Chapter 4

A growing economy

In the previous chapter we ignored technological progress. An incontestable fact of real life in industrialized countries is, however, the presence of a persistent rise in GDP per capita – on average between 1.5 and 2.5 percent per year since 1870 in many developed economies. In regard to U.K., U.S., and Japan, see Fig. 4.1. And in regard to Denmark, see Fig. 4.2. This fact should be taken into account in a model which, like the Diamond model, aims at dealing with long-run issues. For example, in relation to the question of dynamic inefficiency, cf. Chapter 3, the cut-off value of the steady-state interest rate is the steady-state GDP growth rate of the economy and this growth rate increases one-to-one with the rate of technological progress. We shall therefore now introduce technological progress.

On the basis of a summary of "stylized facts" about growth, Section 4.1 motivates the assumption that technological progress at the aggregate level takes the Harrod-neutral form. In Section 4.2 we extend the Diamond OLG model by incorporating this form of technological progress into the model. Section 4.3 extends the golden rule concept to allow for the existence of technological progress. In Section 4.4 what is known as the neoclassical theory of the functional income distribution is addressed. In this connection an expedient analytical tool, the elasticity of factor substitution, is presented. Section 4.5 goes into detail with the special case of a constant elasticity of factor substitution (the CES production function). Finally, Section 4.6 concludes.

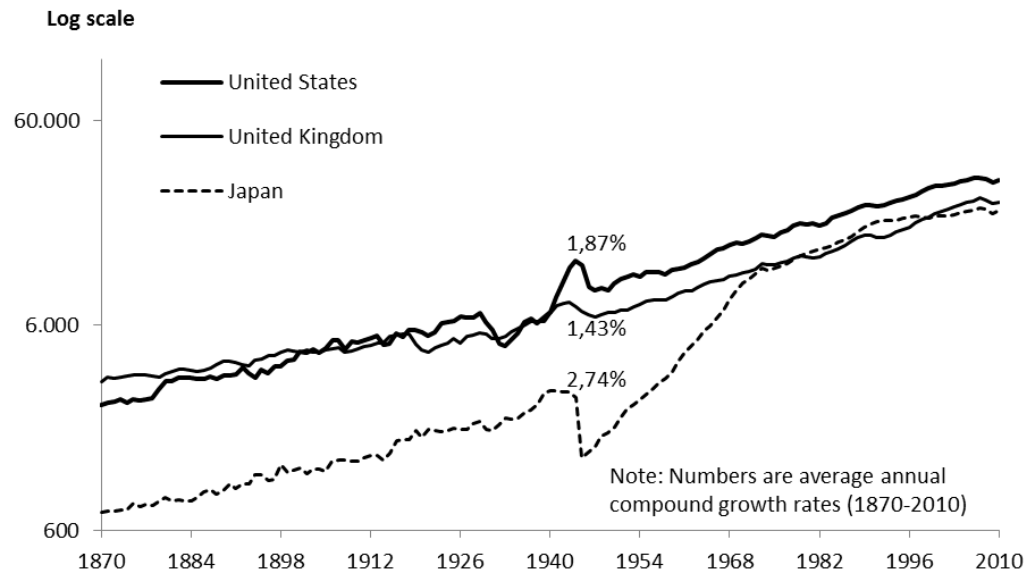

Figure 4.1: GDP per capita in U.S., U.K., and Japan 1870-2010. Source: Bolt and van Zanden (2013).

4.1 Harrod-neutrality and Kaldor's stylized facts

Suppose the technology changes over time in such a way that we can write the aggregate production function as

$$Y_t = F(K_t, T_t L_t), \tag{4.1}$$

where the level of technology is represented by the factor T_t which is growing over time, and where Y_t, K_t, and L_t stand for output, capital input, and labor input, respectively. When technological change takes this purely "labor-augmenting" form, it is known as *Harrod-neutral technological progress.*

Kaldor's stylized facts

The reason that macroeconomists often assume that technological change at the aggregate level takes the Harrod-neutral form as in (4.1) and not for example the form $Y_t = F(X_t K_t, T_t L_t)$ (where both X and T are changing over time), is the following. You want the long-run properties of the model to comply with Kaldor's list of "stylized facts" (Kaldor 1961) concerning the long-run evolution of industrialized economies. Abstracting from short-run fluctuations,

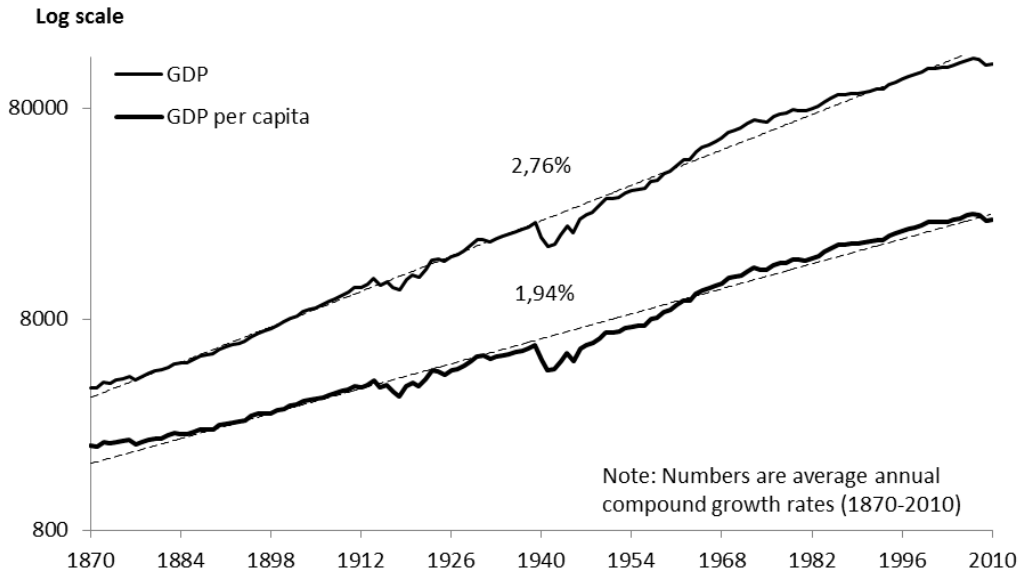

Figure 4.2: GDP and GDP per capita. Denmark 1870-2006. Sources: Bolt and van Zanden (2013); Maddison (2010); and The Conference Board Total Economy Database (2013).

1. the growth rates in K/L and Y/L are roughly constant;

2. the output-capital ratio, Y/K, the income share of labor, wL/Y, and the average rate of return, $(Y - wL - \delta K)/K$,[1] are roughly constant;

3. the growth rate of Y/L can vary substantially across countries for quite long time.

Ignoring the conceptual difference between the path of Y/L and that of Y *per capita* (a difference not so important in this context), the figures 4.1 and 4.2 illustrate Kaldor's "fact 1" about the long-run property of the Y/L path for the more developed countries. Japan had an extraordinarily high growth rate for a couple of decades after World War II, usually explained by fast technology transfer from the most developed countries (the catching-up process which can only last until the technology gap is eliminated). Fig. 4.3 gives rough support for a part of Kaldor's "fact 2", namely that about the labor income share. The third fact is a fact well documented empirically.[2]

[1] In this formula, land (and/or similar natural resources) is ignored. For countries where land is a quantitatively important production factor, the denominator should be replaced by $K + p_J J$, where p_J is the real price of land, J.

[2] For a summary, see Pritchett (1997).

Figure 4.3: Labor's share of GDP in USA (1950-2011) and Denmark (1970-2011). Source: Feenstra, Inklaar and Timmer (2013), www.ggdc.net/pwt.

It is fair to add, however, that the claimed regularities 1 and 2 do not fit all developed countries equally well. Although Solow's growth model (Solow, 1956) can be seen as the first successful attempt at building a model consistent with Kaldor's "stylized facts", Solow himself once remarked about them: "There is no doubt that they are stylized, though it is possible to question whether they are facts" (Solow, 1970). Several empiricists have questioned the methods standard national income accounting apply to separate the income of entrepreneurs, sole proprietors, and unincorporated businesses into labor and capital income, claiming these methods obscure a tendency in recent decades of the labor income share to fall (Gollin, 2002; Karabarbounis and Neiman, 2013).

Balanced growth requires Harrod-neutrality

Notwithstanding these ambiguities, it is definitely a fact that many long-run models are constructed so a to comply with Kaldor's "stylized facts". To do that a model must be capable of generating a *balanced growth path;* as we shall see, this in turn requires that technological change takes the Harrod-neutral form.

With K_t and Y_t defined in connection with (4.1) and C_t being aggregate consumption, we have the following definition of a balanced growth path.

DEFINITION 1 A *balanced growth path* is a path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ along which the variables K_t, Y_t, and C_t are positive and grow at constant rates (not necessarily positive).

Before describing the exact association between Harrod-neutral technological change and balanced growth, we shall draw attention to a general equivalence relationship in a closed economy, namely the equivalence between balanced growth and constancy of certain key ratios like Y/K and C/Y. This relationship is an implication of accounting based on the aggregate dynamic resource constraint,

$$K_{t+1} - K_t = I_t - \delta K_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad (4.2)$$

where I_t is gross investment, which in a closed economy equals gross saving, $S_t \equiv Y_t - C_t$, and δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$.[3]

We will denote the growth rate of a positive variable, x, between t and $t+1$, $g_{x,t+1}$ i.e., $g_{x,t+1} \equiv (x_{t+1} - x_t)/x_t \equiv \Delta x_{t+1}/x_t$. When there is no risk of confusion, we suppress the explicit dating and write $g_x \equiv \Delta x/x$.

PROPOSITION 1 *(the balanced growth equivalence theorem).* Let $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which K_t, Y_t, C_t, and S_t ($\equiv Y_t - C_t$) are positive for all $t = 0, 1, 2, \ldots$. Then, given the dynamic resource constraint (4.2), the following holds:
(i) if there is balanced growth, then $g_Y = g_K = g_C$ and so the ratios Y/K and C/Y are constant;
(ii) if Y/K and C/Y are constant, then $Y, K,$ and C grow at the same constant rate, i.e., not only is there balanced growth but the growth rates of Y, K, and C are the same.

Proof Consider a path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ along which K, Y, C, and $S_t \equiv Y - C_t$ are positive for all $t = 0, 1, 2, \ldots$. (i) Suppose the path is a balanced growth path. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (4.2), $S/K = g_K + \delta$ must be constant, implying[4]

$$g_S = g_K. \qquad (*)$$

By (4.2), $Y \equiv C + S$, and so

$$\begin{aligned} g_Y &= \frac{\Delta Y}{Y} = \frac{\Delta C}{Y} + \frac{\Delta S}{Y} = \frac{C}{Y} g_C + \frac{S}{Y} g_S = \frac{C}{Y} g_C + \frac{S}{Y} g_K \qquad \text{(by (*))} \\ &= \frac{C}{Y} g_C + \frac{Y - C}{Y} g_K = \frac{C}{Y}(g_C - g_K) + g_K. \qquad (**) \end{aligned}$$

[3] In both (4.1) and (4.2) it is implicitly assumed, as is usual in simple macroeconomic models, that technological progress is *disembodied* rather than *embodied,* a distinction described in Section 2.2 of Chapter 2.

[4] The ratio between two positive variables is constant if and only if the variables have the same growth rate (not necessarily constant or positive). For this and similar simple growth-arithmetic rules, see Appendix A.

Let us provisionally assume that $g_C \neq g_K$. Then (**) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{***}$$

a constant since g_Y, g_K, and g_C are constant. Constancy of C/Y implies $g_C = g_Y$, hence, (***), $C/Y = 1$. In view of $Y \equiv C + S$, however, this result contradicts the given condition that $S > 0$. Hence, our provisional assumption is falsified, and we have $g_C = g_K$, so that (***) is not valid and $C/Y = 1$ is not implied. Now (**) gives $g_Y = g_K = g_C$. It follows that Y/K and C/Y are constant.

