In this chapter we analyze models where the generally held current expectation of the future value of an endogenous variable has an influence on the current value of this variable. Under the hypothesis of rational expectations such models lead to a difference equation involving expectation terms. These are important for many topics in conomics, including the theory of asset price bubbles. Both at the formal level and in substance the framework departs from the simpler framework, considered in the preceding chapter, where only *past* expectations of current and future variables influence current variables.

After the introductory Section 26.1, the set of solutions to linear expectational difference equations, in the "normal case", where the expected future has only a "moderate influence", is studied in Section 26.2, based on the method of repeated forward substitution; the section concludes with a discussion of examples of asset price bubbles. The complementary case, where the expected future has a "large influence", is considered in Section 26.3. Finally, Section 26.4 concludes. Some of the more technical aspects are dealt with in the appendix.

26.1 Forward-looking rational expectations

In the preceding chapter we studied stochastic models where rational expectations entered in one of the following ways:

$$y_t = \alpha\ E(y_t|I_{t-1}) + \beta\ x_t,$$

or

$$y_t = \alpha\ E(y_t|I_{t-1}) + \gamma E(y_{t+1}|I_{t-1}) + \beta\ x_t, \qquad t = 0, 1, 2, ...$$

Here y_t is the endogenous stochastic variable, α, β, and γ are given constants, x_t is an exogenous stochastic variable, and $E(y_t|I_{t-1})$ and $E(y_{t+1}|I_{t-1})$ are mathematical expectations, given the model and conditional on the cumulative information available in period $t-1$. The distinctive feature is that the only way y_t here is affected by expectations is through agents' expectations formed in the *preceding* period. This type of models are called models with *past expectations* affecting current endogenous variables.

In many macroeconomic problems, however, there is an important role for agents' *current* expectations of *future* values of the endogenous variables. Even in the previous chapter, where small stochastic AD-AS models were considered, we had to drastically simplify by assuming money demand is independent of the interest rate. In reality money demand depends on the nominal interest rate. When this is taken into account, output in an AD-AS model depends on the expected real interest rate, which in turn depends on expected inflation between the current and the *next* period. This typically gives rise to an equation of the form

$$y_t = aE(y_{t+1}|I_t) + c\ x_t, \qquad t = 0, 1, 2, ..., \tag{26.1}$$

where $a \neq 0$ (otherwise the model is uninteresting) and $E(y_{t+1}|I_t)$ is the mathematical expectation of y_{t+1} conditional on information available at the end of period t.[1] Here the expectation of a future value of an endogenous variable has an impact on the current value of this variable. We call this a model with *current expectations of future variable values* or, for short, a model with *forward-looking expectations.* Thinking of an appropriate AD-AS model, it will have investment demand (and therefore also aggregate demand) depending on both the expected real interest rate and expected future aggregate demand, *two* forward-looking variables. Or we might think of equity shares. Their market value today will depend on the expectations, formed today, of the market value tomorrow.

The conditioning information, I_t, is assumed to contain knowledge of the realized values of y and x up to and including period t. The hypothesis is that the "generally held" subjective expectation conditional on I_t coincides with the objective conditional expectation based on the model (including knowledge of the exact values of the parameters a and c and knowledge of the stochastic process which x_t follows). As we discussed in Chapter 25, the assumption of rational expectations should generally be seen as just a simplification which may under certain conditions lead to useful approximative conclusions. Assuming rational expectations implies that the results which emerge from the model cannot depend on *systematic* expectation errors from the economic agents' side.

For ease of exposition we will use the notation $E_t y_{t+1} \equiv E(y_{t+1}|I_t)$ and thus write

$$y_t = aE_t y_{t+1} + c\ x_t, \qquad t = 0, 1, 2, ... \tag{26.2}$$

A stochastic difference equation of this form is called a *linear expectational difference equation of first order* with constant coefficients a and c.[2] A *solution* of the equation is a stochastic process $\{y_t\}$ which satisfies (26.2), given the stochastic process followed by x_t. In many economic applications there is no *given* initial value, y_0. On the contrary, the interpretation is that y_t depends, for all t, on expectations about the future.[3] So y_t can be a *jump variable* that can immediately shift its value in response to the emergence of new information about the future

[1] We imagine that all agents have the same information, hence also the same rational expectation.

[2] Later we allow the coefficients a and c to be time-dependent.

[3] The reason we say "depends on" is that it would be inaccurate to say that y_t is *determined* (in a one-way-sense) by expectations about the future. Rather there is *mutual dependence.* In view of y_t being an element in the information I_t, the expectation of y_{t+1} in (26.2) may depend on y_t just as much as y_t depends on the expectation of y_{t+1}.

x's. For example, a share price may immediately jump to a new value when the accounts of the firm become publicly known (often even before, due to sudden rumors).

Owing to the lack of an initial condition for y_t, there can easily be infinitely many processes for y_t satisfying our expectational difference equation. We have an infinite forward-looking "regress", where a variable's value today depends on its expected value tomorrow, this value depending on the expected value the day after tomorrow and so on. Then usually there are infinitely many expected sequences which can be self-fulfilling in the sense that if only the agents expect a particular sequence, then the aggregate outcome of their behavior will be that the sequence is realized. It "bites its own tail" so to speak. Yet, when an equation like (26.2) is part of a larger model, there will often (but not always) be conditions that allow us to select *one* of the many solutions to (26.2) as the only *economically* relevant one. For example, an economy-wide transversality condition or another general equilibrium condition may rule out divergent solutions and leave a unique convergent solution as the final solution.

We assume $a \neq 0$, since otherwise (26.2) itself is already the unique solution. It turns out that the set of solutions to (26.2) takes a different form depending on whether $|a| < 1$ or $|a| > 1$:

The case $|a| < 1$. In general, there is a unique *fundamental solution* (to be defined below) and infinitely many explosive solutions ("bubble solutions").

The case $|a| > 1$. In general, there is no fundamental solution but infinitely many non-explosive solutions. (The case $|a| = 1$ resembles this.)

In the case $|a| < 1$, the expected future has modest influence on the present. Here we will concentrate on this case, since it is the case most frequently appearing in macroeconomic models with rational expectations.

26.2 Solutions when $|a| < 1$

Various solution methods are available. *Repeated forward substitution* is the most easily understood method.

26.2.1 Repeated forward substitution

Repeated forward substitution consists of the following steps. We first shift (26.2) one period ahead:

$$y_{t+1} = a\ E_{t+1} y_{t+2} + c\ x_{t+1}.$$

Then we take the conditional expectation on both sides to get

$$E_t y_{t+1} = a\ E_t(E_{t+1} y_{t+2}) + c\ E_t x_{t+1} = a\ E_t y_{t+2} + c\ E_t x_{t+1}, \tag{26.3}$$

where the second equality sign is due to the *law of iterated expectations*, which says that

$$E_t(E_{t+1} y_{t+2}) = E_t y_{t+2}. \tag{26.4}$$

see Box 1. Inserting (26.3) into (26.2) then gives

$$y_t = a^2 E_t y_{t+2} + ac\ E_t x_{t+1} + c\ x_t. \tag{26.5}$$

The procedure is repeated by forwarding (26.2) two periods ahead; then taking the conditional expectation and inserting into (26.5), we get

$$y_t = a^3 E_t y_{t+3} + a^2 c\ E_t x_{t+2} + ac\ E_t x_{t+1} + c\ x_t.$$

We continue in this way and the general form (for $n = 0, 1, 2, ...$) becomes

$$y_{t+n} = a\ E_{t+n}(y_{t+n+1}) + c\ x_{t+n},$$

$$E_t y_{t+n} = a\ E_t y_{t+n+1} + c\ E_t x_{t+n},$$

$$y_t = a^{n+1} E_t y_{t+n+1} + c x_t + c \sum_{i=1}^{n} a^i E_t x_{t+i}. \tag{26.6}$$

Box 1. The law of iterated expectations

The method of repeated forward substitution applies the law of iterated expectations. This law says that $E_t(E_{t+1} y_{t+2}) = E_t y_{t+2}$, as in (26.4). The logic is the following. Events in period $t + 1$ are stochastic as seen from period t and so $E_{t+1} y_{t+2}$ (the expectation conditional on information including these events) is a stochastic variable. Then the law of iterated expectations says that the conditional expectation of this stochastic variable as seen from period t is the same as the conditional expectation of y_{t+2} itself as seen from period t. So, given that expectations are rational, then an earlier expectation of a later expectation of y is just the earlier expectation of y. Put differently: my best forecast today of how I am going to forecast tomorrow a share price the day after tomorrow, will be the same as my best forecast today of the share price the day after tomorrow. If beforehand we have good reasons to expect that we will revise our expectations upward, say, when next period's additional information arrives, the original expectation would be biased, hence not rational.[4]

[4] A detailed account of the law of iterated expectations is given in Appendix B of Chapter 25.

26.2.2 The fundamental solution

PROPOSITION 1 Consider the expectational difference equation (26.2), where $a \neq 0$. If

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \text{ exists,} \tag{26.7}$$

then

$$y_t = c\sum_{i=0}^{\infty} a^i E_t x_{t+i} = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i} \equiv y_t^*, \qquad t = 0, 1, 2, ..., \tag{26.8}$$

is a solution to the equation.

Proof Assume (26.7). Then the formula (26.8) is meaningful. In view of (26.6), it satisfies (26.2) if and only if $\lim_{n\to\infty} a^{n+1} E_t y_{t+n+1} = 0$. Hence, it is enough to show that the process (26.8) satisfies this latter condition.

