interior optimal path.

Let a given path $\left\{\left(\tilde{k}_t, c_t\right)\right\}_{t=0}^{\infty}$ be an interior optimal path. This will be our "reference path". Since the reference path is optimal, no welfare improving reallocation of resources is possible. An example of a technically feasible reallocation of resources is the following. We increase c_0, c_1,. . . , c_{T-1} so that $\tilde{k}_1$, $\tilde{k}_2$,. . . , $\tilde{k}_T$ are all decreased by h units where h is a small positive number (for h sufficiently small this is always possible since we consider an interior path implying that K_t

[26] This draws upon Becker (2008).

is positive for all t). Finally, in period T per capita consumption is decreased and gross investment thereby increased sufficiently so as to bring $\tilde{k}_{T+1}$ back to its level in the reference path.

The implied changes in per capita consumption, c_t, relative to the reference path, can be calculated on the basis of the dynamic resource constraint expressed in growth-corrected variables:

$$\tilde{c}_t \equiv \frac{c_t}{(1+g)^t} = f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - (1+g)(1+n)\tilde{k}_{t+1},$$

cf. (8.27). For any given changes in $\tilde{k}_t$ and $\tilde{k}_{t+1}$, $\Delta\tilde{k}_t$ and $\Delta\tilde{k}_{t+1}$, the differential of $\tilde{c}_t$ is

$$d\tilde{c}_t = \left[f'(\tilde{k}_t) + 1 - \delta\right]\Delta\tilde{k}_t - (1+g)(1+n)\Delta\tilde{k}_{t+1}.$$

The actual changes in c_t are then

$$\Delta c_t \approx dc_t = (1+g)^t d\tilde{c}_t, \quad t = 0, 1, \ldots, T.$$

The considered reallocation is such that $\Delta\tilde{k}_0 = 0$, $\Delta\tilde{k}_t = -h$ for $t = 1, 2, \ldots,$ T, and $\Delta\tilde{k}_{T+1} = 0$. It follows that

$$\Delta c_t \approx \begin{cases} 1 \cdot (1+g)(1+n)h & \text{for } t = 0, \\ -(1+g)^t \left[f'(\tilde{k}_t) + 1 - \delta - (1+g)(1+n)\right] h & \text{for } t = 1, 2, \ldots, T-1, \\ -(1+g)^T \left(f'(\tilde{k}_T) + 1 - \delta\right) h & \text{for } t = T. \end{cases}$$

For period 0 this reallocation implies a utility *gain* approximately equal to

$$v'(c_0)\Delta c_0 \approx v'(c_0)(1+g)(1+n)h > 0.$$

In each of the periods $1, 2, \ldots, T-1$ there tends to be a utility *loss,* although two countervailing forces are in play. On the one hand the marginal product of the h units of growth-corrected capital is lacking. On the other hand, the lower needed investment than otherwise gives scope for higher consumption. In any event, the total discounted utility loss incurred in these periods is approximately

$$\sum_{t=1}^{T-1} \beta^t v'(c_t)(1+g)^t \left[f'(\tilde{k}_t) + 1 - \delta - (1+g)(1+n)\right] h.$$

Finally, in period T there is a utility loss because the marginal product of the h units of growth-corrected capital is lacking. The discounted utility loss incurred by this is approximately

$$\beta^T v'(c_T)(1+g)^T \left(f'(\tilde{k}_T) + 1 - \delta\right) h.$$

Since the reference path is assumed optimal, the gain and the losses should for small h cancel so that, approximately,

$$v'(c_0) = \sum_{t=1}^{T-1} \beta^t v'(c_t)(1+g)^t \left[\frac{f'(\tilde{k}_t) + 1 - \delta}{(1+g)(1+n)} - 1 \right] + \beta^T v'(c_T)(1+g)^T \frac{f'(\tilde{k}_T) + 1 - \delta}{(1+g)(1+n)}, \tag{8.71}$$

where we have divided through by $(1+g)(1+n)h$. Like the Euler equation (8.36), (8.71) is a necessary condition for optimality. The argument used in its derivation is called a $T-1$ periods *reversed arbitrage* argument. For $T = 1$ the first term in (8.71) disappears and (8.71) reduces to (8.36).