(ii) Suppose Y/K and C/Y are positive constants. Applying that the ratio between two variables is constant if and only if the variables have the same (not necessarily constant or positive) growth rate, we can conclude that $g_Y = g_K = g_C$. By constancy of C/Y follows that $S/Y \equiv 1 - C/Y$ is constant. So $g_S = g_Y = g_K$, which in turn implies that S/K is constant. By (4.2),

$$\frac{S}{K} = \frac{\Delta K + \delta K}{K} = g_K + \delta,$$

so that also g_K is constant. This, together with constancy of Y/K and C/Y, implies that also g_Y and g_C are constant. □

Remark. It is part (i) of the proposition which requires the assumption $S > 0$ for all $t \geq 0$. If $S = 0$, we would have $g_K = -\delta$ and $C \equiv Y - S = Y$, hence $g_C = g_Y$ for all $t \geq 0$. Then there would be balanced growth if the common value of g_C and g_Y had a constant growth rate. This growth rate, however, could easily differ from that of K. Suppose $Y = AK^\alpha L^{1-\alpha}$, $0 < \alpha < 1$, $g_A = \gamma$ and $g_L = n$, where γ and n are constants. We would then have $1 + g_C = 1 + g_Y = (1+\gamma)(1-\delta)^\alpha(1+n)^{1-\alpha}$, which could easily be larger than 1 and thereby different from $1 + g_K = 1 - \delta \leq 1$ so that (i) no longer holds. □

For many long-run closed-economy models, including the Diamond OLG model, it holds that if and only if the dynamic system implied by the model is in steady state, will the economy feature balanced growth. The reason is that when the dynamic resource constraint (4.2) applies, a steady state is predominantly equivalent to Y/K and C/Y being constant, cf. Proposition 3 below.

Note that Proposition 1 pertains to *any* model for which (4.2) is valid. No assumption about market form and economic agents' behavior are involved. And except for the assumed constancy of the capital depreciation rate δ, no assumption about the technology is involved, not even that constant returns to scale is present.

Proposition 1 suggests that if one accepts Kaldor's stylized facts as a rough description of more than a century's growth experience and therefore wants the model to be consistent with them, one should construct the model so that it can generate balanced growth. Our next proposition states that for a model to be capable of doing that, technological progress must take the Harrod-neutral form (i.e., be labor-augmenting). This proposition also holds in a fairly general environment, but not as general as that of Proposition 1. Constant returns to scale and constant growth in the labor force, two aspects about which Proposition 1 is silent, will now have a role to play.[5]

Consider an aggregate production function

$$Y_t = \tilde{F}(K_t, L_t, t), \qquad \frac{\partial \tilde{F}}{\partial t} > 0, \tag{4.3}$$

where $\tilde{F}$ is homogeneous of degree one w.r.t. the first two arguments (CRS). The third argument, t, represents technological progress: as time proceeds, unchanged inputs of capital and labor result in more and more output. Let the labor force grow at a constant rate n,

$$L_t = L_0(1+n)^t, \quad n > -1, \tag{4.4}$$

where $L_0 > 0$. The Japanese economist Hirofumi Uzawa (1928-) is famous for several contributions, not least his balanced growth theorem (Uzawa 1961), which we here state in a modernized form.

PROPOSITION 2 *(Uzawa's balanced growth theorem).* Let $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$, and satisfy the dynamic resource constraint (4.2), given the production function (4.3) and the labor force (4.4). Then:
(i) a *necessary* condition for this path to be a balanced growth path is that along the path it holds that

$$Y_t = \tilde{F}(K_t, T_t L_t, 0), \tag{4.5}$$

where $T_t = (1+g)^t$ with $1 + g \equiv (1 + g_Y)/(1 + n)$ and g_Y being the constant growth rate of output along the balanced growth path;
(ii) for any $g \geq 0$ such that there is a $q > (1+g)(1+n)+\delta$ with the property that $\tilde{F}(1, k^{-1}, 0) = q$ for some $k > 0$ (i.e., at any t, hence also at $t = 0$, the production function $\tilde{F}$ in (4.3) allows an output-capital ratio equal to q), a *sufficient* condition for the existence of a balanced growth path with

[5] On the other hand we do *not* imply that CRS is *always* necessary for a balanced growth path (see Exercise 4.??).

output-capital ratio q is that the technology can be written as in (4.5) with $T_t = (1+g)^t$.

Proof See Appendix B.

The form (4.5) indicates that along a balanced growth path (BGP from now), technological progress must be purely labor augmenting, that is, Harrod-neutral. Moreover, by defining a new CRS production function F by $F(K_t, T_tL_t) \equiv \tilde{F}(K_t, T_tL_t, 0)$, we see that (i) of the proposition implies that at least along a BGP with technology growth, we can rewrite the original production function this way:

$$Y_t = \tilde{F}(K_t, L_t, t) = \tilde{F}(K_t, T_tL_t, 0) \equiv F(K_t, T_tL_t). \tag{4.6}$$

where $T_t = (1+g)^t$.

As also noted in Chapter 2, presence of Harrod-neutrality says nothing about what the *source* of technological progress is. Harrod-neutrality does not mean that technological change emanates specifically from the labor input. It only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest* itself such that we can rewrite the aggregate production function as in (4.6).

What is the intuition behind the Uzawa result that for balanced growth to be possible, technological progress must at the aggregate level have the purely labor-augmenting form? First, notice that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output. In contrast, labor is a non-produced production factor which in the present context grows in an exogenous way. Second, because of CRS, the original production function, (4.3), implies that

$$1 = \tilde{F}(\frac{K_t}{Y_t}, \frac{L_t}{Y_t}; t). \tag{4.7}$$

Now, since capital is accumulated non-consumed output, it tends to inherit the trend in output such that K_t/Y_t must be constant along a BGP (this is what Proposition 1 is about). Labor does not inherit the trend in output; indeed, the ratio L_t/Y_t is free to adjust as t proceeds. When there is technological progress ($\partial\tilde{F}/\partial t > 0$) along a BGP, this progress must manifest itself in the form of a changing L_t/Y_t in (4.7) as t proceeds, precisely because K_t/Y_t *must* be constant along the path. In the "normal" case where $\partial\tilde{F}/\partial L > 0$, the needed change in $L(t)/Y(t)$ is a *fall* (i.e., rise in $Y(t)/L(t)$). This is what (4.7) shows. Indeed, the fall in L_t/Y_t must exactly offset the effect on $\tilde{F}$ of the rising t, when there is a fixed capital-output ratio. It follows that along the BGP, Y_t/L_t is an increasing implicit function of t. If we denote this function T_t, we end up with (4.6).

The generality of Uzawa's theorem is noteworthy. Like Proposition 1, Uzawa's theorem is about technically feasible paths, while economic institutions, market forms, and agents' behavior are not involved. The theorem presupposes CRS, but does not need that the technology has neoclassical properties not to speak of satisfying the Inada conditions. And the theorem holds for exogenous as well as endogenous technological progress.

A simple implication of the theorem is the following. Let y_t denote "labor productivity" in the sense of Y_t/L_t, k_t denote the capital-labor ratio, K_t/L_t, and c_t the consumption-labor ratio, C_t/L_t. We have:

COROLLARY Along a BGP with positive gross saving and the technology level, T, growing at a constant rate, g, output grows at the rate $(1+g)(1+n)-1$ ($\approx g+n$ for g and n "small") while labor productivity, y, capital-labor ratio, k, and consumption-labor ratio, c, all grow at the rate g.

Proof That $g_Y = (1+g)(1+n) - 1$ follows from (i) of the proposition. As to g_y we have

$$y_t \equiv \frac{Y_t}{L_t} = \frac{Y_0(1+g_Y)^t}{L_0(1+n)^t} = y_0(1+g)^t,$$

showing that y grows at the rate g; (i) of the proposition also implies that Y/K is constant along the BGP. As $y/k = Y/K$, k then grows at the same rate as y. Finally, also $c/y \equiv C/Y$ is constant along the BGP, implying that also c grows at the same rate as y. □

Factor income shares

There is one facet of Kaldor's stylized facts which we have not yet related to Harrod-neutral technological progress, namely the claimed long-run "approximate" constancy of the income share of labor and the rate of return on capital. It turns out that, assuming a neoclassical technology, profit maximizing firms, and perfect competition in the output and factor markets, these constancies are inherent in the combination of constant returns to scale and balanced growth.

To see this, let the aggregate production function be $Y_t = F(K_t, T_tL_t)$ where F is neoclassical and has CRS. With w_t denoting the real wage at time t, in equilibrium under *perfect competition* the labor income share will be

$$\frac{w_tL_t}{Y_t} = \frac{\frac{\partial Y_t}{\partial L_t}L_t}{Y_t} = \frac{F_2(K_t, T_tL_t)T_tL_t}{Y_t}. \tag{4.8}$$

When the capital good is nothing but non-consumed output, the rate of return on capital at time t can be written

$$r_t = \frac{Y_t - w_t L_t - \delta K_t}{K_t}. \tag{4.9}$$

Since land as a production factor is ignored, gross capital income equals non-labor income, $Y_t - w_t L_t$. Denoting the gross capital income share by α_t, we thus have

$$\begin{aligned} \alpha_t &= \frac{(r_t+\delta)K_t}{Y_t} = \frac{Y_t - w_t L_t}{Y_t} = \frac{F(K_t, T_t L_t) - F_2(K_t, T_t L_t)T_t L_t}{Y_t} \\ &= \frac{F_1(K_t, T_t L_t)K_t}{Y_t} = \frac{\frac{\partial Y_t}{\partial K_t}K_t}{Y_t}, \end{aligned} \tag{4.10}$$

where the second equality comes from (4.9), the third from (4.8), and the fourth from Euler's theorem.[6]

PROPOSITION 3 *(factor income shares)* Suppose the path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ is a BGP with positive saving in this competitive economy. Then $\alpha_t = \alpha$, a constant $\in (0,1)$. The labor income share will be $1-\alpha$ and the rate of return on capital $\alpha q - \delta$, where q is the constant output-capital ratio along the BGP.

Proof We have $Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t)$. From Proposition 1 follows that along the BGP, Y_t/K_t is some constant, q. Since $Y_t/K_t = f(\tilde{k}_t)/\tilde{k}_t$ and $f'' < 0$, this implies $\tilde{k}_t$ constant, say equal to $\tilde{k}^*$. Along the BGP, $\partial Y_t/\partial K_t$ $(= f'(\tilde{k}_t))$ thus equals the constant $f'(\tilde{k}^*)$. From (4.10) then follows that $\alpha_t = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, since $0 < \alpha$ is implied by $f' > 0$ and $\alpha < 1$ is implied by the fact that $q = Y/K = f(\tilde{k}^*)/\tilde{k}^* > f'(\tilde{k}^*)$, in view of $f'' < 0$ and $f(0) \geq 0$. So, by the first equality in (4.10), the labor income share can be written $w_t L_t / Y_t = 1 - \alpha_t = 1 - \alpha$ whereby the rate of return on capital is $r_t = (1 - w_t L_t/Y_t) Y_t/K_t - \delta = \alpha q - \delta$. □

Although this proposition implies constancy of the factor income shares under balanced growth, it does not *determine* them. The proposition expresses the factor income shares in terms of the unknown constants α and q. These constants will generally depend on the effective capital-labor ratio in steady state, $\tilde{k}^*$, which will generally be an unknown as long as we have not formulated a theory of saving. This takes us back to Diamond's OLG model which provides such a theory.

[6] Indeed, from Euler's theorem follows that $F_1 K + F_2 TL = F(K, TL)$, when F is homogeneous of degree one.