In (26.8), replace t by $t+n+1$ to get $y_{t+n+1} = c\sum_{i=0}^{\infty} a^i E_{t+n+1} x_{t+n+1+i}$. Using the law of iterated expectations, this yields

$$\begin{aligned} E_t y_{t+n+1} &= c\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} \qquad \text{so that} \\ a^{n+1} E_t y_{t+n+1} &= c\, a^{n+1} \sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} = c \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}. \end{aligned}$$

It remains to show that $\lim_{n\to\infty} \sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = 0$. From the identity

$$\sum_{j=1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{n} a^j E_t x_{t+j} + \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}$$

follows

$$\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{n} a^j E_t x_{t+j}.$$

Letting $n \to \infty$, this gives

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{\infty} a^j E_t x_{t+j} = 0,$$

which was to be proved. □

The solution (26.8) is called the *fundamental solution* of (26.2), and we mark fundamental solutions by an asterisk *. In the present case, the fundamental solution is (for $c \neq 0$) defined only when the condition (26.7) holds. In general this condition requires that $|a| < 1$. In addition, (26.7) requires that the absolute value of the expectation of the exogenous variable does not increase "too fast". More precisely, the requirement is that $|E_t x_{t+i}|$, when $i \to \infty$, has a growth factor less than $|a|^{-1}$. As an example, let $0 < a < 1$ and $g > 0$, and suppose that $E_t x_{t+i} > 0$ for $i = 0, 1, 2, ...$, and that $1 + g$ is an upper bound for the growth factor of $E_t x_{t+i}$. Then

$$E_t x_{t+i} \leq (1+g) E_t x_{t+i-1} \leq (1+g)^i E_t x_t = (1+g)^i x_t.$$

Multiplying by a^i, we get $a^i E_t x_{t+i} \leq a^i (1+g)^i x_t$. By summing from $i = 1$ to n,

$$\sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \sum_{i=1}^{n} [a(1+g)]^i .$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \lim_{n\to\infty} \sum_{i=1}^{n} [a(1+g)]^i = x_t \frac{a(1+g)}{1 - a(1+g)} < \infty,$$

if $1 + g < a^{-1}$, using the sum rule for an infinite geometric series.

As noted in the proof of Proposition 1, the fundamental solution, (26.8), has the property that

$$\lim_{n\to\infty} a^n E_t y_{t+n} = 0. \tag{26.9}$$

That is, the expected value of y is not "explosive": its absolute value has a growth factor less than $|a|^{-1}$. Given $|a| < 1$, the fundamental solution is the only solution of (26.2) with this property. Indeed, it is seen from (26.6) that whenever (26.9) holds, (26.8) must also hold. In Example 1 below, y_t is interpreted as the market price of a share and x_t as dividends. Then the fundamental solution gives the share price as the present value of the expected future flow of dividends.

EXAMPLE 1 *(the fundamental value of an equity share)* Consider arbitrage between shares of stock and a risk-free asset paying the constant rate of return $r > 0$. Let period t be the current period. Let p_{t+i} be the market price (in real terms, say) of the share at the beginning of period $t + i$ and d_{t+i} the dividend paid out at the end of that period, $t+i$, $i = 0, 1, 2, ...$ As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, ...$ An investor who buys n_t shares at time t (the beginning of period t) thus invests $V_t \equiv p_t n_t$ units of account at time t. At the end of period t the gross return comes out as the known dividend $d_t n_t$ plus the sales value, $p_{t+1} n_t$, of the shares at the beginning of next period. We here follow the same dating convention as elsewhere in this book which. As mentioned before, this dating of the variables is unlike standard *accounting* and *finance* notation in discrete time, where V_t would be the end-of-period-t market value of the stock of shares that begins to yield dividends in period $t + 1$.[5]

Suppose investors have rational expectations and care only about expected return. Then the no-arbitrage condition reads

$$d_t + E_t p_{t+1} - p_t = p_t r. \tag{26.10}$$

The expected return on the share appears on the left-hand side, and the risk-free return on the right-hand side. For $p_t > 0$, the condition can also be expressed as the requirement that the two *rates* of return, $(d_t + E_t p_{t+1} - p_t)/p_t$ and r, should be the same. Of particular interest is that the condition can be written

$$p_t = \frac{1}{1+r} E_t p_{t+1} + \frac{1}{1+r} d_t, \tag{26.11}$$

which is of the same form as (26.2) with $a = c = 1/(1+r) \in (0,1)$. Assuming dividends do not grow "too fast", we find the fundamental solution, denoted p_t^*, as

$$p_t^* = \sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} E_t d_{t+i} = E_t \left(\sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} d_{t+i} \right). \tag{26.12}$$

The fundamental solution thus equals the mathematical expectation, conditional on all information available at time t, of the present value of actual subsequent dividends.

If the dividend process is $d_{t+1} = d_t + \varepsilon_{t+1}$, where ε_{t+1} is white noise, then the dividend process is known as a *random walk* and $E_t d_{t+i} = d_t$ for $i = 1, 2, ...$. Thus $p_t^* = d_t/r$, by the sum rule for an infinite geometric series. In this case the fundamental value is thus itself a random walk. More generally, the dividend process could be a *martingale,* that is, a sequence of stochastic variables with the property that the expected value next period exists and equals the current actual value, i.e., $E_t d_{t+1} = d_t$. In contrast to a random walk, in a martingale

[5] Whereas our p_t stands for the (real) value of a share of *stock* bought at *the beginning* of period t, throughout this book we use P_t to denote the nominal price per unit of consumption (flow) in period t, but paid for at the *end* of the period. At the beginning of period t, after the uncertainty pertaining to period t has been resolved and available information thereby been updated, the consumer-investor decides the assets and the debt to hold through the period and the consumption flow for the period. But only the investment expence, say p_t, is disbursed immediately.

It is convenient to think of the course of actions such that receipt of the previous period's dividend, d_{t-1}, and payment for that period's consumption, at the price P_{t-1}, occur right before period t begins and the new information arrives. Indeed, the resolution of uncertainty at discrete points in time motivates a *distinction* between "end of" period $t-1$ and "beginning of" period t, where the new information has just arrived.

$\varepsilon_{t+1} \equiv d_{t+1} - d_t$ need not be white noise; it is enough that $E_t\varepsilon_{t+1} = 0$.[6] Given the constant required return r, we still have $p_t^* = d_t/r$. So the fundamental value itself is in this case a martingale. □

In finance theory the present value of the expected future flow of dividends on an equity share is referred to as the *fundamental value* of the share. It is by analogy with this that the general designation *fundamental solution* has been introduced for solutions of the form (26.8).

We could also think of real assets. Thus p_t could be the market price of a house rented out and d_t the rent. Or p_t could be the market price of an oil well and d_t the revenue (net of extraction costs) from the extracted oil in period t. Broadly interpreted, the d's in the formula (26.12) represent the "fundamentals". Depending on the particular asset, "fundamentals" may be the dividends from a financial asset, the rents from owning a house or land, the services rendered by a car, etc.; sometimes also factors behind these elements (technology, market conditions etc.) are subsumed under the heading "fundamentals".

26.2.3 Bubble solutions

Other than the fundamental solution, the expectational difference equation (26.2) has infinitely many *explosive solutions*. In view of $|a| < 1$, these are characterized by violating the condition (26.9). That is, they are solutions whose expected value explodes over time.

It is convenient to first consider the *homogenous* expectation equation associated with (26.2). This is defined as the equation emerging by setting $c = 0$ in (26.2):

$$y_t = aE_t y_{t+1}, \qquad t = 0, 1, 2, \ldots. \tag{26.13}$$

Every stochastic process $\{b_t\}$ of the form

$$b_{t+1} = a^{-1}b_t + u_{t+1}, \qquad \text{where } E_t u_{t+1} = 0, \tag{26.14}$$

has the property that

$$b_t = aE_t b_{t+1}, \tag{26.15}$$

and is thus a solution to (26.13). The "disturbance" u_{t+1} represents "new information" which may be related to movements in "fundamentals", x_{t+1}. But it does not have to. In fact, u_{t+1} may be related to conditions that *per se* have no economic relevance whatsoever (see Section 26.2.6 below).

For ease of notation, from now on we just write b_t even if we think of the whole process $\{b_t\}$ rather than the value taken by b in the specific period t. The

[6] A random walk is thus a special case of a martingale.

meaning should be clear from the context. A solution to (26.13) is referred to as a *homogenous solution* associated with (26.2). Let b_t be a given homogenous solution and let K be an arbitrary constant. Then $B_t = Kb_t$ is also a homogenous solution (try it out for yourself). Conversely, any homogenous solution b_t associated with (26.2) can be written in the form (26.14). To see this, let b_t be a given homogenous solution, that is, $b_t = aE_t b_{t+1}$. Let $u_{t+1} = b_{t+1} - E_t b_{t+1}$. Then

$$b_{t+1} = E_t b_{t+1} + u_{t+1} = a^{-1} b_t + u_{t+1},$$

where $E_t u_{t+1} = E_t b_{t+1} - E_t b_{t+1} = 0$. Thus, b_t is of the form (26.14).

For convenience we here repeat our original expectational difference equation (26.2) and name it (*):

$$y_t = aE_t y_{t+1} + c\ x_t, \ldots t = 0, 1, 2, \ldots, \quad a \neq 0. \tag{*}$$

PROPOSITION 2 Consider the expectational difference equation (*), where $a \neq 0$. Let $\tilde{y}_t$ be a particular solution to the equation. Then:
(i) every stochastic process of the form

$$y_t = \tilde{y}_t + b_t, \tag{26.16}$$

where b_t satisfies (26.14), is a solution to (*);
(ii) every solution to (*) can be written in the form (26.16) with b_t being an appropriately chosen homogenous solution associated with (*).

Proof. Let some particular solution $\tilde{y}_t$ be given. (i) Consider $y_t = \tilde{y}_t + b_t$, where b_t satisfies (26.14). Since $\tilde{y}_t$ satisfies (*), we have $y_t = a\ E_t \tilde{y}_{t+1} + c\ x_t + b_t$. Consequently, by (26.13),

$$y_t = a\ E_t \tilde{y}_{t+1} + c\ x_t + a\ E_t b_{t+1} = a\ E_t(\tilde{y}_{t+1} + b_{t+1}) + c\ x_t = a\ E_t y_{t+1} + c\ x_t,$$

saying that (26.16) satisfies (*). (ii) Let Y_t be an arbitrary solution to (*). Define $b_t = Y_t - \tilde{y}_t$. Then we have

$$\begin{aligned} b_t &= Y_t - \tilde{y}_t = aE_t Y_{t+1} + cx_t - (aE_t \tilde{y}_{t+1} + cx_t) \\ &= aE_t(Y_{t+1} - \tilde{y}_{t+1}) = aE_t b_{t+1}, \end{aligned}$$

where the second equality follows from the fact that both Y_t and $\tilde{y}_t$ are solutions to (*). This shows that b_t is a solution to the homogenous equation (26.13) associated with (*). Since $Y_t = \tilde{y}_t + b_t$, the proposition is hereby proved. □

Proposition 2 holds for any $a \neq 0$. In case the fundamental solution (26.8) exists and $|a| < 1$, it is convenient to choose this solution as the particular solution in (26.16). Thus, referring to the right-hand side of (26.8) as y_t^*, we can use the particular form,

$$y_t = y_t^* + b_t. \tag{26.17}$$

When the component b_t is different from zero, the solution (26.17) is called a *bubble solution* and b_t is called the *bubble component.* In the typical economic interpretation the bubble component shows up only because it is expected to show up next period, cf. (26.15). The name bubble springs from the fact that the expected value of b_t, conditional on the information available in period t, explodes over time when $|a| < 1$. To see this, as an example, let $0 < a < 1$. Then, from (26.13), by repeated forward substitution we get

$$b_t = a\ E_t(aE_{t+1}b_{t+2}) = a^2 E_t b_{t+2} = ... = a^i E_t b_{t+i}, \qquad i = 1, 2, ...$$

It follows that $E_t b_{t+i} = a^{-i} b_t$, and from this follows that the bubble, for t going to infinity, is unbounded in expected value:

$$\lim_{i\to\infty} E_t b_{t+i} = \begin{cases} \infty, \text{ if } b_t > 0 \\ -\infty, \text{ if } b_t < 0 \end{cases}. \tag{26.18}$$

Indeed, the absolute value of $E_t b_{t+i}$ will for rising i grow *geometrically* towards infinity with a growth factor equal to $1/a > 1$.