An alternative reallocation − an *unreversed arbitrage* − is one where the h units of growth-corrected capital are *permanently* sacrificed.[27] In this case the total discounted utility loss pertaining to period 1 and onward is approximately

$$\sum_{i=1}^{\infty} \beta^t v'(c_t)(1+g)^t \left[(1+g)(1+n) - \left(f'(\tilde{k}_t) + 1 - \delta \right) \right] h.$$

Ignoring approximation errors, the optimal reference path must satisfy

$$v'(c_0) = \sum_{i=1}^{\infty} \beta^t v'(c_t)(1+g)^t \left[\frac{f'(\tilde{k}_t) + 1 - \delta}{(1+g)(1+n)} - 1 \right]. \tag{8.72}$$

But both (8.72) and (8.71), as $T \to \infty$, can hold only if

$$\lim_{T\to\infty} \beta^T v'(c_T)(1+g)^T \frac{f'(\tilde{k}_T) + 1 - \delta}{(1+g)(1+n)} = 0.$$

In view of the first-order condition (8.36), this unreversed arbitrage argument implies

$$\lim_{T\to\infty} \beta^{T-1} v'(c_{T-1})(1+g)^{T-1} = 0,$$

which is the transversality condition (8.70) as was to be shown. In the limit, for $h \to 0$, the approximation errors implicit in the equations become negligible.[28]

[27] Here the argument presupposes that there is scope for this permanent reduction in $\tilde{k}$, i.e., that our reference path does not have $\lim_{T\to\infty} \tilde{k}_T = 0$. If it does, we can use a symmetric reasoning with $h < 0$, again leading to the conclusion that (8.70) must hold along an interior optimal path.

[28] In case $\tilde{k}_T$ is unbounded from above for $T \to \infty$ (because (A2') and/or (A3') are violated), the transversality condition (*) is *stronger* than (8.70) and requires an independent proof. See Literature notes.

The "generations-oriented" format of the transversality condition Instead of the above "period-oriented" format based on the social planner's optimized period-utility function, $v(c_t)$, consider the social planner's problem formulated in a "generations-oriented" format based on the individual's period-utility functions, $u(c_{1t})$ and $u(c_{2t+1})$, as in Section 8.1. For this format the transversality condition reads

$$\lim_{T\to\infty} [\beta(1+g)]^T u'(c_{1T-1})(1+n)\tilde{k}_T = 0, \tag{8.73}$$

cf. (8.13) where we have entered $\beta \equiv (1+\bar{R})^{-1}$. By inserting the "envelope condition" $v'(c_T) = \beta u'(c_{1T-1})$, from (8.23), we see that (8.73) is equivalent to (*).

Application to Proposition 1 (equivalence) Proposition 1 in Section 8.1.1 compares the resource allocation in the Barro model of a market economy with positive bequests (Chapter 7) to that of a social planner facing the same technology and initial resources as in the market economy and having an effective intergenerational discount factor, $\beta \equiv (1+\bar{R})^{-1}$, equal to the private one. This β is assumed to satisfy assumption (A2') saying that $0 < \beta < (1+g)^{\theta-1}$.

We may write the intertemporal utility function of the altruistic parent belonging to generation 0 (Section 7.2.1) as

$$U_0 = \sum_{T=0}^{\infty} \beta^T \left[u(c_{1T}) + (1+\rho)^{-1}u(c_{2T+1})\right].$$

This intertemporal utility function is closely related to the social welfare function of the social planner. Indeed, from (8.68) we see that

$$W_0 = (1+\rho)^{-1}u(c_{20}) + \beta U_0.$$

This is exactly what the old in period 0 in the Barro model maximizes by choosing $c_{20} \geq 0$ and $b_0 \geq 0$ subject to the budget constraint $c_{20} + (1+n)b_0 = (1+r_0)s_{-1}$ and taking into account that the chosen b_0 indirectly affects the maximum lifetime utility to be achieved by the next generation, cf. Section 7.2.1.