4.2 The Diamond model with Harrod-neutral technological progress

Recall from the previous chapter that in the Diamond model people live in two periods, as young and as old. Only the young work and each young supplies one unit of labor inelastically. The period utility function, $u(c)$, satisfies the No Fast Assumption. The saving function of the young is $s_t = s(w_t, r_{t+1})$. We now include Harrod-neutral technological progress in the aggregate production function:

$$Y_t = F(K_t, T_t L_t), \tag{4.11}$$

where F is neoclassical with CRS and T_t represents level of technology in period t. We assume that T_t grows at a constant exogenous rate, that is,

$$T_t = T_0(1+g)^t, \qquad g > 0. \tag{4.12}$$

The initial level of technology, T_0, is historically given.[7] Employment equals L_t which is the number of young, growing at the constant exogenous rate $n > -1$.

Suppressing for a while the explicit dating of the variables, in view of CRS w.r.t. K and TL, we have

$$\tilde{y} \equiv \frac{Y}{TL} = F(\frac{K}{TL}, 1) = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0,$$

where TL is *labor input in efficiency units* and $\tilde{k} \equiv K/(TL)$ is known as the *effective* or *technology-corrected capital-labor ratio* - also sometimes called the effective capital intensity. There is perfect competition in all markets. In each period the representative firm maximizes profit, $\Pi = F(K, TL) - \hat{r}K - wL$. With respect to capital this leads to the first-order condition

$$\frac{\partial Y}{\partial K} = \frac{\partial \left[TLf(\tilde{k})\right]}{\partial K} = f'(\tilde{k}) = r + \delta, \tag{4.13}$$

where δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$. With respect to labor we get the first-order condition

$$\frac{\partial Y}{\partial L} = \frac{\partial \left[TLf(\tilde{k})\right]}{\partial L} = \left[f(\tilde{k}) - f'(\tilde{k})\tilde{k}\right] T = w. \tag{4.14}$$

[7] In connection with (4.3), measurement units were chosen such that $T_0 = 1$ in (4.5). But transparency is improved when we let measurement units be arbitrary. This corresponds to substituting $Y_t = \tilde{F}(K_t, T_0 L_t, t)$ for (4.3).

In view of $f'' < 0$, a $\tilde{k}$ satisfying (4.13) is unique. Let us denote its value in period t, $\tilde{k}_t^d$. Assuming equilibrium in the factor markets, this desired effective capital-labor ratio equals the effective capital-labor ratio from the supply side, $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$, which is predetermined in every period. The equilibrium interest rate and real wage in period t are thus given by

$$r_t = f'(\tilde{k}_t) - \delta \equiv r(\tilde{k}_t), \quad \text{where} \quad r'(\tilde{k}_t) = f''(\tilde{k}_t) < 0, \tag{4.15}$$

$$w_t = \left[f(\tilde{k}_t) - f'(\tilde{k}_t)\tilde{k} \right] T_t \equiv \tilde{w}(\tilde{k}_t) T_t, \quad \text{where } \tilde{w}'(\tilde{k}_t) = -\tilde{k}_t f''(\tilde{k}_t) > 0. \tag{4.16}$$

Here, $\tilde{w}(\tilde{k}_t) = w_t/T_t$ is known as the *technology-corrected real wage.*

The equilibrium path

The aggregate capital stock at the beginning of period $t + 1$ must still be owned by the old generation in that period and thus equal the aggregate saving these people did as young in the previous period. Hence, as before, $K_{t+1} = s_t L_t = s(w_t, r_{t+1})L_t$. In view of $K_{t+1} \equiv \tilde{k}_{t+1} T_{t+1} L_{t+1} = \tilde{k}_{t+1} T_t (1 + g) L_t (1 + n)$, together with (4.15) and (4.16), we get

$$\tilde{k}_{t+1} = \frac{s(\tilde{w}(\tilde{k}_t)T_t, r(\tilde{k}_{t+1}))}{T_t(1 + g)(1 + n)}. \tag{4.17}$$

This is the general version of the law of motion of the Diamond OLG model with Harrod-neutral technological progress.

For the model to comply with Kaldor's "stylized facts", the model should be capable of generating balanced growth. Essentially, this capability is equivalent to being able to generate a steady state. In the presence of technological progress this latter capability requires a restriction on the lifetime utility function, U. Indeed, we see from (4.17) that the model is consistent with existence of a steady state only if the time-dependent technology level, T_t, in the numerator and denominator cancels out. This requires that the saving function is homogeneous of degree one in its first argument such that $s(\tilde{w}(\tilde{k}_t)T_t, r(\tilde{k}_{t+1})) = s(\tilde{w}(\tilde{k}_t), r(\tilde{k}_{t+1}))T_t$. In turn this is so if and only if the lifetime utility function of the young is *homothetic;* this property entails that if wealth (here w_t) is multiplied by a positive factor, then the chosen c_{1t} and c_{2t+1} are also multiplied by this factor (see Appendix C); it then follows that s_t is multiplied by this factor as well.

In addition to the No Fast Assumption from Chapter 3 we thus impose the Homotheticity Assumption:

$$\text{the lifetime utility function } U \text{ is homothetic.} \tag{A4}$$

Then we can write

$$s_t = s(1, r(\tilde{k}_{t+1}))\tilde{w}(\tilde{k}_t)T_t \equiv \hat{s}(r(\tilde{k}_{t+1}))\tilde{w}(\tilde{k}_t)T_t, \tag{4.18}$$

where $\hat{s}(r(\tilde{k}_{t+1}))$ is the saving-wealth *ratio* of the young. The distinctive feature is that this saving-wealth ratio is independent of wealth (but in general it depends on the interest rate). By (4.17), the law of motion of the economy reduces to

$$\tilde{k}_{t+1} = \frac{\hat{s}(r(\tilde{k}_{t+1}))}{(1+g)(1+n)}\tilde{w}(\tilde{k}_t). \tag{4.19}$$

The equilibrium path of the economy can be analyzed in a similar way as in the case of no technological progress. In the assumptions (A2) and (A3) from Chapter 3 we replace k by $\tilde{k}$ and $1+n$ by $(1+g)(1+n)$. As a generalization of Proposition 4 from Chapter 3, these generalized versions of (A2) and (A3), together with the No Fast Assumption (A1) and the Homotheticity Assumption (A4), guarantee that there exists at least one locally asymptotically stable steady state $\tilde{k}^* > 0$. That is, given these assumptions, we have $\tilde{k}_t \to \tilde{k}^*$ for $t \to \infty$ and so the system will sooner or later settle down in a steady state. The convergence of $\tilde{k}$ implies convergence of many key variables, for instance the equilibrium factor prices given in (4.15) and (4.16). We see that

$$\begin{aligned} r_t &= f'(\tilde{k}_t) - \delta \to f'(\tilde{k}^*) - \delta \equiv r^*, \quad \text{and} \\ w_t &= \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \to \left[f(k^*) - k^* f'(k^*)\right] T_t \equiv \tilde{w}^* T_t = \tilde{w}^* T_0 (1+g)^t, \end{aligned}$$

for $t \to \infty$.

The prediction of the model is now that the economy will in the long run behave in accordance with Kaldor's stylized facts. Indeed, in many models, including the present one, convergence toward a steady state is equivalent to saying that the time path of the economy converges toward a BGP. In the present case, with perfect competition, the implication is that in the long run the economy will be consistent with Kaldor's stylized facts.

The claimed equivalence follows from:

PROPOSITION 4 Consider a Diamond economy with Harrod-neutral technological progress at the constant rate $g > 0$ and positive gross saving for all t. Then:
(i) if the economy features balanced growth, then it is in a steady state;
(ii) if the economy is in a steady state, then it features balanced growth.

Proof (i) Suppose the considered economy features balanced growth. Then, by Proposition 1, Y/K is constant. As $Y/K = \tilde{y}/\tilde{k} = f(\tilde{k})/\tilde{k}$, also $\tilde{k}$ is constant. Thereby the economy is in a steady state. (ii) Suppose the considered economy is in a steady state, i.e., given (4.19), $\tilde{k}_t = \tilde{k}_{t+1} = \tilde{k}^*$ for some $\tilde{k}^* > 0$. The constancy of $\tilde{k} \equiv K/(TL)$ and $\tilde{y} \equiv Y/(TL) = f(\tilde{k})$ implies that both g_K and g_Y equal $g_{TL} = (1+g)(1+n) - 1 > 0$. As K and Y thus grow at the same rate, Y/K is constant. With $S \equiv Y - C$, constancy of $S/K = (\Delta K + \delta K)/K = g_K + \delta$, implies constancy of S/K so that S also grows at the rate g_K and thereby at the same rate as output. Hence S/Y is constant. Because $C/Y \equiv 1 - S/Y$, also C grows at the constant rate g_Y. All criteria for a balanced growth path are thus satisfied. □

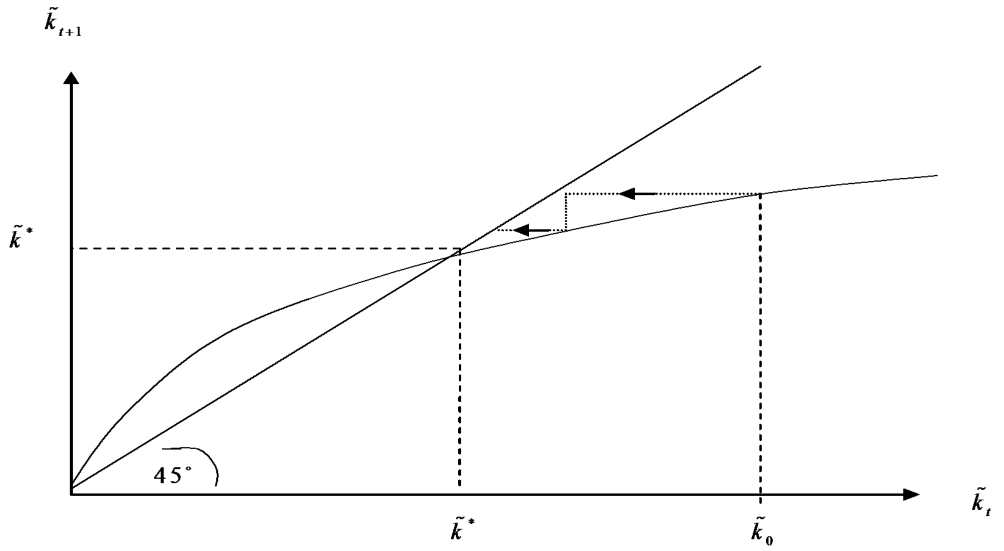

Figure 4.4: Transition curve for a well-behaved Diamond OLG model with Harrod-neutral technical progress.

Let us portray the dynamics by a transition diagram. Fig. 4.4 shows a "well-behaved" case in the sense that there is only one steady state. In the figure the initial effective capital-labor ratio, $\tilde{k}_0$, is assumed to be relatively large. This need not be interpreted as if the economy is highly developed and has a high initial capital-labor ratio, K_0/L_0. Indeed, the reason that $\tilde{k}_0 \equiv K_0/(T_0 L_0)$ is large relative to its steady-steady value may be that the economy is "backward" in the sense of having a relatively low initial level of technology. Growing at a given rate g, the technology will in this situation grow faster than the capital-labor ratio, K/L, so that the effective capital-labor ratio declines over time. The process continues until the steady state is essentially reached with a real interest rate $r^* = f'(\tilde{k}^*) - \delta$. This is to remind the reader that from an empirical point of view, the adjustment towards a steady state can be from above as well as from below.