Let us consider a special case of (*) that allows a simple graphical illustration of both the fundamental solution and some bubble solutions.

When x_t has constant mean

Suppose the stochastic process x_t (the "fundamentals") takes the form $x_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise. Then

$$y_t = a\ E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad 0 < |a| < 1. \tag{26.19}$$

The fundamental solution is

$$y_t^* = c\ x_t + c\sum_{i=1}^{\infty} a^i \bar{x} = c\bar{x} + c\varepsilon_t + c\frac{a\bar{x}}{1-a} = \frac{c\bar{x}}{1-a} + c\varepsilon_t.$$

Referring to (i) of Proposition 2,

$$y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t + b_t \tag{26.20}$$

is thus also a solution of (26.19) if b_t is of the form (26.14).

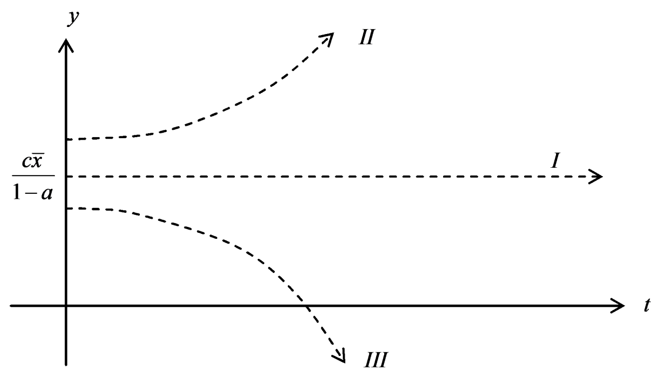

Figure 26.1: Deterministic bubbles (the case $0 < a < 1$, $c > 0$, and $x_t = \bar{x}$).

It may be instructive to consider the case where all stochastic features are eliminated. So we assume $u_t \equiv \varepsilon_t \equiv 0$. Then we have a model with perfect foresight; the solution (26.20) simplifies to

$$y_t = \frac{c\bar{x}}{1-a} + b_0 a^{-t}, \tag{26.21}$$

where we have used repeated *backward* substitution in (26.14). By setting $t = 0$ we see that $y_0 - \frac{c\bar{x}}{1-a} = b_0$. Inserting this into (26.21) gives

$$y_t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})a^{-t}. \tag{26.22}$$

In Fig. 26.1 we have drawn three trajectories for the case $0 < a < 1$, $c > 0$. Trajectory I has $y_0 = c\bar{x}/(1-a)$ and represents the fundamental solution. Trajectory II, with $y_0 > c\bar{x}/(1-a)$, and trajectory III, with $y_0 < c\bar{x}/(1-a)$, are bubble solutions. Since we have imposed no boundary condition apriori, one y_0 is as good as any other. The interpretation is that there are infinitely many trajectories with the property that if only the economic agents expect the economy will follow that particular trajectory, the aggregate outcome of their behavior will be that this trajectory is realized. This is the potential indeterminacy arising when y_t is not a predetermined variable. However, as alluded to above, in a complete economic model there will often be restrictions on the endogenous variable(s) not visible in the basic expectational difference equation(s), here (26.19). It may be that the economic meaning of y_t precludes negative values (a share certificate would be an example). In that case no-one can rationally expect a path such as III in Fig. 26.1. Or perhaps, for some reason, there is an upper bound on y_t (think of the full-employment ceiling for output in a situation where the "natural" growth factor for output is smaller than a^{-1}). Then no one can rationally expect a trajectory like II in the figure.

To sum up: in order for a solution of a first-order linear expectational difference equation with constant coefficient a, where $|a| < 1$, to differ from the fundamental solution, the solution must have the form (26.17) where b_t has the form described in (26.14). This provides a clue as to what asset price bubbles might look like.

Asset price bubbles

A stylized fact of stock markets is that stock price indices are quite volatile on a month-to-month, a year-to-year, and, not least, a decade-to-decade scale, cf. Fig. 26.2. There are different views about how these swings should be understood. According to the *Efficient Market Hypothesis* the swings just reflect unpredictable changes in the "fundamentals", that is, arrival of new information relevant for the present value of rationally expected future dividends. This is for instance the view of Nobel laureate Eugene Fama (1970, 2003) from University of Chicago.

In contrast, Nobel laureate Robert Shiller (1981, 2003, 2005) from Yale University, and others, have pointed to the phenomenon of *excess volatility.* The view is that asset prices tend to fluctuate more than can be rationalized by shifts in information about fundamentals (present values of dividends). Although in no way a verification, graphs like those in Fig. 26.2 and Fig. 26.3 are suggestive. Fig. 26.2 shows the monthly real Standard and Poors (S&P) composite stock prices and real S&P composite earnings for the period 1871-2008. The unusually large increase in real stock prices since the mid-90's, which ended with the collapse in 2000, is known as the "dot-com bubble". Fig. 26.3 shows, on a monthly basis, the ratio of real S&P stock prices to an average of the previous ten years' real S&P earnings along with the long-term real interest rate. It is seen that this ratio reached an all-time high in 2000, by many observers considered as "the year the dot-com bubble burst".

Shiller's interpretation of the large stock market swings is that they are due to fads, herding, and shifts in fashions and "animal spirits" (the latter being a notion from Keynes).

A third possible source of large stock market swings was pointed out by Blanchard (1979) and Blanchard and Watson (1982). They argued that bubble phenomena need not be due to irrational behavior and non-rational expectations. This lead to the theory of *rational bubbles* – the idea that excess volatility can be the result of speculative bubbles arising from self-fulfilling – and thus *rational* – expectations.

Consider an asset which yields either dividends or services in production or consumption in every period in the future. The fundamental value of the asset is, at the theoretical level, defined as the present value of the expected subsequent flow of dividends or services. An *asset price bubble* is then defined as a positive

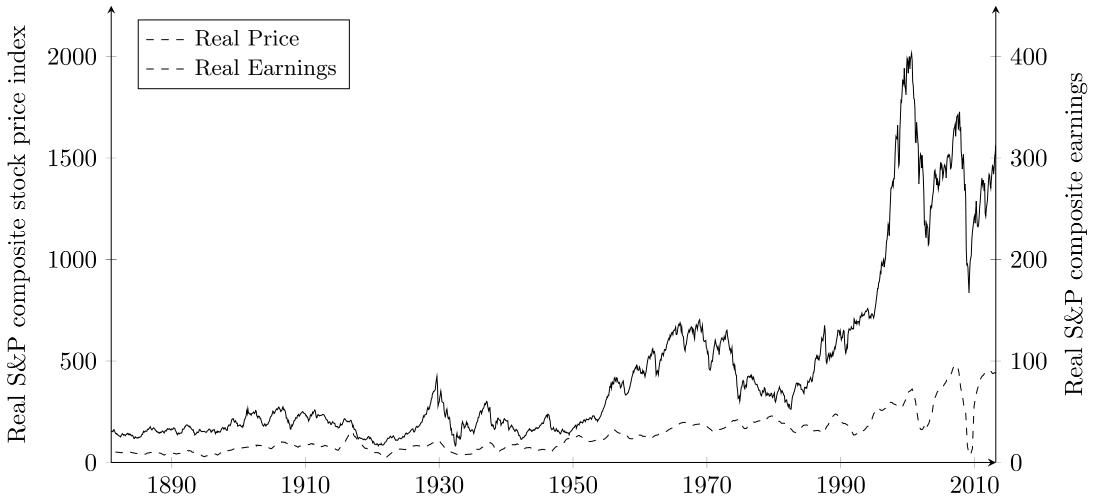

Figure 26.2: Monthly real S&P composite stock prices from January 1871 to March 2013 (left) and monthly real S&P composite earnings from January 1871 to March 2013 (right). Source: http://www.econ.yale.edu/ shiller/data.htm.

deviation, over some stretch of time, of the market price, p_t, of the asset from its fundamental value, p_t^*:

$$p_t = p_t^* + b_t. \tag{26.23}$$

With a required rate of return, r, as in Example 1, an asset price bubble that emerges in a setting where the no-arbitrage condition,

$$d_t + p_{t,t+1}^e - p_t = p_t r, \tag{26.24}$$

holds under rational expectations, i.e., $p_{t,t+1}^e = E_t p_{t+1}$, is called a *rational bubble.* The bubble emerges because there is in the market a self-fulfilling belief that it will appreciate at a rate high enough to warrant the overcharge involved.

In real-world situations market participants observe only the market price, p_t. For shares of stock in a firm both the basic concept of a "true" conditional mathematical expectation and the idea of an objective decomposition of p_t into p_t^* and b_t may be questionable concepts because of the inherently unknown future. Owing to this kind of difficulties, the development of the theory of rational bubbles has resulted in different theory varieties. Until further notice, we here concentrate on the simplest – but unfortunately not most convincing – variety. That is, the variety where all market participants have rational expectations, share the same information, and are able to decipher p_t into p_t^* and b_t, given this information.

Let us consider some potential examples.