From the perspective of generation 0, i.e., the young in period 0 in the Barro model, the transversality condition is

$$\lim_{T\to\infty} \beta^{T-1}(1+\rho)^{-1}u'(c_{2T})(1+n)b_T = 0, \tag{8.74}$$

which follows from (7.8) by inserting $\beta \equiv (1+\bar{R})^{-1}$, letting $t = 0$, and replacing i by T.

For the social planner's problem formulated in "generations-oriented" format as in Section 8.1, the transversality condition is given by (8.73) above. For comparison with (8.74), we substitute $\tilde{k}_T \equiv k_T/(1+g)^T$ and divide through by β to get

$$\lim_{T\to\infty} \beta^{T-1}u'(c_{1T-1})(1+n)k_T = 0. \tag{8.75}$$

To complete the proof of Proposition 1, we now show:

LEMMA C1 Barro's and the social planner's transversality conditions, (8.74) and (8.75), are equivalent.

Proof. (incomplete) From the budget constraint, (7.5), of the old parent in the Barro model, with t replaced by $T-1$, we have

$$0 \leq b_T = (1+r_T)k_T - \frac{c_{2T}}{1+n} < (1+r_T)k_T = (1+f'(\tilde{k}_T)-\delta)k_T,$$

in view of $c_{2T} > 0$ and $r_T = f'(\tilde{k}_T) - \delta$ in the competitive market economy. Multiplying through by $\beta^{T-1}(1+\rho)^{-1}u'(c_{2T})(1+n) > 0$ gives

$$\begin{aligned} 0 &\leq \beta^{T-1}(1+\rho)^{-1}u'(c_{2T})(1+n)b_T \\ &< \beta^{T-1}(1+\rho)^{-1}u'(c_{2T})(1+n)(1+f'(\tilde{k}_T)-\delta)k_T = \beta^{T-1}u'(c_{1T-1})(1+n)k_T, \end{aligned}$$

where the equality follows from (8.10) with t replaced by $T-1$. Hence, letting $T\to\infty$, a technically feasible path satisfying (8.75) will also satisfy (8.74).

That the inverse also holds, follows from ...?? □

D. Saddle-point dynamics

We shall here be more detailed about formal aspects of the solution to the social planner's problem with infinite horizon in sections 8.3 and 8.4. The claim is that the solution coincides with the unique converging path, cf. figures 8.3 and 8.4, if the parameters satisfy either (A2') or (A2") (from Section 8.3 and 8.4, respectively), that is, if

$$0 < \beta \leq (1+g)^{\theta-1} \tag{A2*}$$

holds. Strict inequality here leads to the modified golden rule, whereas the limiting case with equality leads to the golden rule.

Of crucial importance is that the non-trivial steady state of the dynamic system is a saddle point. A steady state of a two-dimensional dynamic system in discrete time is a *saddle point* if a certain matrix, known as the *Jacobian*, evaluated in the steady state has one eigenvalue with absolute value below one, the *stable eigenvalue*, and one eigenvalue with absolute value above one, the *unstable eigenvalue.* Our dynamic system is given by (8.44) and (8.47). We start with the simple case where there is no technological progress.

The case with no technological progress

With $g = 0$ the dynamic system can be written

$$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t - c_t}{1+n} \equiv h(k_t, c_t), \tag{8.76}$$

$$c_{t+1} = \left(\beta \frac{f'(h(k_t, c_t)) + 1 - \delta}{1+n}\right)^{1/\theta} c_t, \tag{8.77}$$

where k_0 is *predetermined*, whereas c_0 is a *jump variable*. A variable which is not predetermined and can immediately jump to another value, is called a jump variable. Such a jump may be triggered by the arrival of new information. Control variables are generally jump variables.