The output growth rate in steady state, $(1+g)(1+n)-1$, is sometimes called the "natural rate of growth". Since $(1+g)(1+n)-1 = g+n+gn \approx g+n$ for g and n "small", the natural rate of growth approximately equals the sum of the rate of technological progress and the growth rate of the labor force. *Warning*: When measured on an *annual* basis, the growth rates of technology and labor force, $\bar{g}$ and $\bar{n}$, do indeed tend to be "small", say $\bar{g} = 0.02$ and $\bar{n} = 0.005$, so that $\bar{g}+\bar{n}+\bar{g}\bar{n} = 0.0251 \approx 0.0250 = \bar{g}+\bar{n}$. But in the context of models like Diamond's, the period length is, say, 30 years. Then the corresponding g and n will satisfy the equations $1+g = (1+\bar{g})^{30}$ $= 1.02^{30} = 1.8114$ and $1+n = (1+\bar{n})^{30} = 1.005^{30} = 1.1614$, respectively. We get $g+n = 0.973$, which is about 10 per cent smaller than the true output growth rate over 30 years, which is $g+n+gn = 1.104$.

We end our account of Diamond's OLG model with some remarks on a popular special case of a homothetic utility function.

An example: CRRA period utility

An example of a homothetic lifetime utility function is obtained by letting the period utility function take the CRRA form introduced in the previous chapter. Then

$$U(c_1, c_2) = \frac{c_1^{1-\theta}-1}{1-\theta} + (1+\rho)^{-1}\frac{c_2^{1-\theta}-1}{1-\theta}, \qquad \theta > 0. \tag{4.20}$$

Recall that the CRRA utility function with parameter θ has the property that the (absolute) elasticity of marginal utility of consumption equals the constant $\theta > 0$ for all $c > 0$. Up to a positive linear transformation it is, in fact, the only period utility function with this property. A proof that the utility function (4.20) is indeed homothetic is given in Appendix C.

One of the reasons that the CRRA function is popular in macroeconomics is that in *representative* agent models, the period utility function *must* have this form to obtain consistency with balanced growth and Kaldor's stylized facts (this is shown in Chapter 7). In contrast, a model with heterogeneous agents, like the Diamond model, does not need CRRA utility to comply with the Kaldor facts. CRRA utility is just a convenient special case leading to homothetic lifetime utility. And *this* is what is needed for a BGP to exist and thereby for compatibility with Kaldor's stylized facts.

Given the CRRA assumption in (4.20), the saving-wealth ratio of the young becomes

$$\hat{s}(r) = \frac{1}{1+(1+\rho)^{1/\theta}(1+r)^{(\theta-1)/\theta}}. \tag{4.21}$$

It follows that $\hat{s}'(r) \gtreqless 0$ for $\theta \lesseqgtr 1$.

When $\theta = 1$ (the case $u(c) = \ln c$), $\hat{s}(r) = 1/(2+\rho) \equiv \hat{s}$, a constant, and the law of motion (4.19) thus simplifies to

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)(2+\rho)}\tilde{w}(\tilde{k}_t).$$

We see that in the $\theta = 1$ case, whatever the production function, $\tilde{k}_{t+1}$ enters only at the left-hand side of the fundamental difference equation, which thereby reduces to a simple transition function. Since $\tilde{w}'(\tilde{k}) > 0$, the transition curve is positively sloped everywhere. If the production function is Cobb-Douglas, $Y_t = K_t^\alpha (T_t L_t)^{1-\alpha}$, then $\tilde{w}(\tilde{k}_t) = (1-\alpha)\tilde{k}_t^\alpha$. Combining this with $\theta = 1$ yields a "well-behaved" Diamond model (one and only one steady steady), cf. Fig. 4.4 above. In fact, as noted in Chapter 3, in combination with Cobb-Douglas technology, CRRA utility results in "well-behavedness" whatever the value of $\theta > 0$.

4.3 The golden rule under Harrod-neutral technological progress

Given that there is technological progress, consumption per unit of labor is likely to grow over time. Therefore the definition of the golden-rule capital-labor ratio from Chapter 3 has to be extended to cover the case of growing consumption per unit of labor. To allow existence of steady states and balanced growth paths, we maintain the assumption that technological progress is Harrod-neutral, that is, we maintain (4.11) where the technology level, T, grows at a constant rate, g.

DEFINITION 2 The golden-rule capital intensity, $\tilde{k}_{GR}$, is that level of $\tilde{k} \equiv K/(TL)$ which gives the highest sustainable path for consumption per unit of labor in the economy.

Consumption per unit of labor is

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} = \frac{F(K_t, T_t L_t) - S_t}{L_t} = \frac{f(\tilde{k}_t)T_t L_t - (K_{t+1} - K_t + \delta K_t)}{L_t} \\ &= f(\tilde{k}_t)T_t - (1+g)T_t(1+n)\tilde{k}_{t+1} + (1-\delta)T_t\tilde{k}_t \\ &= \left[f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - (1+g)(1+n)\tilde{k}_{t+1}\right] T_t. \end{aligned}$$

In a steady state we have $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}$ and therefore

$$c_t = \left[f(\tilde{k}) + (1-\delta)\tilde{k} - (1+g)(1+n)\tilde{k}\right] T_t \equiv \tilde{c}(\tilde{k})T_t,$$

where $\tilde{c}(\tilde{k})$ is the "technology-corrected" level of consumption per unit of labor in steady state. We see that in steady state, consumption per unit of labor will grow at the same rate as the technology. Thus, $\ln c_t = \ln \tilde{c}(\tilde{k}) + \ln T_0 + t\ln(1+g)$. Fig. 4.5 illustrates.

Since the evolution of technology, parameterized by T_0 and g, is exogenous, the highest possible path of c_t is found by maximizing $\tilde{c}(\tilde{k})$. This gives the first-order condition

$$\tilde{c}'(\tilde{k}) = f'(\tilde{k}) + (1-\delta) - (1+g)(1+n) = 0. \tag{4.22}$$

Assuming, for example, $n \geq 0$, we have $(1+g)(1+n)-(1-\delta) > 0$ since $g > 0$. Then, by continuity the equation (4.22) necessarily has a unique solution in $\tilde{k} > 0$, if the production function satisfies the condition

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > (1+g)(1+n) - (1-\delta) > \lim_{\tilde{k}\to\infty} f'(\tilde{k}),$$

which is a milder condition than the Inada conditions. Considering the second-order condition $\tilde{c}''(\tilde{k}) = f''(\tilde{k}) < 0$, the $\tilde{k}$ satisfying (4.22) does indeed maximize $\tilde{c}(\tilde{k})$. By definition, this $\tilde{k}$ is the golden-rule capital intensity, $\tilde{k}_{GR}$. Thus

$$f'(\tilde{k}_{GR}) - \delta = (1+g)(1+n) - 1 \approx g + n, \tag{4.23}$$

where the right-hand side is the "natural rate of growth". This says that the golden-rule capital intensity is that level of the capital intensity at which the net marginal productivity of capital equals the output growth rate in steady state.

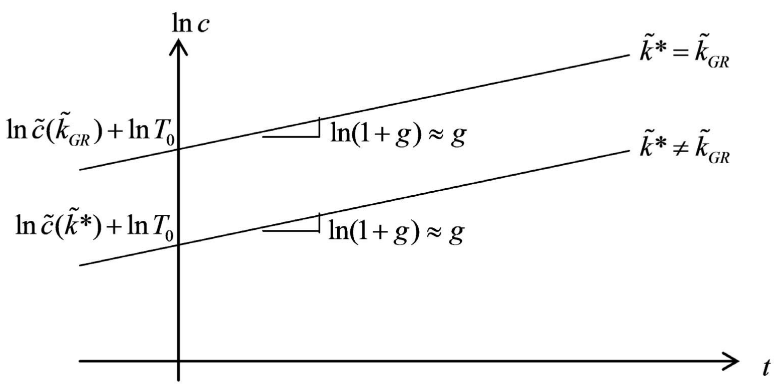

Figure 4.5: The highest sustainable path of consumption is where $\tilde{k}^* = \tilde{k}_{GR}$.

Is dynamic inefficiency a problem in practice? As in the Diamond model without technological progress, it is theoretically possible that the economy ends up in a steady state with $\tilde{k}^* > \tilde{k}_{GR}$.[8] If this happens, the economy is dynamically inefficient and $r^* < (1+g)(1+n) - 1 \approx g + n$. To check whether dynamic inefficiency is a realistic outcome in an industrialized economy or not, we should compare the observed average GDP growth rate over a long stretch of time to the average real interest rate or rate of return in the economy. For the period after the Second World War the average GDP growth rate ($\approx g + n$) in Western countries is typically about 3 per cent per year. But what interest rate should one choose? In simple macro models, like the Diamond model, there is no uncertainty and no need for money to carry out trades. In such models all assets earn the same rate of return, r, in equilibrium. In the real world there is a spectrum of interest rates, reflecting the different risk and liquidity properties of the different assets. The expected real rate of return on a short-term government bond is typically less than 3 per cent per year (a relatively safe and liquid asset). This is much lower than the expected real rate of return on corporate stock, 7-9 per cent per year. Our model cannot tell which rate of return we should choose, but the conclusion hinges on that choice.

Abel et al. (1989) study the problem on the basis of a model with *uncertainty*. They show that a sufficient condition for dynamic efficiency is that gross investment, I, does not exceed the gross capital income in the long run, that is $I \leq Y - wL$. They find that for the U.S. and six other major OECD nations this seems to hold. Indeed, for the period 1929-85 the U.S. has, on average, $I/Y = 0.15$ and $(Y - wL)/Y = 0.29$. A similar difference is found for other industrialized countries, suggesting that they are dynamically efficient. At least in these countries, therefore, the potential coordination failure laid bare by OLG models does not seem to have been operative in practice.

4.4 The functional distribution of income

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The neoclassical theory

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[8] The proof is analogue to that in Chapter 3 for the case $g = 0$.

How the labor income share depends on the capital-labor ratio

To begin with we ignore technological progress and write aggregate output as $Y = F(K, L)$, where F is neoclassical and CRS obtains. From Euler's theorem follows that $F(K, L) = F_1K + F_2L = f'(k)K + (f(k) - kf'(k))L$, where $k \equiv K/L$. In equilibrium under perfect competition we have

$$Y = \hat{r}K + wL,$$

where $\hat{r} = r + \delta$ is the cost per unit of capital input and w is the real wage, i.e., the cost per unit of labor input. The labor income share is

$$\frac{wL}{Y} = \frac{f(k) - kf'(k)}{f(k)} \equiv \frac{w(k)}{f(k)} \equiv SL(k) = \frac{wL}{\hat{r}K + wL} = \frac{\frac{w/\hat{r}}{k}}{1 + \frac{w/\hat{r}}{k}},$$

where the function $SL(\cdot)$ is the share of labor function and $w/\hat{r}$ is the factor price ratio.

Suppose that capital tends to grow faster than labor so that k rise over time. Unless the production function is Cobb-Douglas, this will under perfect competition affect the labor income share. But apriori it is not obvious in what direction. If the proportionate rise in the factor price ratio $w/\hat{r}$ is greater (smaller) than that in k, then SL goes up (down). Indeed, if we let $\mathrm{E}\ell_x g(x)$ denote the elasticity of a function $g(x)$ w.r.t. x, then

$$SL'(k) \gtreqless 0 \text{ for } \mathrm{E}\ell_k \frac{w}{\hat{r}} \gtreqless 1,$$

respectively.

Usually, however, the inverse elasticity is considered, namely $\mathrm{E}\ell_{w/\hat{r}}k$. This elasticity, which indicates how sensitive the cost minimizing capital-labor ratio, k, is to a given factor price ratio $w/\hat{r}$, coincides with the *elasticity of factor substitution* (for a general definition, see below). The latter is often denoted σ. Since in the CRS case, σ will be a function of only k, we write $\mathrm{E}\ell_{w/\hat{r}}k = \sigma(k)$. We therefore have

$$SL'(k) \gtreqless 0 \text{ for } \sigma(k) \lesseqgtr 1,$$

respectively.