EXAMPLE 2 *(an ever-expanding rational bubble)* Consider again an equity share for which the no-arbitrage condition (26.24) holds under rational expectations. As

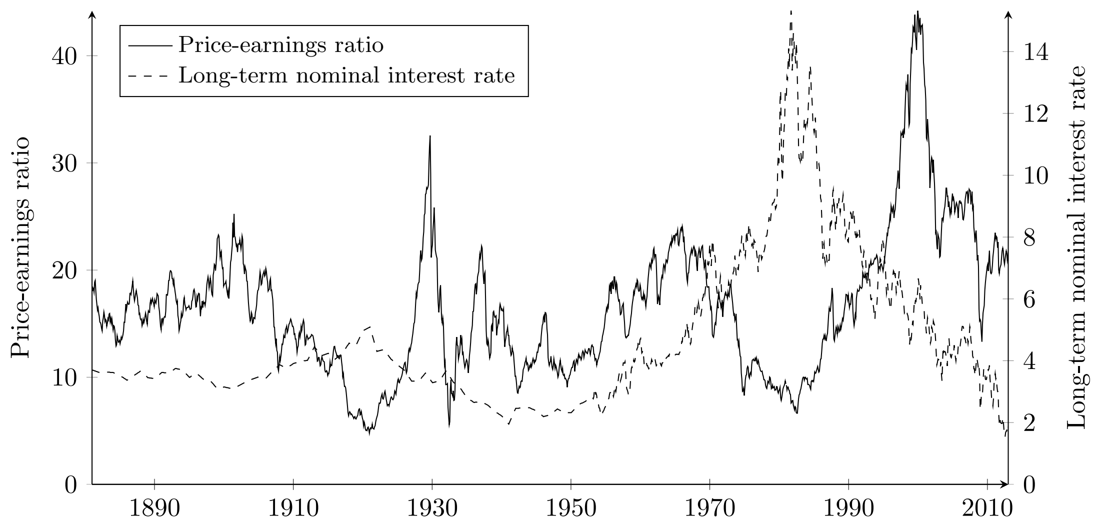

Figure 26.3: S&P price-earnings ratio and long-term nominal interest rate from January 1881 to March 2013. The earnings are calculated as a moving average of the S&P composite earnings data over the preceding ten years. The long-term nominal interest rate from before 1953 is the government bond yield from Homer (2005) and from 1953 it is the 10-year Treasury rate. Source: http://www.econ.yale.edu/ shiller/data.htm.

in Example 1, the implied expectational difference equation is $p_t = aE_t p_{t+1} + cd_t$, with $a = c = 1/(1+r) \in (0,1)$. Let the price of the share at time t be $p_t = p_t^* + b_t$, where p_t^* is the fundamental value and $b_t > 0$ a bubble component that follows the *deterministic* process, $b_{t+1} = (1+r)b_t$, $b_0 > 0$, so that $b_t = b_0(1+r)^t$. This is called a *deterministic rational bubble*. The sum $p_t^* + b_t$ will satisfy the no-arbitrage condition (26.24) exactly as much as p_t^* itself, because we just add something which equals the discounted value of itself one period later.

Agents may be ready to pay a price over and above the fundamental value (whether or not they know the "true" fundamental value) if they expect they can sell at a sufficiently higher price later; trading with such motivation is called *speculative behavior*. If generally held and lasting for some time, this expectation may be self-fulfilling. Note that (26.24) implies that the asset price ultimately grows at the rate r. Indeed, let $d_t = d_0(1+\gamma)^t$, $0 \leq \gamma < r$ (if $\gamma \geq r$, the asset price would be infinite). By the rule of the sum of an infinite geometric series, we then have $p_t^* = d_t/(r-\gamma)$, showing that the fundamental value grows at the rate γ. Consequently, $p_t/b_t = (p_t^* + b_t)/b_t = p_t^*/b_t + 1 \to 1$, as $\gamma < r$. It follows that the asset price in the long run grows at the same rate as the bubble, the rate r. □

We are not acquainted with *ever*-expanding incidents of that caliber in real world situations, however. A deterministic rational bubble thus appears implausible. Let us now consider an example of a *stochastic* rational bubble which sooner or later *bursts.*

EXAMPLE 3 *(a bursting bubble)* Once again we consider the no-arbitrage condition (26.24) where for simplicity we still assume the required rate of return, r, is a constant, though possibly including a risk premium. Following Blanchard (1979), we assume that the market price, p_t, of the share contains a stochastic bubble of the following form:

$$b_{t+1} = \varepsilon_{t+1} + \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases} \tag{26.25}$$

where $t = 0, 1, 2, \ldots$, ε_{t+1} is white noise, and $b_0 > 0$. In addition it seems natural to assume $q_t = f(t, p_t^*, b_t)$, $f_t \leq 0$, $f_{p^*} \geq 0$, $f_b \leq 0$. If $f_t < 0$, the probability that the bubble will persist at least one period ahead decreases as time proceeds. If $f_{p^*} > 0$, the probability that the bubble persists at least one period ahead is higher the greater the fundamental value has become. If $f_b < 0$, the probability that the bubble persists at least one period ahead is less, the greater the bubble has already become. In this way the probability of a crash becomes greater and greater as the share price comes further and further away from fundamentals. As a compensation, the longer time the bubble has lasted, the higher is the expected growth rate of the bubble in the absence of a collapse.

This bubble satisfies the criterion for a rational bubble. Indeed, (26.25) implies

$$E_t\ b_{t+1} = 0 + (\frac{1+r}{q_{t+1}} b_t) q_{t+1} + 0 \cdot (1 - q_{t+1}) = (1+r) b_t.$$

This is of the form (26.14) with $a^{-1} = 1 + r$, and the bubble is therefore a stochastic rational bubble. The stochastic component is $u_{t+1} = b_{t+1} - E_t b_{t+1}$ $= b_{t+1} - (1+r) b_t$ and has conditional expectation equal to zero. Although u_{t+1} must have zero conditional expectation, it need not be white noise (it can for instance have varying variance). $\square$

As Example 3 illustrates, a stochastic rational bubble does not have the implausible ever-expanding form of a deterministic rational bubble. The market participants understand that an eventual collapse is inevitable, but nobody knows when. The expected return may warrant staying in the market until the crash suddenly occurs

Nevertheless, there are many cases where even stochastic rational bubbles can be ruled out or at least be judged implausible. The next section reviews some arguments.

26.2.4 When rational bubbles in asset prices can or can not be ruled out

Consider the broad class of assets whose services are valued independently of the price.[7] Let p_t be the market price of the particular asset considered and let p_t satisfy the no-arbitrage equation (26.24) which we will here write in the form

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r. \tag{26.26}$$

We still assume that p_t^*, the fundamental value of the asset at time t, is known to the market participants. Even if the asset yields services rather than dividends, we think of p_t^* and d_t as the same for all agents. This is because a user who, in a given period, values the service flow of the asset relatively low can hire it out to the one who values it highest (the one with the highest willingness to pay).

Partial equilibrium arguments

The principle of reasoning to be used is called *backward induction:* If we know something about an asset price in the future, we can conclude something about the asset price today.

(a) Assets which can be freely disposed of ("free disposal") On such assets a *negative* rational bubble cannot exist. The logic can be illustrated on the basis of Example 2 above. We simplify by letting the dividend be a constant $d > 0$ and start by ignoring stochastic elements altogether. Then, from the formula (26.22) with $a = (1 + r)^{-1}$, we have

$$p_t - p^* = (p_0 - p^*)(1 + r)^t,$$

where $r > 0$ and $p^* = d/r$. Suppose there is a negative bubble in period 0, i.e., $b_0 \equiv p_0 - p^* < 0$. In period 1, since $1 + r > 1$, the bubble has become greater in absolute value. The downward movement of p_t continues and sooner or later p_t becomes negative. The intuition is that the low p_0 in period 0 implies a low return, $p_0 r$, on the right-hand side of (26.26), and so a *negative* capital gain ($p_{t+1} - p_t < 0$) is required for the no-arbitrage condition to be still satisfied.[8] Thereby $p_1 < p_0$, and so on. After some time, $p_t < 0$.

A negative price means that the "seller" has to *pay* to dispose of the object. In a market with self-interested rational agents nobody will do that if the object can

[7] This is in contrast to assets that serve as means of payment, cf. Chapter 17.4.

[8] Or, what amounts to the same: the low p_0 causes a high dividend-price ratio, and so a negative capital gain is needed for the rate of return to still equal r.

just be thrown away. An asset which can be freely disposed of (share certificates for instance) can therefore never have a negative price. We conclude that a *negative* rational bubble can not be consistent with rational expectations in this case. Adding stochastic elements to the setup only means that a negative rational bubble would imply that *in expected value* the share price becomes negative at some point in time, cf. (26.18). Again, given free disposal, rational expectations rule this out.

Hence, if we imagine that for a short moment we have $p_t < p_t^*$, then everyone will want to *buy* the asset and hold it because by own use or by hiring out a discounted value equal to p_t^* is obtained. There is thus excess demand until p_t has risen to p_t^*.

A less obvious point is that when a negative rational bubble can be ruled out, then, if at the first date of trading of the asset there were no positive bubble, neither can a positive bubble arise later on the same asset. Let us make this precise:

PROPOSITION 3 Assume free disposal of a given asset. Then, if a rational bubble in the asset price is present today, it must be positive and must have been present also yesterday and so on back to the first date of trading the asset. And if a rational bubble bursts, it will not restart later.

Proof As argued above, in view of free disposal, a negative rational bubble in the asset price can be ruled out. It follows that $b_t = p_t - p_t^* \geq 0$ for $t = 0, 1, 2, ...$, where $t = 0$ is the first date of trading the asset. We now show by contradiction that if, for an arbitrary $t = 1, 2, ...$, it holds that $b_t > 0$, then $b_{t-1} > 0$. Let $b_t > 0$. Then, if $b_{t-1} = 0$, we have $E_{t-1}b_t = E_{t-1}u_t = 0$ (from (26.14) with t replaced by $t - 1$). Since $b_t \geq 0$, this implies that $b_t = 0$ with probability *one* as seen from period $t - 1$. Ignoring zero probability events, this rules out $b_t > 0$ and so $b_{t-1} = 0$ implies a contradiction. Hence $b_{t-1} > 0$. Replacing t by $t - 1$ and so on backward in time, we end up with $b_0 > 0$. This reasoning also implies that if a bubble bursts in period t, it can not restart in period $t + 1$, nor, by extension, in any subsequent period. □

This proposition (due to Diba and Grossman, 1988) claims that a rational bubble in an asset price must have been there since trading of the asset began. Yet, at least if the asset is a share of stock in a production firm, such a conclusion is not without limitations. If the firm invents and introduces a new technology at some point in time, is a share in the firm then the same asset as before? In a judicial sense the firm is the same, but the asset, as defined by its statistical pay-off properties, will generally not be the same. Even if an earlier bubble has crashed, a later emergence of a new rational bubble on the firm's equity can not be ruled out, at least not on the basis of Proposition 3.