We now linearize around the non-trivial steady state. A convenient approach is based on taking the differential on both sides of (8.76) and (8.77), respectively:

$$\begin{aligned} dk_{t+1} &= \frac{(f'(k^*) + 1 - \delta)\, dk_t - dc_t}{1+n} \equiv a_1 dk_t + a_2 dc_t, \quad (a_1 \equiv h_1, a_2 \equiv h_2) \\ dc_{t+1} &= \frac{\frac{c^*}{\theta}(f'(k^*) + 1 - \delta)^{\frac{1}{\theta}-1} f''(k^*)(a_1 dk_t + a_2 dc_t) + (f'(k^*) + 1 - \delta)^{\frac{1}{\theta}} dc_t}{\left[\beta^{-1}(1+n)\right]^{\frac{1}{\theta}}} \\ &\equiv a_3(a_1 dk_t + a_2 dc_t) + a_4 dc_t = a_1 a_3 dk_t + (a_2 a_3 + a_4) dc_t. \end{aligned}$$

Using $k^* = k_{MGR}$ and (8.49), we have

$$\begin{aligned} a_1 &= \beta^{-1} \geq 1, \text{ by (A2*) for } g = 0, \quad a_2 = \frac{-1}{1+n} < 0, \\ a_3 &= \frac{c^* f''(k^*)}{\theta \beta^{-1}(1+n)} < 0, \qquad a_1 a_3 = \frac{c^* f''(k^*)}{\theta(1+n)} < 0, \\ a_2 a_3 &= \frac{-c^* f''(k^*)}{\theta \beta^{-1}(1+n)^2} > 0, \qquad a_4 = \left(\frac{f'(k^*) + 1 - \delta}{\beta^{-1}(1+n)}\right)^{\frac{1}{\theta}} = 1. \end{aligned} \tag{8.78}$$

The linear approximation The approximating linear dynamic system in deviations from the steady state, $\binom{k^*}{c^*}$, is

$$\begin{pmatrix} dk_{t+1} \\ dc_{t+1} \end{pmatrix} = A \begin{pmatrix} dk_t \\ dc_t \end{pmatrix},$$

where the matrix A is the mentioned *Jacobian*, given by

$$A = \begin{bmatrix} a_1 & a_2 \\ a_1 a_3 & a_2 a_3 + a_4 \end{bmatrix} = \begin{bmatrix} \beta^{-1} & \frac{-1}{1+n} \\ \frac{c^* f''(k^*)}{\theta(1+n)} & 1 + b \end{bmatrix},$$

with

$$b \equiv \frac{-c^* f''(k^*)}{\theta \beta^{-1}(1+n)^2} > 0.$$

The determinant and trace of A are:

$$\det A = a_1(a_2 a_3 + a_4) - a_1 a_3 a_2 = a_1 a_4 = a_1 \geq 1, \quad \text{(by (8.78))} \tag{8.79}$$

$$\text{tr}A = a_1 + a_2 a_3 + a_4 = a_1 + 1 + b > 2. \tag{8.80}$$

Let ε_1 and ε_2 be the *eigenvalues* of A. Then

$$\begin{aligned} \varepsilon_1 &= \frac{1}{2}(\text{tr}A - \sqrt{\Delta}), \\ \varepsilon_2 &= \frac{1}{2}(\text{tr}A + \sqrt{\Delta}), \end{aligned}$$

where $\Delta \equiv (\text{tr}A)^2 - 4\det A$ (the discriminant of A). From matrix algebra we know the rules $\varepsilon_1 \cdot \varepsilon_2 = \det A$ and $\varepsilon_1 + \varepsilon_2 = \text{tr}A$. In view of (8.79), $\det A > 0$, hence the eigenvalues are of the same sign. And in view of (8.80), $\text{tr}A > 0$, hence the eigenvalues are both positive. Further,

$$\begin{aligned} \Delta &= {a_1}^2 + 1 + 2a_1 + b^2 + 2(a_1+1)b - 4a_1 = (a_1-1)^2 + b^2 + 2(a_1+1)b \\ &= (a_1-1)^2 + b^2 + 2(a_1-1)b + 4b = (a_1 - 1 + b)^2 + 4b > 0. \end{aligned}$$