We may put forward another example that illustrates the importance of the size of $\sigma(k)$. Consider a closed economy with perfect competition and a given aggregate capital stock K. Suppose that for some reason aggregate labor supply, L, goes down. In what direction will aggregate labor income wL then change? The effect of the smaller L is to some extent offset by a higher w brought about by the higher capital-labor ratio. Indeed, $dw/dk = -kf''(k) > 0$. So we cannot apriori sign the change in wL. Since $w = w(k)$ and $k \equiv KL^{-1}$, we can write wL as a function $wL = W(K, L)$. In Exercise 4.?? the reader is asked to show that

$$\frac{\partial W}{\partial L} = (1 - \frac{\alpha(k)}{\sigma(k)})w \gtreqless 0 \text{ for } \alpha(k) \lesseqgtr \sigma(k), \tag{4.24}$$

respectively, where $a(k) \equiv kf'(k)/f(k)$ is the output elasticity w.r.t. capital which under perfect competition equals the gross capital income share. It follows that the lower L will not be fully offset by the higher w as long as the elasticity of factor substitution, $\sigma(k)$, exceeds the gross capital income share, $\alpha(k)$. This condition seems confirmed by most of the empirical evidence (see, e.g., Antras, 2004, and Chirinko, 2008).

The elasticity of factor substitution*

We shall here discuss the concept of elasticity of factor substitution at a more general level. Fig. 4.6 depicts an isoquant, $F(K, L) = \bar{Y}$, for a given neoclassical production function, $F(K, L)$, which need not have CRS. Let MRS denote the marginal rate of substitution of K for L, i.e., $MRS = F_L(K, L)/F_K(K, L)$.[9] At a given point (K, L) on the isoquant curve, MRS is given by the absolute value of the slope of the tangent to the isoquant at that point. This tangent coincides with that isocost line which, given the factor prices, has minimal intercept with the vertical axis while at the same time touching the isoquant. In view of $F(\cdot)$ being neoclassical, the isoquants are by definition strictly convex to the origin. Consequently, MRS is rising along the curve when L decreases and thereby K increases. Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio K/L, as a function of MRS. If we let MRS rise along the given isoquant, the corresponding value of the ratio K/L will also rise.

The *elasticity of substitution* between capital and labor is defined as the elasticity of the ratio K/L with respect to MRS when we move along a given isoquant, evaluated at the point (K, L). Let this elasticity be denoted $\tilde{\sigma}(K, L)$. Thus,

$$\tilde{\sigma}(K, L) = \frac{MRS}{K/L} \frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} = \frac{\frac{d(K/L)}{K/L}}{\frac{dMRS}{MRS}}_{|Y=\bar{Y}}. \tag{4.25}$$

[9] When there is no risk of confusion as to what is up and what is down, we use MRS as a shorthand for the more correct MRS_{KL}.

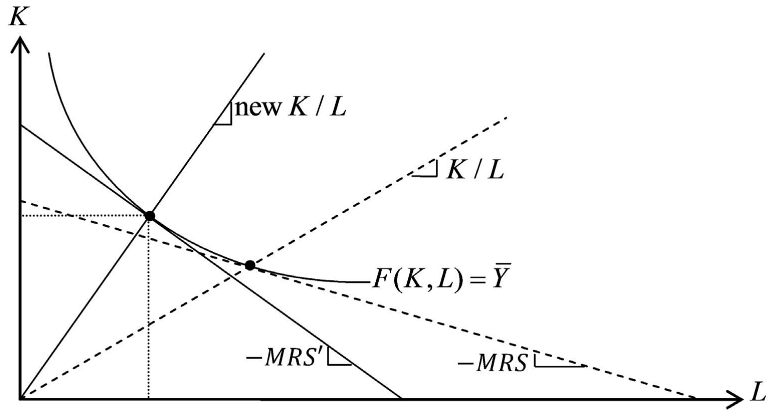


Figure 4.6: Substitution of capital for labor as the marginal rate of substitution increases from MRS to MRS'.

Although the elasticity of factor substitution is a characteristic of the technology as such and is here defined without reference to markets and factor prices, it helps the intuition to refer to factor prices. At a cost-minimizing point, MRS equals the factor price ratio $w/\hat{r}$. Thus, the *elasticity of factor substitution* will under cost minimization coincide with *the percentage increase in the ratio of the cost-minimizing factor ratio induced by a one percentage increase in the inverse factor price ratio, holding the output level unchanged.*[10] The elasticity of factor substitution is thus a positive number and reflects how sensitive the capital-labor ratio K/L is under cost minimization to an increase in the factor price ratio $w/\hat{r}$ for a given output level. The less curvature the isoquant has, the greater is the elasticity of factor substitution. In an analogue way, in consumer theory one considers the elasticity of substitution between two consumption goods or between consumption today and consumption tomorrow, cf. Chapter 3. In that context the role of the given isoquant is taken over by an indifference curve. That is also the case when we consider the intertemporal elasticity of substitution in labor supply, cf. the next chapter.

Calculating the elasticity of substitution between K and L at the point (K, L), we get

$$\tilde{\sigma}(K, L) = -\frac{F_K F_L (F_K K + F_L L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]}, \tag{4.26}$$

where all the derivatives are evaluated at the point (K, L). When $F(K, L)$

[10] This characterization is equivalent to interpreting the elasticity of substitution as the percentage *decrease* in the factor ratio (when moving along a given isoquant) induced by a one-percentage *increase* in the *corresponding* factor price ratio.

has CRS, the formula (4.26) simplifies to

$$\tilde{\sigma}(K,L) = \frac{F_K(K,L)F_L(K,L)}{F_{KL}(K,L)F(K,L)} = -\frac{f'(k)\left(f(k)-f'(k)k\right)}{f''(k)kf(k)} \equiv \sigma(k), \quad (4.27)$$

where $k \equiv K/L$.[11] We see that under CRS, the elasticity of substitution depends only on the capital-labor ratio k, not on the output level. We will now consider the case where the elasticity of substitution is independent also of the capital-labor ratio.

4.5 The CES production function*

It can be shown[12] that if a neoclassical production function with CRS has a constant elasticity of factor substitution different from one, it must be of the form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}}, \quad (4.28)$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function* (CES for Constant Elasticity of Substitution). For a given choice of measurement units, the parameter A reflects efficiency (or what is known as *total factor productivity*) and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (4.28) is not defined when either K or L (or both) equal 0. We can circumvent this problem by extending the domain of the CES function and assign the function value 0 to these points when $\beta < 0$. Continuity is maintained in the extended domain (see Appendix E).

By taking partial derivatives in (4.28) and substituting back we get

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta}\left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}\left(\frac{Y}{L}\right)^{1-\beta}, \quad (4.29)$$

where $Y/K = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and $Y/L = A\left[\alpha k^{\beta} + 1 - \alpha\right]^{\frac{1}{\beta}}$. The marginal rate of substitution of K for L therefore is

$$MRS = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha}k^{1-\beta} > 0.$$

[11] The formulas (4.26) and (4.27) are derived in Appendix D.
[12] See, e.g., Arrow et al. (1961).

Consequently,

$$\frac{dMRS}{dk} = \frac{1-\alpha}{\alpha}(1-\beta)k^{-\beta},$$

where the inverse of the right-hand side is the value of $dk/dMRS$. Substituting these expressions into (4.25) gives

$$\tilde{\sigma}(K,L) = \frac{1}{1-\beta} \equiv \sigma, \tag{4.30}$$

confirming the constancy of the elasticity of substitution. Since $\beta < 1$, $\sigma > 0$ always. A higher substitution parameter, β, results in a higher elasticity of factor substitution, σ. And $\sigma \lesseqgtr 1$ for $\beta \lesseqgtr 0$, respectively.

Since $\beta = 0$ is not allowed in (4.28), at first sight we cannot get $\sigma = 1$ from this formula. Yet, $\sigma = 1$ can be introduced as the *limiting* case of (4.28) when $\beta \to 0$, which turns out to be the Cobb-Douglas function. Indeed, one can show[13] that, for fixed K and L,

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to AK^{\alpha}L^{1-\alpha}, \text{ for } \beta \to 0.$$

By a similar procedure as above we find that a Cobb-Douglas function always has elasticity of substitution equal to 1; this is exactly the value taken by σ in (4.30) when $\beta = 0$. In addition, the Cobb-Douglas function is the *only* production function that has unit elasticity of substitution everywhere.

Another interesting limiting case of the CES function appears when, for fixed K and L, we let $\beta \to -\infty$ so that $\sigma \to 0$. We get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\min(K,L), \text{ for } \beta \to -\infty. \tag{4.31}$$

So in this case the CES function approaches a Leontief production function, the isoquants of which form a right angle, cf. Fig. 4.7. In the limit there is *no* possibility of substitution between capital and labor. In accordance with this the elasticity of substitution calculated from (4.30) approaches zero when β goes to $-\infty$.

Finally, let us consider the "opposite" transition. For fixed K and L we let the substitution parameter rise towards 1 and get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\left[\alpha K + (1-\alpha)L\right], \text{ for } \beta \to 1.$$

Here the elasticity of substitution calculated from (4.30) tends to ∞ and the isoquants tend to straight lines with slope $-(1-\alpha)/\alpha$. In the limit, the production function thus becomes linear and capital and labor become *perfect substitutes.*

[13] Proofs of this and the further claims below are in Appendix E.

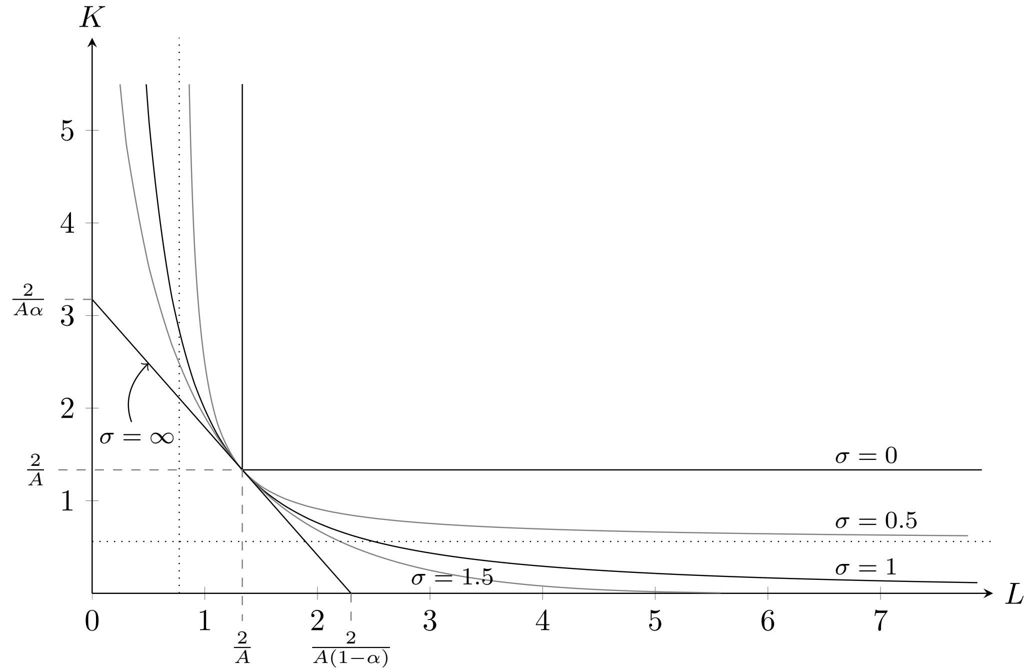


Figure 4.7: Isoquants for the CES function for alternative values of σ ($A = 1.5$, $\bar{Y} = 2$ and $\alpha = 0.42$).