These ambiguities reflect a general difficulty involved in the concepts of rational expectations and rational bubbles when we are dealing with uncertainties about future developments of the economy. The market's evaluation of many assets of macroeconomic importance, not the least shares in firms, depends on vague beliefs about future preferences, technologies, and societal circumstances. Then the fundamental value of the asset can not be determined in any objective way. There is no well-defined probability distribution over the potential future outcomes. *Fundamental uncertainty*, also known as *Knightian uncertainty*, is present (see Box 26.1).

Box 26.1. Calculable uncertainty versus fundamental uncertainty

One form of uncertainty is *calculable uncertainty* which is present when there is a set of well-defined alternative outcomes to which can be associated an "objective" probability distribution (as in dice games or quantum mechanics). Another form is *fundamental uncertainty* which is present in situations where the full "range" of possible outcomes is not even known, hence cannot be endowed with a probability distribution ("it is not known what is unknown"). The latter form of uncertainty is also called *Knightian uncertainty*, so named after the University of Chicago economist Frank Knight who wrote the book *Risk, Uncertainty, and Profit* (Knight, 1921).

(b) Bonds with finite maturity The finite maturity ensures that the value of the bond is given at some finite future date. Therefore, if there were a positive bubble in the market price of the bond, no rational investor would buy just before that date. Anticipating this, no one would buy the date before, and so on. Consequently, nobody will buy in the first place. By this backward-induction argument follows that a positive bubble cannot get started. And since there also is "free disposal", negative rational bubbles are ruled out. So *all* rational bubbles can be precluded.

From now on we take as given that negative rational bubbles are ruled out in the cases we consider. So, the discussion is about whether *positive* rational asset price bubbles may exist or not.

(c) Assets whose supply is elastic Machines and buildings can be reproduced and have relatively stable costs of reproduction at least in the medium run. This precludes rational bubbles, since a potential buyer can avoid the overcharge on the existing machine or building by initiating production of a new one. Notice, however, that building sites with a specific amenity value and apartments in attractive quarters of a city are not easily reproducible. Therefore, rational bubbles on such assets are more difficult to rule out.

Returning to shares of stock in an established firm, an argument against a rational bubble may be that if there were a bubble, the firm would tend to exploit it by issuing more shares. But thereby market participants' mistrust is raised and may pull market evaluation back to the fundamental value. On the other hand, the firm might anticipate this adverse response from the market. So, following the interest of the initial shareholders, the firm chooses instead to "fool" the market by steady financing behavior as if no bubble were present. The initial shareholders calmly enjoy the rising value of their stock. It seems thus not obvious that this kind of argument can rule out rational bubbles on shares.

(d) Assets for which there exists a "backstop-technology" For some articles of trade there exists potential substitutes in elastic supply which will be demanded if the price of the article becomes sufficiently high. Such a substitute is called a "backstop-technology". For example oil and other fossil fuels will, when their prices become sufficiently high, be subject to intense competition from substitutes (renewable energy sources). This precludes an unbounded bubble process in the price of oil.

On account of the arguments (c) and (d), it seems more difficult to rule out rational bubbles when it comes to assets which are not reproducible or substitutable, let alone assets whose fundamentals are difficult to ascertain. For some assets the fundamentals are not easily ascertained. Examples are paintings of past great artists, rare stamps, diamonds, gold etc. Also new firms that introduce completely novel products are potential candidates. Think of the proliferation of radio broadcasting in the 1920s before the Wall Street crash in 1929 and of the internet in the 1990s before the dot-com bubble burst in 2000.

If we stay at the narrow definition of rational bubbles as requiring a well-defined fundamental, dramatic boom-bust events like these may not be considered results of *rational* bubbles. We may then think of a broader class of real-world bubbly phenomena driven by self-reinforcing expectations.

Adding general equilibrium arguments

The above considerations are of a partial equilibrium nature. On top of this, *general equilibrium* arguments can be put forward to further limit the possibility of rational bubbles. We will here consider two such general equilibrium arguments. We still consider assets whose services are valued independently of the price and which, as in (a) above, can be freely disposed of. As above, we may think of assets that yield services in consumption or production or in the form of a dividend stream. Let the particular asset considered have market price p_t and fundamental value, p_t^*, equal to the present value of the flow of services. We shall see that in an economy with a finite number of "neoclassical" households (to be defined below), rational bubbles can be ruled out in general equilibrium. We shall also see that this holds in an overlapping generations economy in general equilibrium *unless* the long-run growth rate in GNP exceeds the equilibrium interest rate.

(e) An economy with a finite number of infinitely-lived households
Assume that the economy consists of a finite number of infinitely-lived households − indexed $i = 1, 2, ..., N$. The households are "neoclassical" in the sense that they save only with a view to utility of future consumption.

At point (a) above we saw that under free disposal, $p_t < p_t^*$ can not be an equilibrium. But suppose there is a positive bubble, i.e., $p_t > p_t^*$. All owners of the bubbly asset who are users will in this case prefer to *sell* and then *rent.* This would imply excess supply and could thus not be an equilibrium. Hence, we turn to households that are not users, but speculators. Assume "short selling" is legal, that is, it is allowed to first rent the asset (for a contracted interval of time) and immediately sell it at p_t. This results in excess supply and so the asset price falls towards p_t^*. Within the contracted interval of time the speculators buy the asset back at the now lower price and return it to the original owners in accordance with the loan accord.[9] Consequently, $p_t > p_t^*$ can not be an equilibrium.

Even ruling out "short selling" (which is sometimes outright forbidden), we can rule out positive bubbles in the present setup with a finite number of households. To assume that owners who are not users would want to hold the bubbly asset forever as a permanent investment will contradict that these owners are "neoclassical". Indeed, their transversality condition would be violated because the value of their wealth would grow at a rate asymptotically equal to the rate of interest, cf. Example 2 above. This state of affairs could not be an equilibrium because it would offer an opportunity to increase consumption now without decreasing it later and without violating the No-Ponzi-Game condition.

We have to instead imagine that the "neoclassical" households who own the bubbly asset, hold it against future sale. This could on the face of it seem rational if there were some probability that the bubble would continue to grow fast enough to warrant holding it until the planned future selling is executed. Let t_i be the point in time where household i wishes to sell the asset and let

$$T = \max\left[t_1, t_2, ..., t_N\right].$$

Then nobody will plan to hold the asset after time T. The household speculator, i, having $t_i = T$ will thus not have anyone to sell to (other than people who will

[9] In brief, by initiating "short selling", the speculator brings herself in a position to gain by a fall in the asset price.

only pay p_T^*). Anticipating this, no-one would buy or hold the asset the period before, and so on.

The conclusion is that $p_t > p_t^*$ cannot be a rational expectations equilibrium in a setup with a finite number of "neoclassical" households, as in, for instance, the Ramsey model or the Barro dynasty model.

The circumstances are different in an overlapping generations model where *new* households − that is, new traders − enter the economy every period. What can we say about that case?

(f) An OLG economy with interest rate above the output growth rate In an overlapping generations (OLG) model with an infinite sequence of new decision makers, rational bubbles are under certain conditions theoretically possible. The argument is that with $N \to \infty$, T as defined above is not bounded. Although this unboundedness is a necessary condition for the possibility of rational bubbles, it is not sufficient, however.

To see why, let us return to the arbitrage examples 1, 2, and 3 where we have $a^{-1} = 1 + r$ so that a hypothetical rational bubble has the form $b_{t+1} = (1 + r)b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. In expected value the hypothetical bubble is growing at a rate equal to the interest rate, r. If at the same time, r is higher than the long-run output growth rate, g_Y, the value of the expanding bubbly asset would sooner or later be larger than GNP and aggregate saving would not suffice to back its continued growth. Agents with rational expectations anticipate this and so the bubble never gets started.

This point is valid when the interest rate in the OLG economy without bubbles is at least as large as the GNP growth rate − which is normally considered the realistic case. Yet, the opposite case, $r < g_Y$, known as dynamic inefficiency, *is* possible (although generally not considered likely). In that situation rational asset price bubbles are feasible in general equilibrium, as we shall see in Chapter 28.

The scope for the emergence of rational bubbles is increased when there are segmented financial markets, and externalities create a wedge between private and social returns on productive investment. Caballero et al. (2006) and Martin and Ventura (2012) show that the existence of financial frictions can make rational bubbles possible even when $r > g_Y$.

26.2.5 Time-dependent coefficients*

In the theory above we assumed that the coefficient a is constant. But the concepts can easily be extended to the case with a time-dependent. Consider the expectational difference equation

$$y_t = a_t E_t y_{t+1} + c_t x_t, \tag{26.27}$$

where $0 < |a_t| < 1$ for all t. We also allow the coefficient, c, to x_t to be time-dependent. This is less crucial, however, because $c_t x_t$ could always be replaced by $c\tilde{x}_t$, where $\tilde{x}_t$ is a new exogenous variable defined by $\tilde{x}_t \equiv c_t x_t / c$.

Repeated forward substitution in (26.27) and use of the law of iterated expectations give, in analogy with (26.6),

$$y_t = (\Pi_{j=0}^{n} a_{t+j}) E_t y_{t+n+1} + c_t x_t + \sum_{i=1}^{n} (\Pi_{j=0}^{i-1} a_{t+j}) c_{t+i} E_t x_{t+i}. \tag{26.28}$$

From now on, to simplify notation for the most used discount factor we define $\beta_{t,i} \equiv \Pi_{j=0}^{i-1} a_{t+j}$. In analogy with Proposition 1 one can show (see Appendix A) that:

> if $\lim_{n\to\infty} \sum_{i=1}^{n} (\Pi_{j=0}^{i-1} a_{t+j}) c_{t+i} E_t x_{t+i}$ exists, then (26.27) has a solution with the property $\lim_{n\to\infty} \left[(\Pi_{j=0}^{n} a_{t+j}) E_t y_{t+n+1}\right] = 0$, namely
>
> $$y_t^* = c_t x_t + \sum_{i=1}^{\infty} (\Pi_{j=0}^{i-1} a_{t+j}) \beta_{t,i} c_{t+i} E_t x_{t+i}. \tag{26.29}$$

This is the *fundamental solution* of (26.27).

In addition, (26.27) has infinitely many bubble solutions of the form $y_t = y_t^* + b_t$, where b_t satisfies $b_{t+1} = a_t^{-1} b_t + u_{t+1}$ with $E_t u_{t+1} = 0$.