Both eigenvalues are thus real and, as $b > 0$, we have $\sqrt{\Delta} > a_1 - 1 + b$. It follows that

$$\begin{aligned} 0 &< \varepsilon_1 = \frac{1}{2}(a_1 + 1 + b - \sqrt{\Delta}) < \frac{1}{2}(1+1) = 1, \\ \varepsilon_2 &= \frac{1}{2}(a_1 + 1 + b + \sqrt{\Delta}) > \frac{1}{2}(2a_1 + 2b) = a_1 + b > a_1 \geq 1, \end{aligned}$$

where the last (weak) inequality comes from (8.78). Hereby we have shown that the steady state, (k^*, c^*), is a saddle point, and ε_1 is the stable eigenvalue while ε_2 is the unstable eigenvalue. The next step is to show that the stability property called saddle-point stability is present.

Saddle-point stability A steady state of a two-dimensional dynamic system is called (locally) *saddle-point stable*, if:

(a) the steady state is a saddle point;

(b) the dynamic system has one predetermined variable and one jump variable;

(c) for any initial value of the predetermined variable in a neighborhood of the steady state, there is a unique value of the jump variable such that the system starts (has initial point) on the saddle path; and

(d) there is a boundary condition or other condition on the system such that the diverging paths are ruled out as solutions.

In our context, (a) and (b) are already established, and (d) follows from the uniqueness of the optimal solution (cf. the end of Section 8.2.2) combined with the fact that the transversality condition (8.50) holds along the converging path. It remains to check point (c). Here the phase diagram in Fig. 8.3 reveals that the saddle path is *not* (at least in a small neighborhood of the steady state) parallel to the jump-variable axis. So also point (c) is satisfied.

An approximative explicit formula for the optimal time path, based on the above linearization of the dynamic system, can be derived as follows. The general solution to the linearized system can be written

$$\begin{pmatrix} k_t \\ c_t \end{pmatrix} = C_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \varepsilon_1^t + C_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} \varepsilon_2^t + \begin{pmatrix} k^* \\ c^* \end{pmatrix}, \tag{8.81}$$

where $\binom{v_{i1}}{v_{i2}}$ is an eigenvector associated with ε_i, $i = 1, 2$, and C_1 and C_2 are constants related to the initial values, k_0 and c_0, in the following way:

$$C_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + C_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} + \begin{pmatrix} k^* \\ c^* \end{pmatrix} = \begin{pmatrix} k_0 \\ c_0 \end{pmatrix}. \tag{8.82}$$

Here, k_0 is predetermined, whereas c_0 is to be chosen such that the system is on the saddle path at time 0. This is equivalent to choosing c_0 such that the unstable eigenvalue, ε_2, is neutralized in (8.81), i.e., such that $C_2 = 0$. Since $a_2 \neq 0$, the eigenvector $\binom{v_{11}}{v_{12}}$ can be written $\binom{1}{(\varepsilon_1 - a_1)/a_2}$. Substituting this together with $C_2 = 0$ into (8.82) and solving for C_1 and c_0 gives

$$\begin{aligned} C_1 &= k_0 - k^*, \\ c_0 &= (k_0 - k^*)\frac{\varepsilon_1 - a_1}{a_2} + c^* = (k_0 - k^*)(\beta^{-1} - \varepsilon_1)(1+n) + c^* > 0, \end{aligned}$$

in view of $\varepsilon_1 < 1 \leq a_1 = \beta^{-1}$ and $a_2 = -(1+n)^{-1}$. So the particular solution we have been looking for is

$$k_t = (k_0 - k^*)\varepsilon_1^t + k^*, \tag{8.83}$$

$$c_t = (k_0 - k^*)(\beta^{-1} - \varepsilon_1)(1+n)\varepsilon_1^t + c^*. \tag{8.84}$$

This is the approximative explicit analytical solution to the social planner's problem.[29] Having found such a solution, we have also, for the approximating linear system, given an algebraic proof of property (c) above. As an implication, the "true" non-linear system is at least *locally* saddle-point stable.