Fig. 4.7 depicts isoquants for alternative CES production functions and their limiting cases. In the Cobb-Douglas case, $\sigma = 1$, the horizontal and vertical asymptotes of the isoquant coincide with the coordinate axes. When $\sigma < 1$, the horizontal and vertical asymptotes of the isoquant belong to the interior of the positive quadrant. This implies that both capital and labor are essential inputs. When $\sigma > 1$, the isoquant terminates in points *on* the coordinate axes. Then neither capital, nor labor are essential inputs. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for industrialized economies. The elasticity also differs across the production sectors. A recent thorough econometric study (Antràs, 2004) of U.S. data indicate the aggregate elasticity of substitution to be in the interval $(0.5, 1.0)$.

The CES production function in intensive form

Dividing through by L on both sides of (4.28), we obtain the CES production function in intensive form,

$$y \equiv \frac{Y}{L} = A(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}, \tag{4.32}$$

where $k \equiv K/L$. The marginal productivity of capital can be written

$$MPK = \frac{dy}{dk} = \alpha A \left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}} = \alpha A^{\beta}\left(\frac{y}{k}\right)^{1-\beta},$$

which of course equals $\partial Y/\partial K$ in (4.29). We see that the CES function violates either the lower or the upper Inada condition for MPK, depending on the sign of β. Indeed, when $\beta < 0$ (i.e., $\sigma < 1$), then for $k \to 0$ both y/k and dy/dk approach an upper bound equal to $A\alpha^{1/\beta} < \infty$, thus violating the lower Inada condition for MPK (see the right-hand panel of Fig. 2.3 of Chapter 2). It is also noteworthy that in this case, for $k \to \infty$, y approaches an upper bound equal to $A(1-\alpha)^{1/\beta} < \infty$. These features reflect the low degree of substitutability when $\beta < 0$.

When instead $\beta > 0$, there is a high degree of substitutability ($\sigma > 1$). Then, for $k \to \infty$ both y/k and $dy/dk \to A\alpha^{1/\beta} > 0$, thus violating the upper Inada condition for MPK (see right panel of Fig. 4.8). It is also noteworthy that for $k \to 0$, y approaches a positive lower bound equal to $A(1-\alpha)^{1/\beta} > 0$. Thus, in this case capital is not essential. At the same time $dy/dk \to \infty$ for $k \to 0$ (so the lower Inada condition for the marginal productivity of capital holds). Details are in Appendix E.

The marginal productivity of labor is

$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha)A(\alpha k^{\beta} + 1 - \alpha)^{(1-\beta)/\beta} \equiv w(k),$$

from (4.29).

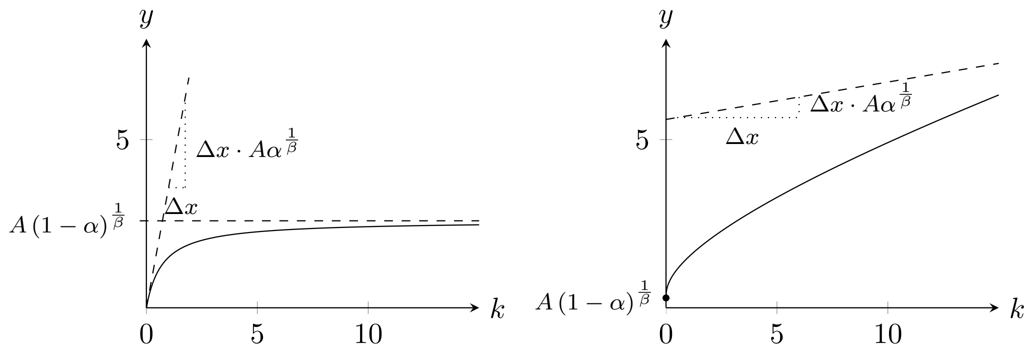
Since (4.28) is symmetric in K and L, we get a series of symmetric results by considering output per unit of capital as $x \equiv Y/K = A\left[\alpha + (1-\alpha)(L/K)^{\beta}\right]^{1/\beta}$. In total, therefore, when there is low substitutability ($\beta < 0$), for fixed input of either of the production factors, there is an upper bound for how much an unlimited input of the other production factor can increase output. And when there is high substitutability ($\beta > 0$), there is no such bound and an unlimited input of either production factor take output to infinity.

The Cobb-Douglas case, i.e., the limiting case for $\beta \to 0$, constitutes in several respects an intermediate case in that *all* four Inada conditions are satisfied and we have $y \to 0$ for $k \to 0$, and $y \to \infty$ for $k \to \infty$.

Generalizations

The CES production function considered above has CRS. By adding an elasticity of scale parameter, γ, we get the generalized form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \qquad \gamma > 0. \tag{4.33}$$



a) The case of $\sigma < 1$. a) The case of $\sigma > 1$.

Figure 4.8: The CES production function in intensive form.

In this form the CES function is homogeneous of degree γ. For $0 < \gamma < 1$, there are DRS, for $\gamma = 1$ CRS, and for $\gamma > 1$ IRS. If $\gamma \neq 1$, it may be convenient to consider $Q \equiv Y^{1/\gamma} = A^{1/\gamma}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}$ and $q \equiv Q/L$ $= A^{1/\gamma}(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$.

The elasticity of substitution between K and L is $\sigma = 1/(1-\beta)$ whatever the value of γ. So including the limiting cases as well as non-constant returns to scale in the "family" of production functions with constant elasticity of substitution, we have the simple classification displayed in Table 4.1.

Table 4.1 The family of production functions
with constant elasticity of substitution.

$\sigma = 0$	$0 < \sigma < 1$	$\sigma = 1$	$\sigma > 1$
Leontief	CES	Cobb-Douglas	CES

Note that only for $\gamma \leq 1$ is (4.33) a *neoclassical* production function. This is because, when $\gamma > 1$, the conditions $F_{KK} < 0$ and $F_{NN} < 0$ do not hold everywhere.

We may generalize further by assuming there are n inputs, in the amounts $X_1, X_2, ..., X_n$. Then the CES production function takes the form

$$Y = A\left[\alpha_1 {X_1}^{\beta} + \alpha_2 {X_2}^{\beta} + ...\alpha_n {X_n}^{\beta}\right]^{\frac{\gamma}{\beta}}, \ \alpha_i > 0 \text{ for all } i, \sum_i \alpha_i = 1, \gamma > 0. \tag{4.34}$$

In analogy with (4.25), for an n-factor production function the *partial elasticity of substitution* between factor i and factor j is defined as

$$\sigma_{ij} = \frac{MRS_{ij}}{X_i/X_j}\frac{d(X_i/X_j)}{dMRS_{ij}}_{|Y=\bar{Y}},$$

where it is understood that not only the output level but also all X_k, $k \neq i, j$, are kept constant. Note that $\sigma_{ji} = \sigma_{ij}$. In the CES case considered in (4.34), all the partial elasticities of substitution take the same value, $1/(1-\beta)$.

4.6 Concluding remarks

(Incomplete)

OLG gives theoretical insights concerning macroeconomic implications of life cycle behavior, allows heterogeneity, provides training in seeing the economy as consisting of a heterogeneous population where the *distribution* of agent characteristics matters for the aggregate outcome.

Farmer (1993), p. 125, notes that OLG models are difficult to apply and for this reason much empirical work in applied general equilibrium theory has regrettably instead taken the representative agent approach.

Outlook: Rational speculative bubbles in general equilibrium, cf. Chapter 27.

4.7 Literature notes and discussion

1. We introduced the assumption that at the macroeconomic level the "direction" of technological progress tends to be Harrod-neutral. Otherwise the model will not be consistent with Kaldor's stylized facts. The Harrod-neutrality of the "direction" of technological progress is in the present model just an exogenous feature. This raises the question whether there are *mechanisms* tending to generate Harrod-neutrality. Fortunately new growth theory provides clues as to the sources of the speed as well as the direction of technological change. A facet of this theory is that the direction of technological change is linked to the same economic forces as the speed, namely profit incentives. See Acemoglu (2003) and Jones (2006).

2. In Section 4.2 we claimed that from an empirical point of view, the adjustment towards a steady state can be from above as well as from below. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

As to the assessment of the role of uncertainty for the condition that dynamic efficiency is satisfied, in addition to Abel et al. (1989) other useful sources include Ball et al. (1998) and Blanchard and Weil (2001).

3. In the Diamond OLG model as well as in many other models, a steady state and a balanced growth path imply each other. Indeed, they are two

sides of the same process. There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technological change, see Groth et al., 2010). Therefore, it is recommendable always to maintain a terminological distinction between the two concepts.

4. Based on time-series econometrics, Attfield and Temple (2010) and others find support for the Kaldor "facts" for the US and UK and thereby for an evolution roughly obeying balanced growth in terms of *aggregate* variables. This does not rule out *structural change. A* changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009).

From here incomplete:

Demange and Laroque (1999, 2000) extend Diamond's OLG model to uncertain environments.

Keeping-up-with-the-Jones externalities. Do we work too much?

Blanchard, O., (2004) The Economic Future of Europe, J. Economic Perspectives, vol. 18 (4), 3-26.

Prescott, E. (2003), Why do Americans work so much more than Europeans? Federal Reserve Bank of Minneapolis Research Department Staff Report No. 321. I Ch. 5?

Chari, V. V., and P. J. Kehoe (2006), Modern macroeconomics in practice: How theory is shaping policy, J. of Economic Perspectives, vol. 20 (4), 3-28.

For expositions in depth of OLG modeling and dynamics in discrete time, see Azariadis (1993), de la Croix and Michel (2002), and Bewley (2007).

Dynamic inefficiency, see also Burmeister (1980).

Two-sector OLG: Galor (1992). Galor's book??

Bewley (2007).

Uzawa's theorem: Uzawa (1961), Schlicht (2006).

The way the intuition behind the Uzawa theorem was presented in Section 4.1 draws upon Jones and Scrimgeour (2008).

La Grandville's normalization of the CES function.

For more general and flexible production functions applied in econometric work, see, e.g., Nadiri (1982).

Other aspects of life cycle behavior: education. OLG where people live three periods.

4.8 Appendix

A. Growth arithmetic in discrete time

Let $t = 0, \pm 1, \pm 2, \ldots$, and consider the variables z_t, x_t, and y_t, assumed positive for all t. Define $\Delta z_t = z_t - z_{t-1}$ and Δx_t and Δy_t similarly. These Δ's need not be positive. The *growth rate* of x_t from period $t-1$ to period t is defined as $g_x(t) \equiv \Delta x_t / x_{t-1} \equiv x_t / x_{t-1} - 1$. The *growth factor* for x_t from period $t-1$ to period t is defined as $1 + g_x(t) = x_t / x_{t-1}$.

PRODUCT RULE If $z_t = x_t y_t$, then $1 + g_z(t) = (1 + g_x(t))(1 + g_y(t))$ and $g_z(t) \approx g_x(t) + g_y(t)$, when $g_x(t)$ and $g_y(t)$ are "small".

Proof. By definition, $z_t = x_t y_t$, which implies $z_{t-1} + \Delta z_t = (x_{t-1} + \Delta x_t)(y_{t-1} + \Delta y_t)$. Dividing by $z_{t-1} = x_{t-1} y_{t-1}$ gives $1 + \Delta z_t / z_{t-1} = (1 + \Delta x_t / x_{t-1})(1 + \Delta y_t / y_{t-1})$ as claimed. By carrying out the multiplication on the right-hand side of this equation, we get $1 + \Delta z_t / z_{t-1} = 1 + \Delta x_t / x_{t-1} + \Delta y_t / y_{t-1} + (\Delta x_t / x_{t-1})(\Delta y_t / y_{t-1}) \approx 1 + \Delta x_t / x_{t-1} + \Delta y_t / y_{t-1}$ when $\Delta x_t / x_{t-1}$ and $\Delta y_t / y_{t-1}$ are "small". Subtracting 1 on both sides gives the stated approximation. □

So the product of two positive variables will grow at a rate approximately equal to the sum of the growth rates of the two variables.