EXAMPLE 4 *(time-dependent required rate of return)* We modify the no-arbitrage condition from Example 1 to

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r_t, \tag{26.30}$$

where r_t is the required rate of return. The corresponding expectational difference equation is $p_t = a_t E_t p_{t+1} + c_t d_t$, with $a_t = c_t = 1/(1 + r_t) \in (0, 1)$. Assuming dividends do not grow "too fast", we find the fundamental solution

$$p_t^* = \frac{1}{1 + r_t} d_t + \sum_{i=1}^{\infty} \frac{1}{\Pi_{j=0}^{i} (1 + r_{t+j})} E_t d_{t+i} = \sum_{i=0}^{\infty} \frac{1}{\Pi_{j=0}^{i} (1 + r_{t+j})} E_t d_{t+i}.$$

A bubble solution is of the form $p_t = p_t^* + b_t$, where b_t could be a bursting bubble like in Example 3 (replace r in (26.25) by r_t); if the probability of a crash is increasing with the size of the bubble, then also the required rate of return is likely to be increasing when agents are risk-averse. □

For now, we shall return to the simpler case with constant coefficients, a and c.

26.2.6 Three classes of bubble processes*

Consider again the expectational difference equation $y_t = aE_t y_{t+1} + cx_t$, where $|a| < 1$ and the exogenous stochastic variable x_t reflects the economic environment ("fundamentals"). As we saw, the defining characteristic of a rational bubble solution associated with this equation is: $b_{t+1} = a^{-1}b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. We classified bubble solutions according to their deterministic or stochastic nature. But bubbles may also be distinguished according to which variables in the economic system they are related to. This leads to the following taxonomy:

1. *Markovian bubbles.* A Markovian bubble is a bubble component that depends only on its own realization in the preceding period. That is, the probability distribution for b_{t+1} is a function only of the previous realization, b_t. A deterministic bubble, $b_{t+1} = a^{-1}b_t$, is an example; similarly, allowing for a time-dependent a,.$b_{t+1} = a(t)^{-1}b_t$ is another example. A stochastic example is the bursting bubble in Example 3 above.

2. *Intrinsic bubbles.* An intrinsic bubble is a bubble that depends on the stochastic variable x_t, which in turn reflects "fundamentals". As an example, consider the stochastic process

$$b_t = a^{-t}x_t, \tag{26.31}$$

where x_{t+1} is a martingale, i.e., $x_{t+1} = x_t + \varepsilon_{t+1}$ with $E_t\varepsilon_{t+1} = 0$. Then $b_{t+1} = a^{-t-1}x_{t+1}$ so that

$$E_t b_{t+1} = a^{-t-1}E_t x_{t+1} = a^{-1}a^{-t}x_t = a^{-1}b_t. \tag{26.32}$$

We see that the process (26.31) satisfies the criterion for a rational bubble. For a financial asset this shows that a rational bubble can be closely related to the dividend process. This is one of the reasons why it is difficult to empirically disentangle rational bubbles from movements in market fundamentals (see Froot and Obstfeld, 1991).

3. *Extrinsic bubbles.* An extrinsic bubble on an asset is a bubble that depends on a stochastic variable which has no connection whatsoever with fundamentals in the economy. This kind of stochastic variables was termed "sunspots" by Cass and Shell (1983), using a metaphorical expression. Let z_t be an example of such a variable and assume z_t is a martingale. Then the process

$$b_t = a^{-t}z_t, \tag{26.33}$$

satisfies the criterion of a rational bubble in that (26.32) holds with x replaced by z. So stochastic variables which are basically irrelevant from a

> strict economic point of view may still have an impact on the economy if only people believe they do or if only every individual believes that most others believe it. The actual level of sunspot activity can be thought of as an economically irrelevant stochastic variable that nevertheless ends up affecting economic behavior.[10] If people believe that this variable has an impact on the course of the economy, this belief may be self-fulfilling.

The hypothesis of extrinsic bubbles has been applied to cases where multiple rational expectations equilibria may exist (like in Diamond's OLG model). In such cases it is possible that agents condition their expectations on some extrinsic phenomenon like the sunspot cycle. In this way expectations may become coordinated such that the resulting aggregate behavior validates the expectations. These notions have proved useful in particular in the case $|a| > 1$ and we shall briefly return to them at the end of the next section.

26.3 Solutions when $|a| > 1^*$

Although $|a| < 1$ is the most common case in economic applications, there exist economic examples where $|a| > 1$.[11] In this case the expected future has "large influence". Generally, there will then be no fundamental solution because the right-hand side of (26.8) will normally equal $\pm\infty$. On the other hand, there are infinitely many non-explosive solutions. Indeed, Proposition 2 still holds, since they were derived independently of the size of a. Any possible bubble component b_t will still satisfy $E_t b_{t+1} = a^{-1} b_t$, but now we get $\lim_{i\to\infty} E_t b_{t+i} = 0$, in view of $|a| > 1$. Consequently, instead of an explosive bubble component we have an implosive one (which is therefore not usually termed a bubble any longer).

Let us consider the case where x_t has constant mean, i.e.,

$$y_t = a\ E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad |a| > 1, \tag{26.34}$$

where $E_t \varepsilon_{t+i} = 0$ for $i = 1, 2, ...$ An educated guess (cf. Appendix B) is that the process

$$\tilde{y}_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t \tag{26.35}$$

satisfies (26.34). That this is indeed a solution is seen by shifting (26.35) one period ahead and taking the conditional expectation: $E_t \tilde{y}_{t+1} = c\bar{x}/(1-a)$. Mul-

[10] In fact, since the level of actual sunspot activity may influence the temperature at the Earth and thereby economic conditions, the sunspot metaphor chosen by Cass and Shell was not particularly felicitous.

[11] See, e.g., Taylor (1986, p. 2009) and Blanchard and Fischer (1989, p. 217).

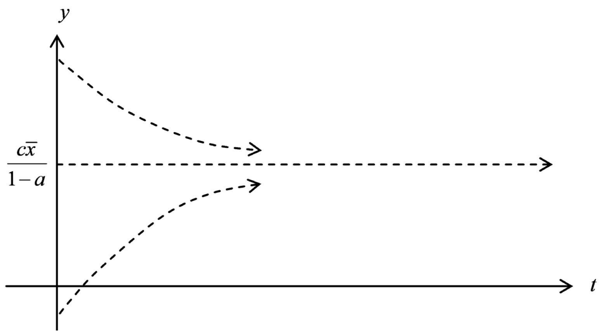

Figure 26.4: Deterministic implosive "bubbles" (the case $a > 1$, $c < 0$, and $x_t = \bar{x}$).

tiplying by a and adding $c(\bar{x} + \varepsilon_t)$ gives

$$aE_t\tilde{y}_{t+1} + c(\bar{x} + \varepsilon_t) = \frac{ac\bar{x} + (1-a)c\bar{x}}{1-a} + c\varepsilon_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t = \tilde{y}_t,$$

which shows that $\tilde{y}_t$ satisfies (26.34).

With this $\tilde{y}_t$ and the process b_t given by (26.14) we have from Proposition 2 that

$$y_t = \frac{c\bar{x}}{1-a} + b_t + c\varepsilon_t \tag{26.36}$$

is also a solution of (26.34). By backward substitution in (26.14) the bubble component b_t can be written as

$$b_t = \sum_{i=0}^{t-1} a^{-i} u_{t-i} + a^{-t} b_0. \tag{26.37}$$

If for example u_t is white noise, this shows that the bubble will gradually die out over time. And if also ε_t is white noise, we see that, as $t \to \infty$, y_t converges towards $c\bar{x}/(1-a)$ except for white noise. If $u_t \equiv 0$ and $\varepsilon_t \equiv 0$, we get again the formula (26.22) which now implies converging paths as illustrated in Fig. 20.2 (for the case $a > 1$, $c < 0$).[12]

Two theoretical implications should be mentioned. On the one hand, the lack of uniqueness (which follows from the fact that y_0 is a forward-looking variable) is much more "troublesome" in this case than in the case $|a| < 1$. When $|a| < 1$, imposing the restriction that the solution be non-explosive (say because of a

[12] The fact that $|a| > 1$ is associated with convergence may seem confusing if one is more accustomed to difference equations on *backward*-looking form. Appendix C relates our forward-looking form to the backward-looking form, common in natural science and math textbooks. The relationship to the associated concepts of *characteristic equation* and *stable* and *unstable roots* is exposed.

transversality condition or some other restriction) removes the ambiguity. But when $|a| > 1$, this is no longer so. As Fig. 20.4 indicates, when $|a| > 1$, there are infinitely many non-explosive solutions. On the other hand, exactly this feature opens up for the existence of non-explosive equilibrium paths with stochastic fluctuations driven by random events that *per se* have no connection whatsoever with fundamentals in the economy. The theory of *extrinsic bubbles* ("sunspot equilibria") has mainly been applied to this case ($|a| > 1$). The hypothesis is that in situations with multiple rational expectations equilibria it may happen that some extraneous stochastic phenomenon *de facto* becomes a coordination device. If people believe that this particular phenomenon has an impact on the economy, then it may end up having an impact due to the behavior induced by the associated conditional expectations. It turns out that when strong nonlinearities are present, cases like $|a| > 1$ may arise. These mechanisms have relevance for business cycle theory and have affinity with themes from Keynes like "animal spirits", "self-justifying beliefs", and "expectations volatility".

26.4 Concluding remarks

(two versions, not integrated)

We have only scratched the surface of bubble theory. Brunnermeier (2008) provides a lexical state-of-the-art account of different types of bubbles and theories about the conditions needed for their occurrence.

The empirical evidence concerning asset price bubbles in general and rational asset price bubbles in particular seems inconclusive. It is very difficult to statistically distinguish between bubbles and mis-specified fundamentals. And rational bubbles can have much more complicated forms than the bursting bubble in Example 3 above. For example Evans (1991) and Hall et al. (1999) study "regime-switching" rational bubbles.

Whatever the possible limits to the plausibility of rational bubbles in asset prices, it is useful to be aware of their logical structure and the variety of forms they can take as logical possibilities. Rational bubbles may serve as a benchmark for a variety of "behavioral asset price bubbles", i.e., bubbles arising through particular psychological mechanisms. This would take us to *behavioral finance* theory. The reader is referred to, e.g., Shiller (2003) and Thaler (   ).