Global saddle-point stability Proposition 2 of Section 8.3 not only claims local saddle-point stability, but *global* saddle-point stability. This claim means that point (c) above can be strengthened to the following. Given an arbitrary positive initial value of the predetermined variable, there is a unique positive value of the jump variable such that the system initially is on the saddle path. That is, global saddle-point stability requires that for *any* $k_0 > 0$, it is possible to start out on the stable arm. Suffice it to say that this condition *is* satisfied by the present model. The proof is analogue to one used for the continuous-time case in Appendix A to Chapter 10.

Including technological progress

With $g > 0$, the original system, (8.44) and (8.47), can be written

$$\tilde{k}_{t+1} = \frac{f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - \tilde{c}_t}{1+n'} \equiv h(\tilde{k}_t, \tilde{c}_t),$$

$$\tilde{c}_{t+1} = \left(\beta' \frac{f'(\tilde{k}_{t+1}) + 1 - \delta}{1+n'}\right)^{1/\theta} \tilde{c}_t,$$

where we have defined $1 + n' \equiv (1+g)(1+n)$ and $\beta' \equiv \beta(1+g)^{1-\theta}$. In view of (A2*), we have $0 < \beta' \leq 1$. In this way we have reduced the system to the same form as in the stationary case above. Thus, the conclusions go through with appropriate reinterpretation of the variables.

Since $\varepsilon_1 > 0$, the solution (8.83) - (8.84) approaches the non-trivial steady state in a non-oscillatory way.

Two technical issues relating to the phase diagram

The first issue is about the point D in Fig. 8.3. This point is located where the $\tilde{c}_{t+1} = \tilde{c}_t$ locus intersects the $\tilde{k}$-axis. Whether it does so for a $\tilde{k} > 0$ or a $\tilde{k} \leq 0$ is immaterial for our stated conclusions. In the case shown in Fig. 8.3, the intersection is at a $\tilde{k} > 0$. This case occurs when capital is essential, i.e., $f(0) = 0$. Indeed, by the definition of h in (8.44), along the $\tilde{c}_{t+1} = \tilde{c}_t$ locus we

[29] Exact explicit analytical solutions are obtainable only in special cases, for example the log utility-Cobb-Douglas case, see Exercise 8.x.

have $\tilde{c}_t = f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - (1+g)(1+n)\tilde{k}_{MGR} \equiv m(\tilde{k}_t)$. Now, $m(\tilde{k}^*) = \tilde{c}^* > 0$ and $m(0) = -(1+g)(1+n)\tilde{k}_{MGR} < 0$ when $f(0) = 0$. Since m is continuous, there is therefore a $\bar{k} \in (0, \tilde{k}^*)$ such that $m(\bar{k}) = 0$.

The second issue relates to the fact that all paths which start above the saddle path, hit the boundary of the dynamic system in finite time. This indicates that they are *not* interior paths. One might then question whether they need at all satisfy the Euler equation in the periods before they hit the boundary. That is, have we really constructed these paths correctly in the phase diagram? The answer is yes. As long as the boundary of the system is not binding, the first-order conditions which lead to the Euler equation *must* hold along an optimal path.

E. Limit inferior and limit superior

Both when discussing infinite horizon transversality conditions and when introducing the catching-up optimality criterion in Section 8.4, we assumed that the relevant limits exist for $T \to \infty$. If full generality were aimed at, we should for example allow non-convergence of D_T for $T \to \infty$. Indeed, the set of feasible paths might in theory be such that D_T fluctuates forever with non-vanishing amplitude. If so, we would have to replace "$\lim_{T\to\infty}$" in (8.54) by "$\liminf_{T\to\infty}$", i.e., the limit inferior.