FRACTION RULE If $z_t = \frac{x_t}{y_t}$, then $1 + g_z(t) = \frac{1+g_x(t)}{1+g_y(t)}$ and $g_z(t) \approx g_x(t) - g_y(t)$, when $g_x(t)$ and $g_y(t)$ are "small".

Proof. By interchanging z and x in Product Rule and rearranging, we get $1 + \Delta z_t / z_{t-1} = \frac{1+\Delta x_t / x_{t-1}}{1+\Delta y_t / y_{t-1}}$, as stated in the first part of the claim. Subtracting 1 on both sides gives $\Delta z_t / z_{t-1} = \frac{\Delta x_t / x_{t-1} - \Delta y_t / y_{t-1}}{1+\Delta y_t / y_{t-1}} \approx \Delta x_t / x_{t-1} - \Delta y_t / y_{t-1}$, when $\Delta x_t / x_{t-1}$ and $\Delta y_t / y_{t-1}$ are "small". This proves the stated approximation. □

So the ratio between two positive variables will grow at a rate approximately equal to the excess of the growth rate of the numerator over that of the denominator. An implication of the first part of Claim 2 is that:

> the ratio between two positive variables is constant if and only if the variables have the same growth rate (not necessarily constant or positive).

POWER FUNCTION RULE If $z_t = x_t^\alpha$, then $1 + g_z(t) = (1 + g_x(t))^\alpha$.

Proof. $1 + g_z(t) \equiv z_{t+1}/z_t = (x_{t+1}/x_t)^\alpha \equiv (1 + g_x(t))^\alpha$. □

Given a time series $x_0, x_1, ..., x_n$ with a more or less monotonous trend, by the *average growth rate* per period (with discrete compounding) is normally meant a g which satisfies $x_n = x_0(1+g)^n$. The solution for g is $g = (x_n/x_0)^{1/n} - 1$.

Finally, the following approximation may be useful if used with caution:

THE GROWTH FACTOR With n denoting a positive integer above 1 but "not too large", the growth factor $(1+g)^n$ can be approximated by $1+ng$ when g is "small". For $g \neq 0$, the approximation error is larger the larger is n.

Proof. (i) We prove the claim by induction. Suppose the claim holds for a fixed $n \geq 2$, i.e., $(1+g)^n \approx 1+ng$ for g "small". Then $(1+g)^{n+1} = (1+g)^n(1+g) \approx (1+ng)(1+g) = 1+ng+g+ng^2 \approx 1+(n+1)g$ for g "small". So the claim holds also for $n+1$. Since $(1+g)^2 = 1+2g+g^2 \approx 1+2g$, for g "small", the claim does indeed hold for $n = 2$. (ii) □

B. Proof of Uzawa's theorem

For convenience we restate the theorem here:

PROPOSITION 2 *(Uzawa's balanced growth theorem).* Let $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$, and satisfy the dynamic resource constraint (4.2), given the production function (4.3) and the labor force (4.4). Then:
(i) a *necessary* condition for this path to be a balanced growth path is that along the path it holds that

$$Y_t = \tilde{F}(K_t, T_t L_t, 0), \tag{*}$$

where $T_t = (1+g)^t$ with $1+g \equiv (1+g_Y)/(1+n)$ and g_Y being the constant growth rate of output along the balanced growth path;
(ii) for any $g \geq 0$ such that there is a $q > (1+g)(1+n)+\delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$ (i.e., at any t, hence also at $t = 0$, the production function $\tilde{F}$ in (4.3) allows an output-capital ratio equal to q), a *sufficient* condition for the existence of a balanced growth path with output-capital ratio q is that the technology can be written as in (4.5) with $T_t = (1+g)^t$.

Proof (i) Suppose the given path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ is a balanced growth path. By definition, g_K and g_Y are then constant, so that $K_t = K_0(1+g_K)^t$ and $Y_t = Y_0(1+g_Y)^t$. With $t = 0$ in (4.3) we then have

$$Y_t(1+g_Y)^{-t} = Y_0 = \tilde{F}(K_0, L_0; 0) = \tilde{F}(K_t(1+g_K)^{-t}, L_t(1+n)^{-t}; 0). \tag{4.35}$$

In view of the assumption that $S_t \equiv Y_t - C_t > 0$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (4.35) then implies

$$Y_t = \tilde{F}(K_t, (1+g_Y)^t(1+n)^{-t}L_t; 0).$$

We see that (*) holds for $T_t = (1+g)^t$ with $g \equiv [(1+g_Y)/(1+n)] - 1$.

(ii) Suppose (*) holds with $T_t = (1+g)^t$. Let $g \geq 0$ be given such that there is a $q > (1+g)(1+n) + \delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$. We claim that with $K_0 = kL_0$, $s \equiv [(1+g)(1+n) - (1-\delta)]/q$, and $S_t = sY_t$, (4.2), (4.4), and (*) imply $Y_t/K_t = q$ for all $t = 0, 1, 2, \ldots$. We use induction to show this.[14] First, by (*)

$$\frac{Y_0}{K_0} = \frac{\tilde{F}(K_0, L_0; 0)}{K_0} = \tilde{F}(1, k^{-1}; 0) = q,$$

where the second equality comes from CRS. Next, suppose that for some t, $Y_t/K_t = q$. From (4.2) and (*) we then get

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta)K_t = \left(s\frac{Y_t}{K_t} + 1 - \delta\right)K_t = (sq + 1 - \delta)K_t \\ &= (1+g)(1+n)K_t. \end{aligned} \tag{4.36}$$

With this K_{t+1}, (*) gives

$$\begin{aligned} Y_{t+1} &= \tilde{F}((1+g)(1+n)K_t, (1+g)^{t+1}L_t(1+n); 0) \\ &= (1+g)(1+n)\tilde{F}(K_t, (1+g)^t L_t; 0) = (1+g)(1+n)Y_t, \end{aligned} \tag{4.37}$$

where the second equality comes from CRS and the last from (*). Thus, $Y_{t+1}/K_{t+1} = Y_t/K_t$ and so $Y_{t+1}/K_{t+1} = q$ when $Y_t/K_t = q$. By induction, $Y_t/K_t = q$ for all $t = 0, 1, 2, \ldots$, as was to be shown. Finally, (4.36) and (4.37) show that g_K and g_Y are constant along the constructed path; since $C_t \equiv Y_t - S_t = (1-s)Y_t$, also g_C is constant. Hence, the constructed path is a balanced growth path. □

It is noteworthy that the proof of the sufficiency part of the theorem is *constructive*. It provides a method for constructing a hypothetical balanced growth path.

[14] *Induction* is the following principle: if we can show that (a) a certain property holds at $t+1$, if it holds at t; and (b) the property holds at $t = 0$, then it must hold for all $t = 0, 1, 2, \ldots$.

C. Homothetic utility functions

Generalities A set C in $\mathbb{R}^n$ is called a *cone* if $x \in C$ and $\lambda > 0$ implies $\lambda x \in C$. A function $f(\mathbf{x}) = f(x_1,\ldots,x_n)$ is *homothetic* in the cone C if for all $\mathbf{x}, \mathbf{y} \in C$ and all $\lambda > 0$, $f(\mathbf{x}) = f(\mathbf{y})$ implies $f(\lambda\mathbf{x}) = f(\lambda\mathbf{y})$.

Consider the continuous utility function $U(x_1, x_2)$, defined in $\mathbb{R}^2_+$. Suppose U is *homothetic* and that the consumption bundles (x_1, x_2) and (y_1, y_2) are on the same indifference curve, i.e., $U(x_1, x_2) = U(y_1, y_2)$. Then for any $\lambda > 0$ we have $U(\lambda x_1, \lambda x_2) = U(\lambda y_1, \lambda y_2)$ so that the bundles $(\lambda x_1, \lambda x_2)$ and $(\lambda y_1, \lambda y_2)$ are also on the same indifference curve.

For a continuous utility function $U(x_1, x_2)$, defined in $\mathbb{R}^2_+$ and increasing in each of its arguments (as is our life time utility function in the Diamond model), one can show that U is homothetic if and only if U can be written $U(x_1, x_2) \equiv F(f(x_1, x_2))$ where the function f is homogeneous of degree one and F is an increasing function. The "if" part is easily shown. Indeed, if $U(x_1, x_2) = U(y_1, y_2)$, then $F(f(x_1, x_2)) = F(f(y_1, y_2))$. Since F is increasing, this implies $f(x_1, x_2) = f(y_1, y_2)$. Because f is homogeneous of degree one, if $\lambda > 0$, then $f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2)$ and $f(\lambda y_1, \lambda y_2) = \lambda f(y_1, y_2)$ so that $U(\lambda x_1, \lambda x_2) = F(f(\lambda x_1, \lambda x_2)) = F(f(\lambda y_1, \lambda y_2)) = U(\lambda y_1, \lambda y_2)$, which shows that U is homothetic. As to the "only if" part, see Sydsaeter et al. (2002).

Using differentiability of our homothetic time utility function $U(x_1, x_2) \equiv F(f(x_1, x_2))$, we find the marginal rate of substitution of good 2 for good 1 to be

$$MRS = \frac{dx_2}{dx_1}_{|U=\bar{U}} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{F' f_1(x_1, x_2)}{F' f_2(x_1, x_2)} = \frac{f_1(1, \frac{x_2}{x_1})}{f_2(1, \frac{x_2}{x_1})}. \tag{4.38}$$

The last equality is due to Euler's theorem saying that when f is homogeneous of degree 1, then the first-order partial derivatives of f are homogeneous of degree 0. Now, (4.38) implies that for a given MRS, in optimum reflecting a given relative price of the two goods, the same consumption ratio, x_2/x_1, will be chosen whatever the budget. For a given relative price, a rising budget (rising wealth) will change the position of the budget line, but not its slope. So MRS will not change, which implies that the chosen pair (x_1, x_2) will move outward along a given ray in $\mathbb{R}^2_+$. Indeed, as the intercepts with the axes rise proportionately with the budget (the wealth), so will x_1 and x_2.

Proof that the utility function in (4.20) is homothetic In Section 4.2 we claimed that (4.20) is a homothetic utility function. This can be proved in the following way. There are two cases to consider. *Case 1:* $\theta > 0$, $\theta \neq 1$.

We rewrite (4.20) as

$$U(c_1, c_2) = \frac{1}{1-\theta}\left[(c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}\right]^{1-\theta} - \frac{1+\beta}{1-\theta},$$

where $\beta \equiv (1+\rho)^{-1}$. The function $x = g(c_1, c_2) \equiv (c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}$ is homogeneous of degree one and the function $G(x) \equiv (1/(1-\theta))x^{1-\theta} - (1+\beta)/(1-\theta)$ is an increasing function, given $\theta > 0$, $\theta \neq 1$. *Case 2:* $\theta = 1$. Here we start from $U(c_1, c_2) = \ln c_1 + \beta \ln c_2$. This can be written

$$U(c_1, c_2) = (1+\beta)\ln\left[(c_1 c_2^{\beta})^{1/(1+\beta)}\right],$$

where $x = g(c_1, c_2) = (c_1 c_2^{\beta})^{1/(1+\beta)}$ is homogeneous of degree one and $G(x) \equiv (1+\beta)\ln x$ is an increasing function. □

D. General formulas for the elasticity of factor substitution

We here prove (4.26) and 4.27. Given the neoclassical production function $F(K, L)$, the slope of the isoquant $F(K, L) = \bar{Y}$ at the point $(\bar{K}, \bar{L})$ is

$$\frac{dK}{dL}_{|Y=\bar{Y}} = -MRS = -\frac{F_L(\bar{K}, \bar{L})}{F_K(\bar{K}, \bar{L})}. \tag{4.39}$$

We consider this slope as a function of the value of $k \equiv K/L$ as we move along the isoquant. The derivative of this function is

$$\begin{aligned} -\frac{dMRS}{dk}_{|Y=\bar{Y}} &= -\frac{dMRS}{dL}_{|Y=\bar{Y}} \frac{dL}{dk}_{|Y=\bar{Y}} \\ &= -\frac{(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}}{F_K^3} \frac{dL}{dk}_{|Y=\bar{Y}} \end{aligned} \tag{4.40}$$

by (2.51) of Chapter 2. In view of $L \equiv K/k$ we have

$$\frac{dL}{dk}_{|Y=\bar{Y}} = \frac{k\frac{dK}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{k\frac{dK}{dL}_{|Y=\bar{Y}}\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{-kMRS\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2}.$$

From this we find

$$\frac{dL}{dk}_{|Y=\bar{Y}} = -\frac{K}{(k+MRS)k},$$

to be substituted into (4.40). Finally, we substitute the inverse of (4.40) together with (4.39) into the definition of the elasticity of factor substitution:

$$\begin{aligned}
\sigma(K,L) &\equiv \frac{MRS}{k}\frac{dk}{dMRS}_{|Y=\bar{Y}} \\
&= -\frac{F_L/F_K}{k}\frac{(k+F_L/F_K)k}{K}\frac{F_K^3}{[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}]} \\
&= -\frac{F_KF_L(F_KK+F_LL)}{KL\left[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}\right]},
\end{aligned}$$

which is the same as (4.26).