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This chapter has studied forward-looking rational expectations giving rise to expectational difference equations of the form $y_t = aE_t y_{t+1} + cx_t$. The case $|a| < 1$ is the most common in macroeconomics. In that case there is only one solution which in expected value n periods ahead does not explode for n going to infinity. This is the *fundamental solution.* On the other hand there are infinitely many solutions which in expected value n periods ahead explode for n going to infinity, the *bubble solutions*. When conditions in the model as a whole allow us to rule out the latter, we are left with the fundamental solution. In the next chapter, we will apply the fundamental solution to a series of New Classical and Keynesian models with forward-looking expectations.

We have considered cases where, if not already from a partial equilibrium point of view, then at least from an general equilibrium point of view, rational asset price bubbles seem unlikely to occur. The latter theme is further explored in Chapter 27.

The empirical evidence concerning asset price bubbles in general and rational asset price bubbles in particular seems inconclusive. It is very difficult to statistically distinguish between bubbles and mis-specified fundamentals. Rational bubbles can also have quite complicated forms. For example Evans (1991) and Hall et al. (1999) study "regime-switching" rational bubbles.

Whatever the possible limits to the emergence of rational bubbles in asset prices, it is useful to be aware of their logical structure and the variety of forms they can take as logical possibilities. Rational bubbles may serve as a benchmark for the analytically harder cases of "irrational asset price bubbles", i.e., bubbles arising when a significant fraction of the market participants do not behave in accordance with the efficient market hypothesis. This would take us to *behavioral finance* theory.

Some of the economic models considered in the next chapter lead to more complicated expectational difference equations than above. An example is the equation $y_t = a_1 E_{t-1} y_t + a_2 E_t y_{t+1} + c\ x_t$. Here forward-looking expectations as well as past expectations of current variables enter into the determination of y_t. As we will see, however, a solution method based on repeated forward substitution can still be used.

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NOTES:

Long-Term Capital Management (LTCM): Members of LTCM's board of directors included Myron S. Scholes and Robert C. Merton, who shared the 1997 Nobel Memorial Prize in Economic Sciences for a "new method to determine the value of derivatives". LTCM collapsed in 1998.

List of historical examples of bubble phenomena to be included.

26.5 Literature notes

(incomplete)

The exposition in section 26.2 is much in debt to Blanchard and Fischer (1989, Chapter 5.1).

For solution methods to more complex expectational difference equations than considered in the text above, the reader is referred to Blanchard and Fischer (1989, Chapter 5, Appendix), Obstfeld and Rogoff (1996), and Gourieroux and Monfort (1997).

Sometimes foreign exchange is added to the list of assets on which rational bubbles are possible. For a collection of theoretical and empirical studies of this candidate, see ...

Flood and Garber (1994).

Tirole, 1982, 1985.

Shleifer, A., 2000, *Efficient Markets: An Introduction to Behavioral Finance*, OUP.

Shleifer, A., and R.W. Vishny, 1997, The limits to arbitrage, *Journal of Finance* 52 (1), 35-55.

For surveys on the theory of rational bubbles and econometric bubble tests, see Salge (1997), Brunnermeier (2008), and Gürkaynak (2008). For discussions of famous historical bubble episodes, see the symposium in *Journal of Economic Perspectives* 4, No. 2, 1990, and Shiller (2005).

Abreu and Brunnermeier (2003) develops a theory of the emergence of bubbles when rational arbitrageurs interact with boundedly rational behavioral traders. A lexical overview of bubble theory is given in Brunnermeier (2008).

The survey by LeRoy (2004) concludes in favor of the tenet that rational bubbles help explain what appears as excess volatility in asset prices.

For discussions of "animal spirits", "self-justifying beliefs", and "expectations volatility", see Keynes (1936, Ch. 12), Farmer (1993), Guesnerie (2001), and Akerlof and Shiller ().

On economic forecasting in practice see Hendry and Clements (1999).

26.6 Appendix

A. Proof of (26.29)

We shall show that if $\lim_{n\to\infty}\sum_{i=1}^{n}(\Pi_{j=0}^{i-1}a_{t+j})c_{t+i}E_t x_{t+i}$ exists, then (26.27) has the solution (26.29). Replace t by $t+n+1$ in (26.29) to get

$$\begin{aligned} y_{t+n+1} &= c_{t+n+1}x_{t+n+1} + \sum_{i=1}^{\infty}(\Pi_{j=0}^{i-1}a_{t+n+1+j})c_{t+n+1+i}E_{t+n+1}x_{t+n+1+i} \Rightarrow \\ E_t y_{t+n+1} &= c_{t+n+1}E_t x_{t+n+1} + \sum_{i=1}^{\infty}(\Pi_{j=0}^{i-1}a_{t+n+1+j})c_{t+n+1+i}E_t x_{t+n+1+i}. \end{aligned} \quad (26.38)$$

Define the "discount factor" D_k by

$$D_k = \Pi_{j=0}^{k-1} a_{t+j}, \quad \text{for } k = 1, 2, \ldots.$$

Multiplying by $D_{n+1} = \Pi_{j=0}^{n} a_{t+j}$ on both sides in (26.38) gives

$$\begin{aligned} D_{n+1} E_t y_{t+n+1} &= \Pi_{j=0}^{n} a_{t+j} \left(c_{t+n+1} E_t x_{t+n+1} + \sum_{i=1}^{\infty} (\Pi_{j=0}^{i-1} a_{t+n+1+j}) c_{t+n+1+i} E_t x_{t+n+1+i} \right) \\ &= D_{n+1} c_{t+n+1} E_t x_{t+n+1} + \sum_{i=1}^{\infty} (\Pi_{j=0}^{n+i} a_{t+n+1+j}) c_{t+n+1+i} E_t x_{t+n+1+i} \\ &= D_{n+1} c_{t+n+1} E_t x_{t+n+1} + \sum_{i=1}^{\infty} D_{n+i+1} c_{t+n+1+i} E_t x_{t+n+1+i} \\ &= \sum_{k=n+1}^{\infty} D_k c_{t+k} E_t x_{t+k}. \end{aligned} \tag{26.39}$$

In view of (26.28) it is enough to show that (26.29) implies $\lim_{n\to\infty} D_{n+1} E_t y_{t+n+1} = 0$. By (26.39) this is equivalent to showing that $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} D_k c_{t+k} E_t x_{t+k} = 0$. We have

$$\begin{aligned} \sum_{k=1}^{\infty} D_k c_{t+k} E_t x_{t+k} &= \sum_{k=1}^{n} D_k c_{t+k} E_t x_{t+k} + \sum_{k=n+1}^{\infty} D_k c_{t+k} E_t x_{t+k} \Rightarrow \\ \sum_{k=n+1}^{\infty} D_k c_{t+k} E_t x_{t+k} &= \sum_{k=1}^{\infty} D_k c_{t+k} E_t x_{t+k} - \sum_{k=1}^{n} D_k c_{t+k} E_t x_{t+k} \Rightarrow \\ \lim_{n\to\infty} \sum_{k=n+1}^{\infty} D_k c_{t+k} E_t x_{t+k} &= \sum_{k=1}^{\infty} D_k c_{t+k} E_t x_{t+k} - \sum_{k=1}^{\infty} D_k c_{t+k} E_t x_{t+k} = 0, \end{aligned}$$

which was to be proved.

B. Repeated backward substitution

When $|a| > 1$, a *particular* solution, $\tilde{y}_t$, of our basic equation

$$y_t = a E_t y_{t+1} + c x_t, \qquad t = 0, 1, 2, ... \tag{26.40}$$

can often be found as a perfect-foresight solution constructed by repeated backward substitution. We will examine whether (26.40) has a solution with perfect foresight. We substitute $y_{t+1} = E_t y_{t+1}$ into (26.40) and write the resulting equation on backward-looking form:

$$y_{t+1} = \frac{1}{a} y_t - \frac{c}{a} x_t. \tag{26.41}$$

Repeated backward substitution gives

$$y_{t+1} = \left(\frac{1}{a}\right)^n y_{t+1-n} - c\left[\left(\frac{1}{a}\right)^n x_{t-n+1} + \left(\frac{1}{a}\right)^{n-1} x_{t-n+2} + \ldots + \frac{1}{a} x_t\right],$$

for $n = 1, 2, \ldots$. By letting $n \to \infty$ in this expression we see that a reasonable *guess* of a particular solution of (26.40) is

$$\tilde{y}_{t+1} = -c \sum_{i=1}^{\infty} (\frac{1}{a})^i x_{t+1-i}, \tag{26.42}$$

if this sum converges (by replacing t by $t-1$, we get the corresponding formula for $\tilde{y}_t$). By (26.42) follows that $E_t \tilde{y}_{t+1} = -c \sum_{i=1}^{\infty} (\frac{1}{a})^i x_{t+1-i} = \tilde{y}_{t+1}$, which corresponds to perfect foresight (reflecting that, by (26.42), $\tilde{y}_{t+1}$ is completely determined by past events which are included in the information on the basis of which expectation is formed in the preceding period). Hence, (26.42) implies

$$\begin{aligned}
\tilde{y}_{t+1} &= E_t \tilde{y}_{t+1} = -c\frac{1}{a} x_t - c \sum_{i=2}^{\infty} (\frac{1}{a})^i x_{t+1-i} = \frac{1}{a}\left(-c x_t - c \sum_{i=2}^{\infty} (\frac{1}{a})^{i-1} x_{t+1-i}\right) \\
&= \frac{1}{a}\left(-c x_t - c \sum_{i=1}^{\infty} (\frac{1}{a})^i x_{t-i}\right) = \frac{1}{a}\left(-c x_t + \tilde{y}_t\right), \quad \text{so that} \\
\tilde{y}_t &= a E_t \tilde{y}_{t+1} + c x_t.
\end{aligned}$$

The process (26.42) therefore satisfies (26.40) and our guess is correct.

Consider the special case (26.34). Here (26.42) takes the form $\tilde{y}_{t+1} = -c\left[\frac{\bar{x}}{a-1} + \sum_{i=1}^{\infty} (\frac{1}{a})^i \varepsilon_{t+1-i}\right.$ where we can replace t by $t-1$. This is the background for the "educated guess", made in the main text, that also the simpler process (26.35) is a (particular) solution of (26.34).

C. The relationship between unstable roots and uniqueness of a converging solution

In the main text we considered stochastic first-order difference equations written on a *forward*-looking form. In math textbooks difference equations are usually written on a *backward*-looking form, suitable for the natural sciences. Concepts such as the *characteristic equation* and *stable* and *unstable roots* are associated with this backward-looking form. There is a link between these concepts and the question of uniqueness or non-uniqueness of a convergent solution to a forward-looking difference equation.