Let "$j \geq t$" be a shorthand for "$j = t, t+1, \ldots$". The *limit inferior* for $t \to \infty$ of a sequence $\{x_t\}_{t=0}^{\infty}$ is defined as $\lim_{t\to\infty} \inf \{x_n | \, n \geq t\}$. Here inf of a set of real numbers, say $S_t = \{x_n | \, n \geq t\}$, means the *infimum* of the set, that is, the greatest lower bound for S_t.[30] Fig. 8.6 illustrates. For $t = t_1$, b_1 is a lower bound, but evidently not the greatest. As $t \to \infty$, the greatest lower bound tends to b_2, which then is the $\liminf_{t\to\infty} x_t$. Analogously, the "lim sup" or *limit superior* for $t \to \infty$ of a sequence $\{x_t\}_{t=0}^{\infty}$ is defined as $\lim_{t\to\infty} \sup \{x_n | \, n \geq t\}$, where sup of the set means the *supremum* of the set, that is, the least upper bound.[31] In Fig. 8.6, for $t = t_1$, the least upper bound for S_t is b_4, but for $t \to \infty$ the least upper bound tends to b_3, which is thus the $\limsup_{t\to\infty} x_t$.

Obviously, $\liminf_{t\to\infty} x_t \leq \limsup x_t$. If $\lim_{t\to\infty} x_t$ exists, then $\lim_{t\to\infty} x_t = \liminf_{t\to\infty} x_t = \limsup_{t\to\infty} x_t$. This is the case where $b_2 = b_3$ in Fig. 8.6. An example of non-convergence is $x_t = (-1)^t$, $t = 0, 1, 2, \ldots$, where $\liminf_{t\to\infty} x_t = -1$ and $\limsup_{t\to\infty} x_t = 1$.

Due to strict concavity in many economic problems, however, infinitely fluctuating paths that do not converge can often be shown to be inferior (see Koopmans, 1965). In "normal" economic optimization problems, as those considered in this

[30] A number less than or equal to all numbers in a set S is called a *lower bound* for S.

[31] A number greater than or equal to all numbers in a set S is called an *upper bound* for S.

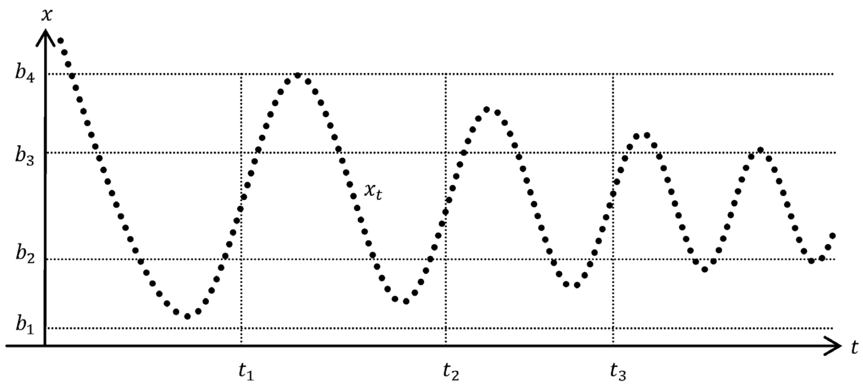

Figure 8.6: $\lim_{t\to\infty} x_t$ does not exist, but $\lim_{t\to\infty} \inf x_t$ and $\lim_{t\to\infty} \sup x_t$ do.

book, infinitely fluctuating paths never turn up. Hence, in our context essentially nothing is lost by using the more narrow specification of both necessary and sufficient transversality conditions and catching-up optimality presented in the text, that is, using "lim" instead of "lim inf".

8.8 Exercises

8.1 See footnote at end of Appendix D.