Under CRS, this reduces to

$$\begin{aligned}
\sigma(K,L) &= -\frac{F_KF_LF(K,L)}{KL\left[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}\right]} \quad \text{(from (2.52) with } h=1) \\
&= -\frac{F_KF_LF(K,L)}{KLF_{KL}\left[-(F_L)^2L/K-2F_KF_L-(F_K)^2K/L\right]} \quad \text{(from (2.58))} \\
&= \frac{F_KF_LF(K,L)}{F_{KL}(F_LL+F_KK)^2}=\frac{F_KF_L}{F_{KL}F(K,L)}, \quad \text{(using (2.52) with } h=1)
\end{aligned}$$

which proves the first part of (4.27). The second part is an implication of rewriting the formula in terms of the production in intensive form.

E. Properties of the CES production function

The generalized CES production function is

$$Y=A\left[\alpha K^{\beta}+(1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \tag{4.41}$$

where A, α, and β are parameters satisfying $A>0$, $0<\alpha<1$, and $\beta<1$, $\beta\neq 0, \gamma>0$. If $\gamma<1$, there is DRS, if $\gamma=1$, CRS, and if $\gamma>1$, IRS. The elasticity of substitution is always $\sigma=1/(1-\beta)$. Throughout below, k means K/L.

The limiting functional forms We claimed in the text that, for fixed $K>0$ and $L>0$, (4.41) implies:

$$\begin{aligned}
\lim_{\beta\to 0} Y &= A(K^{\alpha}L^{1-\alpha})^{\gamma}=AL^{\gamma}k^{\alpha\gamma}, && (*) \\
\lim_{\beta\to -\infty} Y &= A\min(K^{\gamma},L^{\gamma})=AL^{\gamma}\min(k^{\gamma},1). && (**)
\end{aligned}$$

Proof. Let $q \equiv Y/(AL^\gamma)$. Then $q = (\alpha k^\beta + 1 - \alpha)^{\gamma/\beta}$ so that

$$\ln q = \frac{\gamma \ln(\alpha k^\beta + 1 - \alpha)}{\beta} \equiv \frac{m(\beta)}{\beta}, \tag{4.42}$$

where

$$m'(\beta) = \frac{\gamma \alpha k^\beta \ln k}{\alpha k^\beta + 1 - \alpha} = \frac{\gamma \alpha \ln k}{\alpha + (1-\alpha)k^{-\beta}}. \tag{4.43}$$

Hence, by L'Hôpital's rule for "0/0",

$$\lim_{\beta \to 0} \ln q = \lim_{\beta \to 0} \frac{m'(\beta)}{1} = \gamma \alpha \ln k = \ln k^{\gamma\alpha},$$

so that $\lim_{\beta \to 0} q = k^{\gamma\alpha}$, which proves (*). As to (**), note that

$$\lim_{\beta \to -\infty} k^\beta = \lim_{\beta \to -\infty} \frac{1}{k^{-\beta}} \to \begin{cases} 0 \text{ for } k > 1, \\ 1 \text{ for } k = 1, \\ \infty \text{ for } k < 1. \end{cases}$$

Hence, by (4.42),

$$\lim_{\beta \to -\infty} \ln q = \begin{cases} 0 \text{ for } k \geq 1, \\ \lim_{\beta \to -\infty} \frac{m'(\beta)}{1} = \gamma \ln k = \ln k^\gamma \text{ for } k < 1, \end{cases}$$

where the result for $k < 1$ is based on L'Hôpital's rule for "∞/-∞". Consequently,

$$\lim_{\beta \to -\infty} q = \begin{cases} 1 \text{ for } k \geq 1, \\ k^\gamma \text{ for } k < 1, \end{cases}$$

which proves (**). □

Properties of the isoquants of the CES function The absolute value of the slope of an isoquant for (4.41) in (L, K) space is

$$MRS_{KL} = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha} k^{1-\beta} \to \begin{cases} 0 \text{ for } k \to 0, \\ \infty \text{ for } k \to \infty. \end{cases} \tag{*}$$

This holds whether $\beta < 0$ or $0 < \beta < 1$.

Concerning the asymptotes and terminal points, if any, of the isoquant $Y = \bar{Y}$ we have from (4.41) $\bar{Y}^{\beta/\gamma} = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]$. Hence,

$$\begin{aligned} K &= \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A\alpha} - \frac{1-\alpha}{\alpha} L^\beta \right)^{\frac{1}{\beta}}, \\ L &= \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A(1-\alpha)} - \frac{\alpha}{1-\alpha} K^\beta \right)^{\frac{1}{\beta}}. \end{aligned}$$

From these two equations follows, when $\beta < 0$ (i.e., $0 < \sigma < 1$), that

$$\begin{aligned} K &\rightarrow (A\alpha)^{-\frac{1}{\beta}}\bar{Y}^{\frac{1}{\gamma}} \text{ for } L \rightarrow \infty, \\ L &\rightarrow [A(1-\alpha)]^{-\frac{1}{\beta}}\bar{Y}^{\frac{1}{\gamma}} \text{ for } K \rightarrow \infty. \end{aligned}$$

When instead $\beta > 0$ (i.e., $\sigma > 1$), the same limiting formulas obtain for $L \rightarrow 0$ and $K \rightarrow 0$, respectively.

Properties of the CES function on intensive form Given $\gamma = 1$, i.e., CRS, we have $y \equiv Y/L = A(\alpha k^\beta + 1 - \alpha)^{1/\beta}$ from (4.41). Then

$$\frac{dy}{dk} = A\frac{1}{\beta}(\alpha k^\beta + 1 - \alpha)^{\frac{1}{\beta}-1}\alpha\beta k^{\beta-1} = A\alpha\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}}.$$

Hence, when $\beta < 0$ (i.e., $0 < \sigma < 1$),

$$\begin{aligned} y &= \frac{A}{(ak^\beta + 1 - \alpha)^{-1/\beta}} \rightarrow \begin{cases} 0 \text{ for } k \rightarrow 0, \\ A(1-\alpha)^{1/\beta} \text{ for } k \rightarrow \infty. \end{cases} \\ \frac{dy}{dk} &= \frac{A\alpha}{[\alpha + (1-\alpha)k^{-\beta}]^{(\beta-1)/\beta}} \rightarrow \begin{cases} A\alpha^{1/\beta} \text{ for } k \rightarrow 0, \\ 0 \text{ for } k \rightarrow \infty. \end{cases} \end{aligned}$$

If instead $\beta > 0$ (i.e., $\sigma > 1$),

$$\begin{aligned} y &\rightarrow \begin{cases} A(1-\alpha)^{1/\beta} \text{ for } k \rightarrow 0, \\ \infty \text{ for } k \rightarrow \infty. \end{cases} \\ \frac{dy}{dk} &\rightarrow \begin{cases} \infty \text{ for } k \rightarrow 0, \\ A\alpha^{1/\beta} \text{ for } k \rightarrow \infty. \end{cases} \end{aligned}$$

The output-capital ratio is $y/k = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and has the same limiting values as dy/dk, when $\beta > 0$.

Continuity at the boundary of $\mathbb{R}^2_+$ When $0 < \beta < 1$, the right-hand side of (4.41) is defined and continuous also on the boundary of $\mathbb{R}^2_+$. Indeed, we get

$$Y = F(K, L) = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]^{\frac{\gamma}{\beta}} \rightarrow \begin{cases} A\alpha^{\frac{\gamma}{\beta}}K^\gamma \text{ for } L \rightarrow 0, \\ A(1-\alpha)^{\frac{\gamma}{\beta}}L^\gamma \text{ for } K \rightarrow 0. \end{cases}$$

When $\beta < 0$, however, the right-hand side is not defined on the boundary. We circumvent this problem by redefining the CES function in the following way when $\beta < 0$:

$$Y = F(K,L) = \begin{cases} A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}} \text{ when } K > 0 \text{ and } L > 0, \\ 0 \text{ when either } K \text{ or } L \text{ equals } 0. \end{cases} \tag{4.44}$$

We now show that continuity holds in the extended domain. When $K > 0$ and $L > 0$, we have

$$Y^{\frac{\beta}{\gamma}} = A^{\frac{\beta}{\gamma}}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right] \equiv A^{\frac{\beta}{\gamma}} G(K,L). \tag{4.45}$$

Let $\beta < 0$ and $(K,L) \to (0,0)$. Then, $G(K,L) \to \infty$, and so $Y^{\beta/\gamma} \to \infty$. Since $\beta/\gamma < 0$, this implies $Y \to 0 = F(0,0)$, where the equality follows from the definition in (4.44). Next, consider a fixed $L > 0$ and rewrite (4.45) as

$$\begin{aligned} Y^{\frac{1}{\gamma}} &= A^{\frac{1}{\gamma}}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} = A^{\frac{1}{\gamma}} L(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}} \\ &= \frac{A^{\frac{1}{\gamma}} L}{(ak^{\beta} + 1 - \alpha)^{-1/\beta}} \to 0 \text{ for } k \to 0, \end{aligned}$$

when $\beta < 0$. Since $1/\gamma > 0$, this implies $Y \to 0 = F(0,L)$, from (4.44). Finally, consider a fixed $K > 0$ and let $L/K \to 0$. Then, by an analogue argument we get $Y \to 0 = F(K,0)$, (4.44). So continuity is maintained in the extended domain.

4.9 Exercises

4.1 *(the aggregate saving rate in steady state)*

a) In a well-behaved Diamond OLG model let n be the rate of population growth and k^* the steady state capital-labor ratio (until further notice, we ignore technological progress). Derive a formula for the long-run aggregate net saving rate, S^N/Y, in terms of n and k^*. *Hint:* use that for a closed economy $S^N = K_{t+1} - K_t$.

b) In the Solow growth model without technological change a similar relation holds, but with a different interpretation of the causality. Explain.

c) Compare your result in a) with the formula for S^N/Y in steady state one gets in *any* model with the same CRS-production function and no technological change. Comment.

d) Assume that $n = 0$. What does the formula from a) tell you about the level of net aggregate savings in this case? Give the intuition behind the result in terms of the aggregate saving by any generation in two consecutive periods. One might think that people's rate of impatience (in Diamond's model the rate of time preference ρ) affect S^N/Y in steady state. Does it in this case? Why or why not?

e) Suppose there is Harrod-neutral technological progress at the constant rate $g > 0$. Derive a formula for the aggregate net saving rate in the long run in a well-behaved Diamond model in this case.

f) Answer d) with "from a)" replaced by "from e)". Comment.

g) Consider the statement: "In Diamond's OLG model any generation saves as much when young as it dissaves when old." True or false? Why?

4.2 (*increasing returns to scale and balanced growth)*