To clarify, we will for simplicity ignore uncertainty. That is, we assume expected values are always realized. Then the forward-looking form (26.40) reads $y_t = a\ y_{t+1} + c\ x_t$. The corresponding backward-looking form is

$$y_{t+1} - \frac{1}{a}y_t = -\frac{c}{a}x_t, \quad (a \neq 0),$$

or

$$y_{t+1} + ky_t = mx_t. \tag{26.43}$$

This is the standard form for a linear first-order difference equation with constant coefficient $k = -1/a$ and time-dependent right-hand side equal to mx_t, where $m \equiv -c/a$. The *homogeneous* difference equation corresponding to (26.43) is $y_{t+1} + ky_t = 0$, to which corresponds the characteristic equation $\rho + k = 0$. The characteristic root is $\rho = -k$ $(= 1/a)$. Any solution to this difference equation can be written

$$y_t = \tilde{y}_t + C\rho^t, \tag{26.44}$$

where $\tilde{y}_t$ is a particular solution of (26.43) and C is a constant depending on the initial value, y_0. If, for example $x_t = \bar{x}$ for all t, then (26.43) becomes

$$y_{t+1} + ky_t = m\bar{x},$$

and a particular solution is the stationary state

$$\tilde{y}_t = \frac{m\bar{x}}{1+k}. \qquad (k \neq -1)$$

By substitution into (26.44) we get $C = y_0 - m\bar{x}/(1+k)$. Hence, the general solution is

$$y_t = \frac{m\bar{x}}{1+k} + (y_0 - \frac{m\bar{x}}{1+k})\rho^t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})(\frac{1}{a})^t. \tag{26.45}$$

This is the same as (26.22).

Now define case A and case B in the following way:

Case A: $|\rho| < 1$, that is, $|a| > 1$.
Case B: $|\rho| > 1$, that is, $|a| < 1$.

The solution formula (26.45) shows that in case A all solutions converge. In this case the characteristic root is called a *stable root*. In case B the solution diverges unless $y_0 = c\bar{x}/(1-a)$. The characteristic root ρ is in case B called an *unstable root*.[13] Which of the two cases the researcher typically finds most "convenient" depends on whether y_0 is a predetermined or a jump variable:

I. y_0 being predetermined.
Case A: $|\rho| < 1$. The solution for y_t is unique and converges for every y_0.
Case B: $|\rho| > 1$. The solution for y_t is unique but does not converge when $y_0 \neq \frac{c\bar{x}}{1-a}$.

II. y_0 being a jump variable.
Case A: $|\rho| < 1$. Even if we can impose the restriction that y_t must converge, y_0 is not uniquely determined.
Case B: $|\rho| > 1$. If we can impose the restriction that y must converge, y_0 is uniquely determined as $y_0 = \frac{c\bar{x}}{1-a}$.

Hence, the cases I.A and II.B are the more "convenient" ones from the point of view of a researcher preferring unique solutions.

The question of multiplicity of solutions is harder in the case of a *non-linear* expectational difference equation. In this case, even if a condition corresponding to $|a| < 1$ is satisfied close to the steady state, there may be more than one non-explosive solution (for an example, see Blanchard and Fischer, 1989, Ch. 5, and the references therein).

In the appendix to Chapter 27 these matters are generalized to *systems* of first-order difference equations.

26.7 Exercises

25.1 *The housing market in an old city quarter (partial equilibrium analysis)* Consider the housing market in an old city quarter with unique amenity value (for convenience we will speak of "houses" although perhaps "apartments" would fit real world situations better). Let H be the aggregate stock of houses (apartments), measured in terms of some basic unit (a house of "normal size", somehow adjusted for quality) existing at a given point in time. No new construction is allowed, but repair and maintenance is required by law and so H is constant through time. Notation:

p_t = the real price of a house (stock) at the beginning of period t,
m = real maintenance costs of a house (assumed constant over time),
$\tilde{R}_t$ = the real rental rate, i.e., the price of housing services (flow), in period t,
R_t = $\tilde{R}_t - m$ = the *net* rental rate = net revenue to the owner per unit of housing services in period t

[13] In the case $|\rho| = 1$ we have: if $\rho = -1$, the conclusion is as in case B; if $\rho = 1$, then $y_t = y_0 - c\bar{x}t$. Being a "knife-edge case", however, $|\rho| = 1$ is usually less interesting.

Let the housing services in period t be called S_t. Note that S_t is a *flow:* so and so many square meter-months are at the disposal for utilization (accommodation) for the owner or tenant during period t. We assume the rate of utilization of the house stock is constant over time. By choosing proper measurement units the rate of utilization is normalized to 1, and so $S_t = 1 \cdot H$. The prices p_t, m, and R_t are measured in *real* terms, that is, deflated by the consumer price index. We assume perfect competition in both the market for houses and the market for housing services.

Suppose the aggregate demand for housing services in period t is

$$D(\tilde{R}_t, X_t), \qquad D_1 < 0, D_2 > 0, \tag{*}$$

where the stochastic variable X_t reflects factors that in our partial equilibrium framework are exogenous (for example present value of expected future labor income in the region).

a) Set up an equation expressing equilibrium at the market for housing services. In a diagram in $(H, \tilde{R})$ space, for given X_t, illustrate how $\tilde{R}_t$ is determined.

b) Show that the equilibrium *net* rental rate at time t can be expressed as an implicit function of H, X_t, and m, written $R_t = \mathcal{R}(H, X_t, m)$. Sign the partial derivatives w.r.t. H and m of this function. Comment.

Suppose a constant tax rate $\tau_R \in [0, 1)$ is applied to rental income, after allowance for maintenance costs. In case of an owner-occupied house the owner still has to pay the tax $\tau_R R_t$ out of the implicit income, R_t, per house per year. Assume further there is a constant property tax rate $\tau_p \geq 0$ applied to the market value of houses. Finally, suppose a constant tax rate $\tau_r \in [0, 1)$ applies to interest income, whether positive or negative. We assume capital gains are not taxed and we ignore all complications arising from the fact that most countries have tax systems based on nominal income rather than real income. In a low-inflation world this limitation may not be serious.

We assume housing services are valued independently of whether the occupant owns or rents. We further assume that the market participants are risk-neutral and that transaction costs can be ignored. Then in equilibrium,

$$\frac{(1-\tau_R)R_t - \tau_p p_t + p_{t+1}^e - p_t}{p_t} = (1-\tau_r)r, \tag{**}$$

where p_{t+1}^e denotes the expected house price next period as seen from period t, and r is the real interest rate in the loan market. We assume $r > 0$ and all tax rates are constant over time.

c) Interpret (**).

Assume from now the market participants have rational expectations (and know the stochastic process which R_t follows as a consequence of the process of X_t).

d) Derive the expectational difference equation in p_t implied by (**).

e) Find the fundamental value of a house, assuming R_t does not grow "too fast". *Hint:* write (**) on the standard form for an expectational difference equation and use the formula for the fundamental solution.

Denote the fundamental value p_t^*. Assume R_t follows the process

$$R_t = \bar{R} + \varepsilon_t, \qquad (***)$$

where $\bar{R}$ is a positive constant and ε_t is white noise with variance σ^2.

f) Find p_t^* under these conditions.

g) How does $E_{t-1}p_t^*$ (the conditional expectation one period beforehand of p_t^*) depend on each of the three tax rates? Comment.

h) How does $Var_{t-1}(p_t^*)$ (the conditional variance one period beforehand of p_t^*) depend on each of the three tax rates? Comment.

25.2 *A housing market with bubbles (partial equilibrium analysis)* We consider the same setup as in Exercise 25.1, including the equations (*), (**), and (***).

Suppose that until period 0 the houses were owned by the municipality. But in period 0 the houses are sold to the public at market prices. Suppose that by coincidence a large positive realization of ε_0 occurs and that this triggers a stochastic bubble of the form

$$\begin{aligned} b_{t+1} &= [1 + \tau_p + (1 - \tau_r)r]\, b_t + \varepsilon_{t+1}, \qquad t = 0, 1, 2, ..., \qquad (\hat{\ }) \\ \text{where} \quad E_t\varepsilon_{t+1} &= 0 \text{ and } b_0 = \varepsilon_0 > 0. \end{aligned}$$

Until further notice we assume b_0 is large enough relative to the stochastic process $\{\varepsilon_t\}$ to make the probability that b_{t+1} becomes non-positive negligible.

a) Can (^) be a rational bubble? You should answer this in two ways: 1) by using a short argument based on theoretical knowledge, and 2) by directly testing whether the price path $p_t = p_t^* + b_t$ is arbitrage free. Comment.

b) Determine the value of the bubble in period t, assuming ε_{t-i} known for $i = 0, 1, ..., t$.

c) Determine the market price, p_t, and the conditional expectation $E_t p_{t+1}$. Both results will reflect a kind of "overreaction" of the market price to the shock ε_t. In what sense?

d) It may be argued that a stochastic bubble of the described ever-lasting kind does not seem plausible. What kind of arguments could be used to support this view?

e) Still assuming $b_0 > 0$, construct a rational bubble which has a constant probability of bursting in each period $t = 1, 2, ...$

f) What is the expected further duration of the bubble as seen from any period $t = 0, 1, 2, ...,$ given $b_t > 0$? *Hint:* $\sum_{i=0}^{\infty} iq^i(1-q) = q/(1-q)$.[14]

g) If the bubble is alive in period t, what is the probability that the bubble is still alive in period $t+s$, where $s = 1, 2, ...$? What is the limit of this probability for $s \rightarrow \infty$?

h) Assess this last bubble model.

i) Housing prices are generally considered to be a good indicator of the turning points in business cycles in the sense that house prices tend to move in advance of aggregate economic activity, in the same direction. In the language of business cycle analysts housing prices are a *procyclical leading indicator*. Do you think this last bubble model fit this observation? *Hint:* consider how a rise in p affects residential investment and how this affects the economy as a whole.

[14] Here is a proof of this formula. $\sum_{i=0}^{\infty} iq^i(1-q) = (1-q)q\sum_{i=0}^{\infty} iq^{i-1} = (1-q)q\sum_{i=0}^{\infty} dq^i/dq$ $= (1-q)qd\left(\sum_{i=0}^{\infty} q^i\right)/dq = (1-q)qd\,(1-q)^{-1}/dq = (1-q)q\,(1-q)^{-2} = q\,(1-q)^{-1}$. $\square$